

# ONE-DIMENSIONAL GALERKIN METHODS AND SUPERCONVERGENCE AT INTERIOR NODAL POINTS\*

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**Abstract.** In the case of one-dimensional Galerkin methods the phenomenon of superconvergence at the knots has been known for years [5], [7]. In this paper, a minor kind of superconvergence at specific points inside the segments of the partition is discussed for two classes of Galerkin methods: the Ritz-Galerkin method for  $2m$ th order self-adjoint boundary problems and the collocation method for arbitrary  $m$ th order boundary problems. These interior points are the zeros of the Jacobi polynomial  $P_n^{m,m}(\sigma)$  shifted to the segments of the partition;  $n = k + 1 - 2m$ , where  $k$  is the degree of the finite element space. The order of convergence at these points is  $k + 2$ , one order better than the optimal order of convergence. Also, it can be proved that the derivative of the finite element solution is superconvergent of  $O(h^{k+1})$  at the zeros of the Jacobi polynomial  $P_{n+1}^{m-1,m-1}(\sigma)$  shifted to the segments of the partition. This is one order better than the optimal order of convergence for the derivative.

**Key words.** Galerkin methods, collocation methods, finite element method, superconvergence, Jacobi polynomials

**AMS MOS subject classification.** 65N30

**CR category.** 5.17

## 1. Introduction. We consider the two-point boundary problem

$$(1.1) \quad -(p(x)y')' + q(x)y = f(x), \quad x \in [-1, +1] = I, \quad y(\pm 1) = 0,$$

where  $p(x) > 0$ ,  $q(x) \geq 0$  and  $f(x)$  are sufficiently smooth. Let

$$(1.2) \quad \begin{aligned} \Delta &= \{-1 = x_0 < x_1 < \cdots < x_N = 1\}, \\ x_j &= -1 + hj, \quad j = 0, \dots, N, \quad h = 2/N, \\ I_j &= [x_{j-1}, x_j], \quad j = 1, \dots, N \end{aligned}$$

be a uniform partition of  $I$  and define  $M_0^{k,0}(\Delta)$  by

$$(1.3) \quad M_0^{k,0}(\Delta) = \{V \mid V \in C^0(I); V \in P_k(I_j), j = 1, \dots, N; V(\pm 1) = 0\}$$

where for any interval  $E$ ,  $P_k(E)$  denotes the space of polynomials of degree  $k$  restricted to  $E$ . Then the finite element approximation  $Y \in M_0^{k,0}(\Delta)$  of  $y$  is determined by

$$(1.4) \quad (pY', V') + (qY, V) = (f, V), \quad V \in M_0^{k,0}(\Delta),$$

where  $(\cdot, \cdot)$  denotes the  $L^2(I)$  inner product. It has the following convergence properties [7]

$$(1.5) \quad \begin{aligned} \|y - Y\|_l &\leq C_1 h^{k+1-l} \|y\|_{k+1}, \quad l = 0, 1, \\ |(y - Y)(x_j)| &\leq C_2 h^{2k} \|y\|_{k+1}, \quad j = 1, \dots, N-1, \end{aligned}$$

where  $C_1$  and  $C_2$  are positive constants and where

$$(1.6) \quad \begin{aligned} \|v\|_l &= \left[ \sum_{j=0}^l (D^j v, D^j v) \right]^{1/2}, \quad l \geq 0, \\ D^j v &= \frac{d^j v}{dx^j}, \quad j \geq 0. \end{aligned}$$

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Also, it is known [3] that for specific points inside  $I$ ,  $Y$  has the error bound

$$(1.7) \quad \begin{aligned} |(y - Y)(\xi_{jl})| &\leq C(y) h^{k+2}, \\ \xi_{jl} &= x_{j-1} + \frac{h}{2}(1 + \sigma_l), \quad l = 1, \dots, k-1, \quad j = 1, \dots, N, \end{aligned}$$

where  $\sigma_1, \dots, \sigma_{k-1}$  are the zeros of  $P'_k(\sigma)$  and  $P_k(\sigma)$  is the  $k$ th degree Legendre polynomial. This is one order better than the optimal error bound which is of  $O(h^{k+1})$ .

It is this phenomenon of so-called *interior superconvergence* on which we will concentrate our attention. In the next two sections, we will treat two classes of finite element methods where this occurs: the Ritz-Galerkin and the collocation method [8]. Also, we will use that superconvergence to give a new proof of the superconvergence of the derivative at other Gaussian points [9].

Before that, we give some definitions we need throughout this paper.

