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Almost Stabilizability Subspaces and High Gain Feedback

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Abstract—**The class of "almost stabilizability subspaces" is introduced as the state-space analog of the class of stable, but not necessarily proper, transfer functions. Almost stabilizability subspaces can be considered as candidate closed-loop eigenspaces associated with infinitely fast and stable modes. We derive the basic properties of these subspaces, and show that they can be approximated by regular stabilizability subspaces. The relation with high gain feedback is elaborated upon in a number of applications.**

I. Introduction

The concept of "almost-invariance" was introduced by J. C. Willems in a series of recent papers [1]–[3] as a geometric means of studying high gain feedback and more generally, asymptotic phenomena in linear systems. In a sense, almost invariant subspaces provide a state space parallel to the frequency-domain use of nonproper transfer functions. Another such parallel was made quite explicit by Hautus [4], who linked the class of "stabilizability subspaces" (which had already appeared, without being named as such, in the work of Wonham and his colleagues in the 1970's [5]) to the set of stable proper transfer functions, which plays a prominent role in recent research like [6]–[8].

Of course, an important role is also played by the class of stable, but not necessarily proper, transfer functions. In this paper, we shall identify the corresponding state-space concept, which we shall term "almost stabilizability subspace." These subspaces can be thought of as candidate closed-loop eigenspaces associated with infinitely fast and stable modes.

The formal definition will be given in Section II, along with a number of basic properties. In Section III the key result is proven that every almost stabilizability subspace can be obtained as the limit of a sequence of stabilizability subspaces. Applications are given in Section IV. Two examples will be discussed of known results that can be reinterpreted in terms of almost stabilizability subspaces. Most of the section, however, is devoted to new results on the problem of stabilization by high gain feedback.

Throughout this paper, we shall work with a fixed finite-dimensional time-invariant linear system, given by

$$x'(t) = Ax(t) + Bu(t) \qquad (x(t) \in \mathcal{X},\ u(t) \in \mathcal{U}) \quad (1.1)$$

(augmented by an observation equation in Section IV-C). The state-space $\mathcal{X}$, the input space $\mathcal{U}$, the system mapping $A: \mathcal{X} \to \mathcal{X}$, and the input mapping $B: \mathcal{U} \to \mathcal{X}$ are all taken over the real field $\mathbb{R}$, but the obvious complexifications will be used where needed without change of notation. The complex number field is denoted by $\mathbb{C}$. The null space and the range of a linear mapping M will be written as $\ker M$ and $\operatorname{Im} M$, respectively. Direct sums of subspaces are indicated by the symbol $\oplus$.

The word "stable" will be used in connection with some "set of stable points" $\mathbb{C}_g \subset \mathbb{C}$ which has been given in advance. We will assume that:

i) $\mathbb{C}_g$ is symmetric about the real axis
ii) $\mathbb{C}_g \cap \mathbb{R} \neq \emptyset$
iii) there exists $c \in \mathbb{R}$ such that $(-\infty, c] \subset \mathbb{C}_g$.

The conditions i) and ii) also appear in [5], but the third condition is new. It is certainly an important restriction because it throws out systems in discrete time. Nevertheless, the condition iii) is crucial for the present paper, as will become apparent below. Frequently, we shall want to be more specific. The most important special choices for $\mathbb{C}_g$ are the open left half plane $\mathbb{C}_- = \{\alpha \in \mathbb{C} \mid \operatorname{Re} \alpha < 0\}$ and the closed left half plane $\bar{\mathbb{C}}_- = \{\alpha \in \mathbb{C} \mid \operatorname{Re} \alpha \leqslant 0\}$.

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II. Definition and Basic Properties

Recall the following definition from [4].

Definition 2.1: A subspace $\mathscr{S}$ of $\mathscr{X}$ is a *stabilizability subspace* if there exists an $F: \mathscr{X} \to \mathscr{U}$ such that $(A+BF)\mathscr{S} \subset \mathscr{S}$ and the restriction of $A+BF$ to $\mathscr{S}$ is stable.

We have the following characterizations of this concept.

Proposition 2.2 [4]: $\mathscr{S}$ is a stabilizability subspace if and only if for every $x \in \mathscr{S}$ there exist stable strictly proper rational functions $\xi(s)$ and $\omega(s)$ such that $\xi(s) \in \mathscr{S}$(for all s) and

$$x = (sI - A)\xi(s) + B\omega(s). \tag{2.1}$$

Proposition 2.3 [9]: $\mathscr{S}$ is a stabilizability subspace if and only if

$$(sI - A)\mathscr{S} + \operatorname{Im} B = \mathscr{S} + \operatorname{Im} B \tag{2.2}$$

for all $s \in \mathbb{C} \backslash \mathbb{C}_g$.

Almost (A, B)-invariant subspaces and almost controllability subspaces were introduced in [1] (see also [10] for a purely algebraic treatment, based on an interpretation in terms of difference equations). Recall the basic result [1] that a subspace $\mathscr{V}_a$ is almost (A, B)-invariant if and only if it can be written as the sum of an (A, B)-invariant subspace [5] and an almost controllability subspace. This is one motivation for the following definition.

Definition 2.4: A subspace $\mathscr{S}_a$ of $\mathscr{X}$ is an *almost stabilizability subspace* if it can be written in the form $\mathscr{S}_a = \mathscr{S} + \mathscr{R}_a$, where $\mathscr{S}$ is a stabilizability subspace, and $\mathscr{R}_a$ is an almost controllability subspace.

Further motivation is provided by the next result.

Proposition 2.5: The following are equivalent.

i) $\mathscr{S}$ is an almost stabilizability subspace.

ii) For every $x \in \mathscr{S}$ there exist stable rational functions $\xi(s)$ and $\omega(s)$ such that $\xi(s) \in \mathscr{S}$ (for all s) and

$$x = (sI - A)\xi(s) + B\omega(s). \tag{2.3}$$

iii) The inclusion

$$\mathscr{S} \subset (sI - A)\mathscr{S} + \operatorname{Im} B \tag{2.4}$$

holds at all points $s \in \mathbb{C} \backslash \mathbb{C}_g$.

The proof follows closely the lines of [10], and will therefore be omitted. Additional support for Definition 2.4 comes from the following proposition.

Proposition 2.6: A subspace $\mathscr{S}$ is a stabilizability subspace if and only if it is both an (A, B)-invariant subspace and an almost stabilizability subspace.

Proof: Combine Proposition 2.3 and Proposition 2.5 iii) with the observation that a subspace $\mathscr{S}$ is (A, B)-invariant if and only if

$$(sI - A)\mathscr{S} \subset \mathscr{S} + \operatorname{Im} B \tag{2.5}$$

for some $s \in \mathbb{C}$ (cf. [5, p. 88]), which is easily seen to be equivalent to the statement that (2.5) holds for all $s \in \mathbb{C}$.

The following direct-sum decomposition is an immediate consequence of the general decomposition given in [2]. Recall that a *sliding subspace* is an almost controllability subspace that does not contain any nonzero controllability subspace, and a *coasting subspace* is an (A, B)-invariant subspace with the same property. If $\mathscr{S}$ is coasting, then $F|_{\mathscr{S}}$ is fixed for F such that $(A+BF)\mathscr{S} \subset \mathscr{S}$ [2].

