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UNIFORM ASYMPTOTIC EXPANSIONS OF LAPLACE INTEGRALS

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Uniform asymptotic expansions of Laplace integrals *)

by

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ABSTRACT

Three different, although related, expansions are considered for the Laplace integral

$$F_\lambda(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} f(t)\, dt$$

for $z \to \infty$; λ and z are real or complex. The expansions are uniformly valid with respect to $\mu := \lambda/z$ in an unbounded domain, containing the point $\mu = 0$. The asymptotic nature of the uniform expansions is discussed and error bounds are given for the remainders in the expansions. Analogue expansions are given for the loop integral

$$G_\lambda(z) = \frac{\Gamma(\lambda+1)}{2\pi i} \int_{-\infty}^{(0^+)} t^{-\lambda-1} e^{zt} f(t)\, dt.$$

A simple example with $f(t) = 1/(1+t)$ is considered, giving expansions for the well-known exponential integral. Further applications are given for Whittaker functions and the Riemann zeta function. The expansions in this paper can be viewed as uniform versions of the expansions obtained by Watson's lemma.

KEY WORDS & PHRASES: *Uniform asymptotic expansions, Laplace integrals, Watson's lemma, exponential integral, error bounds for remainder*

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

We consider Laplace integrals of the form

$$(1.1)\qquad F_\lambda(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} f(t)\, dt$$

where f is analytic in a domain that contains the non-negative reals in its interior, and λ and z are real or complex variables for which $F_\lambda(z)$ is properly defined. Although $F_\lambda(z)$ is an entire function of λ, we suppose that $\operatorname{Re}\lambda > 0$. We assume that $f(0) \neq 0$ and that there is a finite real number σ such that the integral (1.1) converges for $\operatorname{Re} z > \sigma$. Without loss of generality this number σ may be zero, otherwise the large parameter z and the function f may be adapted. Further conditions on f will be given in the sequel.

We are interested in the asymptotic expansion of $F_\lambda(z)$ for $z \to \infty$ in some sector of the complex z-plane. It is well known that if for some integer $n \geq 0$

$$(1.2)\qquad f(t) = \sum_{s=0}^{n-1} a_s t^s + \mathcal{O}(t^n), \qquad t \to 0^+,$$

then the asymptotic expansion of $F_\lambda(z)$ is given by Watson's lemma (see, for instance, OLVER (1974)), that is,

$$(1.3)\qquad F_\lambda(z) = \sum_{s=0}^{n-1} a_s (\lambda)_s\, z^{-s-\lambda} + \mathcal{O}(z^{-n-\lambda}),$$

as $z \to \infty$ in the sector $|\arg z| \leq \frac{1}{2}\pi - \delta < \frac{1}{2}\pi$. More general forms of (1.2) and (1.3) are also possible. In (1.3) we have used Pochhammer's notation

$$(1.4)\qquad (\lambda)_s = \frac{\Gamma(\lambda+s)}{\Gamma(\lambda)}, \qquad s = 0,1,2,\ldots,$$

or $(\lambda)_0 = 1$, $(\lambda)_s = \lambda(\lambda+1)\ldots(\lambda+s-1)$, $s \geq 1$.

The expansion (1.3) looses its asymptotic character when λ is large. For instance, if $\lambda = \mathcal{O}(z)$ then the ratio of consecutive terms satisfies

$$\frac{a_{s+1}}{a_s}\,\frac{s+\lambda}{z} = \mathcal{O}(1) \qquad (a_s \neq 0).$$

In this paper we shall modify Watson's lemma and we obtain an expansion in which large values of λ are allowed. We replace (1.2) by an expansion at the saddle point $t = \mu := \lambda/z$ of the function $- zt + \lambda \ln t$. For positive λ and z it is the interior point of $[0,\infty)$ at which $\exp(-zt+\lambda \ln t)$ attains its maximal value. For bounded values of λ the saddle point μ tends to zero, and Watson's lemma gives a good description of the asymptotic behaviour in that event. When μ is not tending to zero the lemma fails. In fact, μ is the uniformity parameter and our expansions are holding uniformly with respect to μ in a domain that contains the non-negative reals in its interior.

The asymptotic phenomenon is that the saddle point μ may range over the interval $[0,\infty)$ and that it hence may coalesce with $t = 0$. The latter is an end-point of the interval of integration and a singularity of $\exp(-zt+\lambda \ln t)$ if $\lambda \neq 0,1,\ldots$. When it coincides, this singularity disappears and the saddle point dissappears as well. It should be remarked that this combination of phenomena cannot be described by the case of a moving saddle point due to a polynomial in the exponential function. For the well-studied case of quadratic and cubic polynomials we refer to OLVER (1974, p.344 and p.351).

When μ is positive we can use Laplace's method, or, what is the same, the transformation $t \mapsto w(t)$ given by

$$(1.5) \qquad t - \mu \ln t = \tfrac{1}{2}w^2 + \mu - \mu \ln \mu, \qquad \mu = \lambda/z,$$

with correspondences $t = 0^+ \leftrightarrow w = -\infty$, $t = \mu \leftrightarrow w = 0$, $t = +\infty \leftrightarrow w = +\infty$. Then (1.1) becomes

$$F_\lambda(z) = \frac{z^{-\lambda} e^{-\lambda} \lambda^\lambda}{\Gamma(\lambda)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}zw^2}\, g(w)\, dw,$$

$$g(w) = f(t)\,\frac{w}{t-\mu} = \sum_{s=0}^{\infty} c_s(\mu)\, w^s,$$

$$c_0(\mu) = f(\mu)/\sqrt{\mu}, \qquad c_1(\mu) = f'(\mu) - \tfrac{1}{3}f(\mu)/\mu.$$

By computing more coefficients $c_s(\mu)$, it becomes clear that this representation cannot be used uniformly in $\mu \in (0,\infty)$. The reason is that the above map $t \mapsto w(t)$ gives singular points $t_n \leftrightarrow w_n$, given by

$$t_n = \mu\, e^{2\pi i n}, \qquad w_n^2 = -\, 4\pi i n \mu, \quad n \in \mathbb{Z}\setminus\{0\}.$$

When $\mu \to 0$ the singular points w_n of $g(w)$ approach zero; hence, the singularities coincide with the saddle point at $w = 0$. For values of μ bounded away from zero (uniformly), this method is useful. In our approach, however, μ may range over an unbounded (complex) domain, which contains the point $\mu = 0$ in its interior. Thus we combine Watson's lemma and (for $\mu \to 0^+$) and Laplace's method (for $\mu \geq \delta > 0$) into one expansion.

Our methods will also be applied to loop integrals of the type

$$(1.6) \qquad G_\lambda(z) = \frac{\Gamma(\lambda+1)}{2\pi i} \int_{-\infty}^{(0^+)} e^{zt}\, t^{-\lambda-1}\, f(t)\, dt,$$

where the path of integration starts at $-\infty$ on the real axis, encircles the origin in the positive direction and returns to the starting point. Here $f(t)$ is analytic in some domain of the t-plane and z is the large parameter. The saddle point of $e^{zt}t^{-\lambda}$ is $t = \mu = \lambda/z$ and also in this case we shall give asymptotic expansion holding uniformly in an unbounded μ-domain that contains $\mu = 0$. This modifies Watson's lemma for loop integrals as considered in OLVER (1974, p.120).

When the function $f(t)$ of (1.1) can be written as the Laplace transform

$$(1.7) \qquad f(t) = \int_0^\infty e^{-xt}\, \psi(x)\, dx, \qquad \mathrm{Re}\; t > 0,$$

then, under certain conditions on ψ, $F_\lambda(z)$ of (1.1) can be written as

$$(1.8) \qquad F_\lambda(z) = \int_0^\infty \psi(x)\, (z+x)^{-\lambda} dx.$$

This integral can be viewed as a generalized Stieltjes transform. For $\lambda = 1$ the asymptotic expansion of such integrals is considered by by WONG (1980, section 8), where ψ meets other conditions than those needed for the transformations (1.7) and (1.8). It seems interesting to take (1.8) as a starting point for obtaining an asymptotic expansion for $z \to \infty$ and to obtain error bounds by using generalized functions, as Wong did for (1.8) with $\lambda = 1$.

Applications of the results can be found for high-order derivatives of Laplace transforms. Namely, when the Laplace transform

$$\hat{f}(z) = \int_0^\infty e^{-zt} f(t)dt$$

is differentiated n-times we obtain

$$\hat{f}^{(n)}(z) = (-1)^n \int_0^\infty t^n e^{-zt} f(t)dt,$$

which is of the type (1.1). Other applications are found for certain special functions. A simple example gives the function $f(t) = 1/(1+t)$. Then both (1.1) and (1.6) can be expressed in terms of functions related to exponential integrals.

