

Tight Inefficiency Bounds for Perception-Parameterized Affine Congestion Games

Pieter Kleer¹ and Guido Schäfer^{1,2}(✉)

¹ Centrum Wiskunde & Informatica (CWI), Amsterdam, The Netherlands
{kleer,schaefer}@cwi.nl

² Vrije Universiteit Amsterdam, Amsterdam, The Netherlands

Abstract. We introduce a new model of congestion games that captures several extensions of the classical congestion game introduced by Rosenthal in 1973. The idea here is to parameterize both the perceived cost of each player and the social cost function of the system designer. Intuitively, each player perceives the load induced by the other players by an extent of $\rho \geq 0$, while the system designer estimates that each player perceives the load of all others by an extent of $\sigma \geq 0$. For specific choices of ρ and σ , we obtain extensions such as altruistic player behavior, risk sensitive players and the imposition of taxes on the resources. We derive tight bounds on the price of anarchy and the price of stability for a large range of parameters. Our bounds provide a complete picture of the inefficiency of equilibria for these games. As a result, we obtain tight bounds on the price of anarchy and the price of stability for the above mentioned extensions. Our results also reveal how one should “design” the cost functions of the players in order to reduce the price of anarchy. Somewhat counterintuitively, if each player cares about all other players to the extent of $\rho = 0.625$ (instead of 1 in the standard setting) the price of anarchy reduces from 2.5 to 2.155 and this is best possible.

1 Introduction

Congestion games constitute an important class of non-cooperative games which was introduced by Rosenthal in 1973 [13]. In a congestion game, we are given a set of resource from which a set of players can choose. Each resource is associated with a cost function which specifies the cost of this resource depending on the total number of players using it. Every player chooses a subset of resources (from a set of resource subsets available to her) and experiences a cost equal to the sum of the costs of the chosen resources. Congestion games are both theoretically appealing and practically relevant. For example, they have applications in network routing, resource allocation and scheduling problems.

Rosenthal [13] proved that every congestion game has a pure Nash equilibrium, i.e., a strategy profile such that no player can decrease her cost by unilaterally deviating to another feasible set of resources. This result was established through the use of an exact potential function (known as *Rosenthal potential*)

satisfying that the cost difference induced by a unilateral player deviation is equal to the potential difference of the respective strategy profiles. In fact, Monderer and Shapley [11] showed that the class of games admitting an exact potential function is isomorphic to the class of congestion games.

One of the main research directions in algorithmic game theory focusses on quantifying the inefficiency caused by selfish behavior. The idea is to assess the quality of a Nash equilibrium relative to an optimal outcome. Here the quality of an outcome is measured in terms of a given *social cost* objective (e.g., the sum of the costs of all players). Koutsoupias and Papadimitriou [10] introduced the *price of anarchy* as the ratio between the worst social cost of a Nash equilibrium and the social cost of an optimum. Anshelevich et al. [1] defined the *price of stability* as the ratio between the best social cost of a Nash equilibrium and the social cost of an optimum.

In recent years, several extensions of Rosenthal's congestion games were proposed to incorporate aspects which are not captured by the standard model. For example, these extensions include risk sensitivity of players in uncertain settings [12], altruistic player behavior [4, 5] and congestion games with taxes [3]. We elaborate in more detail on these extensions in Sect. 2. These games were studied intensively with the goal to obtain a precise understanding of the price of anarchy.

In this paper, we introduce a new model of congestion games, which we term *perception-parameterized congestion games*, that captures all these extensions (and more) in a unifying way. The key idea here is to parameterize both the perceived cost of each player and the social cost function. Intuitively, each player perceives the load induced by the other players by an extent of $\rho \geq 0$, while the system designer estimates that each player perceives the load of all others by an extent of $\sigma \geq 0$. The above mentioned extensions reduce to special cases of our model by choosing the parameters ρ and σ accordingly.

Despite the fact that we deal with a more general class of congestion games, we manage to derive tight bounds on the price of anarchy and the price of stability for a large range of parameters. Our bounds provide a complete picture of the inefficiency of equilibria for these perception-parameterized congestion games. As a consequence, we obtain tight bounds on the price of anarchy and the price of stability for the above mentioned extensions. While the price of anarchy bounds are (mostly) known from previous results, the price of stability results are new. As in [3–5, 12], we focus on congestion games with affine cost functions.

We illustrate our model by means of a simple example; formal definitions of our perception-parameterized congestion games are given in Sect. 2. Suppose we are given a set of m resources and that every player has to choose precisely one of these resources. The cost of a resource $e \in [m]$ ¹ is given by a cost function c_e that maps the *load* on e to a real value. In the classical setting, the load of a resource e is defined as the total number of players x_e using e . That is, the cost that player i experiences when choosing resource e is $c_e(x_e)$. In contrast, in

¹ Given a positive integer m , we use $[m]$ to refer to the set $\{1, \dots, m\}$.