Proposition 2.7: Every almost stabilizability subspace can be written in the form $\mathscr{S}_a = \mathscr{R} \oplus \mathscr{R}_a \oplus \mathscr{S}$, where $\mathscr{R}$ is a controllability subspace, $\mathscr{R}_a$ is a sliding subspace, and $\mathscr{S}$ is a coasting subspace such that the restriction of $A+BF$ to $\mathscr{S}$ is stable for any F such that $(A+BF)\mathscr{S} \subset \mathscr{S}$.

It is easily checked that the sum of two almost stabilizability subspaces is again an almost stabilizability subspace. Hence, there is a unique largest almost stabilizability subspace in any given subspace $\mathscr{K}$, and we shall denote it by $\mathscr{S}_a^*(\mathscr{K})$. The following characterization closely parallels the one given by Hautus [4] for the largest stabilizability subspace in $\mathscr{K}$, which we shall denote by $\mathscr{S}^*(\mathscr{K})$. The proof follows the lines of [4] and [10] and will be omitted.

Proposition 2.8: $\mathscr{S}_a^*(\mathscr{K})$ equals the set of all $x \in \mathscr{K}$ for which there exist stable rational functions $\xi(s)$ and $\omega(s)$ such that $\xi(s) \in \mathscr{K}$ and

$$x = (sI - A)\xi(s) + B\omega(s). \tag{2.6}$$

Removing the restriction "$x \in \mathscr{K}$," we obtain the following subspace, which will turn out to be more useful.

Definition 2.9: $\mathscr{S}_b^*(\mathscr{K})$ equals the set of all $x \in \mathscr{X}$ for which there exist stable rational functions $\xi(s)$ and $\omega(s)$ such that $\xi(s) \in \mathscr{K}$ and (2.6) holds.

This subspace can be interpreted as the set of all vectors that can serve as initial values for stable and possibly impulsive trajectories that stay in $\mathscr{K}$ for all time [2], [10]. Geometrically, $\mathscr{S}_b^*(\mathscr{K})$ can be characterized as follows. We write $\mathscr{R}_a^*(\mathscr{K})$ for the largest almost controllability subspace in $\mathscr{K}$, and define $\mathscr{R}_b^*(\mathscr{K}) = A\mathscr{R}_a^*(\mathscr{K}) + \operatorname{Im} B$ as in [2].

Proposition 2.10: $\mathscr{S}_b^*(\mathscr{K}) = \mathscr{S}^*(\mathscr{K}) + \mathscr{R}_b^*(\mathscr{K})$.

The proof can be given without difficulty, using the methods of [2] and/or those of [10]. Actually, [2] uses the above formula as the definition of the subspace $\mathscr{S}_b^*(\mathscr{K})$ [see Theorem 18, where one should read $\mathscr{V}^+_{b,\ker H}$ ($\mathscr{S}_b^*(\ker H)$ in our notation) rather than $\mathscr{V}^*_{b,\ker H}$; also, replace $\mathscr{V}^*_{\ker H}$ by $\mathscr{V}^+_{\ker H}$ ($\mathscr{S}^*(\ker H)$ in our notation)].

It follows from the proposition that we can compute $\mathscr{S}_b^*(\mathscr{K})$ if we can compute $\mathscr{S}^*(\mathscr{K})$ and $\mathscr{R}_b^*(\mathscr{K})$. For $\mathscr{R}_b^*(\mathscr{K})$, one can use the algorithm *ACSA* [3]; the subspace $\mathscr{S}^*(\mathscr{K})$ can be obtained by the method described in [5]. Subspace computations as well as eigenvalue evaluations are involved. For the numerical side of this, see, e.g., [33].

A convenient way of finding out whether a given subspace is an almost stabilizability subspace is given by the following rank test (cf. [10]).

Proposition 2.11: Let $\mathscr{S}_a$ be a given subspace. Write $\dim \mathscr{S}_a = k$, $\dim(\mathscr{S}_a + A\mathscr{S}_a + \operatorname{Im} B) = r$. Choose a basis for $\mathscr{X}$ such that the first k basis vectors span $\mathscr{S}_a$ and the first r basis vectors span $\mathscr{S}_a + A\mathscr{S}_a + \operatorname{Im} B$. Let the matrices of A and B with respect to this basis and a given basis in $\mathscr{U}$ be

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \\ 0 \end{pmatrix}. \tag{2.7}$$

Then $\mathscr{S}_a$ is an almost stabilizability subspace if and only if

$$\operatorname{rank}\begin{pmatrix} sI - A_{11} & B_1 \\ -A_{21} & B_2 \end{pmatrix} = r \tag{2.8}$$

for all $s \in \mathbb{C} \backslash \mathbb{C}_g$.

An alternative route leads via the definition and the algorithms given in [2] and [5].

Finally, let us consider what can be said in terms of trajectories about the class of almost stabilizability subspaces, which we defined here in an algebraic way. We present two results: one for the open left half plane; and one for the closed left half plane.

Theorem 2.12: The following are equivalent.

i) $\mathscr{S}$ is an almost stabilizability subspace with respect to the open left half plane $\mathbb{C}_-$.

ii) For every $x_0 \in \mathscr{S}$, there exist constants $C \geq 0$, $\delta > 0$, and $T \geq 0$ such that the following is true. For every $\epsilon > 0$, there exists a smooth control $u_\epsilon(\cdot)$ such that the corresponding state trajectory $x_\epsilon(\cdot)$ satisfies

$$d(x_\epsilon(t), \mathcal{S}) \leqslant \epsilon \qquad \text{for all } t \geqslant 0 \tag{2.9}$$

$$\|x_\epsilon(t)\| \leqslant Ce^{-\delta t} \qquad \text{for all } t \geqslant T. \tag{2.10}$$

Proof: To prove that i) implies ii), we have to show that the class of almost stabilizability subspaces w.r.t. $\mathbb{C}_-$ is contained in the class of subspaces satisfying property ii). Since the latter class is easily seen to be closed under subspace addition, it is, by Definition 2.4, sufficient to show that each stabilizability subspace w.r.t. $\mathbb{C}_-$ satisfies ii) and that each almost controllability subspace satisfies ii) as well. Let $\mathcal{S}$ be a stabilizability subspace w.r.t. $\mathbb{C}_-$. Then there exists $F: \mathcal{X} \to \mathcal{U}$ such that $(A+BF)\mathcal{S} \subset \mathcal{S}$, and $\sigma(A+BF|_{\mathcal{S}}) \subset \mathbb{C}_-$. Let $\delta > 0$ be such that $\operatorname{Re}\lambda < \delta$ for all $\lambda \in \sigma(A+BF|_{\mathcal{S}})$. Then it is clear that property ii) is fulfilled, for a suitable choice of C, for every $\epsilon > 0$, and with $T = 0$, by the feedback control $u(t) = Fx(t)$. Next, let $\mathcal{S}$ be an almost controllability subspace. By the definition given by J. C. Willems [1], there exists, for every $x_0 \in \mathcal{S}$, a $T \geqslant 0$ such that for all $\epsilon > 0$ there exists a smooth control $u_\epsilon(\cdot)$ such that the corresponding state trajectory satisfies (2.9) and (2.10), even with $C = 0$. So this completes the first part of the proof.