In a supplementary paper we give examples of other special functions (a ratio of gamma functions, Bessel functions and a parabolic cylinder function). For these cases we introduce a specific transformation of integration variables in order to bring the integrals into the standard forms (1.1) or (1.5). This transformation involves logarithmic functions and a remarkable feature is that it cannot be replaced by earlier considered transformations determined by polynomials, just as (1.5) fails for small μ.

The outline of the paper is as follows. In section 2 the conditions on f are given. In section 3 we give an expansion which is obtained by replacing (1.2) by an expansion at the saddle point $\mu = \lambda/z$. It gives a rather simple expansion for $F_\lambda(z)$ and it may be viewed as the simplest modification of Watson's lemma. It includes error bounds.

In section 4 we consider an expansion of FRANKLIN & FRIEDMAN (1957). Their aim was to modify Watson's lemma to obtain a convergent expansion for integrals of the type (1.1). Their expansion is also uniformly valid with respect to μ and it is more efficient than our first expansion (i.e., less terms are needed for the same order of accuracy). A drawback, however, is that the coefficients in the Franklin & Friedman expansion are rather complicated.

In section 5 we give a third analogue. It has the interesting feature that the uniformity parameter μ and the large parameter z are separated in the asymptotic series. The first two expansions lack this property.

In section 6 we mention further applications of our methods for some special functions such as the generalized Riemann zeta function and the Whittaker functions.

Terminology. In this paper we call a variable *fixed* when it is independent of z or μ. The argument or phase of a complex number z will be denoted by ph z.

2. CONDITIONS ON f

For the various expansions in the following sections we suppose that f of (1.1) is holomorphic in a connected domain Ω of the t-plane with the following conditions satisfied:

(i) Ω contains the circle $|t| \le \gamma$ for some fixed $\gamma > 0$,

(ii) Ω contains the sector

$$S_{\alpha',\beta'} = \{t \mid -\alpha' < \text{ph } t < \beta',\ |t| > 0\}, \tag{2.1}$$

where α', β' are fixed positive numbers, $|\beta'-\alpha'| < 2\pi$.

We assume that

$$f(t) = \mathcal{O}(t^p), \tag{2.2}$$

as $t \to \infty$ in $S_{\alpha',\beta'}$, where p is a fixed real number. From a result of Ritt (see OLVER (1974, ChI.4)) it follows that for the derivatives of f we have

$$f^{(m)}(t) = \mathcal{O}(t^{p-m}), \qquad m = 0,1,\ldots \tag{2.3}$$

as $t \to \infty$ in any closed annular sector properly interior to $S_{\alpha',\beta'}$. For this closed sector we take

$$(2.4)\qquad \bar{S}_{\alpha,\beta} = \{t \mid -\alpha \le \mathrm{ph}\ t \le \beta\},$$

with α,β fixed positive and respectively less than α',β'.

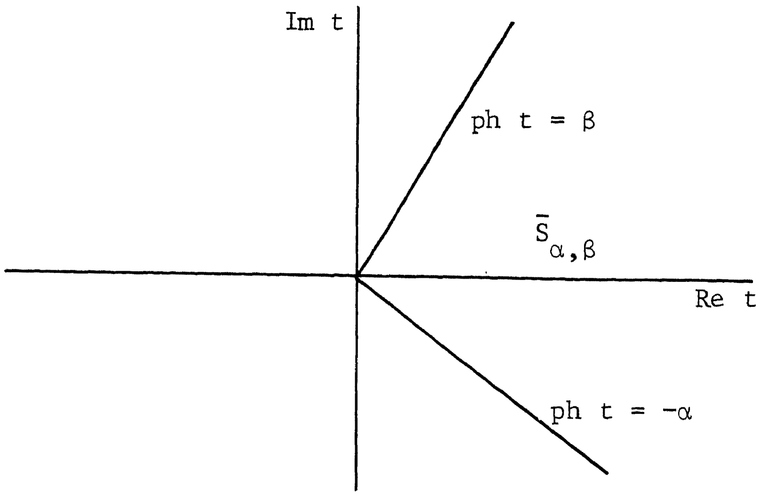

Fig. 2.1 Sector $\bar{S}_{\alpha,\beta}$ in the complex t-plane

The asymptotic expansions of (1.1) given in the following sections are uniformly valid with respect to $\mu \in \bar{S}_{\alpha,\beta}$. Since ph $\lambda \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, this gives a range for the phase of the large parameter $z = \lambda/\mu$. A more extended range for μ will be given in subsection 3.3.

REMARK 2.1. In applications f is usually an analytic function. When we consider just real values of z,λ and t it is obvious to suppose that for some integer n $f \in C^n$ on $[-\gamma,\infty)$ with $\gamma > 0$, and that f and its derivatives should satisfy (2.3) for $0 \le m \le n$. Then the asymptotic expansion can be constructed with terms containing the first n derivatives of f.

REMARK 2.2. Condition (2.2) and its implication (2.3) do not imply that

$$(2.5)\qquad f^{(m+1)}(t) = \mathcal{O}(t^{-1} f^{(m)}(t)),$$

as $t \to \infty$ in $\bar{S}_{\alpha,\beta}$, $m = 0,1,\ldots$. For instance, when $f(t) = 1/(1+\exp(-t))$, we have $f(t) = \mathcal{O}(1)$ in the sector $|\arg t| < \frac{1}{2}\pi$. Furthermore,

$$f^{(m)}(t) = \mathcal{O}(e^{-t}), \qquad t \in \bar{S}_{\alpha,\alpha}, \qquad \alpha < \tfrac{1}{2}\pi.$$

Hence in (2.2) and (2.3) we can take $p = 0$, but (2.5) is not true. For proving the theorems in the next section, (2.3) is sufficient. However, when (2.5) is not fulfilled, the resulting expansions are not very interesting.

3. TAYLOR EXPANSIONS AT THE SADDLE POINT

Let us first suppose that λ and z are real. Complex values of these parameters will be considered in subsection 3.3.

3.1. Formal asymptotic expansion

For each point $\mu \in \Omega$ we can expand

$$(3.1) \qquad f(t) = \sum_{s=0}^{\infty} a_s(\mu)(t-\mu)^s,$$

where $a_s(\mu) = f^{(s)}(\mu)/s!$ and the series is defined and convergent if $|t-\mu|$ is small enough. For μ we take the saddle point of $\exp(-zt+\lambda \ln t)$, that is,

$$(3.2) \qquad \mu = \lambda/z.$$

Substituting (3.1) in (1.1) we obtain after interchanging the order of summation and integration the formal result

$$(3.3) \qquad F_\lambda(z) \sim z^{-\lambda} \sum_{s=0}^{\infty} a_s(\mu) P_s(\lambda) z^{-s}, \qquad z \to \infty,$$

where

$$(3.4) \qquad P_s(\lambda) = \frac{z^{\lambda+s}}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt}(t-\mu)^s dt, \qquad s = 0,1,\ldots .$$

These $P_s(\lambda)$ are polynomials in λ. They follow the recursion relation

$$(3.5) \qquad P_{s+1}(\lambda) = s[P_s(\lambda)+\lambda P_{s-1}(\lambda)],$$

with $P_0(\lambda) = 1$, $P_1(\lambda) = 0$. The next few are

$$P_2(\lambda) = \lambda, \quad P_3(\lambda) = 2\lambda, \quad P_4(\lambda) = 3\lambda(\lambda+2).$$

An explicit representation of $P_s(\lambda)$ is obtained by expanding $(t-\mu)^s$ in powers of t. The result is

$$(3.6) \qquad P_s(\lambda) = \sum_{r=0}^{s} \binom{s}{r} (\lambda)_r (-\lambda)^{s-r},$$

where $(\lambda)_r$ is defined in (1.4). There is a lot of cancellation of high powers of λ in (3.6). From induction it follows that the degree of $P_{2s}(\lambda)$ and $P_{2s+1}(\lambda)$ equals s. The polynomials are non-negative for $\lambda \geq 0$, as follows from (3.5).

Before proving the validity of (3.3) it is possible to give an indication about the asymptotic nature of (3.3). For bounded values of μ it immediately follows from the properties of $P_s(\lambda)$, $\lambda = \mu z$, that $P_{2s+1}(\lambda)z^{-1}/P_{2s}(\lambda) = \mathcal{O}(z^{-1})$, $z \to \infty$. For unbounded μ-values we need the help of the coefficients $a_s(\mu)$. From (2.3) it follows that $a_s(\mu) = \mathcal{O}(\mu^{p-s})$, $\mu \to \infty$, which gives (3.3) the desired properties. Large μ-values do not disturb the asymptotic nature of (3.3).

