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ON OSCILLATION PROPERTIES AND THE INTERVAL OF ORTHOGONALITY OF ORTHOGONAL POLYNOMIALS*

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Abstract. This paper is mainly concerned with the true interval of orthogonality for a sequence of orthogonal polynomials, which is the smallest closed interval containing the limit points of the set of zeros of the polynomials. We give bounds for the endpoints of this interval in terms of the coefficients in the three term recurrence formula and show them to be generalizations of most existing results. Similar findings are reported for the limit interval of orthogonality, which is defined as the smallest closed interval containing the derived set of the set of limit points. Our bounds are based upon an oscillation theorem for orthogonal polynomials which is of independent interest.

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1. Introduction. Let $\{c_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=2}^{\infty}$ be sequences of real numbers and assume that λ_n is positive. Then it is a classical result that the polynomials $P_n(x)$, $n=0,1,\cdots$, defined by the recurrence formula

$$(1)\qquad \begin{aligned} &P_n(x)=(x-c_n)P_{n-1}(x)-\lambda_nP_{n-2}(x), \qquad n=1,2,\cdots,\\ &P_{-1}(x)=0, \qquad P_0(x)=1, \end{aligned}$$

where it is convenient for us to define $\lambda_1=0$, are orthogonal with respect to a (not necessarily unique) mass distribution $d\psi(x)$ on the real line. That is, there is a bounded, nondecreasing function ψ with an infinite spectrum (= support of $d\psi$) such that

$$(2)\qquad \int_{-\infty}^{\infty} P_m(x)P_n(x)\,d\psi(x)=k_n\delta_{nm} \qquad (k_n>0).$$

$P_n(x)$ has n real, distinct zeros $x_{n1}<x_{n2}<\cdots<x_{nn}$ with the property

$$(3)\qquad x_{n+1,i}<x_{ni}<x_{n+1,i+1}, \qquad i=1,2,\cdots,n,$$

so that

$$(4)\qquad \xi_i=\lim_{n\to\infty} x_{ni} \quad \text{and} \quad \eta_j=\lim_{n\to\infty} x_{n,n-j+1}$$

both exist in the extended real number system (see, e.g., [6, §I.5]). The interval $[\xi_1,\eta_1]$ is called the *true interval of orthogonality* since it is the smallest closed interval in which the support of a distribution corresponding to $\{P_n\}$ is concentrated. The *spread* of the true interval of orthogonality is defined as $\eta_1-\xi_1$, while its *centre*, defined only when $\xi_1>-\infty$ or $\eta_1<\infty$, is given by $\frac{1}{2}(\xi_1+\eta_1)$.

Regarding the finiteness of ξ_1, we will have use for a criterion which is essentially due to Stieltjes [20] and elaborated by Chihara [1]. Namely, in order that $\xi_1\geq A>-\infty$, it is necessary and sufficient that there exist numbers γ_n such that

$$(5)\qquad c_n-A=\gamma_{2n-2}+\gamma_{2n-1} \quad \text{and} \quad \lambda_{n+1}=\gamma_{2n-1}\gamma_{2n}, \qquad n>0,$$

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where $\gamma_0 \geq 0$ and $\gamma_n > 0$ for $n > 0$. Here $\gamma_0 \geq 0$ may be replaced by $\gamma_0 = 0$, since the existence of a sequence $\{\gamma_n\}$ satisfying (5) and $\gamma_0 > 0$ implies the existence of a sequence $\{\gamma_n'\}$ satisfying (5) and $\gamma_0' = 0$ (or, in fact, any number between 0 and γ_0). When (5) holds one also has $\eta_1 = \infty$ if and only if $\{\gamma_n\}$ is unbounded.

From (3) and (4) we obviously have $\xi_i \leq \xi_{i+1} < \eta_{j+1} \leq \eta_j$, so that

$$\sigma = \lim_{i \to \infty} \xi_i \quad \text{and} \quad \tau = \lim_{j \to \infty} \eta_j \tag{6}$$

exist, again allowing for $\pm\infty$. It is important to note at this point that

$$\xi_{i+1} = \xi_i \Rightarrow \sigma = \xi_i, \qquad i = 0, 1, \cdots \tag{7}$$

and

$$\eta_{j+1} = \eta_j \Rightarrow \tau = \eta_j, \qquad j = 0, 1, \cdots, \tag{8}$$

where $\xi_0 \equiv -\infty$, $\eta_0 \equiv \infty$ (see, e.g., [6, Thm. II.4.6]).