Table 1. An overview of (tight) price of anarchy and price of stability results for certain values of ρ and σ . Here $h(1) \approx 0.625$ (see Theorem 1 for a formal definition). The respective references where these bounds were established first are given in the column “Ref.”; an asterisk indicates that this result is new.

Model	Parameters	PoA	Ref.	PoS	Ref.
Classical	$\rho = \sigma = 1$	5/2	[6]	1.577	[3]
Altruism (1)	$\sigma = 1, 1 \leq \rho \leq 2$	$\frac{4\rho+1}{1+\rho}$	[4, 5]	$\frac{\sqrt{3+1}}{\sqrt{3+\rho-1}}$	[*]
Altruism (2)	$\sigma = 1, 2 \leq \rho \leq \infty$	$\rho + 1$	[5]	–	–
Risk-neutral players	$\sigma = \rho = 1/2$	5/3	[12]	1.447	[*]
Wald’s minimax	$\sigma = 1/2, \rho = 1$	2	[2, 12]	1	[*]
Constant universal taxes	$\sigma = 1, \rho = h(1)$	2.155	[3]	2.013	[*]
Generalized affine CG	–	∞	[*]	2	[*]

our setting players have different perceptions of the load induced by the other players. More precisely, the *perceived load* of player i choosing resource e is $1 + \rho(x_e - 1)$, where $\rho \geq 0$ is a parameter. Consequently, the perceived cost of player i for choosing e is $c_e(1 + \rho(x_e - 1))$. Note that as ρ increases players care more about the presence of other players.² In addition, we introduce a similar parameter $\sigma \geq 0$ for the social cost objective. Intuitively, this can be seen as the system designer’s estimate of how each player perceives the load of the other players. In our example, the social cost is defined as $\sum_{e \in [m]} c_e(1 + \sigma(x_e - 1))x_e$.

Our Results. We prove the following bounds on the price of anarchy (PoA) and the price of stability (PoS) of affine congestion games for a large range of parameters (ρ, σ) (specified below):

$$\text{PoA} \leq \max \left\{ \rho + 1, \frac{2\rho(1 + \sigma) + 1}{\rho + 1} \right\} \quad \text{and} \quad \text{PoS} \leq \frac{\sqrt{\sigma(\sigma + 2)} + \sigma}{\sqrt{\sigma(\sigma + 2)} + \rho - \sigma}. \quad (1)$$

We prove that these bounds are tight for general affine congestion games. Further, for the special case of symmetric network congestion games we show that the bound of $(2\rho(1 + \sigma) + 1)/(\rho + 1)$ on the price of anarchy is asymptotically tight. In contrast, for this case we derive a better (tight) bound on the price of stability for $\sigma = 1$ and $\rho \geq 0$. An overview of the price of anarchy and the price of stability results that we obtain from (1) for several applications known in the literature is given in Table 1; see Fig. 2 for an illustration of our PoA bound. The connection between these applications and our model is discussed at the end of Sect. 2.

In light of the above bounds, we obtain an (almost) complete picture of the inefficiency of equilibria (parameterized by ρ and σ); for example, see Fig. 1 for

² In this work, we concentrate on the homogeneous player case.

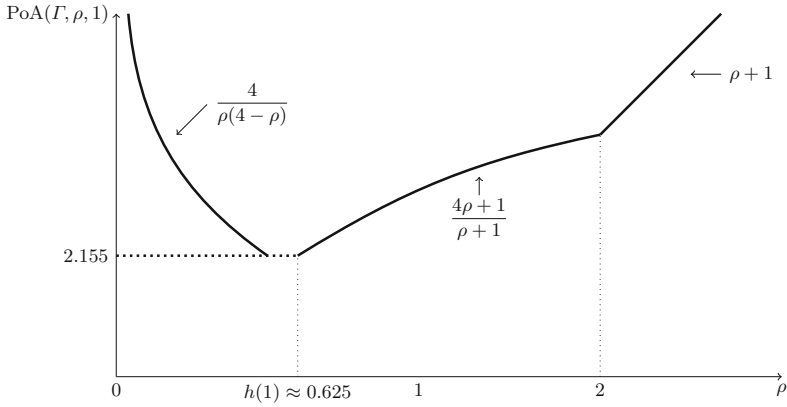


Fig. 1. Lower bounds on the price of anarchy for $\sigma = 1$. The bounds $(4\rho + 1)/(\rho + 1)$ and $\rho + 1$ are also tight upper bounds. The dotted horizontal line indicates the lower bound following from [4, Theorem 3.7]. The bound $4/(\rho(4 - \rho))$ is a lower bound for symmetric singleton congestion games given in the proof of Theorem 5. A tight bound for $0 < \rho \leq h(1)$ remains an open problem.