For proving the asymptotic properties of (3.3) it is useful to introduce the functions

$$(3.7) \qquad \tilde{P}_s(\lambda) = \frac{z^{\lambda+s}}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} |t-\mu|^s dt, \qquad s = 0,1,2,\ldots .$$

For $s = 0,2,\ldots$, we have $\tilde{P}_s(\lambda) = P_s(\lambda)$; for general $s \geq 0$ we can write

$$\tilde{P}_s(\lambda) = \frac{\lambda^s s!}{\Gamma(\lambda)} [u_s(\lambda) + (-1)^s v_s(\lambda)],$$

where u_s and v_s both satisfy the recursion

$$(3.8) \qquad s\lambda w_s = (s-1)w_{s-1} + w_{s-2}$$

with initial values $u_0(\lambda) = \Gamma(\lambda,\lambda)$, $v_0 = \gamma(\lambda,\lambda)$, $u_1(\lambda) = -v_1(\lambda) = e^{-\lambda}\lambda^{\lambda-1}$. Here $\Gamma(\lambda,z)$ and $\gamma(\lambda,z)$ are the well-known incomplete gamma functions.

By applying Laplace's method to (3.8) it is found that for large positive values of λ

$$(3.9) \qquad \tilde{P}_s(\lambda) \sim \pi^{-\frac{1}{2}}(2\lambda)^{s/2}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s), \qquad s = 0,1,2,\dots .$$

For even s the right-hand side equals the term of highest order of the polynomial $P_s(\lambda)$.

3.2. Error bounds, interpretation of the expansion

In this subsection we give error bounds for the asymptotic expansion (3.3), from which the nature of the expansion follows. First we introduce a remainder for (3.1) by writing

$$(3.10) \qquad f(t) = \sum_{s=0}^{n-1} a_s(\mu)(t-\mu)^s + R_n(t,\mu)(t-\mu)^n, \qquad n = 0,1,\dots .$$

Then we obtain for (3.3)

$$(3.11) \qquad F_\lambda(z) = z^{-\lambda}\Big[\sum_{s=0}^{n-1} a_s(\mu)P_s(\lambda)z^{-s} + E_n(z,\lambda)z^{-n}\Big],$$

where the remainder E_n is defined by

$$(3.12) \qquad E_n(z,\lambda) = \frac{z^{\lambda+n}}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-zt}(t-\mu)^n R_n(t,\mu)\,dt.$$

By Taylor's theorem, the remainder in (3.10) can be written as

$$(3.13) \qquad R_n(t,\mu) = \frac{1}{n!}\, f^{(n)}(\xi), \qquad \xi \text{ between } t \text{ and } \mu.$$

From (2.3) it follows that we can assign positive numbers M_n, not depending on t, such that for $n = 0,1,2,\dots$

$$(3.14) \qquad |f^{(n)}(t)| \le M_n|1+t|^{p-n}, \qquad t \ge 0.$$

Let us introduce for positive x,y and real ν the function

$$H_\nu(x,y) = \max(x^\nu, y^\nu). \tag{3.15}$$

Then an error bound for (3.3) is given by the following theorem.

THEOREM 3.1. *For* $z > 0$ *and* $\mu \ge 0$ *the error term* (3.12) *is bounded as follows*

$$|E_n(z,\lambda)| \le \frac{M_n}{n!}\frac{z^{\lambda+n}}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-zt}|t-\mu|^n H_{p-n}(1+t,1+\mu)\,dt. \tag{3.16}$$

PROOF. Follows by combining (3.12), (3.13) and (3.14). □

Next we investigate the asymptotic nature of the expansion (2.3) and for that purpose we introduce the functions

$$\psi_s(z,\mu) = (1+\mu)^{p-s}\,\tilde{P}_s(z\mu)z^{-s}, \qquad s = 0,1,2,\ldots, \tag{3.17}$$

where $\tilde{P}_s(\lambda)$ is defined in (3.7) and p is a real fixed number.

THEOREM 3.2. $\{\psi_s(z,\mu)\}$ *is an asymptotic sequence for* $z \to \infty$, *uniformly with respect to* $\mu \ge 0$.

PROOF. We have to prove that $\psi_{s+1}(z,\mu) = \mathcal{O}[\psi_s(z,\mu)]$ ($z \to \infty$, uniformly in μ), $s \ge 0$. From (3.9) it follows that we can find numbers c_s, independent of λ, such that $\tilde{P}_{s+1}(\lambda) \le c_s(\lambda+1)^{\frac{1}{2}}\tilde{P}_s(\lambda)$, $\lambda \ge 0$. Hence

$$\psi_{s+1}(z,\mu) \le \frac{c_s(\lambda+1)^{\frac{1}{2}}}{z+\lambda}\,\psi_s(z,\mu). \tag{3.18}$$

It is easily verified that $(\lambda+1)^{\frac{1}{2}}(z+\lambda)^{-1} = \mathcal{O}(1)$ as $z \to \infty$, uniformly in $\mu \ge 0$. This proves the theorem. □

Now we are ready to prove the main result. First we rewrite (3.3) in the form

$$z^\lambda F_\lambda(z) \sim \sum_{s=0}^\infty a_s(\mu)P_s(\lambda)z^{-s}; \quad \{\psi_s(z,\mu)\}. \tag{3.19}$$

THEOREM 3.3. *The expansion (3.19) is a uniform asymptotic expansion for* $z \to \infty$, *the uniformity holding with respect to* $\mu \in [0,\infty)$.

PROOF. According to (3.11) we have to show

$$z^{-n}E_n(z,\lambda) = O[\psi_n(z,\mu)], \tag{3.20}$$

as $z \to \infty$, uniformly in $\mu \geq 0$. Consider (3.16) and assume first that $p-n \geq 0$. We split up the interval of integration $[0,\infty)$ into $[0,2\mu+1]$ and $[2\mu+1,\infty)$. On the first interval we have

$$H_{p-n}(t,\mu) \leq 2^{p-n}(1+\mu)^{p-n}.$$

It follows that (3.16) becomes

$$|z^{-n}E_n(z,\lambda)| \leq \frac{M_n}{n!}\left[2^{p-n}\psi_n(z,\mu)+\frac{z^\lambda}{\Gamma(\lambda)}\int_{2\mu+1}^{\infty} t^{\lambda-1}e^{-zt}(t-\mu)^n(1+t)^{p-n}dt\right]. \tag{3.21}$$

We will prove that the second term is exponentially small with respect to the first term $2^{p-n}\psi_n(z,\mu)$, thus proving (3.20).

To bound the integral in (3.20) we choose fixed numbers A_p such that $(t-\mu)^n(1+t)^{p-n} \leq A_p t^p$, $t \geq 2\mu + 1$. So we obtain an integral which is written as

$$\int_{2\mu+1}^{\infty} t^{\lambda+p-1}e^{-(z-1)t}e^{-t}dt.$$

The maximal value of $t^{\lambda+p-1}e^{-(z-1)t}$ is attained at $t_0 = \frac{\lambda+p-1}{z-1}$. When $z > \max(2,p)$ we have $t_0 < 2\mu+1$. So we can bound the integral by

$$(2\mu+1)^{\lambda+p-1}e^{-(z-1)(2\mu+1)}\int_{2\mu+1}^{\infty} e^{-t}dt = (2\mu+1)^{\lambda+p-1}e^{-z(2\mu+1)}.$$

Thus, for the integral in (3.21) we obtain the bound

$$\frac{z^\lambda}{\Gamma(\lambda)}\int_{2\mu+1}^{\infty} t^{\lambda-1}e^{-zt}(t-\mu)^n(1+t)^{p-n}dt \leq \frac{A_p z^\lambda}{\Gamma(\lambda)}(2\mu+1)^{\lambda+p-1}e^{-z(2\mu+1)}.$$

Using $1/\Gamma(\lambda) \leq (2\pi)^{\frac{1}{2}}e^\lambda\lambda^{-\lambda+\frac{1}{2}}$ $(\lambda>0)$ we obtain for the right-hand side (apart from a few fixed quantities)

$$\lambda^{\frac{1}{2}}(2\mu+1)^{p-1}2^{\lambda}(1+\frac{1}{2\mu})^{\mu z}e^{-\lambda-z} \le \lambda^{\frac{1}{2}}(2\mu+1)^{p-1}(2/e)^{\lambda}e^{-\frac{1}{2}z} = \mathcal{O}(\rho^{\lambda}e^{-\frac{1}{2}z}),$$

for some fixed $\rho \in (2/e,1)$, as $z \to \infty$, uniformly in $\mu \ge 0$. This last $\mathcal{O}$-term is exponentially small with respect to $\psi_n(z,\mu)$ (see (3.9) and (3.17)). This proves the case $p-n \ge 0$.