It can be shown [6, Thm. III.4.2] that the sets of orthogonal polynomials $\{P_n^{(k)}(x)\}_n$, $k = 0, 1, \cdots$, which are determined through the recurrence formula (1) by the sequences $\{c_n^{(k)} = c_{n+k}\}_{n=1}^{\infty}$ and $\{\lambda_n^{(k)} = \lambda_{n+k}\}_{n=2}^{\infty}$, have true intervals of orthogonality $[\xi_1^{(k)}, \eta_1^{(k)}]$ with the properties

$$\xi_1^{(k)} \leq \xi_1^{(k+1)} \leq \sigma \quad \text{and} \quad \tau \leq \eta_1^{(k+1)} \leq \eta_1^{(k)}, \qquad k = 0, 1, \cdots. \tag{9}$$

Further, the next theorem is easily seen to hold as a consequence of [6, Thms. IV.2.1 and IV.3.2].

THEOREM 1.

$$\xi_1^{(k)} \to \sigma \quad \textit{and} \quad \eta_1^{(k)} \to \tau, \qquad k \to \infty.$$

We emphasize that σ and τ are determined only by the limiting behaviour of the parameter sequences $\{c_n\}$ and $\{\lambda_n\}$, so that any finite number of changes in the parameter values has no influence on the values of σ and τ. In view of this fact, we are justified in calling $[\sigma, \tau]$ the *limit interval of orthogonality*. The *spread* and the *centre* of the limit interval of orthogonality are defined as $\tau - \sigma$ and $\frac{1}{2}(\sigma + \tau)$, respectively, provided these quantities are meaningful.

It is the purpose of this paper to give bounds on the true and limit intervals of orthogonality in terms of the parameters c_n and λ_n. Our main tool will be the oscillation theorem for orthogonal polynomials given in §2, which is of independent interest. An extension of this result will be derived in the Appendix.

We note that any result on ξ_1 (or σ), e.g., Stieltjes' criterion (5), may be transformed into a result on η_1 (or τ) and vice versa by considering the polynomials $\bar{P}_n(x) = (-1)^n P_n(-x)$, which satisfy the recurrence relation (1) with parameter sequences $\{\bar{c}_n = -c_n\}$ and $\{\bar{\lambda}_n = \lambda_n\}$. Therefore, as far as the endpoints are concerned, we shall concentrate only on one side of the intervals of orthogonality. In fact, upper bounds on ξ_1 and σ will be given in §3 and lower bounds in §4. Several known results will appear as corollaries to our theorems. We remark that some of these known results are given in the literature under the condition that the distribution $d\psi$ with respect to which the polynomials P_n are orthogonal is unique. This is because they are stated (or derived) in terms of supporting points of $d\psi$ instead of limit points of zeros of the polynomials P_n, while both points of view are equivalent only if $d\psi$ is unique (cf. [3] and [6, Chap. II]).

In the final section, some bounds will be derived on spread and centre of the true and limit intervals of orthogonality and these will be compared with existing results.

2. The basic oscillation theorem. We need some preliminary results and notation first. Let $\mathbf{u}=\{u_0,u_1,\cdots,u_n,\cdots\}$ be an infinite sequence of real numbers. The finite sequence consisting of the first $n+1$ elements of $\mathbf{u}$ will be denoted by $\mathbf{u}_{(n)}$, i.e., $\mathbf{u}_{(n)}=\{u_0,u_1,\cdots,u_n\}$. By $S(\mathbf{u}_{(n)})$, we denote the number of sign changes in the sequence $\mathbf{u}_{(n)}$ by deleting all zero terms, with the special convention $S(\mathbf{0}_{(n)})=-1$, $\mathbf{0}_{(n)}$ denoting the sequence consisting of $n+1$ zeros. We let $S(\mathbf{u})=\lim_{n\to\infty}S(\mathbf{u}_{(n)})$, which exists but, of course, may be infinite.

Our next prerequisite concerns Sturmian sequences of polynomials. We recall the definition (see [17, pp. 7–8]).

DEFINITION 1. A sequence of $n+1$ polynomials $\{R_0,R_1,\cdots,R_n\}$, $n>0$, is called a *Sturmian sequence* on the interval (a,b) if these four conditions are satisfied:

(i) $R_n(x)\neq 0$ for $x=a,\ b$,

(ii) $R_0(x)\neq 0$ for all $x\in[a,b]$,

(iii) $R_i(x)=0\ (0<i<n)\ \&\ x\in[a,b]\Rightarrow R_{i-1}(x)R_{i+1}(x)<0$,

(iv) $R_n(x)=0\ \&\ x\in[a,b]\Rightarrow R_{n-1}(x)R_n'(x)>0$.