For any  $E \subset I$  and  $m \geq 0$ , we define

$$(1.8) \quad \begin{aligned} \|v\|_{H^m(E)} &= \left[ \sum_{l=0}^m (D^l v, D^l v)_{L^2(E)} \right]^{1/2}, \\ \|v\|_{W^m(E)} &= \sum_{l=0}^m \|D^l v\|_{L^\infty(E)}, \\ W^m(E) &= \{v | D^l v \in L^\infty(E), l = 0, \dots, m\}; \\ H^m(E) &= \{v | D^l v \in L^2(E), l = 0, \dots, m\}. \end{aligned}$$

Also, we define the  $\Delta$ -related norms

$$(1.9) \quad \begin{aligned} \|v\|_{m,\Delta} &= \left[ \sum_{j=1}^N \sum_{l=0}^m (D^l v, D^l v)_{L^2(I_j)} \right]^{1/2}, \\ \|v\|_{W^m(\Delta)} &= \max_{j=1, \dots, N} \|v\|_{W^m(I_j)}. \end{aligned}$$

Finally, throughout this paper,  $C$ ,  $C_1$ , etc. will be positive constants, not the same at each occurrence.

**2. The Ritz-Galerkin method.** Consider the  $2m$ th order two-point boundary problem

$$(2.1) \quad \begin{aligned} Lu &\equiv \sum_{l=0}^m (-1)^l D^l [p_l(x) D^l u] = f(x), \quad x \in I, \\ D^l u(\pm 1) &= 0, \quad l = 0, \dots, m-1, \end{aligned}$$

where  $p_0, \dots, p_m$  and  $f$  are such that  $u \in H^s(I)$ , for some  $s \geq 2m$ , and that there exists some  $C > 0$  with the property

$$(2.2) \quad \begin{aligned} B(v, v) &\geq C \|v\|_m^2, \quad v \in H_0^m(I), \\ B(u, v) &= \sum_{l=0}^m (p_l D^l u, D^l v), \quad u, v \in H_0^m(I), \\ H_0^m(I) &= \{v | v \in H^m(I); D^l v(\pm 1) = 0, l = 0, \dots, m-1\}; \end{aligned}$$

in other words,  $B(\cdot, \cdot)$  is strongly coercive.

For some partition  $\Delta$  of  $I$  defined by (1.2) and some integer  $k \geq 2m - 1$ , we define the finite element space

$$(2.3) \quad M_0^{k,m}(\Delta) = \{V \mid V \in H_0^m(I); V \in P_k(I_j), j = 1, \dots, N\}.$$

The solution  $u$  of (2.1) can be approximated in  $M_0^{k,m}(\Delta)$  by the solution  $U$  of the weak Galerkin form

$$(2.4) \quad B(U, V) = (f, V), \quad V \in M_0^{k,m}(\Delta).$$

If  $u \in H^{k+1}(I) \cap H_0^m(I)$ , the error function  $e = u - U$  has the bounds [2], [4]

$$(2.5) \quad \begin{aligned} \|e\|_l &\leq Ch^{k+1-l} \|u\|_{k+1}, \quad l = 0, \dots, m, \\ |D^l e(x_j)| &\leq Ch^{2r} \|u\|_{k+1}, \quad l = 0, \dots, m-1, \quad j = 1, \dots, N-1, \\ r &= k+1-m. \end{aligned}$$

What we want to prove is the fact that *inside* each segment  $I_j$  there exist  $n = k+1-2m$  distinct and specific points where  $|e(x)|$  is of  $O(h^{k+2})$ , one order better than the optimal order of convergence. This is, of course, only true, if  $n \geq 1$  or  $k \geq 2m$ . These points are shown to be the zeros of the Jacobi polynomial  $P_n^{m,m}(\sigma)$ , which will be introduced in the next section.

*Remark.* For reasons of convenience, we confine ourselves to the case that  $L$  is a self-adjoint operator and that  $\Delta$  is a uniform partition of  $I$ . The results in § 2, however, are also valid if  $\Delta$  is quasi-uniform (i.e.  $\max h_i \geq \lambda \min h_i$ ,  $\lambda$  independent of the mesh) and if  $L$  is a skew-adjoint operator of the form

$$Lu = \sum_{i=0}^m (-1)^i D^i \left[ \sum_{j=0}^i p_{ij}(x) D^j u \right],$$

provided that, of course  $B(\cdot, \cdot)$  defined by

$$B(u, v) = \sum_{i=0}^m \sum_{j=0}^i (p_{ij} D^j u, D^i v)$$

is strongly coercive.