When $p-n \le 0$ we introduce $\bar{\mu} = \max(0, \frac{\mu-1}{2})$ and the interval of integration of (3.16) is split up into $[0,\bar{\mu}]$ and $[\bar{\mu},\infty)$. Then we have

$$|z^{-n}E_n(z,\mu)| \le \frac{M_n}{n!}[2^{n-p}\psi_n(z,\mu)+\frac{z^{\lambda}}{\Gamma(\lambda)}\int_0^{\bar{\mu}} t^{\lambda-1}e^{-zt}(\mu-t)^n(1+t)^{p-n}dt].$$

When $\mu \le 1$ the integral vanishes. The maximal value of $t^{\lambda-1}e^{-zt}$ is attained at $t_0 = \mu - 1/z$. When $\mu > 1$ and $z > 1$, we have $t_0 > \bar{\mu}$. Therefore

$$\frac{z^{\lambda}}{\Gamma(\lambda)}\int_0^{\bar{\mu}} t^{\lambda-1}e^{-zt}(\mu-t)^n(1+t)^{p-n}dt \le \frac{z^{\lambda}}{\Gamma(\lambda)}\mu^n\bar{\mu}^{-\lambda}e^{-z\bar{\mu}}$$
$$\le (2\pi)^{-\frac{1}{2}}\mu^n\lambda^{\frac{1}{2}}(\tfrac{1}{2}e^{\frac{1}{2}})^{\lambda}(1-1/\mu)^{\mu z}e^{\frac{1}{2}z} = \mathcal{O}(\rho^{\lambda}e^{-\frac{1}{2}z}),$$

for some fixed $\rho \in (\frac{1}{2}e^{\frac{1}{2}},1)$, as $z \to \infty$, uniformly in $\mu \ge 0$. This proves the theorem. □

<u>REMARK 3.1</u>. In (3.19) we can use a simpler asymptotic scale. Consider (3.17) and (3.9), then it follows that

$$(3.22)\qquad \tilde{\psi}_s(z,\mu) = (1+\mu)^p\frac{(1+\lambda)^{s/2}}{(z+\lambda)^s}, \qquad s = 0,1,\ldots,$$

can be used as well.

<u>REMARK 3.2</u>. From the proof of Theorem 3.3 and from the above remark we conclude that the remainder $E_n(z,\lambda)$ defined in (3.11) can be bounded as follows

$$(3.23)\qquad |z^{-n}E_n(z,\lambda)| \le \kappa_n\tilde{\psi}_n(z,\mu), \qquad n = 0,1,\ldots,$$

where κ_n is fixed and is approximately equal to $\pi^{-\frac{1}{2}}2^{n/2}M_n\Gamma(\frac{1}{2}+\frac{1}{2}n)/n!$

3.3. Extension to complex variables

For complex λ, z and $\mu = \lambda/z$ we introduce

$$(3.24)\qquad \theta = \mathrm{ph}\ z, \quad \nu = \mathrm{ph}\ \lambda, \quad \kappa = \mathrm{ph}\ \mu$$

with $\nu = \theta + \kappa$. For the convergence of (1.1) at $t = 0$ we need $-\frac{1}{2}\pi < \nu < \frac{1}{2}\pi$. The coefficients $a_s(\mu)$ in (3.1), appearing also in (3.3), are defined for $\mu \in \bar{S}_{\alpha,\beta}$, i.e., for $-\alpha \le \kappa \le \beta$. In this subsection we investigate whether these two domains of ν and κ can be used for the asymptotic expansion.

By rotating the path of integration we can write for any $\bar{\kappa} \in [-\alpha,\beta]$

$$(3.25)\qquad F_\lambda(z) = \frac{1}{\Gamma(\lambda)} \int_0^{\infty e^{i\bar{\kappa}}} t^{\lambda-1} e^{-zt} f(t)\,dt,$$

where the choice $\bar{\kappa} = \kappa$ gives a line of integration through the saddle point $t = \mu$. Except for $\theta = 0$, $\kappa = 0$ it is not a path of steepest descent. Still it is a useful path for obtaining the expansion for complex values of z and μ.

Let us introduce

$$p(t) = t - \mu \ln t, \qquad t \ne 0, \qquad |\mathrm{ph}\ t| < \pi$$

$$P = \{t \mid \mathrm{ph}\ t = \mathrm{ph}\ \mu = \kappa, \quad |t| \ge 0\}.$$

Hence P is the line through the saddle point. On P we have for $\mu \ne 0$

$$p(t) - p(\mu) = \mu(s-1-\ln s), \qquad t = \mu s,\ s > 0.$$

Hence for $t \in P$, $\nu \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ and $t = \mu s\ (s>0)$

$$(3.26)\qquad \mathrm{Re}\ e^{i\theta}[p(t)-p(\mu)] = \mathrm{Re}\ e^{i\nu}|\mu|(s-1-\ln s)$$

vanishes at $s = 1$ $(t=\mu)$ and is otherwise positive on P. Furthermore, (3.26) is unbounded as $t \to 0$ or ∞ along P.

With this result we have the following complex version of Theorem 3.3. For a clear interpretation of the scale functions ψ_s for complex parameters we prefer the simpler functions $\tilde{\psi}_s$ of (3.22).

THEOREM 3.4. *The expansion* (3.3) *is a uniform asymptotic expansion with respect to the scale* $\{\tilde{\psi}_s(z,\mu)\}$, *the uniformity holding with respect to* $\mu \in \bar{S}_{\alpha,\beta}$ *and* ph μz = ph λ *in fixed closed subsets of* $(-\frac{1}{2}\pi,\frac{1}{2}\pi)$.

PROOF. The line of integration in (3.25) is the path P introduced earlier ($\bar{\kappa} = \kappa$). The proof of Theorem 3.3 needs some modifications, the details of which are left out. $\square$

REMARK 3.3. The restriction on λ, i.e., ph $\lambda \in (-\frac{1}{2}\pi,\frac{1}{2}\pi)$ or Re $\lambda > 0$ is quite natural for the convergence of (1.1) at $t = 0$. By partial integration we can replace the condition Re $\lambda > 0$ by Re $\lambda > -n$, where n is any natural number. In fact, $F_\lambda(z)$ is an entire function of λ and at the moment a restriction of the form Re $\lambda > 0$ seems not relevant. We expect that (3.3) makes sense for, say, negative μ or λ values.

REMARK 3.4. Given any θ in $(-\frac{1}{2}\pi-\beta,\frac{1}{2}\pi+\alpha)$, we can find a $\bar{\kappa} \in (-\alpha,\beta)$ such that (3.25) converges at $t = \infty e^{i\bar{\kappa}}$. Fixing this direction at $\infty e^{i\bar{\kappa}}$, we try to deform the line of integration into a contour P so that it has the following properties:

(i) $0 \in P$, $\mu \in P$, $\infty e^{i\bar{\kappa}} \in P$;

(ii) P lies within Ω, the domain of holomorphy of f;

(iii) $\mathrm{Re}\{e^{i\theta}[p(t)-p(\mu)]\}$ is positive on P, except at $t = \mu$, and is bounded away from zero as $t \to 0$ or ∞ along P.

Let $D(\theta) \subset \Omega$ be the subset of points μ for which a path P can be constructed having the above three properties. Then for fixed values of $\mu \in D(\theta)$ a theorem of OLVER (1974, p.127) can be used to prove that (3.3) is valid for these values of z and μ. In this way the ranges for the parameters μ and z prescribed by Theorem 3.4 may be enlarged considerably.

3.4. Expansion for loop integrals

In this subsection we consider integrals of the form

$$(3.27)\qquad G_\lambda(z) = \frac{\Gamma(\lambda+1)}{2\pi i}\int_{-\infty}^{(0^+)} t^{-\lambda-1}\, e^{zt}\, f(t)dt.$$

We first consider real positive values of λ and z. Furthermore we suppose that f meets the same conditions as given in section 2. For the domain Ω we need an extra condition such that the convergence of (3.27) at $t = \infty e^{\pm i\pi}$ is guaranteed. For instance, apart from the conditions (i) and (ii) of Ω, we need

(iii) Ω contains the lines $\{t \mid |t| \geq 0,\ \mathrm{ph}\ t = \pm(\pi-\varepsilon)\}$, for sufficiently small positive ε.

By substituting (3.1) we obtain for (3.27) the formal expansion

$$(3.28)\qquad G_\lambda(z) \sim z^\lambda \sum_{s=0}^{\infty} a_s(\mu)Q_s(\lambda)z^{-s}, \qquad \mu = \lambda/z,$$

with

$$(3.29)\qquad Q_s(\lambda) = \frac{z^{s-\lambda}\Gamma(\lambda+1)}{2\pi i}\int_{-\infty}^{(0^+)} t^{-\lambda-1}e^{zt}(t-\mu)^s dt.$$

The $Q_s(\lambda)$ are, again, polynomials in λ. It is easily verified that

$$(3.30)\qquad Q_s(\lambda) = (-1)^s P_s(-\lambda), \qquad s = 0,1,2,\ldots\ .$$

This makes the expansions (3.3) and (3.28) quite similar.