For the second part, let $\mathcal{S}$ be a subspace that satisfies property ii), and take $x_0 \in \mathcal{S}$. By Proposition 2.5 iii), it will be sufficient to show that, for all $s \in \mathbb{C}\backslash\mathbb{C}_-$, we have

$$x_0 \in (sI - A)\mathcal{S} + \operatorname{Im} B. \tag{2.11}$$

So, take a fixed $s \in \mathbb{C}$ with $\operatorname{Re} s \geqslant 0$. Let C, δ, and T be the constants mentioned in property ii); take $\epsilon > 0$, and let $u_\epsilon(\cdot)$ be a control such that the corresponding state trajectory $x_\epsilon(\cdot)$ satisfies (2.9) and (2.10). Now, consider the following relation:

$$\frac{d}{dt}\left(e^{-st}x_\epsilon(t)\right) + (sI - A)\left(e^{-st}x_\epsilon(t)\right) = e^{-st}Bu_\epsilon(t). \tag{2.12}$$

It follows from the estimate (2.10) and the assumption $\operatorname{Re} s \geqslant 0$ that the following integrals converge:

$$\int_0^\infty \frac{d}{dt}\left(e^{-st}x_\epsilon(t)\right) dt = -x_0 \tag{2.13}$$

$$\int_0^\infty e^{-st}x_\epsilon(t)\, dt \overset{\text{def}}{=} x_\epsilon. \tag{2.14}$$

From the equality (2.12), we conclude that the integral

$$\int_0^\infty e^{-st}Bu_\epsilon(t)\, dt \overset{\text{def}}{=} Bu_\epsilon \tag{2.15}$$

also converges, and that one has

$$x_0 = (sI - A)x_\epsilon - Bu_\epsilon. \tag{2.16}$$

By elementary calculus, one finds that

$$\int_\tau^\infty e^{-\delta t}\, dt = \epsilon \tag{2.17}$$

for $\tau = -\delta^{-1}\log(\epsilon\delta)$. Set

$$T(\epsilon) = -\delta^{-1}(\log\epsilon + \log\delta). \tag{2.18}$$

For ϵ small enough so that $T(\epsilon) \geqslant T$, we can estimate

$$\begin{aligned} d(x_\epsilon, \mathcal{S}) &= d\left(\int_0^\infty e^{-st}x_\epsilon(t)\, dt, \mathcal{S}\right) \\ &\leqslant \int_0^{T(\epsilon)} |e^{-st}| d(x_\epsilon(t), \mathcal{S})\, dt \\ &\quad + \int_{T(\epsilon)}^\infty |e^{-st}| d(x_\epsilon(t), \mathcal{S})\, dt \\ &\leqslant \int_0^{T(\epsilon)} \epsilon\, dt + \int_{T(\epsilon)}^\infty Ce^{-\delta t}\, dt = \epsilon T(\epsilon) + C\epsilon. \end{aligned} \tag{2.19}$$

Using (2.16), we can now write

$$\begin{aligned} d(x_0, (sI - A)\mathcal{S} + \operatorname{Im} B) &\leqslant d((sI - A)x_\epsilon, (sI - A)\mathcal{S}) \\ &\leqslant \|sI - A\| d(x_\epsilon, \mathcal{S}) \leqslant \|sI - A\|(\epsilon T(\epsilon) + C\epsilon). \end{aligned} \tag{2.20}$$

The right-hand side in this inequality can be made smaller than any given positive amount by taking ϵ sufficiently small. Hence, we must have

$$d(x_0, (sI - A)\mathcal{S} + \operatorname{Im} B) = 0 \tag{2.21}$$

or, $x_0 \in (sI - A)\mathcal{S} + \operatorname{Im} B$. This is what we wanted to prove.

Theorem 2.13: The following are equivalent.

i) $\mathcal{S}$ is an almost stabilizability subspace with respect to the closed left half plane $\overline{\mathbb{C}}_-$.

ii) For every $x_0 \in \mathcal{S}$, there exists an integer $k \geqslant 0$ and constants $C \geqslant 0$, $T \geqslant 0$ such that the following is true. For every $\epsilon > 0$, there exists a smooth control $u_\epsilon(\cdot)$ such that the corresponding state trajectory satisfies

$$d(x_\epsilon(t), \mathcal{S}) \leqslant \epsilon \qquad \text{for all } t \geqslant 0 \tag{2.22}$$

$$\|x_\epsilon(t)\| \leqslant Ct^k \qquad \text{for all } t \geqslant T. \tag{2.23}$$

Proof: The implication i) $\Rightarrow$ ii) can be proved in the same way as above. The proof of the reverse implication is also analogous to the corresponding part of the proof of Theorem 2.12, but a little bit easier. We can again write, for $x_0 \in \mathcal{S}$ and $s \in \mathbb{C}\backslash\overline{\mathbb{C}}_-$ (so $\operatorname{Re} s > 0$):

$$x_0 = (sI - A)x_\epsilon - Bu_\epsilon. \tag{2.24}$$

Here, $u_\epsilon \in \mathcal{U}$ and x_ϵ is obtained from

$$x_\epsilon = \int_0^\infty e^{-st}x_\epsilon(t)\, dt, \tag{2.25}$$

with $x_\epsilon(\cdot)$ satisfying (2.22) and (2.23). This time, we can directly estimate

$$d(x_\epsilon, \mathcal{S}) \leqslant \int_0^\infty |e^{-st}| d(x_\epsilon(t), \mathcal{S})\, dt \leqslant (\operatorname{Re} s)^{-1}\epsilon \tag{2.26}$$

and the proof is completed in the same way as above.

Remark 1: Roughly speaking, what the above results show is that an almost stabilizability subspace with respect to the open left half plane is a subspace having the property that from each point in that subspace there starts an exponentially decaying trajectory staying arbitrarily close to the given subspace. A similar statement, with "exponentially decaying," replaced by "polynomially bounded," holds for the class of almost stabilizability subspaces w.r.t. $\overline{\mathbb{C}}_-$. These statements have to be interpreted carefully, however. In particular, it is not generally possible to give an ϵ-independent bound on $x_\epsilon(t)$ for small t. The point can be illustrated by the example of the triple integrator $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, $\dot{x}_3 = u$. The plane $x_1 = 0$ is an almost controllability subspace, and so we can, for every $\epsilon > 0$, steer to zero in finite time from any given $x(0)$ with $x_1(0) = 0$, without ever making $\|x_1(t)\|$ larger than ϵ. In order to do this, however, it is clear that the derivative of $x_1(\cdot)$ [of which the value at zero is given, $\dot{x}_1(0) = x_2(0)$] must change increasingly fast for ϵ getting smaller and smaller. As a consequence, the second derivative of $x_1(\cdot)$, which is equal to $x_3(\cdot)$, will show a "peaking" behavior near $t = 0$ as ϵ tends to zero. This aspect of almost invariance is studied in more detail in [20].