**2.1. The Jacobi polynomial.** The Jacobi polynomial  $P_n^{\alpha,\beta}(\sigma)$  is defined by Rodrigues' formula [1] as

$$(2.6) \quad \begin{aligned} P_n^{\alpha,\beta}(\sigma) &= [w(\sigma)]^{-1} D^n [(1-\sigma^2)^n w(\sigma)] A_n^{\alpha,\beta}, \quad n \geq 0, \\ w(\sigma) &= (1-\sigma)^\alpha (1+\sigma)^\beta, \quad \alpha, \beta > -1, \end{aligned}$$

where  $A_n^{\alpha,\beta}$  is some normalizing factor, e.g., such that  $P_n^{\alpha,\beta}(1) = 1$  or  $P_n^{\alpha,\beta}(1) = (1+\alpha)(1+\alpha/2) \cdots (1+\alpha/n)$ . It has the important property

$$(2.7) \quad (w P_i^{\alpha,\beta}, P_j^{\alpha,\beta}) = \delta_{ij} (w P_i^{\alpha,\beta}, P_i^{\alpha,\beta}), \quad 0 \leq i, j,$$

where  $\delta_{ij}$  is the Kronecker symbol.

From now on, we are only interested in the case  $\alpha = \beta = m$ , where  $m$  is some nonnegative integer. In that case, we replace the double superscript  $m, m$  by the single superscript  $m$ .

LEMMA 1. Let the linear interpolation  $\Pi: C^{-1}(I) \rightarrow P_{n+2m-1}(I)$  be determined by

$$(2.8) \quad \begin{aligned} D^l(\Pi f)(\pm 1) &= D^l f(\pm 1), \quad l = 0, \dots, m-1, \\ (\Pi f)(\sigma_{in}^m) &= f(\sigma_{in}^m), \quad i = 1, \dots, n, \end{aligned}$$

where  $\sigma_{in}^m, i = 1, \dots, n$  are the zeros of  $P_n^m(\sigma)$ . Then, for  $f \in W^{2(m+n)}(I)$ , we have the approximation

$$(2.9) \quad \int_{-1}^{+1} f(\sigma) d\sigma = \int_{-1}^{+1} (\Pi f)(\sigma) d\sigma + R_{mn} D^{2r} f(\xi), \quad \xi \in (-1, +1),$$

$$R_{mn} = (-1)^m 2^{2r+1} \frac{n!(r!)^2 (2m+n)!}{[(2r)!]^3 (2r+1)},$$

$$r = m + n = k + 1 - m.$$

This is a generalization of Legendre ( $m = 0$ ) and Lobatto quadrature ( $m = 1$ ).

*Proof.* From (2.8), it follows that there exists a function  $g(\sigma)$  such that

$$(2.10) \quad (f - \Pi f)(\sigma) = (1 - \sigma^2)^m P_n^m(\sigma) g(\sigma).$$

From the orthogonality relation (2.7), we learn that

$$(f - \Pi f, I) = 0, \quad \text{if } g \in P_{n-1}(I),$$

which means that for  $f \in P_{2r-1}(I)$ , the quadrature error is zero. For any other  $f \in W^{2r}(I)$ , it is clear that

$$(2.11) \quad \int_{-1}^{+1} (f - \Pi f)(\sigma) d\sigma = \int_{-1}^{+1} (f - F) d\sigma,$$

where  $F \in P_{2r-1}(I)$  is some Hermite approximation of  $f$  which satisfies the relations (2.8) with  $\Pi f$  replaced by  $F$ . The rest of the proof follows from the theory of Ciarlet and Raviart [6]. For the evaluation of  $R_{mn}$  we refer to the appendix.  $\square$

Elaboration of (2.8) gives the formula

$$(2.12) \quad \int_{-1}^{+1} (\Pi f)(\sigma) d\sigma = \sum_{l=0}^{m-1} [\theta_{l1} D^l f(-1) + \theta_{l2} D^l f(+1)] + \sum_{i=1}^n \omega_i f(\sigma_{in}^m),$$

with

$$(2.13) \quad \omega_i = \int_{-1}^{+1} \Phi_i(\sigma) d\sigma, \quad \Phi_i(\sigma) = \frac{(1 - \sigma^2)^m P_n^m(\sigma)}{(\sigma - \sigma_{in}^m) [(1 - \sigma^2)^m dP_n^m(\sigma)/d\sigma]_{\sigma=\sigma_{in}^m}},$$

$$\theta_{li} = \int_{-1}^{+1} \psi_{li}(\sigma) d\sigma, \quad \psi_{li} \in P_k(I),$$

$$\psi_{li}(\sigma_{jn}^m) = 0, \quad l = 0, \dots, m-1, \quad i = 1, 2, \quad j = 1, \dots, n,$$

$$D^s \psi_{li}((-1)^j) = \delta_{ij} \delta_{ls}, \quad 1 \leq i, j \leq 2, \quad \xi \leq l, s \leq m-1.$$

Note that in (2.13),  $\Phi$  and  $\Psi$  are natural basis functions for Hermite interpolation and  $\sigma_{jn}^m, j = 1, \dots, n$  are the zeros of  $P_n^m(\sigma)$ .