For (1.1) the "natural" path of integration is the real positive t-axis $(0,\infty)$, especially when λ and z are positive real numbers. For (3.27) such a natural path is the contour through $t = \mu$ defined by $\mathrm{Im}\ p(t) = 0$, where $p(t) = t - \mu \ln t$. This equation can be written as $\rho \sin\phi - \mu\phi = 0$, where we have written $t = \rho e^{i\phi}$. The solution $\phi = 0$ gives the real line $(0,\infty)$, the ideal contour for (1.1). Another solution gives the ideal contour for (3.27), the steepest descent path

$$(3.31)\qquad \rho(\phi) = \frac{\mu\phi}{\sin\phi}, \qquad -\pi < \phi < \pi$$

with indeed $\rho(0) = \mu$, as prescribed. For small μ the conditions on Ω guarantee that the contour defined by (3.31) lies wholly inside this domain. For any positive μ the path may meet the boundary of $\bar{S}_{\alpha,\beta}$. From the points of intersection the contour has to be deformed such that it remains inside Ω and ends at $\infty e^{\pm i\pi}$. The main contributions to (3.27) come from the parts around $t = \mu$, denoted by C^+ and C^- in the following figure.

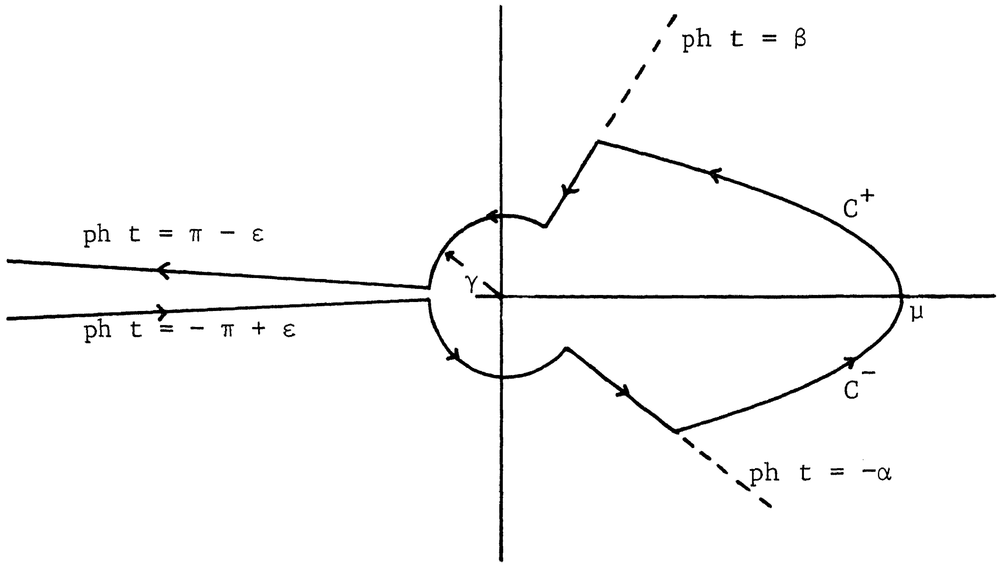

Figure 3.1. Contour for (3.27)

The length of the parts $C^{\pm}$ is $O(\mu)$, μ large, and therefore the technique of Theorem 3.3 is applicable. On $C^{\pm}$ the growth of f is given by (2.2). On the remaining parts we may assume $f(t) = O(e^{\sigma t})$, for a fixed real σ. It can be proved that on the remaining parts the contribution is exponentially small (in z *and* in λ) with respect to the contributions of $C^{\pm}$. See again the proof of Theorem 3.3, which has to be adapted of course for this case.

Consequently, the reaminder $\tilde{E}_n(z,\lambda)$ in

$$G_\lambda(z) = z^\lambda\Big[\sum_{s=0}^{n-1} a_s(\mu)Q_s(\lambda)z^{-s} + \tilde{E}_n(z,\lambda)z^{-n}\Big]$$

can be estimated by

$$\tilde{E}_n(z,\lambda) = O\left\{\frac{z^{n-\lambda}\Gamma(\lambda+1)}{2\pi i}\int_{-\infty}^{(0^+)} e^{zt}t^{-\lambda-1}|t-\mu|^n dt\right\}.$$

where the contour of integration is given by (3.31). For large values of λ, the integral

$$\frac{z^{n-\lambda}\Gamma(\lambda+1)}{2\pi i}\int_{-\infty}^{(0^+)} e^{zt}t^{-\lambda-1}|t-\mu|^n dt$$

has the same asymptotic behaviour as $\tilde{P}_s(\lambda)$ of (3.7), see (3.9). Therefore, it follows that $\tilde{E}_n(z,\lambda)$ can be estimated as in (3.23).

For complex values of λ and z, those points $\mu \in \Omega$ have to be considered for which a contour P can be found meeting similar conditions as in Remark (3.4). In (iii) the expression $\mathrm{Re}\{e^{i\theta}[p(t)-p(\mu)]\}$ has to be negative on P (except at $t = \mu$).

3.5. Application to exponential integrals

When we take $f(t) = 1/(1+t)$ in (1.1) and (1.6) the integrals represent functions that can be expressed in terms of incomplete gamma functions or (generalized) exponential integrals. It is easily shown that

$$(3.32)\qquad F_\lambda(z) = e^z\int_z^\infty e^{-\rho}\,\rho^{-\lambda}d\rho,\quad G_\lambda(z) = e^{-z}\int_0^z e^{\rho}\rho^{\lambda}d\rho,$$

where we suppose that $|\arg z| < \pi$ and in the second integral that $\mathrm{Re}\,\lambda > -1$.

The uniform asymptotic expansion of the first function is obtained by GAUTSCHI (1959), complete with error bounds. Our expansion based on (3.3) is somewhat different. Gautschi's expansion can be obtained by the method of section 5.

For the annular sector $\bar{S}_{\alpha,\beta}$ of (2.4) we take $\alpha = \beta = \pi-\varepsilon$ $(0<\varepsilon<\pi)$. The relation (2.2) is satisfied with $p = -1$. The coefficients $a_s(\mu)$ of (3.1) are given by

$$a_s(\mu) = \frac{(-1)^s}{(1+\mu)^{s+1}},\qquad \mu \in \bar{S}_{\pi-\varepsilon,\pi-\varepsilon},\qquad s = 0,1,\ldots .$$

The first function in (3.32) is usually written as the generalized exponential integral $E_\lambda(z)$. We have $E_\lambda(z) = z^{\lambda-1}e^{-z}F_\lambda(z)$ and we obtain

$$(3.33)\qquad e^{z}E_{\lambda}(z) = \sum_{s=0}^{n-1} \frac{(-1)^{s}P_{s}(\lambda)}{(z+\lambda)^{s+1}} + z^{-n-1}E_{n}(z,\lambda),$$

where $P_s(\lambda)$ are given in subsection 3.1. For complex values of z and λ this expansion holds for

$$-\tfrac{1}{2}\pi < \mathrm{ph}\,\lambda < \tfrac{1}{2}\pi, \qquad \mu = \lambda/z \in \bar{S}_{\pi-\varepsilon,\pi-\varepsilon}.$$

Gautschi's expansion is given for real positive variables and it has very convenient error bounds, which are obtained numerically. In the above example the remainder $R_n(t,\mu)$ of (3.10) is given by

$$(3.34)\qquad R_{n}(t,\mu) = \frac{(-1)^{n}}{(1+\mu)^{n}(1+t)},$$

from which we obtain for E_n of (3.33) (see (3.12))

$$(3.25)\qquad E_{n}(z,\lambda) = \frac{(-1)^{n}z^{\lambda+n}}{(1+\mu)^{n}\Gamma(\lambda)} \int_{0}^{\infty} t^{\lambda-1}e^{-zt}(t-\mu)^{n}(1+t)^{-1}dt.$$

Majorizing $1/(1+t) \le 1$ we have for $z > 0$, $\lambda \ge 0$,

$$(3.36)\qquad |E_{n}(z,\lambda)| \le \frac{\tilde{P}_{n}(\lambda)}{(1+\mu)^{n}},$$

where $\tilde{P}_n(\lambda)$ is defined in (3.7). This bound for $|E_n(z,\lambda)|$ is not as sharp as (3.16). However, it is rather simple, especially for n even, in which case $\tilde{P}_n(\lambda) = P_n(\lambda)$, and the error is expressed in terms of the first neglected term of the expansion.

For the function $G_\lambda(z)$ of (3.32) we apply the method of subsection 3.4. It follows that

$$G_{\lambda}(z) = z^{\lambda+1}\left[\sum_{s=0}^{n-1} \frac{P_{s}(-\lambda)}{(z+\lambda)^{s+1}} + z^{-n-1}\, E_{n}(z,\lambda)\right],$$

$$E_{n}(z,\lambda) = \frac{(-1)^{n}z^{n-\lambda}\Gamma(\lambda+1)}{(1+\mu)^{n}2\pi i} \int_{-\infty}^{(0^{+})} e^{zt}t^{-\lambda-1}(t-\mu)^{n}(1+t)^{-1}dt.$$

For a further estimate, the contour can be replaced by the saddle point contour defined in (3.31).

