On the Uniqueness of a Certain Thin Octagon (or Partial 2-Geometry, or Parallelism) Derived from the Binary Golay Code

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Abstract—The question of the uniqueness of a certain combinatorial structure has arisen in three contexts: a) is the regular near octagon with parameters \((s, t_1, t_2, t_3, t_4) = (1, 1.2, 2, 3)\) unique [5]? b) is the partial 2-geometry with nexus three and blocks of size 24 unique [2]? c) is there a unique graph such that it is the graph of a parallelism of \(\binom{24}{4}\) with respect to any vertex [1]? We observe that these questions are equivalent and give an affirmative answer. In fact, we prove a more general theorem, showing the truth of a conjecture by Cameron.

I. INTRODUCTION

For the definition of near \(n\)-gon (see [5], [6]). We shall only use thin regular near \(2d\)-gons which are nothing but distance regular bipartite graphs of diameter \(d\). If \(d(x, y) = i\) then let \(c_i = |\{z \mid z \sim y\}\) and \(d(z, x) = i - 1\}\) (where \(\sim\) denotes adjacency). By distance regularity, \(c_i\) does not depend on the choice of \(x\) and \(y\). Write \(k = c_d\), the valency of the graph.

A partial \(\lambda\)-geometry with nexus \(e\) is an incidence structure with \(b\) blocks and \(v\) points which satisfies:

a) two distinct points are on 0 or \(\lambda\) common blocks;

b) two distinct blocks have 0 or \(\lambda\) common points;

c) given a point \(x\) not on a block \(B\), there are precisely \(e\) blocks on \(x\) meeting \(B\);

d) no point is on all blocks and no block contains all points;

e) all blocks have size \(k\) and all points are in \(r\) blocks, where \(k, r > \lambda\), (see [1], [2]).

If \(\lambda > 1\) then one may weaken e) to each block has more than \(\lambda\) points and each point is on more than \(\lambda\) blocks and obtain the existence of \(k\) and \(r\) and moreover that \(k = r\). Assume \(\lambda > 1\) and \(e < k\) (when \(e = k\) we have a symmetric BIBD).

One immediately verifies that the bipartite graph with points and blocks as vertices and incidence as adjacency is distance regular with \((c_1, c_2, c_3, c_4) = (1, \lambda, e, k)\), i.e., we have a thin regular near octagon. The converse also being clear we see that the concepts of thin regular near 8-gon (with \(c_2 > 1\)) and partial \(\lambda\)-geometry with nexus \(e\) (with \(\lambda > 1\) and \(e < k\)) are equivalent.

II. UNIQUENESS

Let \(\Gamma\) be a distance regular bipartite graph with diameter \(d \geq 4\) and \(c_i = i (1 \leq i \leq d - 1)\).

Lemma: Let 2\(j\) be a \(j\)-cube (vertices: binary vectors of length \(j\), edges: pairs of vertices differing by a unit vector) and \(\pi\) be a map sending 0 to \(x_0 \in \Gamma\) and the unit vectors to (distinct) neighbors of \(x_0\) in \(\Gamma\). Then \(\pi\) can be extended in a unique way to a map \(\pi: 2^j \rightarrow \Gamma\) preserving adjacency and squares. Moreover, \(\pi\) is injective when restricted to \((2d - 1)\)-cubes.

Proof (cf. [1, th. 5.11 (i)]): First observe that two points at distance three in \(\Gamma\) determine a 3-cube: each has three neighbors at distance two from the other, and on these six points we have a regular bipartite graph of valency two, hence a hexagon, completing the cube. It follows easily that a 2-claw determines a square and a 3-claw a 3-cube.

We define \(\pi(v)\) by induction on the number of nonzero coordinates (the weight of \(v\). If the weight is zero or one \(\pi(v)\) is prescribed. If the weight is two so that \(e = e'\) for two unit vectors \(e, e'\), then let \(\pi(v)\) be the common neighbor of \(\pi(e)\) and \(\pi(e')\) distinct from \(\pi(0)\).