This definition is justified by the following theorem [17, Satz 7].

THEOREM 2 (Sturm's theorem). *If the sequence of polynomials $\{R_0,R_1,\cdots,R_n\}$ is a Sturmian sequence on the interval (a,b), then the number of zeros of R_n in the interval (a,b) equals $S(\mathbf{R}(a))-S(\mathbf{R}(b))$, where $\mathbf{R}(x)=\{R_0(x),R_1(x),\cdots,R_n(x)\}$.*

The relevance of this theorem for this paper resides in the next lemma, which concerns the sequence of orthogonal polynomials $\{P_0,P_1,\cdots,P_n,\cdots\}$ defined by the recurrence relation (1).

LEMMA 1. *The sequence $\mathbf{P}_{(n)}=\{P_0,P_1,\cdots,P_n\}$, where $n>0$, is a Sturmian sequence on any interval (a,b) where $P_n(a)\neq 0$ and $P_n(b)\neq 0$.*

Proof. See [21, p. 45].

We are now in a position to state our basic result.

THEOREM 3 (basic oscillation theorem). *For the polynomials $\{P_n\}_{n=0}^{\infty}$ defined by the recurrence relation (1) one has:*

(i) $S(\mathbf{P}(x))=k\Leftrightarrow\eta_{k+1}\leq x<\eta_k$, $k=0,1,\cdots$,

(ii) $S(\mathbf{P}(x))=\infty\Leftrightarrow x<\tau$ *or* $x=\tau<\eta_j$ *for all* j,

(iii) $S(\tilde{\mathbf{P}}(x))=k\Leftrightarrow\xi_k<x\leq\xi_{k+1}$, $k=0,1,\cdots$,

(iv) $S(\tilde{\mathbf{P}}(x))=\infty\Leftrightarrow x>\sigma$ *or* $x=\sigma>\xi_i$ *for all* i,

where $\mathbf{P}(x)=\{P_0(x),P_1(x),\cdots\}$, $\tilde{\mathbf{P}}(x)=\{\tilde{P}_0(x),\tilde{P}_1(x),\cdots\}$ *and* $\tilde{P}_n(x)=(-1)^nP_n(x)$.

Proof. It is evident that (ii) and (iv) are implied by (i) and (iii), respectively, while (iii) readily follows from (i) by considering the polynomials $\bar{P}_n(x)=(-1)^nP_n(-x)$ mentioned in the introduction. So it remains to prove (i).

To this end, let x and n be such that $P_n(x)\neq 0$. Choose η such that $\max(x,x_{nn})<\eta<\eta_0\equiv\infty$. By (3) we then have $\eta>x_{ii}$ $(i=1,2,\cdots,n)$, and (1) subsequently implies $P_i(\eta)>0$ for $i=0,1,\cdots,n$, whence $S(\mathbf{P}_{(n)}(\eta))=0$. Now applying Sturm's theorem to $\mathbf{P}_{(n)}$ in the interval (x,η), we get $S(\mathbf{P}_{(n)}(x))-S(\mathbf{P}_{(n)}(\eta))=$ number of zeros of P_n in (x,η), i.e.,

$$S\big(\mathbf{P}_{(n)}(x)\big)=\text{number of zeros of } P_n \text{ in } (x,\infty). \tag{10}$$

Letting n tend to infinity in (10), (i) emerges as a consequence of (3) and (4). □

Aspects of the basic oscillation theorem may be found in the literature under various guises. Thus a special case of it was employed by Stieltjes [20, p. 564] in the context of continued fractions, while parts (ii) and (iv) of the theorem are essentially contained in [23, Thm. 8(a)] in the context of difference equations. Further, by making the identification

$$P_n(x)=\det(A_n-xI_n), \tag{11}$$

where I_n is the $n\times n$ identity matrix and

$$A_n=\begin{pmatrix} c_1 & \sqrt{\lambda_2} & & & 0 \\ \sqrt{\lambda_2} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{\lambda_{n-1}} \\ 0 & & & \sqrt{\lambda_{n-1}} & c_n \end{pmatrix}, \tag{12}$$

our questions regarding (essentially) the zeros x_{nk} may be put in terms of eigenvalues of symmetric tridiagonal matrices for which the Sturmian approach is well known (see, e.g., [16, Chap. 7]). Indeed, we shall repeatedly make use of this identification to obtain new results or point out alternative proofs.