4. THE EXPANSION OF FRANKLIN AND FRIEDMAN

In FRANKLIN & FRIEDMAN (1957) a modification of Watson's lemma is given, such that the resulting expansion of (1.1) in many cases is both asymptotic and convergent for $z > 0$. Their procedure is as follows (they use a different notation).

Introduce $\mu_n = (\lambda+n)/z$, $n = 0,1,\ldots$, and suppose that $\mu_n \in \Omega$ (see section 2). Then (1.1) is written as

$$F_\lambda(z) = z^{-\lambda} f(\mu_0) + \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt}[f(t)-f(\mu_0)]dt$$

$$= z^{-\lambda} f(\mu_0) + \frac{1}{z\Gamma(\lambda)} \int_0^\infty t^{\lambda} e^{-zt} f_1(t)dt,$$

where

$$(4.1) \qquad f_1(t) = \frac{d}{dt} \frac{f(t)-f(\mu_0)}{t-\mu_0} .$$

Continuing the above procedure we obtain

$$(4.2) \qquad F_\lambda(z) = z^{-\lambda}[\sum_{s=0}^{n-1} f_s(\mu_s)(\lambda)_s z^{-2s} + \bar{E}_n(z,\lambda)],$$

with

$$(4.3) \qquad \bar{E}_n(z,\lambda) = \frac{z^{\lambda-n}}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+n-1} e^{-zt} f_n(t)dt,$$

$$(4.4) \qquad f_{s+1}(t) = \frac{d}{dt} \frac{f_s(t)-f_s(\mu_s)}{t-\mu_s}, \qquad f_0(t) = f(t).$$

Note that f_s has the same analytic properties as f, it is holomorphic in Ω and hence in the sector (2.1) where (2.2) holds with p replaced by $p-2s$, that is $f_s(t) = \mathcal{O}(t^{p-2s})$, $t \in \bar{S}_{\alpha,\beta}$.

FRANKLIN & FRIEDMAN (1957) proved that under certain conditions on f

$$(\text{i}) \qquad \bar{E}_n(z,\lambda) = \mathcal{O}(z^{-2n}),$$

(ii) $$\lim_{n\to\infty} \bar{E}_n(z,\lambda) = 0, \qquad z > 0. \text{ That is,}$$

$$\sum_{s=0}^{\infty} f_s(\mu_s)(\lambda)_s z^{-2s} \qquad \text{converges to } F_\lambda(z).$$

We are not interested here in the second result. We prove that their expansion is uniformly valid with respect to $\mu = \lambda/z$. First we derive a suitable representation for $f_s(t)$.

LEMMA 4.1. *Let* f_s *be defined by* (4.4) *and let* $t > 0$, $\mu_n > 0$. *Then*

$$f_s(t) = \int_0^1 \cdots \int_0^1 v_s v_{s-1}^3 \cdots v_1^{2s-1} f^{(2s)}[\mu_0+v_1(z^{-1}+\ldots v_s(t-\mu_{s-1})\ldots)]\, dv_s \ldots dv_1. \tag{4.5}$$

PROOF. For $s = 1$ we have

$$f_1(t) = \frac{d}{dt}\,\frac{f(t)-f(\mu_0)}{t-\mu_0} = \frac{d}{dt}\int_0^1 f'[\mu_0+v(t-\mu_0)]dv$$
$$= \int_0^1 vf''[\mu_0+v(t-\mu_0)]dv.$$

Hence, the lemma holds true for $s = 1$. The remaining part of the proof follows by induction with respect to s. □

LEMMA 4.2. *There is a number* ξ_s *between* t *and one of* $\mu_0,\ldots,\mu_{s-1}$ *such that*

$$f_s(t) = \frac{f^{(2s)}(\xi_s)}{2^s s!}.$$

PROOF. Follows by repeated application of the mean value theorem on (4.5). □

REMARKS.

(i) A similar result is obtained by WONG (1980) in the event that all μ_s are equal. His method is slightly different.

(ii) The construction of $f_s(\mu_s)$ and the results of the lemmas are related to Lagrange's interpolation formula.

(iii) It is not necessary that f is holomorphic in these lemmas; $f \in C^{2s}(\mathbb{R}^+)$ is enough.

When we use (2.3) and Lemma 4.2 it follows that for real positive μ_s

$$|f_s(\mu_s)| \le K_s |1+\mu_s|^{p-2s},$$

and it is obvious that this inequality can be extended to $\mu_s \in \bar{S}_{\alpha,\beta}$. Furthermore,

$$\{(1+\mu_s)^{p-2s}(\lambda)_s z^{-2s}\}$$

is an asymptotic sequence for $z \to \infty$ uniformly in μ. The remainder (4.3) can be estimated as in subsection 3.2, although the representation of the bound is somewhat different. Here we have for positive z and λ, using Lemma 4.2 and (3.14),

$$|\bar{E}_n(z,\lambda)| \le \frac{2^{-n}M_n}{n!} \bar{\psi}_n(z,\mu), \tag{4.6}$$

$$\bar{\psi}_n(z,\mu) = \frac{z^{\lambda-n}}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+n-1} e^{-zt} H_{p-2n}(1+t,1+\bar{\mu})dt, \tag{4.7}$$

where $H_\nu(x,y)$ is defined in (3.15), and $\bar{\mu} = \mu_n$ when $p-2n \ge 0$, $\bar{\mu} = \mu_0$ when $p-2n \le 0$.

Using the methods of subsection 3.2, especially those of Theorem 3.2, we can prove that

$$\bar{\psi}_n(z,\mu) = (1+\mu)^{p-2n}(\lambda)_n z^{-2n} \mathcal{O}(1), \qquad z \to \infty \tag{4.8}$$

uniformly in $\mu = \lambda/z \ge 0$, from which follows that $\{\bar{\psi}_s(z,\mu)\}$ is an asymptotic sequence for $z \to \infty$, uniformly in $\mu \ge 0$. From this result we obtain

THEOREM 4.1. *The expansion* (4.2) *is a uniform asymptotic expansion for* $z \to \infty$ *with respect to the scale* $\{\bar{\psi}_s(z,\mu)\}$, *the uniformity holding with respect to* $\mu \in [0,\infty)$. *In other words,*

$$z^\lambda F_\lambda(z) \sim \sum_{s=0}^\infty f_s(\mu_s)(\lambda)_s z^{-2s}; \quad \{\bar{\psi}_s(z,\mu)\} \tag{4.9}$$

as $z \to \infty$, *uniformly in* $\mu \ge 0$.

Again it is possible to extend the results to complex values of λ and z. A complication may arise here when the numbers μ_n leave the sector $\bar{S}_{\alpha,\beta}$ for larger values of n. When this is not the case, the expansion (4.9) holds uniformly with respect to $\mu \in \bar{S}_{\alpha,\beta}$ and with respect to ph λ in fixed closed subintervals of $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

To conclude this section we point out that (4.2) can also be obtained by writing

$$(4.10) \qquad f(t) = \gamma_0 + \gamma_1(t-\mu_0) + t(t-\mu_0)g_1(t),$$

where γ_0 and γ_1 are not depending on t. It follows that $\gamma_0 = f(\mu_0)$, $\gamma_1 = [f(\mu_0)-f(0)]/\mu_0$. Setting (4.10) in (1.1) we arrive at (4.3) with n = 1, since $\frac{d}{dt}[tg_1(t)] = f_1(t)$ of (4.1). The following step is

$$(4.11) \qquad f_1(t) = \gamma_2 + \gamma_3(t-\mu_1) + t(t-\mu_1)g_2(t)$$

and it is clear that we arrive at (4.2). Representation (4.10) is used in BLEISTEIN (1966) for a different class of integrals. Note that in (4.10) and (4.11), and in the corresponding further steps, the functions f_s are interpolated in different points μ_s. These points are the saddle points in the new integrals. Hence, the contributions due to $\gamma_1, \gamma_3, \ldots$ vanish, just as $P_1(\lambda)$ of (3.4) vanishes. In Bleistein's paper the interpolating point is fixed, but that is quite natural for the integrals considered there.

5. A MODIFICATION OF THE EXPANSION OF FRANKLIN AND FRIEDMAN

The changing interpolation point makes the expansion of the previous section rather powerful. It is useful to give a slightly different expansion, in which the coefficients can be obtained in a simpler way. In this expansion the interpolating point is μ.

An interesting feature of the exapnsion is that the uniformity parameter μ and the large parameter z are uncoupled. In the expansions (3.3) and (4.2) the parameter $\lambda = \mu z$ occurs explicitly in the expansion.