In the next section, we will use (2.9)–(2.12) to establish superconvergence of  $O(h^{k+2})$  at the Jacobi points.

**2.2. Superconvergence at Jacobi points.** We return to problem (2.1) and its Ritz–Galerkin solution (2.4). It is standard that

$$(2.14) \quad B(e, V) = 0, \quad V \in M_0^{k,m}(\Delta).$$

For  $k \geq 2m$ , we define for any  $I_j$  the  $n$ -dimensional subspace  $S_0(I_j)$  of  $M_0^{k,m}(\Delta)$  by

$$(2.15) \quad S_0(I_j) = \{V \mid V \in H_0^m(I) \cap P_k(I_j); \text{supp}(V) = I_j\}.$$

For  $S_0(I_j)$ , a basis can be constructed, consisting of the Lagrange polynomials  $\phi_i(x)$  defined by

$$(2.16) \quad \phi_i(x) = \Phi_i(1 + 2(x - x_j)/h), \quad i = 1, \dots, n,$$

where  $\Phi_i$  is defined by (2.13).

If we apply (2.14) to  $\phi_i$ , we obtain after partial integration

$$(2.17) \quad (e, L\phi_i) = \sum_{l=1}^m \sum_{\nu=0}^{l-1} [(-1)^{\nu+1} D^{l-\nu-1} e(x) D^\nu (p_l(x) D^l \phi_i(x))]_{x_{j-1}}^{x_j}, \quad i = 1, \dots, n.$$

We now define the interior nodal points  $\xi_{jl}$  by

$$(2.18) \quad \xi_{jl} = x_{j-1} + \frac{h}{2}(1 + \sigma_{ln}^m), \quad l = 1, \dots, n,$$

where  $\sigma_{ln}^m$  is the  $l$ th zero of  $P_n^m(\sigma)$ , as defined in § 2.1.

Application of Lemma 1 to  $(e, L\phi_i)$  combined with the use of (2.17) gives

$$(2.19) \quad \begin{aligned} \frac{h}{2} \sum_{l=1}^n \omega_l e(\xi_{jl}) L\phi_i(\xi_{jl}) &= \sum_{l=1}^m \sum_{\nu=0}^{l-1} [(-1)^{\nu+1} D^{l-\nu-1} e(x) D^\nu (p_l(x) D^l \phi_i(x))]_{x_{j-1}}^{x_j} \\ &\quad - \sum_{l=0}^{m-1} \sum_{\nu=1}^2 \theta_{lv} D^l (e L\phi_i)(x_{j-2+\nu}) \left(\frac{h}{2}\right)^{l+1} \\ &\quad - \left(\frac{h}{2}\right)^{2r+1} R_{mn} [D^{2r} (e L\phi_i)]_{x=\xi \in I_j}, \quad i = 1, \dots, n, \end{aligned}$$

where  $R_{mn}$  is defined by (2.9). If we multiply both sides of (2.19) by  $2h^{2m-1}$  and apply formula (2.5), we have

$$(2.20) \quad \begin{aligned} \left| \sum_{l=1}^n [\omega_l L\phi_i(\xi_{jl}) h^{2m}] e(\xi_{jl}) \right| &\leq C_1 \sum_{l=0}^{m-1} (|D^l e(x_{j-1})| + |D^l e(x_j)|) + C_2 h^{2k+2} \|e L\phi_i\|_{W^{2r}(I_j)} \\ &\leq C_1 h^{2r} \|u\|_{k+1} + C_2 h^{k+2} \|e\|_{W^{2r}(I_j)}. \end{aligned}$$

We need an estimate of  $\|e\|_{W^{2r}(I_j)}$ . To that end, we define the projection  $\Pi_\Delta: H_0^m(I) \cap W^{k+1}(I) \rightarrow M_0^{k,m}(\Delta)$  by

$$(2.21) \quad \begin{aligned} (\Pi_\Delta u)(\xi_{jl}) &= u(\xi_{jl}), \quad j = 1, \dots, N, \quad l = 1, \dots, n, \\ D^l (\Pi_\Delta u)(x_j) &= D^l u(x_j), \quad j = 0, \dots, N, \quad l = 0, \dots, m-1. \end{aligned}$$

Then we have

$$(2.22) \quad \begin{aligned} \|e\|_{W^{2r}(I_j)} &\leq \|e\|_{W^{2r}(I_j)} + \|\delta\|_{W^k(I_j)}, \\ \varepsilon &= u - \Pi_\Delta u, \quad \delta = U - \Pi_\Delta u, \quad \delta \in M_0^{k,m}(\Delta). \end{aligned}$$