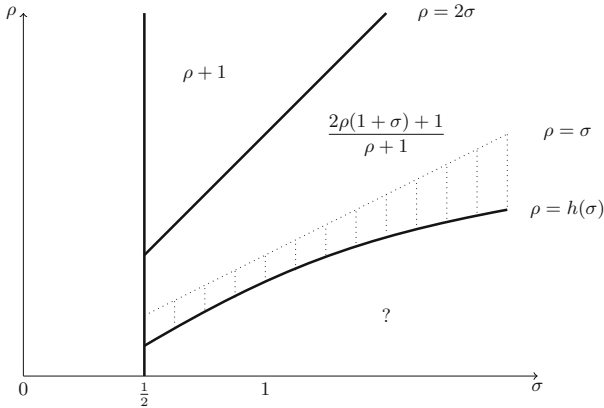


Fig. 2. The bound $\rho + 1$ holds for $\rho \geq 2\sigma \geq 1$. The bound $(2\rho(1 + \sigma) + 1)/(1 + \rho)$ holds for $\sigma \leq \rho \leq 2\sigma$. Roughly speaking, this bound also holds for $h(\sigma) \leq \rho \leq \sigma$, but our proof of Theorem 1 only works for a discretized range of σ (hence the vertical dotted lines in this area). The function h is given in Theorem 1.

an illustration of the price of anarchy if $\sigma = 1$. Note that the price of anarchy *decreases* from $\frac{5}{2}$ for $\rho = 1$ to 2.155 for $\rho = h(1) \approx 0.625$.³

³ The price of anarchy for $\rho = h(1)$ was first established by Caragiannis et al. [3]. However, our bounds reveal that the price of anarchy is in fact minimized at $\rho = h(1)$ (see also Fig. 1).

2 Our Model, Applications and Related Work

We first formally introduce our model of congestion games with parameterized perceptions. We then show that our model subsumes several other models that were studied in the literature as special cases.

A *congestion game* Γ is given by a tuple $(N, E, (\mathcal{S}_i)_{i \in N}, (c_e)_{e \in E})$ where $N = [n]$ is the set of players, E the set of resources (or facilities), $\mathcal{S}_i \subseteq 2^E$ the set of strategies of player i , and $c_e : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ the cost function of facility e . Given a strategy profile $s = (s_1, \dots, s_n) \in \times_i \mathcal{S}_i$, we define x_e as the number of players using resource e , i.e., $x_e = x_e(s) = |\{i \in N : e \in s_i\}|$. If $\mathcal{S}_i = \mathcal{S}_j$ for all $i, j \in N$, the game is called *symmetric*. For a given graph $G = (V, E)$, we call Γ a (*directed*) *network congestion game* if for every player i there exist $s_i, t_i \in V$ such that \mathcal{S}_i is the set of all (directed) (s_i, t_i) -paths in G . An *affine congestion game* has cost functions of the form $c_e(x) = a_e x + b_e$ with $a_e, b_e \geq 0$. If $b_e = 0$ for all $e \in E$, the game is called *linear*.

We introduce our unifying model of *perception-parameterized congestion games* with affine latency functions. For a fixed parameter $\rho \geq 0$, we define the cost of player $i \in N$ by

$$C_i^\rho(s) = \sum_{e \in s_i} c_e(1 + \rho(x_e - 1)) = a_e[1 + \rho(x_e - 1)] + b_e \quad (2)$$

for a given strategy profile $s = (s_1, \dots, s_n)$. For a fixed parameter $\sigma \geq 0$, the social cost of a strategy profile s is given by

$$C^\sigma(s) = \sum_{i \in N} C_i^\sigma(s) = \sum_{e \in E} x_e(a_e[1 + \sigma(x_e - 1)] + b_e). \quad (3)$$

We refer to the case $\rho = \sigma = 1$ as the *classical congestion game* with cost functions $c_e(x) = a_e x + b_e$ for all $e \in E$.

A strategy profile s is a *Nash equilibrium* if for all players $i \in N$ it holds that $C_i^\rho(s) \leq C_i^\rho(s'_i, s_{-i})$ for all $s'_i \in \mathcal{S}_i$, where (s'_i, s_{-i}) denotes the strategy profile in which player i plays s'_i and all the other players their strategy in s . The price of anarchy (PoA) and price of stability (PoS) of a game Γ are defined as

$$\text{PoA}(\Gamma, \rho, \sigma) = \frac{\max_{s \in \text{NE}} C^\sigma(s)}{\min_{s^* \in \times_i \mathcal{S}_i} C^\sigma(s^*)} \quad \text{and} \quad \text{PoS}(\Gamma, \rho, \sigma) = \frac{\min_{s \in \text{NE}} C^\sigma(s)}{\min_{s^* \in \times_i \mathcal{S}_i} C^\sigma(s^*)},$$

where $\text{NE} = \text{NE}(\rho)$ denotes the set of Nash equilibria with respect to the player costs as defined in (2). For a collection of games \mathcal{H} , $\text{PoA}(\mathcal{H}, \rho, \sigma) = \sup_{\Gamma \in \mathcal{H}} \text{PoA}(\Gamma, \rho, \sigma)$ and $\text{PoS}(\mathcal{H}, \rho, \sigma) = \sup_{\Gamma \in \mathcal{H}} \text{PoS}(\Gamma, \rho, \sigma)$. Unless stated otherwise, our results refer to the class of perception-parameterized congestion games with affine latency functions; we therefore drop the parameter \mathcal{H} below. Rosenthal [13] shows that classical congestion games (i.e., $\rho = \sigma = 1$) have an exact potential function: $\Phi : \times_i \mathcal{S}_i \rightarrow \mathbb{R}$ is an *exact potential function* for a congestion game Γ if for every strategy profile s , for every $i \in N$ and every $s'_i \in \mathcal{S}_i$: $\Phi(s) - \Phi(s_{-i}, s'_i) = C_i(s) - C_i(s_{-i}, s'_i)$. The Rosenthal potential

$\Phi(s) = \sum_{e \in E} \sum_{k=1}^{x_e} c_e(k)$ is an exact potential function for classical congestion games.