Remark 2: In [1], the main problem is to prove the equivalence between a characterization of almost controllability subspaces in terms of trajectories (based on the differential equation $\dot{x} = Ax + Bu$) and one in the terms of linear algebra [based on the pair of linear mappings (A, B)]. To go from the "analytic" characterization to the "algebraic" one, we used a technique based on Proposition 2.5 and on integration of (2.12) from zero to infinity. The same technique applies to almost controllability subspaces, and so it provides an alternative to the method of [1] (where the proof was, in fact, not worked out in full detail). The equivalent of Proposition 2.5 for almost controllability subspaces has been proved in [10], and the connection with the "geometric" characterization of [1] has been laid in [21].

Remark 3: There are some obvious questions related to the above results that are not completely answered here. For instance, a natural property to consider would be the following. Let us say that $\mathscr{S}$ satisfies $(R1)$ (for Reviewer no. 1, who suggested this property) if for all $x_0 \in \mathscr{S}$ and for all $\epsilon > 0$ there exists a smooth control $u_\epsilon(\cdot)$ such that the corresponding trajectory satisfies $d(x_\epsilon(t), \mathscr{S}) \leqslant \epsilon$ for all $t \geqslant 0$ and $x_\epsilon(t) \to 0$ as $t \to \infty$. It follows from the two theorems above that the class of subspaces satisfying $(R1)$ contains the class of almost stabilizability subspace w.r.t. $\mathbb{C}_-$, and that it is contained in the class of almost stabilizability subspaces w.r.t. $\overline{\mathbb{C}}_-$. Both inclusions can be proper. To see this, consider first the double integrator $\dot{x}_1 = x_2$, $\dot{x}_2 = u$ (x_1 is "position," x_2 is "velocity"), and let $\mathscr{S}$ be the zero-velocity subspace $\{x \in \mathbb{R}^2 | x_2 = 0\}$. This subspace is not almost stabilizable w.r.t. $\mathbb{C}_-$, as can be verified using either of the characterizations given above, but it does satisfy property $(R1)$. For any positive maximum velocity $\epsilon > 0$, one can steer to zero from any initial state, even in finite time; but note that the amount of time needed grows without bound as ϵ tends to zero. On the other hand, the state space $\mathscr{X}$ for the rather trivial system described by $A = 0$, $B = 0$ is an (almost) stabilizability subspace with respect to $\overline{\mathbb{C}}_-$, but it is certainly not possible to steer any nonzero initial state to the origin. We conjecture, however, that if the pair (A, B) is stabilizable with respect to the open left half plane, then the class of subspaces satisfying $(R1)$ coincides with the class of almost stabilizability subspaces w.r.t. $\overline{\mathbb{C}}_-$.

III. Approximation

The common notion of convergence for sequences of subspaces, which can be derived from the Grassmannian topology, is the following.

Definition 3.1: A sequence of subspaces $\{\mathscr{V}_n\}_n$ is said to converge to a k-dimensional subspace $\mathscr{V}$ if $\dim \mathscr{V}_n = k$ for all sufficiently large n, and the following holds. For every basis $\{x_1, \cdots, x_k\}$ of $\mathscr{V}$, there exist k sequences of vectors $\{x_1^n\}_n, \cdots, \{x_k^n\}_n$ such that $\{x_1^n, \cdots, x_k^n\}$ is a basis for $\mathscr{V}_n$ for sufficiently large n, and $x_j^n \to x_j$ as $n \to \infty$ for each $j \in \{1, \cdots, k\}$.

The main result of this section is as follows.

Theorem 3.2: For every almost stabilizability subspace $\mathscr{V}_a$, there exists a sequence of stabilizability subspaces $\{\mathscr{V}_n\}_n$ converging to it.

For the proof, we need some preliminary lemmas. The first one of these can be proved by standard means.

Lemma 3.3: Suppose that $\mathscr{V} = \mathscr{V}_1 \oplus \cdots \oplus \mathscr{V}_r$, and suppose also that we have sequences $\{\mathscr{V}_n^j\}_n$ converging to $\mathscr{V}_j$, for each $j \in \{1, \cdots, r\}$. Then the subspaces $\mathscr{V}_n^1, \cdots, \mathscr{V}_n^r$ are linearly independent for all sufficiently large n. Moreover, if we define $\mathscr{V}_n = \mathscr{V}_n^1 \oplus \cdots \oplus \mathscr{V}_n^r$ for these n, then the sequence $\{\mathscr{V}_n\}_n$ converges to $\mathscr{V}$.

It has been shown in [1] that every subspace $\mathscr{L}$ of the form

$$\mathscr{L} = \operatorname{span}\left\{b, (A+BF)b, \cdots, (A+BF)^k b\right\} \tag{3.1}$$

(with $b \in \operatorname{Im} B$, $F\colon \mathscr{X} \to \mathscr{U}$, $k \in \mathbb{Z}_+$) is an almost controllability subspace. Let us call subspaces of this form *singly-generated* almost controllability subspaces. The next lemma is a direct consequence of the results of [1].

Lemma 3.4: Every almost controllability subspace $\mathscr{R}_a$ can be written as a direct sum of singly generated almost controllability subspaces.

We now proceed to the proof of the main result.

Proof (of Theorem 3.2): In view of Proposition 2.7, Lemma 3.3, and Lemma 3.4, it is sufficient to prove that every singly generated almost controllability subspace can be approximated by a sequence of stabilizability subspaces. Since the set of stabilizability subspaces for the pair $(A + BF, B)$ is the same as that for the pair (A, B), we may restrict ourselves to the case of an almost controllability subspace given by

$$\mathscr{R}_a = \operatorname{span}\{b, Ab, \cdots, A^{k-1}b\} \tag{3.2}$$

with $b \in \operatorname{Im} B$ and $k \in \mathbb{N}$. For this subspace, an approximating sequence of stabilizability subspaces can be constructed in the following way.

Clearly, the mapping $I + (1/n)A$ will be invertible for all sufficiently large n, so that we can define a sequence $\{\mathscr{V}_n\}_n$ by

$$\mathscr{V}_n = \left(I + \frac{1}{n}A\right)^{-k} \mathscr{R}_a. \tag{3.3}$$

It is immediate that $\mathscr{V}_n \to \mathscr{R}_a$ and $n \to \infty$, so it remains to show that $\mathscr{V}_n$ is a stabilizability subspace for all sufficiently large n. To do this, we use Proposition 2.3. First note that, for all n,

$$\mathscr{R}_a = \operatorname{span}\left\{b, \left(I + \frac{1}{n}A\right)b, \cdots, \left(I + \frac{1}{n}A\right)^{k-1} b\right\} \tag{3.4}$$

so that we have

$$\mathscr{V}_n = \operatorname{span}\left\{\left(I + \frac{1}{n}A\right)^{-k} b, \cdots, \left(I + \frac{1}{n}A\right)^{-1} b\right\}. \tag{3.5}$$

Noting that

$$sI - A = n\left[\left(1 + \frac{s}{n}\right)I - \left(I + \frac{1}{n}A\right)\right] \tag{3.6}$$

one obtains

$$(sI - A)\mathscr{V}_n = \operatorname{span}\left\{\left(1 + \frac{s}{n}\right)\left(I + \frac{1}{n}A\right)^{-k} b - \left(I + \frac{1}{n}A\right)^{-k+1} b, \right.$$
$$\left. \cdots, \left(1 + \frac{s}{n}\right)\left(I + \frac{1}{n}A\right)^{-1} b - b\right\}. \tag{3.7}$$

From this formula, we can read off that

$$(sI - A)\mathscr{V}_n + \operatorname{span}\{b\} = \mathscr{V}_n + \operatorname{span}\{b\} \tag{3.8}$$

for all $s \neq -n$. Since $-n \in \mathbb{C}_g$ for all sufficiently large n, we have, for these values of n,

$$(sI - A)\mathscr{V}_n + \operatorname{Im} B = \mathscr{V}_n + \operatorname{Im} B \tag{3.9}$$

for all $s \in \mathbb{C} \setminus \mathbb{C}_g$. This is what we needed to prove.