In closing this section, we remark that Chihara ([1], [4], see also [6]) has obtained characterizations for ξ_1, η_1, σ and τ which are in appearance quite different from the basic oscillation theorem. A third characterization, which may be conceived as a consequence of Chihara's results, has been stated and given an independent proof by Whitehurst [22, Chap. 4]. It is not very difficult to prove directly the equivalence of Chihara's or Whitehurst's results and the basic oscillation theorem.

3. Upper bounds on ξ_1 and σ. Our starting point in this section will be a lemma concerning the system of equations

$$z_n+a_nz_{n-1}+b_nz_{n-2}=0, \qquad n=1,2,\cdots. \tag{13}$$

LEMMA 2. *If the system of equations* (13), *where* $b_n>0$, *possesses a solution* $z_{-1},z_0,z_1,\cdots$ *satisfying* $z_nz_{n+1}<0$ *for* $n\geq N\geq 0$, *then*

$$a_M+\sum_{m=M+1}^{M+k}\left(a_m-2\sqrt{b_m}\right)>0 \tag{14}$$

for any two integers $k\geq 0$ *and* $M>N+1$ ($M\geq N+1$ *if* $z_{N-1}=0$).

Proof. Assuming that a given solution has $z_m\neq 0$ for $m=M-1,M,\cdots,M+k-1$, we can write down the equalities

$$a_M=-\frac{z_M}{z_{M-1}}-b_M\frac{z_{M-2}}{z_{M-1}},$$

and, for $m=M,M+1,\cdots,M+k-1$,

$$a_{m+1}-2\sqrt{b_{m+1}}=\frac{z_m}{z_{m-1}}-\frac{z_{m+1}}{z_m}-\frac{\left(z_m+z_{m-1}\sqrt{b_{m+1}}\right)^2}{z_{m-1}z_m}.$$

Summing these $k+1$ equalities yields

$$a_M+\sum_{m=M+1}^{M+k}\left(a_m-2\sqrt{b_m}\right)=-\left\{\frac{z_{M+k}}{z_{M+k-1}}+b_M\frac{z_{M-2}}{z_{M-1}}+\sum_{m=M}^{M+k-1}\frac{\left(z_m+z_{m-1}\sqrt{b_{m+1}}\right)^2}{z_{m-1}z_m}\right\},$$

from which the lemma follows at once. □

Returning to the recurrence formula (1), we let x be any real number,

$$(15)\qquad y_n=P_n(x),\qquad n=-1,0,1,\cdots,$$

and $y=\{y_0,y_1,\cdots\}$. Further, let $\{\chi_1,\chi_2,\cdots\}$ be any sequence of positive numbers and define

$$(16)\qquad z_{-1}=0,\quad z_0=1\quad\text{and}\quad z_n=(\chi_1\chi_2\cdots\chi_n)^{-1}y_n,\ n>0.$$

If we let b_1 be positive but otherwise arbitrary,

$$(17)\qquad a_n=(c_n-x)/\chi_n\quad\text{and}\quad b_{n+1}=\lambda_{n+1}/(\chi_n\chi_{n+1}),\qquad n>0,$$

then $\{z_n\}_{n=-1}^{\infty}$ satisfies the recurrence relation (13) with $b_n>0$, so that Lemma 2 applies. Translating this result in terms of y_n, c_n, λ_n, χ_n and x yields

$$(18)\qquad \frac{c_M}{\chi_M}+\sum_{m=M+1}^{M+k}\left(\frac{c_m}{\chi_m}-2\left(\frac{\lambda_m}{\chi_{m-1}\chi_m}\right)^{1/2}\right)>x\sum_{m=M}^{M+k}\frac{1}{\chi_m}$$

for $k\geq 0$ and $M>N+1$ ($M\geq N+1$ if $y_{N-1}=0$), whenever $y_ny_{n+1}<0$ for $n\geq N\geq 0$.

By the basic oscillation theorem one has $x\leq\xi_1$ if and only if $S(\tilde{y})=0$. That is, $x\leq\xi_1$ if and only if $y_ny_{n+1}<0$ for $n\geq 0$, since $y_n=0$ is clearly impossible when $x\leq\xi_1$. Further noting that $y_{-1}=0$, we conclude that the inequality $x\leq\xi_1$ implies the inequalities (18) for all $k\geq 0$ and $M>0$. From this result one easily deduces the following theorem.