We review various models that fall within, or are related to, the framework proposed above (for certain values of ρ and σ). These models sometimes interpret the parameters differently than explained above.

Altruism [4,5]. We can rewrite the cost of player i as $C_i^\rho(s) = \sum_{e \in s_i} (a_e x_e + b_e) + (\rho - 1)a_e(x_e - 1)$. The term $(\rho - 1)a_e(x_e - 1)$ can be interpreted as a “dynamic” (meaning load-dependent) tax that players using resource e have to pay. For $1 \leq \rho \leq \infty$ and $\sigma = 1$, this model is equivalent to the altruistic player setting proposed by Caragiannis et al. [4]. Chen et al. [5] also study this model of altruism for $1 \leq \rho \leq 2$ and $\sigma = 1$.

Constant Taxes [3]. We can rewrite the cost of player i as $C_i^\rho(s) = \sum_{e \in s_i} \rho a_e x_e + (1 - \rho)a_e + b_e$. Dividing by ρ gives that s is a Nash equilibrium with respect to C_i^ρ if and only if s is a Nash equilibrium with respect to $T_i^\rho(s) = C_i^\rho/\rho = \sum_{e \in s_i} (a_e x_e + b_e/\rho) + \sum_{e \in s_i} (1 - \rho)/\rho a_e$. That is, s is a Nash equilibrium in a classical congestion game in which players take into account constant resource taxes of the form $(1 - \rho)/\rho \cdot a_e$. Caragiannis, Kaklamanis and Kanellopoulos [3] study this type of taxes, which they call *universal tax functions*, for ρ satisfying $(1 - \rho)/\rho = 3/2\sqrt{3} - 2$. They consider these taxes to be *refundable*, i.e., they are not taken into account in the social cost, which is equivalent to the case $\sigma = 1$. Note that the function $\tau : (0, 1] \rightarrow [0, \infty)$ defined by $\tau(\rho) = (1 - \rho)/\rho$ is bijective.⁴ Caragiannis et al. [3] showed that the price of anarchy can be decreased to 2.155 by the usage of universal tax functions, which improves significantly the classical bound of 2.5. Furthermore, from [3, Theorem 3.7] it follows that the price of anarchy can never be better than 2.155 for $0 \leq \rho \leq h(1)$. However, in this work we show that the price of stability increases from 1.577 (for classical games) to 2.013, for this specific set of tax functions.

Risk Sensitivity Under Uncertainty [12]. Nikolova, Piliouras and Shamma [12] consider congestion games in which there is a (non-deterministic) order of the players on every resource. A player is only affected by players in front of her. That is, the load on resource e for player i in a strict ordering r , where $r_e(i)$ denotes the position of player i , is given by $x_e(i) = |\{j \in N : r_e(j) \leq r_e(i)\}|$. The cost of player i is then $C_i(s) = \sum_{e \in s_i} c_e(x_e(i))$. Note that $x_e(i)$ is a random variable if the ordering is non-deterministic. The social cost of the model is defined by the sum of all player costs $C^{\frac{1}{2}}(s) = \sum_{e \in E} \frac{1}{2} a_e x_e(x_e + 1) + b_e$ which is independent of the ordering r .⁵ Note that the social cost corresponds to the case $\sigma = \frac{1}{2}$ in our framework. Nikolova et al. [12] study various risk attitudes towards the ordering r that is assumed to have a uniform distribution over all possible orderings. The two relevant attitudes are that of *risk-neutral* players and players applying *Wald’s minimax principle*. Risk-neutral players define their cost as the expected cost under the ordering r , which correspond to the case $\rho = \frac{1}{2}$ in (2). This

⁴ This relation between altruism (or spite) and constant taxes is also mentioned by Caragiannis et al. [4].

⁵ In every ordering there is always one player first, one player second, and so on.

can roughly be interpreted as that players expect to be scheduled in the middle on average. Wald's minimax principle implies that players assume a worst-case scenario, i.e., being scheduled last on all the resources. This corresponds to the case $\rho = 1$.