Remark 1: If we think of almost controllability subspaces as invariant subspaces for the infinite modes of a closed-loop system (called into existence by infinite-gain feedback), then the theorem can be interpreted as saying that, in this context, "infinity" can always be read as "minus infinity."

Remark 2: It is essential for the proof that the stable part of the complex plane contains points of arbitrarily large modulus. If this assumption is not satisfied, the argument breaks down and the theorem no longer holds true, as can be seen from the following example:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.10}$$

It is easily verified that the one-dimensional stabilizability subspaces for (A, B) are spanned by vectors of the form $(1 \quad \mu)^T$, with $\mu \in \mathbb{C}_g$. So, if $\mathbb{C}_g$ would be bounded, it would not be possible to find a sequence of stabilizability subspaces converging to the almost controllability subspace Im B.

Remark 3: In a discrete-time context, one cannot define the concept of almost controllability subspaces as in [1]—or rather, the concept one obtains in this way is the same as the old concept of controllability subspaces. However, the class of almost controllability subspaces is described by an algorithm ("*ACSA*"—see [1]) that does not depend on a continuous-time interpretation, and so one may ask what the class of subspaces described by *ACSA* means in discrete time. J. C. Willems [3], [22] has shown that one should relate this class in discrete time to *anticipating action*, just as it relates to *differentiating action* in continuous time. Theorem 3.2 and the above remark can then be interpreted as a special instance of the general statement: there is a stable way to approximate differentiation, but not to approximate anticipation. We may conclude that the combination of concepts proposed in Definition 2.4 probably does not make much sense for systems in discrete time.

Remark 4: An interesting question is, whether the approximation that was employed above can also be used to obtain a suitable feedback control in the sense of Theorems 2.12 and 2.13. That is, if one constructs a sequence of feedback mappings corresponding to a sequence of stabilizability subspaces approximating a given almost stabilizability subspace $\mathscr{S}_a$, will the corresponding trajectories (with initial points in $\mathscr{S}_a$) keep closer and closer to the subspace $\mathscr{S}_a$? The main issue here is that the effect of the decrease of the angle between the given subspace and the approximating subspaces is counteracted by the "peaking" effect discussed in Remark 1 following Theorem 2.13. One has to compute in order to find out what the overall result will be. Computations in [23] lead to the conclusion that the maximal distance will converge to zero provided a proper approximation scheme is used. Although the formulation in [23] differs slightly from ours in Theorems 2.12 and 2.13, the result still indicates that high-gain feedback is feasible as a method to obtain the trajectories that are required in these theorems, as one would hope would be the case. Note that the control laws used by J. C. Willems in [1] are open-loop in an essential way, since they are able to take an initial state to zero in finite time.

IV. Applications

A. Singular Optimal Control

Consider the linear system

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ z(t) = Hx(t) \end{cases} \tag{4.1}$$

with associated cost functional

$$J_\epsilon(x_0) = \min_u \int_0^\infty (\|z(t)\|^2 + \epsilon^2 \|u(t)\|^2)\, dt. \tag{4.2}$$

It is assumed that the pair (A, B) is stabilizable, the pair (H, A) is detectable, B is full column rank, and H is full row rank. Under these conditions, the following result has been given by Francis [11].

Theorem 4.1: $J_\epsilon(x_0)$ tends to 0 as $\epsilon \downarrow 0$ if and only if

$$x_0 \in \mathscr{S}_b^*(\ker H) \tag{4.3}$$

where $\mathscr{S}_b^*(\ker H)$ has to be defined with respect to $\overline{\mathbb{C}}_-$ (i.e., $\mathbb{C}_g = \overline{\mathbb{C}}_-$).

A similar result is given in [2]. Of course, Francis did not formulate his theorem in terms of almost invariant subspaces, but it can be seen from the algorithms he uses that the subspace constructed in [11] is exactly $\mathscr{S}_b^*(\ker H)$, in the above interpretation. The fact that the closed left half-plane is important here (rather than the open LHP) is quite interesting in the light of Theorems 2.12 and 2.13; direct connections remain to be worked out.

B. Solvability of a Rational Matrix Equation

Let $R_1(s)$ and $R_2(s)$ be strictly proper rational transfer matrices, of sizes $p \times m$ and $p \times r$, respectively. Under various circumstances, it is important to know whether the equation

$$R_1(s)X(s) = R_2(s) \tag{4.4}$$

has a solution in the set of stable rational transfer matrices. Suppose that we have a realization for the transfer matrix $[R_1(s) \quad R_2(s)]$:

$$[R_1(s) \quad R_2(s)] = H(sI - A)^{-1}[B \quad G]. \tag{4.5}$$

Then we should be able to state the solvability conditions for (4.4) in terms of the matrices H, A, B, and G. Indeed, the following result was essentially proved by Bengtsson [12].

Theorem 4.2: Suppose that the realization given by (4.5) is observable. Then the equation (4.4) has a stable rational solution if and only if

$$\operatorname{Im} G \subset \mathscr{S}_b^*(\ker H). \tag{4.6}$$

This condition is quite closely related to the condition of Theorem 4.1, and, in fact, the proof of Theorem 4.1 in [11] is based on Bengtsson's result. For more on connections between control problems in the time domain and solvability of certain rational matrix equations, see, e.g., [3] and [4].

C. High Gain Feedback

We shall now concentrate on an application of a different type, involving the notion of "almost stabilizability subspace" itself rather than an $\mathscr{S}_b^*$-space. Our concern will be with dynamic output feedback rather than state feedback, so we consider the controlled and observed linear system

$$\begin{cases} x'(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t). \end{cases} \tag{4.7}$$

In addition to our earlier notational conventions, we denote the output space by $\mathscr{Y}$, and we set $\dim \mathscr{Y} = p$. We shall need the following concept. A subspace $\mathscr{T}$ of $\mathscr{X}$ will be called a *minimum-phase input subspace* if there exists a mapping $T: \mathscr{Y} \to \mathscr{X}$ such that:

$$\mathscr{T} = \operatorname{Im} T \tag{4.8}$$

$$\det CT \neq 0 \tag{4.9}$$

$$\det(sI - A)\det C(sI - A)^{-1}T \neq 0 \qquad \text{for } s \in \mathbb{C}_+. \tag{4.10}$$

The condition (4.10) says that the system with input mapping T, state mapping A, and output mapping C should have no unstable zeros (cf. [13, p. 41]), and this motivates the terminology. Note, however, that we also require CT to be invertible. A characterization in terms of the subspace $\mathscr{T}$ itself can be given as follows.