Suppose \(\pi(v)\) defined for vectors \(v\) of weight less than \(i\), and let \(v\) have weight \(i \geq 3\): \(v = \sum_{h=1}^{i} e_h\) for distinct unit vectors \(e_h\). Then we can define \(\pi(v)\) as the common neighbor of \(\pi(v - e_1)\) and \(\pi(v - e_2)\) distinct from \(\pi(v - e_1 - e_2)\). For \(h > 2\) the points \(\pi(v)\) and \(\pi(v - e_1 - e_2 - e_h)\) have distance three, hence determine a 3-cube, and we find that \(\pi(v)\) is adjacent to \(\pi(v - e_h)\) as was required. Thus, using only \(c_2 = 2\) and \(c_3 = 3\) we find a unique map \(\pi\) satisfying all requirements and injective on 3-cubes. If we
also have \( c_i = i \) \((2 \leq i \leq h)\) then one easily sees that \( \pi \) is still injective on \((2h + 1)\)-cubes: if two vectors have the same image then by choosing the origin appropriately one may suppose that they have disjoint supports of equal weights at most \( h \) (note that \( \Gamma \) is bipartite and \( 2^j \) is connected so that vectors with the same image have even distance), but inside a radius of \( h \) nothing can happen.

Now let \( k \) be the valency of \( \Gamma \) and apply the Lemma to find a labeling of \( \Gamma \) with vectors from \( 2^k \). The number of vertices of \( \Gamma \) is

\[
|V(\Gamma)| = 1 + k + \binom{k}{2} + \cdots + \binom{k}{d - 1} + \frac{d}{k} \binom{k}{d},
\]

so that on the average each point gets \( 2^k / |V(\Gamma)| \) labels. But the collection of labels of a point \( x \) is a binary code \( \mathcal{C}_x \) with word length \( k \) and minimum distance at least \( 2d \), and this average is just the trivial upper bound for the cardinality of such a code when one observes that spheres of radius \( d - 1 \) around codewords are disjoint, and a vector can have distance \( d \) to at most \( k/d \) distinct codewords. Consequently each point has exactly \( 2^k / |V(\Gamma)| \) labels, and each \( \mathcal{C}_x \) is an extended perfect code. By the classification of perfect codes (see e.g., [4, ch. 6, th. 33]) it follows that we have one of the following three possibilities:

\begin{enumerate}
  \item \( |\mathcal{C}_x| = 1 \), the code is trivial. Now \( d = k \), \( |V(\Gamma)| = 2^k \) and we have a \( k \)-cube.
  \item \( |\mathcal{C}_x| = 2 \), a repetition code. Now \( d = \frac{1}{2} k \), \( |V(\Gamma)| = 2^{k - 1} \) and we have a half \( k \)-cube (i.e., a \( k \)-cube with antipodal vertices identified).
  \item \( |\mathcal{C}_x| = 2^{12} \), a code isomorphic to the extended binary Golay code \( \mathcal{C} \). Now \( k = 24 \), \( d = 4 \) and all codes \( \mathcal{C}_x \) are translates of each other (for: suppose \( x \sim y \). For each \( v \in \mathcal{C}_x \) there is a unique \( v' \in \mathcal{C}_y \) with \( d(v, v') = 1 \). The map \( v \to v' \) changes distances by at most two, but since all distances are multiples of four it must preserve distances. Considering triples of vectors of weight \( 8 \ u, v, w \) in \( \mathcal{C}_x \) such that \( u + v + w \) is the all-one vector one sees that this map is a translation over a vector of weight one). Our graph \( \Gamma \) is the graph with as vertices \( 2^{12} \) cosets of \( \mathcal{C} \), and two vertices are adjacent if and only if the corresponding cosets contain vectors differing by a unit vector.
\end{enumerate}

This settles question 5.7 in [1, p. 94]—there also \( d = 3 \) is allowed, but in this case it is given that 3-claws generate 3-cubes and our reasoning still works.

Let us formulate a theorem.

**Theorem:** Let \( \Gamma \) be a distance regular bipartite graph with diameter \( d \geq 3 \), valency \( k \) and parameters \( c_i = i \) \((1 \leq i \leq d + 1)\). If \( d = 3 \) then suppose additionally that any 3-claw generates a 3-cube (in the sense of [1, p. 91]). Then \( \Gamma \) is a \( d \)-cube, a half \( 2d \)-cube, or we have \( d = 4 \), \( k = 24 \), and \( \Gamma \) is the graph of the sextet parallelism (or, in other words, is the thin near octagon derived from the extended binary Golay code as described above).

**References**