Approximate Nash Equilibria [7]. Suppose that s is a Nash equilibrium under the cost functions defined in (2). Then, in particular, we have $C_i^1(s) \leq C_i^\rho(s) \leq C_i^\rho(s'_i, s_{-i}) \leq \rho C_i^1(s'_i, s_{-i})$ for any player i and $s'_i \in \mathcal{S}_i$ and $\rho \geq 1$. That is, we have $C_i^1(s) \leq \rho C_i^1(s'_i, s_{-i})$ which means that the profile s is a ρ -approximate equilibrium, as studied by Christodoulou, Koutsoupias and Spirakis [7]. In particular, this implies that any upper bound on the price of anarchy, or price of stability, in our framework yields an upper bound on the price of stability for ρ -approximate equilibria for the same class of games. For $\sigma = 1$ and $1 \leq \rho \leq 2$, we obtain a bound of $(\sqrt{3} + 1)/(\sqrt{3} + \rho - 1)$ on the price of stability. In particular, this also yields the same bound on the price of stability for ρ -approximate equilibria. This bound was previously obtained by Christodoulou et al. [7]. Conceptually our approach is different: We prove our bound by observing that every Nash equilibrium in our framework yields an approximate equilibrium. In particular, this gives rise to a potential function that can be used to carry out the technical details (namely the potential function that is exact for our congestion game).⁶

Generalized Affine Congestion Games. Let \mathcal{A}' denote the class of all congestion games Γ for which all resources have the same cost function $c(x) = ax + b$, where $a = a(\Gamma)$ and $b = b(\Gamma)$ satisfy $a \geq 0$ and $a + b > 0$. The class of affine congestion games with non-negative coefficients is contained in \mathcal{A}' since every such game can always be transformed⁷ into a game Γ' with $a_e = 1$ and $b_e = 0$ for all resources $e \in E'$, where E' is the resource set of Γ' . Without loss of generality we can assume that $a + b = 1$, since the cost functions can be scaled by $1/(a + b)$. The cost functions of $\Gamma \in \mathcal{A}'$ can then equivalently be written as $c(x) = \rho x + (1 - \rho)$ for $\rho \geq 0$. This is precisely the definition of $C_i^\rho(s)$ (with $a_e = 1$ and $b_e = 0$ taken there). In particular, if we take $\sigma = \rho$, meaning that $C^\rho(s) = \sum_{i \in N} C_i^\rho(s)$, we have $\text{PoA}(\mathcal{A}') = \sup_{\rho \geq 0} \text{PoA}(\mathcal{A}, \rho, \rho)$ and $\text{PoS}(\mathcal{A}') = \sup_{\rho \geq 0} \text{PoS}(\mathcal{A}, \rho, \rho)$, where \mathcal{A} denotes the class of affine congestion games with non-negative coefficients.

Due to page limitations some material is omitted below. All missing details can be found in the full version of this paper [9].

3 Price of Anarchy

We derive the upper bound on the price of anarchy given in (1). We start with the bound of $(2\rho(1 + \sigma) + 1)/(\rho + 1)$.

⁶ Nevertheless, the framework of Christodoulou et al. [7] is somewhat more general and might be used to obtain a tight bound for the price of stability of approximate equilibria (which is not known to the best of our knowledge).

⁷ This transformation can be done in such a way that both PoA and PoS of the game do not change. For a proof the reader is referred to, e.g., [5, Lemma 4.3].

We need the following technical lemma for the proof of Theorem 1:

Lemma 1. *Let s be a Nash equilibrium under the cost functions $C_i^\rho(s)$ and let s^* be a minimizer of $C^\sigma(\cdot)$. For $\rho, \sigma \geq 0$ fixed, if there exist $\alpha(\rho, \sigma), \beta(\rho, \sigma) \geq 0$ such that*

$$(1 + \rho x)y - \rho(x - 1)x - x \leq -\beta(\rho, \sigma)(1 + \sigma(x - 1))x + \alpha(\rho, \sigma)(1 + \sigma(y - 1))y$$

for all non-negative integers x and y , then $\beta(\rho, \sigma)C^\sigma(s) \leq \alpha(\rho, \sigma)C^\sigma(s^*)$.

Theorem 1. *We have $PoA(\rho, \sigma) \leq (2\rho(1 + \sigma) + 1)/(\rho + 1)$ if*

- (i) $\frac{1}{2} \leq \sigma \leq \rho \leq 2\sigma$, or
- (ii) $\sigma = 1$ and $h(\sigma) \leq \rho \leq 2\sigma$, where $h(\sigma) = g(1 + \sigma + \sqrt{\sigma(\sigma + 2)}, \sigma)$ is the optimum of the function

$$g(a, \sigma) = \frac{\sigma(a^2 - 1)}{(1 + \sigma)a^2 - (2\sigma + 1)a + 2\sigma(\sigma + 1)}.$$

Further, there exists a function $\Delta = \Delta(\sigma)$ satisfying for every fixed $\sigma_0 \geq 1/2$: if $\Delta(\sigma_0) \geq 0$, then the stated bound is true for all $h(\sigma_0) \leq \rho \leq 2\sigma_0$.