Lemma 4.3: A subspace $\mathscr{T}$ is a minimum-phase input subspace if and only if $\mathscr{T} \oplus \ker C = \mathscr{X}$ and the restriction of PA to $\ker C$ is stable, where P is the projection onto $\ker C$ along $\mathscr{T}$.

Proof: Suppose that $\mathscr{T}$ is a minimum-phase input subspace, and let T be a mapping satisfying (4.8)–(4.10). It follows from

(4.9) that $\mathcal{T}\oplus\ker C=\mathcal{X}$. We see that $T(CT)^{-1}$ also satisfies (4.8)–(4.10) so we may as well assume that $CT=I$. Then the projection P onto $\ker C$ along $\mathcal{T}$ is simply given by

$$P=I-TC. \tag{4.11}$$

The subspace $\ker C$ is, of course, invariant for PA, and the factor mapping induced by PA on the quotient space $\mathcal{X}/(\ker C)$ is clearly 0. Therefore, we have

$$\det(sI-PA)=s^p\det(sI-PA|_{\ker C}). \tag{4.12}$$

Now, consider the following manipulations, in which we use the determinant equality $\det(I+MN)=\det(I+NM)$, the rule $A(sI-A)^{-1}=s(sI-A)^{-1}-I$, and the fact that $CT=I$.

$$\begin{aligned}\det(sI-PA)&=\det(sI-(I-TC)A)\\&=\det(sI-A)\det\left(I+TCA(sI-A)^{-1}\right)\\&=\det(sI-A)\det\left(I+CA(sI-A)^{-1}T\right)\\&=\det(sI-A)\det\left(I-CT+sC(sI-A)^{-1}T\right)\\&=s^p\det(sI-A)\det C(sI-A)^{-1}T.\end{aligned} \tag{4.13}$$

Comparing this to (4.12), we see that (4.10) implies that $PA|_{\ker C}$ is stable.

For the converse, suppose that we have a subspace $\mathcal{T}$ that satisfies the condition of the lemma. Then it is easily verified that there exists a (unique) mapping T: $\mathcal{Y}\to\mathcal{X}$ such that $TC=I-P$, and that this mapping satisfies (4.8) and (4.9). (In fact, $CT=I$.) Moreover, it is immediate from (4.12) and (4.13) that (4.10) holds.

The following result was proved in [14, Lemma 2.12]; see also [15, Theorem 4.4 and Lemma 5.1].

Theorem 4.4: Suppose that, for the system (4.7), we have a stabilizability subspace $\mathcal{V}$ that contains a minimum-phase input subspace. Let the dimension of $\mathcal{V}$ be k. Then the system (4.7) can be stabilized by dynamic output feedback of the form

$$\begin{cases}w'(t)=A_cw(t)+G_cy(t), & w(t)\in\mathcal{W}\\u(t)=F_cw(t)+Ky(t)\end{cases} \tag{4.14}$$

where the order of the feedback dynamics (i.e., $\dim\mathcal{W}$) is equal to $k-p$. In particular, if $k=p$, then the system (4.7) can be stabilized by direct output feedback alone. We now want to prove the same theorem, but with "stabilizability subspace" replaced by "almost stabilizability subspace." The idea is that there are stabilizability subspaces arbitrarily close to a given almost stabilizability subspace (Theorem 3.2) and, on the other hand, any subspace that is close enough to a minimum-phase input subspace will itself be a minimum-phase input subspace. Let us first formally establish the latter fact.

Lemma 4.5: Let $\mathcal{T}$ be a minimum-phase input subspace, and let $\{\mathcal{T}_n\}$ be a sequence of subspaces converging to $\mathcal{T}$. Then $\mathcal{T}_n$ is a minimum-phase input subspace for all sufficiently large n.

Proof: Let T be a mapping satisfying (4.8)–(4.10), as in the proof of Lemma 4.3, we may assume that $CT=I$. By the definition of convergence, there exists a basis $\{x_1,\cdots,x_p\}$ for $\mathcal{T}$ and a corresponding basis $\{x_1^n,\cdots,x_p^n\}$ for each $\mathcal{T}_n$, such that $\{x_i^n\}$ converges to x_i for each $i=1,\cdots,p$. Define $y_i=Cx_i$ ($i=1,\cdots,p$). Then $\{y_1,\cdots,y_p\}$ is a basis for $\mathcal{Y}$, and $x_i=Ty_i$ ($i=1,\cdots,p$). Define T_n: $\mathcal{Y}\to\mathcal{X}$ for each n by $T_ny_i=x_i^n$ ($i=1,\cdots,p$). Obviously, we have $T_n\to T$. Hence, we also have $CT_n\to CT=I$ which implies that CT_n will be invertible for all sufficiently large n. In other words, $\mathcal{T}_n\oplus\ker C=\mathcal{X}$ for these values of n. The projection along $\mathcal{T}_n$ onto $\ker C$ is given by

$$P_n=I-T_n(CT_n)^{-1}C. \tag{4.15}$$

By the continuity of matrix inversion and multiplication, it follows that $P_n\to I-T(CT)^{-1}C=I-TC=P$, the projection onto $\ker C$ along $\mathcal{T}$. This implies, in particular, that the sequence of mappings $\{P_nA|_{\ker C}\}$ converges to $PA|_{\ker C}$. By the continuity property of the eigenvalues [16, p. 191], it follows that $P_nA|_{\ker C}$ is stable for all sufficiently large n. An appeal to Lemma 4.3 now completes the proof.

We are now in a position to prove the main result of this section.

Theorem 4.6: Suppose that, for the system (4.7), we have an almost stabilizability subspace $\mathcal{V}$ that contains a minimum-phase input subspace $\mathcal{T}$. Let the dimension of $\mathcal{V}$ be k. Then the system (4.7) can always be stabilized by dynamic output feedback of the form (4.14), where the order of the feedback dynamics is equal to $k-p$.

Proof: Let $\{\mathcal{V}_n\}$ be a sequence of stabilizability subspaces converging to $\mathcal{V}$. Then there exists a sequence $\{\mathcal{T}_n\}$, with $\mathcal{T}_n\subset\mathcal{V}_n$ for each n, such that $\{\mathcal{T}_n\}$ converges to $\mathcal{T}$. According to Lemma 4.5, $\mathcal{T}_n$ will be a minimum-phase input subspace for all sufficiently large n. Take such an n, and apply Theorem 4.4 to the corresponding $\mathcal{V}_n$ and $\mathcal{T}_n$.

We immediately have the following corollary.

Corollary 4.7: Suppose that the system (4.7) is square and minimum-phase, and also suppose that the matrix CB is invertible. Then the system (4.7) can be stabilized by direct output feedback alone, i.e., there exists K such that $A+BKC$ is stable.

Proof: If suffices to note that $\operatorname{Im}B$ is an almost controllability subspace and that the assumptions of the corollary imply that $\operatorname{Im}B$ is also a minimum-phase input subspace. The result then follows from an application of Theorem 4.6.