Since for any  $x \in I_j$ , we have (see [6])

$$(2.23) \quad |D^l \varepsilon(x)| \leq \begin{cases} |D^l(u)x|, & l > k, \\ Ch^{k+1-l} \|D^{k+1} u\|_{L^\infty(I_j)}, & l \leq k, \end{cases}$$

and since

$$(2.24) \quad \begin{aligned} \|\delta\|_{W^k(I_j)} &\leq Ch^{-k} \|\delta\|_{L^\infty(I_j)} \\ &\leq Ch^{-k} [\|e\|_{L^\infty(I_j)} + \|\varepsilon\|_{L^\infty(I_j)}] \quad (\text{Poincaré's inequality}) \\ &\leq Ch^{-k} [\|e\|_1 + C_1 h^{k+1} \|u\|_{W^{k+1}(I_j)}] \\ &\leq C [\|u\|_{k+1} + C_1 h \|u\|_{W^{k+1}(I_j)}], \end{aligned}$$

it turns out that

$$(2.25) \quad \|e\|_{W^{2r}(I_j)} \leq \|\varepsilon\|_{W^{2r}(I_j)} + \|\delta\|_{W^k(I_j)} \leq C[\|u\|_{k+1} + \|u\|_{W^{2r}(I_j)}]$$

and hence

$$(2.26) \quad \left| \sum_{l=1}^n [\omega_l L\phi_i(\xi_{je}) h^{2m}] e(\xi_{je}) \right| \leq Ch^{k+2}(\|u\|_{k+1} + \|u\|_{W^{2r}(I_j)}).$$

*Remark.* In (2.24), we used the property

$$h^l \|V\|_{W^l(I_j)} \leq Ch^i \|V\|_{W^i(I_j)}, \quad V \in P_k(I_j), \quad 0 \leq i, l \leq k.$$

On the other hand, if we apply Lemma 1 to the inner product

$$(2.27) \quad 2h^{2m-1}(\phi_i, L\phi_i) = 2h^{2m-1}B(\phi_i, \phi_i),$$

we find that

$$(2.28) \quad |2h^{2m-1}B(\phi_i, \phi_i) - h^{2m}\omega_l L\phi_i(\xi_{il})| \leq Ch^{2k+2}\|\phi_l L\phi_i\|_{W^{2r}(I_j)} \leq Ch^2,$$

which means that  $(h^{2m}\omega_l L\phi_i(\xi_{il}))$  is an  $O(h^2)$  perturbation of a positive definite matrix whose entries are of  $O(1)$ .

If we present  $(h^{2m}\omega_l L\phi_i(\xi_{je}))$  and  $(2h^{2m-1}B(\phi_i, \phi_e))$  by  $M_1$  and  $M_2$  respectively, we find that, since  $M_2^{-1}$  exists and is of  $O(1)$  (this follows from the strong coercivity of  $B$ ),

$$(2.29) \quad M_1 = M_2 + h^2 M_3 = M_2(I + h^2 M_2^{-1} M_3) = M_2(I + h^2 M_4),$$

where the entries of  $M_4$  are of  $O(1)$ . Elementary matrix calculus shows that  $(I + h^2 M_4)^{-1}$  exists, if  $h$  is small enough, and can be expanded in a power series:

$$(2.30) \quad (I + h^2 M_4)^{-1} = \sum_{l=0}^{\infty} (-1)^l h^{2l} M_4^l.$$

This implies that  $(h^{2m}\omega_l L\phi_i(\xi_{il}))^{-1}$  exists and has entries of  $O(1)$ . This completes the proof of

**THEOREM 1.** *Let  $u \in H_0^m(I) \cap H^{k+1}(I) \cap W^{2r}(\Delta)$  be the solution of (2.1) and let  $U \in M_0^{k,m}(\Delta)$  be the solution of (2.4). Then  $e = u - U$  has the bounds (2.5) and the additional bound*

$$(2.31) \quad |e(\xi_{jl})| \leq C(u) h^{k+2}, \quad j = 1, \dots, N, \quad l = 1, \dots, n,$$

where  $\xi_{jl}$  is defined by (2.18).

We can use the local convergence properties (2.5) and (2.31) to establish superconvergence properties of  $De$  at interior points of  $I_j$ . Let  $\varepsilon(x)$  be defined by (2.22). Then on any  $I_j$ ,  $\varepsilon(x)$  has the representation

$$(2.32) \quad \varepsilon(x) = h^{k+1}(1 - \sigma^2)^m P_n^m(\sigma) E_j(x),$$

$$\sigma = \frac{2}{h}(x - \bar{x}_j), \quad \bar{x}_j = \frac{1}{2}(x_{j-1} + x_j),$$

where  $E_j(x)$  and  $E_j'(x)$  have bounds depending on  $j$  only. This property can be proved by expanding  $u$  and  $\Pi_\Delta u$  as Taylor series around  $\bar{x}_j$ .