Proof (Sketch). For the functions $\alpha(\rho, \sigma) = (2\rho(1 + \sigma) + 1)/(1 + 2\sigma)$ and $\beta(\rho, \sigma) = (1 + \rho)/(1 + 2\sigma)$, we prove the inequality in Lemma 1. We show that for certain functions f_1 and f_2 , the smallest ρ satisfying the inequality of Lemma 1 is given by the quantity

$$h(\sigma) = \sup_{x, y \in \mathbb{N}: f_1(x, y, \sigma) > 0} \frac{f_2(x, y, \sigma)}{f_1(x, y, \sigma)}.$$

We divide the set $(x, y) \in \mathbb{N} \times \mathbb{N}$ in lines of the form $x = ay$ and determine the supremum over every line. After that we take the supremum over all lines, which then gives the desired result. We first show that the case $x \leq y$ is trivial. We then focus on $y < x$. In this case, we show that $h(\sigma) = \max\{\gamma_1(\sigma), \gamma_2(\sigma)\}$ for certain functions γ_1 and γ_2 . Numerical experiments suggest that $\Delta(\sigma) := \gamma_1(\sigma) - \gamma_2(\sigma) \geq 0$, that is, the maximum is always attained for γ_1 (which is the definition of h given in the statement). In particular, this means that if for a fixed σ the non-negativity of $\Delta(\sigma)$ is satisfied, then this yields an exact proof of the inequality of Lemma 1 for $h(\sigma) \leq \rho \leq 2\sigma$. The function Δ is specified in the full version of this paper [9]. □

Numerical experiments suggest that $\Delta(\sigma)$ is non-negative for all $\sigma \geq 1/2$. We emphasize that for a fixed σ , with $\Delta(\sigma) \geq 0$, the proof that the inequality holds for all $h(\sigma) \leq \rho \leq 2\sigma$ is exact in the parameter ρ . The first two cases of Theorem 1 capture all the price of anarchy results from the literature.

We next show that the bound of Theorem 1 is also an (asymptotic) lower bound for linear symmetric network congestion games.⁸ This improves a result

⁸ In the the full version [9] we show tightness for general congestion games.

in the risk-uncertainty model of Piliouras et al. [12], who only prove asymptotic tightness for symmetric linear congestion games (for their respective values of ρ and σ). It also improves a result in the altruism model by Chen et al. [5], who show tightness only for general congestion games.

Christodoulou and Koutsoupias [6] showed that for symmetric congestion games ($\rho = \sigma = 1$) the bound of $\frac{5}{2}$ on the price of anarchy is asymptotically tight. More recently, Correa et al. [8] proved that the bound of $\frac{5}{2}$ is tight for symmetric network congestion games. Our lower bound proof is a generalization of their construction.

Theorem 2. *For $\rho, \sigma > 0$ fixed, there exists a symmetric network linear congestion game such that for every $\epsilon > 0$, $PoA(\rho, \sigma) \geq (2\rho(1 + \sigma) + 1)/(\rho + 1) - \epsilon$.*

For $\rho \geq 2\sigma$, we can obtain a tight bound of $\rho + 1$ on the price of anarchy. Remarkably, the bound itself does not depend on σ , only the range of ρ and σ for which it holds. For the parameters $\sigma = 1$ and $\rho \geq 2$ in the altruism model of Caragiannis et al. [4], this bound is known to be tight for non-symmetric singleton congestion games (where all strategies consist of a single resource). We only provide tightness for general congestion games, but the construction is significantly simpler.

Theorem 3. *We have $PoA(\rho, \sigma) \leq \rho + 1$ for $1 \leq 2\sigma \leq \rho$ and this bound is tight.*

4 Price of Stability

We show the bound given in (1) to be an (asymptotically) tight bound for the price of stability for a large range of pairs (ρ, σ) . We need the following technical lemma.

Lemma 2. *For all non-negative integers x and y , and $\sigma \geq 0$ arbitrary, we have*

$$\left(x - y + \frac{1}{2}\right)^2 - \frac{1}{4} + 2\sigma x(x - 1) + (\sqrt{\sigma(\sigma + 2)} + \sigma)[y(y - 1) - x(x - 1)] \geq 0.$$

Theorem 4. *We have*

$$PoS(\rho, \sigma) \leq \frac{\sqrt{\sigma(\sigma + 2)} + \sigma}{\sqrt{\sigma(\sigma + 2)} + \rho - \sigma} \quad \text{for } \sigma > 0 \text{ and } \frac{2\sigma}{1 + \sigma + \sqrt{\sigma(\sigma + 2)}} \leq \rho \leq 2\sigma$$

and this bound is asymptotically tight.

Our proof is similar to a technique of Christodoulou, Koutsoupias and Spirakis [7] for upper bounding the price of stability of ρ -approximate equilibria. However, for a general σ the analysis is more involved. The main technical contribution comes from establishing the inequality in Lemma 2. The proof of the asymptotic tightness is also based on a construction due to Christodoulou et al. [7] used for obtaining a (non-tight) lower bound on the price of stability of approximate equilibria. The lower bound proof is omitted here.