Remark 1: The proof of the theorem is constructive, once the pair $(\mathcal{T},\mathcal{V})$ has been given. To illustrate this, consider the situation of the corollary. First of all, we have to find a sequence of stabilizability subspaces approximating $\operatorname{Im}B$. This subspace is the direct sum of the singly generated almost controllability subspaces spanned by the vectors from a basis for $\operatorname{Im}B$. The simplest approximation scheme suggested by the proof of Theorem 3.2 is given by

$$\mathcal{V}_n=\left(I+\frac{1}{n}A\right)^{-1}(\operatorname{Im}B). \tag{4.16}$$

Other options are also available: see [23]. Now we have to find a mapping T_n such that $CT_n=I$ and $\operatorname{Im}T_n=\mathcal{V}_n$. This is of course solved by

$$T_n=\left(I+\frac{1}{n}A\right)^{-1}B\left[C\left(I+\frac{1}{n}A\right)^{-1}B\right]^{-1}. \tag{4.17}$$

Note that it follows from the invertibility of CB that the inverse at the right-hand side does exist, for n large enough. By an easy calculation, one finds that F_n is such that $(A+BF_n)\mathcal{V}_n\subset\mathcal{V}_n$ if and only if

$$F_n\left(I+\frac{1}{n}A\right)^{-1}B=-nI. \tag{4.18}$$

In this case, the restricted mapping $A+BF_n|\mathcal{V}_n$ has a single eigenvalue at $-n$, with geometric (and algebraic) multiplicity $m=\dim\operatorname{Im}B$. As is indicated in [15] (see the proof of Lemma 5.1 of that paper), the corresponding gain matrix K_n is given by

$$K_n=F_nT_n=-\frac{1}{n}\left[C\left(I+\frac{1}{n}A\right)^{-1}B\right]^{-1}. \tag{4.19}$$

From the theory above, it follows that the closed-loop mapping $A+BK_nC$ will have an eigenvalue of multiplicity m at $-n$, while the other eigenvalues will approach the zeros of $\det(sI-A)\det C(sI-A)^{-1}B$ as n goes to infinity. So, for n large enough, a stabilizing feedback gain matrix will have been obtained. Of course, this can also be verified by direct computation.

Remark 2: The problem is, then, to find a suitable pair of minimum-phase input subspace $\mathcal{T}$ and an almost stabilizability subspace $\mathcal{V}$. For instance, one may fix $\mathcal{T}$ and look for an almost stabilizability subspace $\mathcal{V}$ containing it. (Minimum-phase input subspaces exist for (4.7) if and only if the pair (C, A) is detectable—see [15, Lemmas 5.2 and 5.3].) Of course, one can always take $\mathcal{V} = \mathcal{X}$, at least under the assumption that the pair (A, B) is stabilizable, but one would be more interested in finding low-dimensional solutions. This is a version of the so-called "stable cover problem" (see, e.g., [24]), which is known to be difficult. This version is somewhat relaxed with respect to the original version, since it asks for an almost stabilizability subspace rather than a proper stabilizability subspace "covering" a given subspace, but this does not seem to make the problem much easier.

Conversely, given an almost stabilizability subspace $\mathcal{V}$, when does there exist a minimum-phase input subspace $\mathcal{T}$ that is contained in $\mathcal{V}$? Of course, a necessary condition is that $\mathcal{V} + \ker C = \mathcal{X}$. Given this, one can find a direct sum decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ with $\mathcal{X}_1 = \ker C$ and $\mathcal{X}_2 \subset \mathcal{V}$. With respect to this decomposition, write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$
$$C = (0 \quad C_2), \qquad \mathcal{V} = \operatorname{Im}\begin{pmatrix} V_1 & 0 \\ 0 & I \end{pmatrix}. \tag{4.20}$$

By Lemma 4.3, there exists a minimum-phase input subspace $\mathcal{T}$ contained in $\mathcal{V}$ if and only if there is a projection

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \tag{4.21}$$

such that $\operatorname{Im} P = \ker C$, $\ker P \subset \mathcal{V}$ and $PA|_{\ker C}$ is stable. The matrix P given above is a projection onto $\ker C$ if and only if $P_{11} = I$, $P_{21} = 0$, and $P_{22} = 0$. The restriction of PA to $\ker C$ is then easily computed as

$$PA|_{\ker C} = A_{11} + P_{12}A_{21}. \tag{4.22}$$

Moreover, one has

$$\ker P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| x_1 + P_{12}x_2 = 0 \right\} = \left\{ \begin{pmatrix} -P_{12}x_2 \\ x_2 \end{pmatrix} \middle| x_2 \in \mathcal{X}_2 \right\} \tag{4.23}$$

and so $\ker P \subset \mathcal{V}$ if and only if $\operatorname{Im} P_{12} \subset \operatorname{Im} V_1$ [see (4.20)]. So, given that $\mathcal{V} + \ker C = \mathcal{X}$, there exists a minimum-phase input subspace $\mathcal{T}$ contained in the given subspace $\mathcal{V}$ if and only if there exists a matrix M such that $A_{11} + V_1MA_{21}$ is stable.

Therefore, it turns out that the problem of finding a minimum-phase input subspace within a given subspace (which is, in fact, a version of the dualized cover problem—cf. [15]) is equivalent to a problem of stabilization by static output feedback. Unfortunately, this is again an unsolved problem. In some special cases, however, a solution is known. An example is provided by Corollary 4.7 above, and an even simpler example occurs in the following situation. Consider a system $\Sigma(A, B, C)$, and suppose that $\operatorname{Im} B$ is A-invariant. Decomposing $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ with $\mathcal{X}_2 = \operatorname{Im} B$, we can write

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}, \quad C = (C_1 \quad C_2) \tag{4.24}$$

with B_2 invertible. We have

$$A + BKC = \begin{pmatrix} A_{11} & 0 \\ A_{21} + B_2KC_1 & A_{22} + B_2KC_2 \end{pmatrix} \tag{4.25}$$

and it is clear that there exists K such that $A + BKC$ is stable if and only if A_{11} is stable and the pair (A_{22}, C_2) is detectable. This may seem rather contrived, but consider the following application, which generalizes Corollary 4.7.

Corollary 4.8: Suppose that the system (4.7) is square and minimum-phase. Suppose also that, for some $l \in \mathbb{N}$, we have $CB = 0$, $CAB = 0, \cdots, CA^{l-2}B = 0$ and $CA^{l-1}B$ is invertible. Then the system (4.7) can be stabilized by a compensator of order $(l-1)m$, where m is the number of inputs (= the number of outputs).