THEOREM 4. *For any sequence of positive numbers* $\{\chi_1,\chi_2,\cdots\}$ *and integers* $k\geq 0$ *and* $M>0$ *one has*

$$(19)\qquad \xi_1<\left(\frac{c_M}{\chi_M}+\sum_{m=M+1}^{M+k}\left(\frac{c_m}{\chi_m}-2\left(\frac{\lambda_m}{\chi_{m-1}\chi_m}\right)^{1/2}\right)\right)\left(\sum_{m=M}^{M+k}\frac{1}{\chi_m}\right)^{-1}.$$

Taking $k=0$ and $\chi_n=1$ for all n, we obtain Corollary 4.1, which is also a direct consequence of Stieltjes' criterion (5) and therefore well known (see, e.g., [6, p. 109]).

COROLLARY 4.1.

$$\xi_1<c_n,\qquad n=1,2,\cdots.$$

Letting $k=1$ and $\chi_n=1$ for all n, a result emerges which was first given (with an error) by Maki [11] and later improved by Chihara [5].

COROLLARY 4.2.

$$\xi_1<\frac{1}{2}(c_n+c_{n+1})-\sqrt{\lambda_{n+1}},\qquad n=1,2,\cdots.$$

We remark that the other part of the Maki–Chihara result to the effect that $\frac{1}{2}(c_n+c_{n+1})-\sqrt{\lambda_{n+1}}$ is unbounded when $\xi_1>-\infty$ and $\eta_1=\infty$, can also be generalized in the spirit of Theorem 4, at least when $\chi_n=1$ for all n. One should simply use Maki's argument on the basis of which lies the result of Stieltjes mentioned in the introduction.

Assuming that $\inf\{c_n\} > -\infty$, we can choose $k=1$ and $\chi_n = c_n - c$ in (19), where c is any number smaller than c_n for all n. After some rearranging, we then get

$$(20)\quad \xi_1 < c + 2\frac{(c_n-c)(c_{n+1}-c)-(\lambda_{n+1}(c_n-c)(c_{n+1}-c))^{1/2}}{c_n+c_{n+1}-2c}, \qquad n=1,2,\cdots.$$

In combination with Corollary 4.1, this result yields a useful third corollary. Namely, if there are values of $\zeta_n \equiv \frac{1}{2}(c_n+c_{n+1}-((c_n-c_{n+1})^2+4\lambda_{n+1})^{1/2})$, $n=1,2,\cdots$, with the property $\zeta_n < c_m$ for all m, we can choose c equal to any of those ζ_n, ζ_1 say, after which the choice $n=1$ yields that $\xi_1 < \zeta_1$. Hence, in this case, $\xi_1 < \zeta_n$ for all n. If, on the other hand, $\zeta_n > c_m$ for some m and all n, Corollary 4.1 implies that the same conclusion holds. Thus, we have the following result, which is sharper than Corollary 4.2, while involving the same parameters.

COROLLARY 4.3.

$$\xi_1 < \frac{1}{2}\left(c_n + c_{n+1} - \left((c_{n+1}-c_n)^2 + 4\lambda_{n+1}\right)^{1/2}\right), \qquad n=1,2,\cdots.$$

We note that upper bounds for ξ_1 can be obtained on the basis of the interpretation (11) for $P_n(x)$. Namely, considering that the eigenvalues of A_n equal those of $K_nA_nK_n$, where K_n is the $n\times n$ matrix consisting of elements $k_{ij}=1$ when $i+j=n+1$ $(i,j=1,2,\cdots,n)$ and 0 elsewhere, one also has

$$(21)\qquad P_n(x) = \det(K_nA_nK_n - xI_n).$$

Hence, we can identify $P_n(x)$ with the nth polynomial in an orthogonal sequence $\{\hat{P}_m(x)\}$ determined by the recurrence formula (1) through the parameters $\hat{c}_m = c_{n+1-m}$ $(m\le n)$, $\hat{c}_m = c_m$ $(m>n)$, $\hat{\lambda}_m = \lambda_{n+2-m}$ $(m \le n+1)$ and $\hat{\lambda}_m = \lambda_m$ $(m>n+1)$. It now follows from (3) and (4) that

$$(22)\qquad \xi_1 < x_{n1} = \hat{x}_{n1} < \hat{x}_{k1}, \qquad k=1,2,\cdots,n-1,$$

where $\hat{x}_{m1}$ denotes the smallest zero of $\hat{P}_m(x)$. However, the only practical bounds obtained by this approach are $\xi_1 < \hat{x}_{11}$, but this gives Corollary 4.1, and $\xi_1 < \hat{x}_{21}$, which amounts to Corollary 4.3.