Proof. Note that we can write $C_i^\rho(s) = a_e x_e + b_e + (\rho - 1)a_e x_e$. By Rosenthal [13],

$$\Phi^\rho(s) := \sum_{e \in E} a_e \frac{x_e(x_e + 1)}{2} + b_e x_e + (\rho - 1) \sum_{e \in E} a_e \frac{(x_e - 1)x_e}{2}$$

is an exact potential for $C_i^\rho(s)$. The idea of the proof is to combine the Nash inequalities and the fact that the global minimum of $\Phi^\rho(\cdot)$ is a Nash equilibrium.

Let s denote the global minimum of Φ^ρ and s^* a socially optimal solution. We can without loss of generality assume that $a_e = 1$ and $b_e = 0$. The Nash inequalities (as in the price of anarchy analysis) yield

$$\sum_{e \in E} x_e(1 + \rho(x_e - 1)) \leq \sum_{e \in E} (1 + \rho x_e)x_e^*.$$

The fact that s is a global optimum of $\Phi^\rho(\cdot)$ yields $\Phi^\rho(s) \leq \Phi^\rho(s^*)$, which reduces to

$$\sum_{e \in E} \rho x_e^2 + (2 - \rho)x_e \leq \sum_{e \in E} \rho(x_e^*)^2 + (2 - \rho)x_e^*.$$

If we can find $\gamma, \delta \geq 0$, and some $K \geq 1$, for which

$$(0 \leq) \gamma [\rho(x_e^*)^2 + (2 - \rho)x_e^* - \rho x_e^2 - (2 - \rho)x_e] + \delta [(1 + \rho x_e)x_e^* - x_e(1 + \rho(x_e - 1))] \leq K \cdot x_e^*[1 + \sigma(x_e^* - 1)] - x_e[1 + \sigma(x_e - 1)], \tag{4}$$

then this implies that $C^\sigma(s)/C^\sigma(s^*) \leq K$. We take $\delta = (K - 1)/\rho$ and $\gamma = ((\rho - 1)K + 1)/(2\rho)$. It is not hard to see that $\delta \geq 0$ always holds, however, for γ we have to be more careful. We will later verify for which combinations of ρ and σ the parameter γ is indeed non-negative. Rewriting the expression in (4) yields that we have to find K satisfying $K \geq f_2(x_e, x_e^*, \sigma)/f_1(x_e, x_e^*, \rho, \sigma)$, where

$$f_2(x_e, x_e^*, \sigma) := (x_e^*)^2 - 2x_e x_e^* + (1 + 2\sigma)x_e^2 - x_e^* + (1 - 2\sigma)x_e$$

$$f_1(x_e, x_e^*, \rho, \sigma) := (1 - \rho + 2\sigma)(x_e^*)^2 - 2x_e x_e^* + (1 + \rho)x_e^2 + (\rho - 1 - 2\sigma)x_e^* - (\rho - 1)x_e.$$

Note that this reasoning is correct only if $f_1(x_e, x_e^*, \rho, \sigma) \geq 0$. This is true because

$$f_1(x_e, x_e^*, \rho, \sigma) = \left(x_e - x_e^* + \frac{1}{2}\right)^2 - \frac{1}{4} + (2\sigma - \rho)x_e^*(x_e^* - 1) + \rho x_e(x_e - 1)$$

is non-negative for all $x_e, x_e^* \in \mathbb{N}$, $\sigma \geq 0$ and $0 \leq \rho \leq 2\sigma$. Furthermore, the expression is zero if and only if $(x_e, x_e^*) \in \{(0, 1), (1, 1)\}$. But for these pairs the nominator is also zero, and hence, the expression in (4) is therefore satisfied for those pairs. We can write

$$f_2(x_e, x_e^*, \sigma) = \left(x_e - x_e^* + \frac{1}{2}\right)^2 - \frac{1}{4} + 2\sigma x_e(x_e - 1)$$

and therefore $f_2/f_1 = \frac{A}{A+(2\sigma-\rho)B}$, where

$$A = \left(x_e - x_e^* + \frac{1}{2}\right)^2 - \frac{1}{4} + 2\sigma x_e(x_e - 1) \quad \text{and} \quad B = x_e^*(x_e^* - 1) - x_e(x_e - 1).$$

Note that if $\rho = 2\sigma$, we have $f_2/f_1 = 1$, and hence we can take $K = 1$. Otherwise,

$$\frac{A}{A + (2\sigma - \rho)B} \leq \frac{\sqrt{\sigma(\sigma + 2)} + \sigma}{\sqrt{\sigma(\sigma + 2)} + \rho - \sigma} =: K \Leftrightarrow A + (\sqrt{\sigma(\sigma + 2)} + \sigma)B \geq 0.$$

The inequality on the right is true by Lemma 2.