Proof: We first show that, under the given circumstances,

$$\mathcal{X} = \mathcal{V}^* \oplus \mathcal{B} \oplus A\mathcal{B} \oplus \cdots \oplus A^{l-1}\mathcal{B} \qquad (\mathcal{B} = \operatorname{Im} B) \tag{4.26}$$

where $\mathcal{V}^*$ denotes, as usual, the largest (A, B)-invariant subspace contained in $\ker C$. The transfer function of the given system is invertible, and this means (see [25]) that $\mathcal{X} = \mathcal{V}^* \oplus \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the smallest (C, A)-invariant subspace containing $\operatorname{Im} B$. The subspace $\mathcal{S}^*$ can be computed by the algorithm of [26]:

$$\mathcal{S}^0 = \{0\}$$
$$\mathcal{S}^{k+1} = A(\mathcal{S}^k \cap \ker C) + \operatorname{Im} B. \tag{4.27}$$

Under the given conditions, we obtain

$$\mathcal{S}^* = \mathcal{B} + A\mathcal{B} + \cdots + A^{l-1}\mathcal{B}. \tag{4.28}$$

The transfer function of the given system has m zeros at infinity, each of order l, and this implies [27] that

$$\dim \mathcal{S}^k - \dim \mathcal{S}^{k-1} = m, \qquad k = 1, \cdots, l. \tag{4.29}$$

Consequently, the sum in (4.28) must be direct.

Corresponding to the decomposition (4.26), we can find bases for $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{U}$ such that the matrices of A, B, and C can be given in block form as follows (letting indexes run from 0 to l for convenience):

$$A = \begin{pmatrix} A_{00} & 0 & \cdots & 0 & A_{0l} \\ A_{10} & 0 & & & \\ 0 & I & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \cdots & 0 & I & A_{ll} \end{pmatrix}_{n \times n} \qquad B = \begin{pmatrix} 0 \\ I \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times m}$$
$$C = (0 \quad \cdots \quad 0 \quad I)_{m \times n} \tag{4.30}$$

(cf. also [28]). We select $\mathcal{V} = \mathcal{S}^* = \mathcal{B} \oplus \cdots \oplus A^{l-1}\mathcal{B}$; note that $\dim \mathcal{V} = lm$. With respect to the same basis of $\mathcal{X}$, we can write

$$\mathcal{V} = \operatorname{Im} \begin{pmatrix} 0 & \cdots & & 0 \\ I & \ddots & & \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I \end{pmatrix}_{n \times lm} . \tag{4.31}$$

According to the remarks above, we will be able to construct a compensator of order $(l-1)m$ if the triple

$$\tilde{A}=\begin{pmatrix} A_{00} & 0 & \cdots & \cdots & 0 \\ A_{10} & 0 & & & \\ 0 & I & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & I & 0 \end{pmatrix}_{(n-m)\times(n-m)}$$

$$\tilde{B}=\begin{pmatrix} 0 & \cdots & & 0 \\ I & \ddots & & \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I \end{pmatrix}_{(n-m)\times(l-1)m}$$

$$\tilde{C}=(0 \quad \cdots \quad 0 \quad 0 \quad I)_{m\times(n-m)} \tag{4.32}$$

allows stabilization by static output feedback. Fortunately, the subspace Im $\tilde{B}$ is $\tilde{A}$-invariant. So we have to check whether the pair constituted by

$$\tilde{A}_{22}=\begin{pmatrix} 0 & \cdots & & 0 \\ I & \ddots & & \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I \end{pmatrix}, \quad \tilde{C}_2=[0 \quad \cdots \quad 0 \quad I] \tag{4.33}$$

is detectable. This is obviously true (the pair is even observable). Finally, the matrix $\tilde{A}_{11}=A_{00}$ needs to be stable. From (4.30), it is clear that the matrix of $A+BF|_{\mathscr{V}^*}$, for any F such that $(A+BF)\mathscr{V}^*\subset\mathscr{V}^*$, is given by A_{00}. Since it is well-known that the zeros of the system (4.7) are given by the eigenvalues of this mapping [29], [30], the stability of A_{00} is guaranteed exactly by the minimum-phase condition. This completes the proof.

Remark 3: The nice thing about the above corollary is that the bound it gives for the compensator order does not depend on the order of the given system. Under suitable hypotheses, it should be expected that the result is also valid for a class of linear systems with infinite-dimensional state space.

Remark 4: The hypotheses of Corollary 4.7 are well-known to provide excellent circumstances for high-gain feedback. In root-locus terms, they mean that there are only first-order asymptotic root loci, and that the finite termination points are all stable. Recently, these hypotheses have turned up in a study of robust controller design via LQG techniques [17] and an investigation into controller design for largely unknown systems [18]. The more general hypotheses of Corollary 4.8 are also not new: they have been used in the study of "cheap control" via singular perturbations (see, e.g., [31] and [28]).

Remark 5: For single-input-single-output systems, it is quite easy to prove Corollary 4.7 by a root-locus argument. Indeed, in this case there is only one pole that goes off to infinity as the gain is increased, and so stability can be guaranteed by selecting the right sign of the gain. Still in the SISO situation, Corollary 4.8 says that a minimum-phase system can be stabilized by a compensator of order one less than the pole-zero excess. In the root-locus terminology, dynamic compensation comes down to insertion of extra poles and zeros so as to influence the asymptotic pattern, and one should be able to prove Corollary 4.8 also from this point of view. Modern multivariable root-locus techniques (see, e.g., [19]) are probably capable of extending these arguments to the MIMO case, but the amount of asymptotic analysis involved might become a problem. In our approach, the asymptotic analysis has been locked up in the proofs of Lemma 4.5 and Theorem 4.6, allowing us to derive further results on high-gain feedback using only methods from linear algebra (as developed in the "geometric approach" to linear systems theory). Thus, we were able to derive Corollary 4.8 in a fairly "clean" way.

Remark 6: It is probably worthwhile to look for further corollaries to Theorem 4.6, but one should expect that results of the type of Corollary 4.8 are only obtainable for rather special classes of systems. A more powerful approach can be developed by again bringing in a certain amount of analysis, but this time based on a richer algebraic structure. For instance, one idea would be the following. Suppose that we have a minimum-phase input subspace $\mathscr{T}$ and an almost stabilizability subspace $\mathscr{V}$ (such that $\mathscr{V}+\ker C=\mathscr{X}$), but that we do not have $\mathscr{T}\subset\mathscr{V}$. Then one could try to "match" the two subspaces, using the freedom suggested by Lemma 4.5, by changing the subspace $\mathscr{T}$ to a nearby $\tilde{\mathscr{T}}$ which does satisfy $\tilde{\mathscr{T}}\subset\mathscr{V}$ and which, hopefully, is still a minimum-phase input subspace. Of course, this calls for a discussion of closeness of subspaces and of right directions in which to turn subspaces, and the arguments will probably be reminiscent of those in root-locus analysis. But one would hope that the fact that the analysis is performed using a more extensive algebraic background will allow one to obtain better results. These remarks remain very tentative; we just note that a similar "matching" technique has been applied quite successfully to find low-order compensators for examples of infinite-dimensional systems in [31] and [14].

V. Conclusions

The purpose of this paper has been to introduce the concept of "almost stabilizability subspaces." We gave a number of equivalent characterizations of this class of subspaces, and linked it to the class of stable but not necessarily proper transfer functions. We established the important fact that almost stabilizability subspaces can be viewed as limits of regular stabilizability subspaces. Several applications were discussed, and special emphasis has been placed on the role that almost stabilizability subspaces can play in the study of high gain feedback. The results that we obtain suggest that we might have a way here to develop a general theory, which escapes the one-parameter framework that is so often characteristic both for root-locus and for LQG techniques. However, our results are only preliminary, and much work in this direction remains to be done.

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