To construct the third expansion we write (1.1) as

$$F_\lambda(z) = \lambda^{-z} f(\mu) + \frac{1}{z\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} \tilde{f}_1(t)\,dt,$$

with

$$\tilde{f}_1(t) = t \frac{d}{dt} \frac{f(t)-f(\mu)}{t-\mu} .$$

Continuing this we obtain with $\tilde{f}_0(t) = f(t)$,

$$\tilde{f}_{s+1}(t) = t \frac{d}{dt} \frac{\tilde{f}_s(t)-\tilde{f}_s(\mu)}{t-\mu}, \qquad s = 0,1,\ldots, \tag{5.1}$$

the expansion

$$\begin{aligned} F_\lambda(z) &= z^{-\lambda}\Big[\sum_{s=0}^{n-1} f_s(\mu) z^{-s} + z^{-n} E_n^*(z,\lambda)\Big], \\ E_n^*(z,\lambda) &= \frac{z^\lambda}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} \tilde{f}_n(t)\,dt. \end{aligned} \tag{5.2}$$

The coefficients $\tilde{f}_s(\mu)$ can be expressed in terms of $a_s(\mu)$ of (3.1), and this makes the construction of the coefficients a much simpler problem than in section 4. To show this we write for $\mu \in \Omega$,

$$\tilde{f}_s(t) = \sum_{r=0}^\infty a_r^{(s)} (t-\mu)^r. \tag{5.3}$$

Note that each f_s is holomorphic in Ω and that the above series is defined when $|t-\mu|$ is small enough. The coefficients $\tilde{f}_s(\mu)$ of (5.2) are given by $\tilde{f}_s(\mu) = a_0^{(s)}$. By substituting (5.3) into (5.1) we obtain the recursion

$$a_r^{(s+1)} = \mu(r+1)a_{r+2}^{(s)} + r a_{r+1}^{(s)}, \qquad r \geq 0, \quad s \geq 0, \tag{5.4}$$

with starting values $a_r^{(0)} = a_r(\mu)$. Hence, when the coefficients $\{a_r(\mu)\}$ are available, the coefficients $\tilde{f}_s(\mu) = a_0^{(s)}$ are computed rather straightforwardly. In the expansion of Franklin and Friedman of the previous section such a recursion is not possible, since the interpolating point is different in each following term of the expansion.

For the functions $\tilde{f}_s(t)$ a representation of the form (4.5) becomes rather complicated. It is easily shown by repeated application of

$$\tilde{f}_s(t) = \tfrac{1}{2}t\, \tilde{f}''_{s-1}(\xi), \qquad \xi \text{ between } t \text{ and } \mu,$$

that $\tilde{f}_s(t)$ can be written as

$$\tilde{f}_s(t) = t \sum_{r=1}^{s} \alpha_s^{(r)}\, \mu^{r-1} f^{(r+s)}(t_r), \qquad s > 0,$$

where $\alpha_s^{(r)}$ do not depend on t and μ and with t_r between t and μ (the variables μ and t are positive). Using (3.14) it follows that

$$|\tilde{f}_s(t)| \le t \sum_{r=1}^{s} |\alpha_s^{(r)}|\mu^{r-1} M_{r+s} H_{p-r-s}(1+t,1+\mu), \tag{5.5}$$

where $H_\nu(x,y)$ is given in (3.15). In this way we can construct a bound for $E_n^*(z,\lambda)$ of (5.2). Since

$$\frac{z^\lambda}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt}\, t H_{p-r-s}(1+t,1+\mu)\,dt \sim \mu(1+\mu)^{p-r-s}$$

as $z \to \infty$, uniformly in $\mu \ge 0$ (see the proof of Theorem 3.3 in subsection 3.2), we obtain

$$|E_s^*(z,\lambda)| \le \sum_{r=1}^{s} |\alpha_s^{(r)}| M_{r+s}^* \mu^r (1+\mu)^{p-r-s} \tag{5.6}$$

for some fixed M_s^*, which are slightly larger than M_s. It also follows that

$$E_n^*(z,\lambda) = \mu(1+\mu)^{p-n-1} \mathcal{O}(1) \tag{5.7}$$

as $z \to \infty$, uniformly in $\mu \ge 0$. Consequently, we have

THEOREM 5.1. *For the expansion in* (5.2) *we can write*

$$z^\lambda F_\lambda(z) \sim \sum_{s=0}^{\infty} \tilde{f}_s(\mu) z^{-s}; \quad \{\mu(1+\mu)^{p-s-1} z^{-s}\}, \tag{5.8}$$

as $z \to \infty$, *uniformly in* $\mu \ge 0$.

Remark that the asymptotic scale is rather simple compared to the scales in (3.19) and (4.9).

Theorem 5.1 can be extended to complex values of z and λ such that $\mu \in \bar{S}_{\alpha,\beta}$ and ph λ belongs to fixed closed subintervals of $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

To conclude this section we mention the corresponding expansion for the loop integral $G_\lambda(z)$ considered in subsection 3.4, see (3.27). The method of the present section gives

$$(5.9) \qquad z^{-\lambda} G_\lambda(z) \sim \sum_{s=0}^{\infty} (-1)^s \tilde{f}_s(\mu) z^{-s}; \qquad \{(1+\mu)^{p-s} z^{-s}\}$$

as $z \to \infty$, uniformly in $\mu \geq 0$. The coefficients $\tilde{f}_s(\mu)$ are the same as those in (5.2), (5.7).

6. EXAMPLES AND APPLICATIONS

6.1. Exponential integrals

For the exponential integral $E_\lambda(z)$ defined in subsection (3.5) the first terms of the three different expansions of sections 3,4 and 5 are as follows.

(i) Formula (3.11) becomes with $n = 5$ (see (3.33))

$$E_\lambda(z) = \frac{e^{-z}}{z+\lambda} \left[1 + \frac{\lambda}{(z+\lambda)^2} - \frac{2\lambda}{(z+\lambda)^3} + \frac{3\lambda(\lambda+2)}{(z+\lambda)^4} + \frac{z+\lambda}{z^6} E_5(z,\lambda) \right];$$

$E_n(z,\lambda)$ is given in (3.12), (3.35) and is bounded in (3.16), (3.36).

(ii) Formula (4.2) becomes with $n = 3$

$$E_\lambda(z) = \frac{e^{-z}}{z+\lambda} \left[1 + \frac{\lambda}{(z+\lambda+1)^2} + \frac{\lambda(\lambda+1)(3\lambda+3z+4)}{(z+\lambda+1)^2 (z+\lambda+2)^3} + (z+\lambda)\bar{E}_3(z,\lambda) \right];$$

$\bar{E}_n(z,\lambda)$ is given in (4.3) and is bounded in (4.6).

(iii) Formula (5.2) becomes with $n = 3$

$$E_\lambda(z) = \frac{e^{-z}}{z+\lambda} \left[1 + \frac{\lambda}{(z+\lambda)^2} + \frac{\lambda(\lambda-2z)}{(z+\lambda)^4} + \frac{(z+\lambda)}{z^4} E_3^*(z,\lambda) \right];$$

$E_n^*(z,\lambda)$ is given in (5.2) and is bounded in (5.5).

Inspection of (iii) and of higher order terms shows that this is exactly the expansion of GAUTSCHI (1960). Gautschi used an integration by parts procedure for the standard representation (cf. the first of (3.32))

$$E_\lambda(z) = \int_1^\infty t^{-\lambda} e^{-zt} dt.$$

Computation of the bounds $E_5, \bar{E}_3$ and E_3^* is not a simple matter. For $E_5(z,\lambda)$ in (i) we can use Remark 3.2 at the end of subsection 3.2. With $p = -1$, $M_n = n!$ we obtain in this case approximately

$$|z^{-n} E_n(z,\lambda)| \le \pi^{-\frac{1}{2}} 2^{n/2} \Gamma(\tfrac{1}{2}+\tfrac{1}{2}n)(1+\mu)^{-1}(1+\lambda)^{n/2}(z+\lambda)^{-n}.$$

For $z = \lambda = 10$ the exact error $(z+\lambda)z^{-6}E_5(z,\lambda)$ equals 0.00039... . The above bound gives 0.00080... . The exact error in (iii), viz. $(z+\lambda)z^{-4}E_3^*(z,\lambda)$, equals 0.000015.. and the estimate $E_3^*(z,\lambda) = \mu(1+\mu)^{-5}$ based on (5.6) with the constants replaced by 1 (which is quite reasonable in this example) gives 0.00006... .