Differentiating (2.32), we obtain

$$(2.33) \quad \varepsilon'(x) = h^{k+1} E_j'(x) (1 - \sigma^2)^m P_n^m(\sigma) + 2h^k E_j(x) \frac{d}{d\sigma} (1 - \sigma^2)^m P_n^m(\sigma).$$

From (2.6) and [1, Formula 22.6.1], it can be proved that

$$(2.34) \quad \begin{aligned} P_n^m(\sigma) &= A_{mn} \frac{d}{d\sigma} P_{n+1}^{m-1}(\sigma), \\ \frac{d}{d\sigma} \left[ (1-\sigma^2)^m \frac{d}{d\sigma} P_{m+1}^{m-1}(\sigma) \right] &= B_{mn} (1-\sigma^2)^{m-1} P_{n+1}^{m-1}(\sigma), \end{aligned}$$

where  $A_{mn}$  and  $B_{mn}$  depend on  $m$  and  $n$  only. From (2.33) and (2.34) we can conclude that

$$(2.35) \quad \begin{aligned} |\varepsilon'(x)| &= O(h^{k+1}) \quad \text{if } x = \eta_{jl}, \\ \eta_{jl} &= x_{j-1} + \frac{h}{2}(1 + \sigma_{ln+1}^{m-1}), \quad j = 1, \dots, N, \quad l = 1, \dots, n+1. \end{aligned}$$

Consider now  $\delta$  defined by (2.22). From (2.5) and Theorem 1, it is proved that

$$(2.36) \quad \|\delta\|_{L^\infty(I)} \leq C(u)h^{k+2}, \quad \|\delta'\|_{L^\infty(I)} \leq C(u)h^{k+1}.$$

From (2.35)–(2.36), one easily proves

**THEOREM 2.** *Let the conditions of Theorem 1 hold. Then  $e(x)$  has the additional bound*

$$(2.37) \quad |De(\eta_{jl})| \leq C(u)h^{k+1},$$

where  $\eta_{jl}$  is defined by (2.35). This is one order better than the optimal order of convergence for  $e'(x)$ .

**2.3. Quadrature rules.** Without giving proofs, we state that all the local convergence properties from the Theorems 1 and 2 are preserved whenever  $(\cdot, \cdot)$  is replaced by some approximating quadrature  $(\cdot, \cdot)_h$  which is of  $O(h^q)$ ,  $q \geq 2r$ , i.e.,

$$|(\alpha, \beta) - (\alpha, \beta)_h| \leq C(\alpha, \beta)h^q, \quad q \geq 2r.$$

Examples are the extended  $r$ -point Gauss–Legendre rule or the extended  $(r+1)$ -point Lobatto rule.

**3. Collocation methods.** We consider the  $m$ th order boundary problem

$$(3.1) \quad \begin{aligned} Lu(x) &\equiv D^m u(x) + \sum_{i=0}^{m-1} p_i(x) D^i u(x) = f(x), \quad x \in I, \\ \beta_l[u] &= 0, \quad l = 1, \dots, m, \end{aligned}$$

where  $p_0, \dots, p_{m-1}$  and  $f$  are sufficiently smooth functions and where  $\beta_1, \dots, \beta_m$  are continuous linear functionals over  $C^{m-1}(I)$ . We note that the functions  $p_0, \dots, p_{m-1}$  and  $f$  and the operator  $L$  are not the same as in the previous section. We assume that (3.1) has a unique solution and that  $\beta_1, \dots, \beta_m$  are linearly independent over  $P_{m-1}(I) = \ker(D^m)$ .

Let  $\Delta$  be a partition of  $I$  defined by (1.2). Then, for  $k \geq 2m-1$ , we define the finite element space  $S_0^{k,m}(\Delta)$  by

$$(3.2) \quad \begin{aligned} S_0^{k,m}(\Delta) &= \{V \mid V \in C_0^{m-1}(I); V \in P_k(I_j), j = 1, \dots, N\}, \\ C_0^{m-1}(I) &= \{v \mid v \in C^{m-1}(I); \beta_l[v] = 0, l = 1, \dots, m\}. \end{aligned}$$

The collocation solution  $U \in S_0^{k,m}(\Delta)$  of (3.1) is defined as follows.