To finish the proof, we determine the pairs (ρ, σ) for which the parameter γ is non-negative. This holds if and only if

$$(\rho - 1)K + 1 = (\rho - 1) \frac{\sqrt{\sigma(\sigma + 2)} + \sigma}{\sqrt{\sigma(\sigma + 2)} + \rho - \sigma} + 1 \geq 0.$$

Rewriting this yields the bound on ρ in the statement of the theorem. □

The price of anarchy bound of $(1 + 2\rho(1 + \sigma))/(1 + \rho)$ is tight even for symmetric network congestion games with linear cost functions (see Theorem 2). In contrast, this is not true for the price of stability bound (for $\sigma = 1$):

Theorem 5. *Let Γ be a linear symmetric network congestion game, then*

$$PoS(\Gamma, \rho, 1) \leq \begin{cases} 4/(\rho(4 - \rho)) & \text{if } 0 \leq \rho \leq 1 \\ 4/(2 + \rho) & \text{if } 1 \leq \rho \leq 2 \\ (2 + \rho)/4 & \text{if } 2 \leq \rho < \infty. \end{cases}$$

In particular, if Γ is a symmetric congestion game on an extension-parallel⁹ graph G , then the upper bounds even hold for the price of anarchy. All bounds are tight.

For $\rho \geq 1$, the bounds were previously shown by Caragiannis et al. [4] for the price of anarchy of singleton symmetric congestion games (which can be modeled on an extension-parallel graph).

Since any Nash equilibrium under the player cost $C_i^\rho(\cdot)$ is in particular a ρ -approximate Nash equilibrium, we also obtain the following result.

Corollary 1. *The price of stability for ρ -approximate equilibria, with $1 \leq \rho \leq 2$, is upper bounded by $4/(2 + \rho)$ for linear symmetric network congestion games.*

Acknowledgements. We thank the anonymous referees for their very useful comments.

⁹ A graph G is *extension-parallel* if it consists of (i) a single edge, (ii) a single edge and an extension-parallel graph composed in series, or (iii) two extension-parallel graphs composed in parallel.

References

1. Anshelevich, E., Dasgupta, A., Kleinberg, J., Tardos, E., Wexler, T., Roughgarden, T.: The price of stability for network design with fair cost allocation. In: Proceedings of the 45th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2004, pp. 295–304 (2004)
2. Caragiannis, I., Fanelli, A., Gravin, N., Skopalik, A.: Computing approximate pure Nash equilibria in congestion games. *SIGecom Exch.* **11**(1), 26–29 (2012)
3. Caragiannis, I., Kaklamanis, C., Kanellopoulos, P.: Taxes for linear atomic congestion games. *ACM Trans. Algorithms* **7**(1), 13:1–13:31 (2010)
4. Caragiannis, I., Kaklamanis, C., Kanellopoulos, P., Kyropoulou, M., Papaioannou, E.: The impact of altruism on the efficiency of atomic congestion games. In: Wirsing, M., Hofmann, M., Rauschmayer, A. (eds.) *TGC 2010. LNCS*, vol. 6084, pp. 172–188. Springer, Heidelberg (2010). doi:[10.1007/978-3-642-15640-3_12](https://doi.org/10.1007/978-3-642-15640-3_12)
5. Chen, P.A., de Keijzer, B., Kempe, D., Schäfer, G.: Altruism and its impact on the price of anarchy. *ACM Trans. Econ. Comput.* **2**(4), 17:1–17:45 (2014)
6. Christodoulou, G., Koutsoupias, E.: The price of anarchy of finite congestion games. In: Proceedings of the Thirty-seventh Annual ACM Symposium on Theory of Computing, STOC 2005, pp. 67–73. ACM, New York (2005)
7. Christodoulou, G., Koutsoupias, E., Spirakis, P.G.: On the performance of approximate equilibria in congestion games. *Algorithmica* **61**(1), 116–140 (2011)
8. Correa, J., de Jong, J., de Keijzer, B., Uetz, M.: The curse of sequentiality in routing games. In: Markakis, E., Schäfer, G. (eds.) *WINE 2015. LNCS*, vol. 9470, pp. 258–271. Springer, Heidelberg (2015). doi:[10.1007/978-3-662-48995-6_19](https://doi.org/10.1007/978-3-662-48995-6_19)
9. Kleer, P., Schäfer, G.: Tight inefficiency bounds for perception-parameterized affine congestion games. *CoRR* abs/1701.07614 (2017)
10. Koutsoupias, E., Papadimitriou, C.: Worst-case equilibria. In: Meinel, C., Tison, S. (eds.) *STACS 1999. LNCS*, vol. 1563, pp. 404–413. Springer, Heidelberg (1999). doi:[10.1007/3-540-49116-3_38](https://doi.org/10.1007/3-540-49116-3_38)
11. Monderer, D., Shapley, L.S.: Potential games. *Games Econ. Behav.* **14**(1), 124–143 (1996)
12. Piliouras, G., Nikolova, E., Shamma, J.S.: Risk sensitivity of price of anarchy under uncertainty. In: Proceedings of the Fourteenth ACM Conference on Electronic Commerce, EC 2013, pp. 715–732. ACM, New York (2013)
13. Rosenthal, R.W.: A class of games possessing pure-strategy Nash equilibria. *Int. J. Game Theory* **2**, 65–67 (1973)