6.2. Whittaker functions

A more general choice $f(t) = (1+t)^\sigma$, $\sigma \in \mathbb{C}$, leads to expansions for Whittaker functions, or confluent hypergeometric functions. Using the representation

$$U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} dt \tag{6.1}$$

we have in the notation of the present paper

$$\begin{aligned} F_\lambda(z) &= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} (1+t)^\sigma dt \\ &= U(\lambda,\sigma+\lambda+1,z) = z^{-\sigma-\lambda} U(-\sigma,1-\lambda-\sigma,z), \end{aligned} \tag{6.2}$$

where we used a well-known functional relation. It follows that we can obtain the asymptotic expansion of $U(a,b,z)$ for $z \to \infty$, uniformly in a and b such that a-b belongs to a fixed bounded set. The second U-function in (6.2) tells us that it possible to obtain the asymptotic expansion of $U(a,b,z)$ for $z \to \infty$ uniformly in $b \in (-\infty, 1+a]$ (or a proper complex extension of this set), where a belongs to a fixed bounded set.

The function $F_\lambda(z)$ of (6.1) has the following relation with the Whittaker function $W_{\kappa,\mu}(z)$:

$$F_\lambda(z) = e^{\frac{1}{2}z} z^{-(\sigma+\lambda+1)/2}\, W_{\kappa,\mu}(z)$$

with $\kappa = (\sigma+1-\lambda)/2$, $\mu = (\sigma+\lambda)/2$ (here μ is not the same as in the previous sections). It follows that for $W_{\kappa,\mu}(z)$ an expansion can be given for $z \to \infty$, uniformly with respect to κ and μ in an unbounded domains containing the points $-\infty$ and $+\infty$, respectively, such that $\kappa + \mu$ belongs to a fixed bounded set. Using the transformation formula $W_{\kappa,\mu}(z) = W_{\kappa,-\mu}(z)$ (which is the analogue of the second line of (6.2)) we infer that we can obtain an expansion for $W_{\kappa,\mu}(z)$ for $z \to \infty$ uniformly for κ and μ in unbounded sets containing the point $-\infty$, such that $\kappa - \mu$ belongs to a fixed bounded set.

With $f(t) = (1+t)^\sigma$, the integral $G_\lambda(z)$ of (1.6) becomes the other Whittaker or confluent hypergeometric function. That is,

$$\begin{aligned} G_\lambda(z) &= \frac{z^{\lambda-\sigma}\Gamma(\lambda+1)}{\Gamma(\lambda+1-\sigma)}\ {}_1F_1(\lambda+1,\lambda+1-\sigma,z) \\ &= \frac{z^{\lambda-\sigma}\Gamma(\lambda+1)e^z}{\Gamma(\lambda+1-\sigma)}\ {}_1F_1(-\sigma,\lambda+1-\sigma,-z). \end{aligned}$$

In terms of the Whittaker function $M_{\kappa,\mu}(z)$ we have

$$G_\lambda(z) = e^{\frac{1}{2}z} z^{(\lambda-\sigma-1)/2}\, \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\sigma)}\, M_{\kappa,\mu}(z)$$

with $\kappa = -(\lambda+\sigma+1)/2$, $\mu = (\lambda-\sigma)/2$.

The coefficients $a_s(\mu)$ in the expansion (3.1) are given by

$$a_s(\mu) = (1+\mu)^{\sigma-s}\binom{\sigma}{s}, \qquad s = 0,1,2,\ldots,$$

and for (3.14) we can use

$$f^{(n)}(t) = n!\binom{\sigma}{n}(1+t)^{\sigma-n}, \qquad t \in \Omega$$

where $\Omega = \mathbb{C}\setminus(-\infty,-1]$, giving $M_n = n!|\binom{\sigma}{n}|$ and $t \in \bar{S}_{\pi-\varepsilon,\pi-\varepsilon}$, ε a small positive number. From (3.22) and (3.23) we infer that for real σ, $z > 0$, $\lambda \geq 0$ the remainder $E_n(z,\lambda)$ in (3.11) is bounded by

$$|E_n(z,\lambda)| \leq K_n(1+\lambda)^{n/2}(1+\mu)^{\sigma-n}, \qquad n = 0,1,\ldots,$$

where K_n is approximately equal to $\pi^{-\frac{1}{2}}2^{n/2}\Gamma(\frac{1}{2}+\frac{1}{2}n)|\binom{\sigma}{n}|$.

6.3. The generalized zeta function

The generalized zeta function is defined by

$$(6.3) \qquad \zeta(\lambda,z) = \sum_{n=0}^{\infty} (n+z)^{-\lambda}, \quad z \neq 0, -1, -2, \ldots, \ \mathrm{Re}\ \lambda > 1.$$

This function reduces to the more familiar Riemann zeta function $\zeta(\lambda)$ for $z = 1$. It can be expressed as an integral by using

$$(6.4) \qquad F_\lambda(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1}e^{-zt}[(e^t-1)^{-1}-t^{-1}]dt$$

and the relation is

$$(6.5) \qquad \zeta(\lambda,z) = F_\lambda(z) + z^{-\lambda} - \frac{z^{1-\lambda}}{1-\lambda}.$$

Since $F_\lambda(z)$ is an entire function of λ, this gives the analytic continuation of (6.3) with respect to λ.

For a representation as a loop integral, cf. (1.6), we have

$$(6.6) \qquad G_\lambda(z) = \frac{\Gamma(\lambda+1)}{2\pi i} \int_{-\infty}^{(0^+)} t^{-\lambda-1}e^{zt}[(e^t-1)^{-1}-t^{-1}]dt$$

and

$$(6.7)\qquad \zeta(-\lambda,z) = -G_\lambda(z) - \frac{z^{\lambda+1}}{\lambda+1}\,.$$

Hence, for $z \to \infty$ we can obtain for $\zeta(\lambda,z)$ an asymptotic expansion, which is uniformly valid in $\lambda \in [0,\infty)$ (by using (6.4) and (6.5)) or an expansion which is uniformly valid in $\lambda \in (-\infty,0]$ (by using (6.6) and (6.7)).

For Ω we take $\mathbb{C}\backslash\{\pm 2\pi i, \pm 4\pi i, \ldots\}$, and for the sector $\bar{S}_{\alpha,\beta}$ we take $\alpha = \beta = \frac{1}{2}\pi-\varepsilon$, where ε is a small positive number. In (2.3) we take $p = -1$. The function $1/(e^t-1)$ and its derivatives are exponentially small for $t \to \infty$ in $\bar{S}_{\frac{1}{2}\pi-\varepsilon,\frac{1}{2}\pi-\varepsilon}$; therefore, the asymptotic behaviour of $f(t) = 1/(e^t-1)-1/t$ and its derivatives is given by

$$f^{(n)}(t) \sim (-1)^{n+1} n!\; t^{-n-1}$$

as $t \to \infty$ in $\bar{S}_{\frac{1}{2}\pi-\varepsilon,\frac{1}{2}\pi-\varepsilon}$.

To bound the remainder $R_n(t,\mu)$ of (3.10) we use the representation

$$R_n(t,\mu) = \frac{1}{2\pi i}\int_C \frac{f(\tau)}{(\tau-\mu)^n(\tau-t)}\,d\tau$$

where C is a contour in Ω containing in its interior the points $\tau = \mu$ and $\tau = t$ and none of the singularities of f. Writing the integral as a residue series we obtain for $n > 0$

$$R_n(t,\mu) = \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} \frac{1}{(2m\pi i-\mu)^n(2m\pi i-t)}$$

which representation holds for every $\mu,t \in \Omega$. For $n > 1$ we obtain

$$\begin{aligned}
|R_n(t,\mu)| &\le 2(4\pi^2+t^2)^{-\frac{1}{2}}\sum_{m=1}^{\infty}(4m^2\pi^2+\mu^2)^{-n/2}\\
&\le 2K_n(4\pi^2+t^2)^{-\frac{1}{2}}\sum_{m=1}^{\infty}[4m^2\pi^2+(\mu+1)^2]^{-n/2}\\
&\le 2K_n(4\pi^2+t^2)^{-\frac{1}{2}}\int_0^{\infty}\frac{dx}{[4x^2\pi^2+(\mu+1)^2]^{n/2}}\\
&= \tfrac{1}{2}\pi^{-1/2}K_n(4\pi^2+t^2)^{-\frac{1}{2}}(\mu+1)^{1-n}\Gamma(\tfrac{n-1}{2})/\Gamma(n/2)\,.
\end{aligned}$$

Here K_n is defined by

$$K_n = \sup_{\substack{m\geq 1\\ \mu\geq 0}} \{[4m^2\pi^2+(\mu+1)^2]/[4m^2\pi^2+\mu^2]\}^{n/2}.$$

It is easily verified that $K_n \leq (1.18)^{n/2}$.

An interesting application is found in considering the asymptotic expansion of the sum

$$\sum_{j=1}^{n} j^s,$$

for $n \to \infty$, uniformly in s. This sum can be expressed in terms of the generalized zeta function. The relation is

$$\sum_{j=1}^{n} j^s = n^s + \zeta(-s) - \zeta(-s,n).$$

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