For  $r = k + 1 - m$ , we define the collocation points  $z_{jl}$  by

$$(3.3) \quad z_{jl} = x_{j-1} + \frac{h}{2}(1 + \sigma_{lr}^0), \quad j = 1, \dots, N, \quad l = 1, \dots, r,$$

where  $\{\sigma_{lr}^0\}$  are the zeros of the  $r$ th degree Legendre polynomial  $P_r^0(\sigma)$ . Then  $U$  is determined by the linear system

$$(3.4) \quad LU(z_{jl}) = f(z_{jl}), \quad j = 1, \dots, N, \quad l = 1, \dots, r.$$

If  $u \in C_0^{m-1}(I) \cap C^{2r+m}(I)$ , the error function  $e = u - U$  has the bounds [5]

$$(3.5) \quad \begin{aligned} \|e\|_{W^l(I)} &\leq C_1(u)h^{k+1-l}, \quad l = 0, \dots, m, \\ |D^l e(x_j)| &\leq C_2(u)h^{2r}, \quad l = 0, \dots, m-1, \quad j = 0, \dots, N. \end{aligned}$$

In order to establish superconvergence at interior points of  $I_j$  [8], we recall the  $n$ -dimensional subspace  $S_0(I_j)$  of  $S_0^{k,m}(\Delta)$  defined by (2.15). For any  $V \in S_0(I_j)$ , we have, if we put  $p_m(x) \equiv 1$ ,

$$(3.6) \quad (e, L^T L V) = (Le, L V) + \sum_{l=1}^m \sum_{\nu=0}^{l-1} (-1)^{\nu+l} [D^{l-\nu-1} e D^\nu (p_l L V)]_{x_{j-1}}^{x_j},$$

where the operator  $L^T$  is defined by

$$(3.7) \quad L^T v = \sum_{l=0}^m (-1)^l D^l (p_l v).$$

If we apply Lemma 1 to the left-hand side of (3.6), we have

$$(3.8) \quad \begin{aligned} \frac{h}{2} \sum_{l=1}^n \omega_l e(\xi_{jl}) L^T L V(\xi_{jl}) + \sum_{l=0}^{m-1} [\theta_{l1} D^l (e L^T L V)(x_{j-1}) + \theta_{l2} D^l (e L^T L V)(x_j)] \left(\frac{h}{2}\right)^{l+1} \\ = (e, L^T L V) - R_{mn} \left(\frac{h}{2}\right)^{2r+1} h^{2r+1} D^{2r} (e L^T L V)(\xi \in I_j), \end{aligned}$$

where  $R_{mn}$  is defined by (2.9).

If we apply the  $r$ -point Gauss-Legendre rule to the first term of the right-hand side of (3.6), we obtain ([1, formula 25.4.29])

$$(3.9) \quad \begin{aligned} \frac{h}{2} \sum_{l=1}^r \lambda_{lr}^0 L e(z_{jl}) L V(z_{jl}) &= (Le, L V) - S_r h^{2r+1} D^{2r} (Le L V)(\xi \in I_j), \\ S_r &= \frac{(r!)^4}{(2r+1)[(2r)!]^3}. \end{aligned}$$

In virtue of (3.4), the left-hand side of (3.9) is identically zero. If we combine (3.7)–(3.9) and apply it for the Lagrange basis functions  $\phi_i$  of  $S_0(I_j)$  as defined by (2.16), we get after multiplication by  $2h^{2m-1}$

$$(3.10) \quad \begin{aligned} \left| \sum_{l=1}^n \omega_l L^T L \phi_i(\xi_{jl}) h^{2m} e(\xi_{jl}) \right| \\ \leq C_1 \sum_{l=0}^{m-1} (|D^l e(x_{j-1})| + |D^l e(x_j)|) \\ + h^{2m} \left| \sum_{l=0}^{m-1} [\theta_{l1} D^l (e L^T L \phi_i)(x_{j-1}) + \theta_{l2} D^l (e L^T L \phi_i)(x_j)] \left(\frac{h}{2}\right)^l \right| \\ + C_2 h^{2k+2} [\|e L^T L \phi_i\|_{W^{2r}(I_j)} + \|Le L \phi_i\|_{W^{2r}(I_j)}] \leq C(u) h^{k+2}, \quad i = 1, \dots, n \end{aligned}$$



Analogously to (2.28), we can prove that

$$(3.11) \quad |\omega_l L^T L \phi_i(\xi_{jl}) h^{2m} - 2h^{2m-1} (L \phi_i, L \phi_l)| \leq Ch^2,$$

which means that, for sufficiently small  $h$ , the matrix  $(\omega_l L^T L \phi_i(\xi_{jl}))$  is an  $O(h^2)$  perturbation of the positive definite matrix  $(2h^{2m-1} (L \phi_i, L \phi_l))$ , whose eigenvalues and entries are of  $O(1)$ . This implies that the entries of  $(\omega_l L^T L \phi_i(\xi_{jl}) h^{2m})^{-1}$  are of  $O(1)$ .

**THEOREM 3.** *Let  $u \in C_0^{m-1}(I) \cap C^{2r+m}(I)$  be the solution of (3.1) and let  $U \in S_0^{k,m}(\Delta)$  be the solution of (3.4). Then  $e(x) = u(x) - U(x)$  has the bounds (3.5) plus the bounds*

$$(3.12) \quad \begin{aligned} |e(\xi_{jl})| &\leq C(u) h^{k+2}, & j = 1, \dots, N, \quad l = 1, \dots, n, \\ |De(\eta_{jl})| &\leq C(u) h^{k+1}, & j = 1, \dots, N, \quad l = 1, \dots, n+1, \end{aligned}$$

where  $\xi_{jl}$  and  $\eta_{jl}$  are given by (2.18) and (2.35), respectively.

*Proof.* The first part of (3.12) was already established by (3.9)–(3.11). The second part is proved analogously to Theorem 2.  $\square$

*Remark.* Russell and Christiansen [8] also gave a proof of (3.12); they proved in another way that the first bound of (3.12) occurs at the interior zeros of the polynomial

$$(3.13) \quad \int_{-1}^{\sigma} (t - \sigma)^{m-1} P_r^{0,0}(t) dt, \quad \sigma = \frac{2}{h}(x - x_{j-1}) - 1,$$

which can be shown to be equal to  $(1 - \sigma^2)^m P_n^{m,m}(\sigma)$  up to a constant factor. The proof of this equality can be given by using formula (2.6) with  $\alpha = \beta = 0$  and elaborating the integral (3.13) which gives the desired result.

**4. Conclusions.** In this paper, it was proved for two classes of Galerkin methods that superconvergence also occurs outside the knots of the partition, albeit in a more modest form. Its existence can easily be proved for other classes of problems which are solved by the Ritz–Galerkin or the collocation method. Examples are nonlinear two-point boundary problems and parabolic equations in one space variable [4].

The interior superconvergence is especially important if the finite element space is of degree  $2m$ , because the order of convergence at  $\bar{x}_j$  is then the same as at  $x_j$ .

**Appendix.** For the computation of  $R_{mn}$  from (2.9), we apply that relation to  $f \in P_{2r}(I)$  defined by

$$(A1) \quad f(\sigma) = (1 - \sigma^2)^m [P_n^m(\sigma)]^2,$$

where we assume that  $P_n^m(\sigma)$  is normalized in such a way that (see [1, formula 22.2.1])

$$(A2) \quad P_n^m(1) = \binom{n+m}{n}.$$

From [1, formula 22.2.1], we learn that

$$(A3) \quad \int_{-1}^{+1} f(\sigma) d\sigma = h_{mn} = \frac{2^{2m+1} (r!)^2}{(2r+1)n!(n+2m)!}.$$

From [1, formulas 22.5.42 and 15.1.1], we learn that

$$(A4) \quad \begin{aligned} P_n^m(\sigma) &= \binom{n+m}{n} \sum_{k=0}^n \frac{(-n)_k (n+2m+1)_k}{(1+m)_k k!} \left(\frac{1-\sigma}{2}\right)^k, \\ (a)_0 &= 1, \\ (a)_k &= \frac{a(a+1) \cdots (a+k-1)}{k!}, \quad k > 0, \end{aligned}$$

which means that the coefficient of  $\sigma^{2r}$  in the expression of  $f$  is equal to

$$\begin{aligned}
 b_{2r} &= (-1)^m A_n^2, \\
 A_n &= \binom{n+m}{n} \frac{(-n)_n (n+2m+1)_n}{(1+m)_n n!} (-1)^n 2^{-n} \\
 (A5) \quad &= \frac{(n+m)!}{n! m!} \cdot \frac{(n+2m+1)(n+2m+2) \cdots (2n+2m)}{(m+1)(m+2) \cdots (m+n)} 2^{-n} \\
 &= 2^{-n} \binom{2n+2m}{n}.
 \end{aligned}$$

Application of (2.9) to  $f$  shows that

$$(A6) \quad h_{mn} = \int_{-1}^{+1} f(\sigma) d\sigma = \int_{-1}^{+1} \Pi f d\sigma + R_{mn} D^{2r} f(\xi) = 0 + R_{mn} (2r)! b_{2r},$$

which implies that

$$(A7) \quad R_{mn} = \frac{h_{mn}}{(2r)! b_{2r}}.$$

Application of (A3) and (A5) to (A7) delivers the desired expression for  $R_{mn}$ .

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