Remark. A third proof of Corollary 4.3 may be given on the basis of Chihara's characterization for ξ_1 (cf. [6, Thm. IV.2.1]).

The arguments leading to Theorem 4 need only slight modification to obtain results on the limit interval of orthogonality. For by the basic oscillation theorem we have $x<\sigma$ only if $S(\bar{y})$ is finite; that is, only if $y_n y_{n+1} < 0$ for n sufficiently large (by definition of σ, $y_n=0$ occurs for at most finitely many n if $x<\sigma$). Hence the inequality $x<\sigma$ implies the inequality (18) for M sufficiently large and all $k\ge 0$. From this it is easy to derive Theorem 5, which, however, also derives directly from the Theorems 1 and 4.

THEOREM 5. *For any sequence of positive numbers* $\{\chi_1, \chi_2, \cdots\}$ *and integer* $k \ge 0$, *one has*

$$(23)\quad \sigma \le \lim_{M\to\infty}\inf\left\{\left(\frac{c_M}{\chi_M} + \sum_{m=M+1}^{M+k}\left(\frac{c_m}{\chi_m} - 2\left(\frac{\lambda_m}{\chi_{m-1}\chi_m}\right)^{1/2}\right)\right)\left(\sum_{m=M}^{M+k}\frac{1}{\chi_m}\right)^{-1}\right\}.$$

Taking $k=0$ and χ_n arbitrary, we get the analogue of Corollary 4.1, which has been obtained previously by Wouk [23, last inequality of Thm. 8(e)] and Chihara [1, Thm. 6]; see also [6, Thm. IV.3.1].

COROLLARY 5.1.

$$\sigma \le \liminf_{n\to\infty} \{c_n\}.$$

We also state as a corollary the analogue of Corollary 4.3, although its proof is most conveniently given via Theorem 1 and Corollary 4.3.

COROLLARY 5.2.

$$\sigma \le \liminf_{n\to\infty} \frac{1}{2}\left\{c_n + c_{n+1} - \left((c_n - c_{n+1})^2 + 4\lambda_{n+1}\right)^{1/2}\right\}.$$

An interesting case arises when we let k tend to infinity in Theorem 5. However, we had better do this not in (23), but at at earlier stage in the reasoning leading to Theorem 5. Namely, from Theorem 4 we see that for all $M>0$

$$\xi_1 \le \lim_{k\to\infty} \inf\{f(M,k)\},$$

where $f(M,k)$ denotes the expression between braces in (23). Hence, by Theorem 1,

$$\sigma \le \lim_{M\to\infty} \inf\left\{ \lim_{k\to\infty} \inf\{f(M,k)\}\right\}. \tag{24}$$

Now let us assume that $\Sigma\chi_n^{-1}=\infty$. Then, evidently, $\liminf_{k\to\infty}\{f(M,k)\} = \liminf_{k\to\infty}\{f(1,k)\}$, so that we obtain the next theorem.

THEOREM 6. *For any sequence of positive numbers* $\{\chi_0, \chi_1, \cdots\}$ *such that* $\Sigma\chi_n^{-1}=\infty$, *one has*

$$\sigma \le \lim_{k\to\infty} \inf\left\{\left(\sum_{m=1}^{k}\left(\frac{c_m}{\chi_m} - 2\left(\frac{\lambda_m}{\chi_{m-1}\chi_m}\right)^{1/2}\right)\right)\left(\sum_{m=1}^{k}\frac{1}{\chi_m}\right)^{-1}\right\}. \tag{25}$$

Taking $\chi_n = 1$ for all n, we obtain the important Corollary 6.1, which has been given previously by Wouk [23, Thm. 8(g)].

COROLLARY 6.1.

$$\sigma \le \lim_{k\to\infty} \inf\left\{\frac{1}{k}\sum_{m=1}^{k}\left(c_m - 2\sqrt{\lambda_m}\right)\right\}.$$

4. Lower bounds on ξ_1 and σ. As in the previous section we start our discussion by considering the system of equations (13). If we plot a solution $z_{-1}, z_0, z_1, \cdots$ of this system by joining successive coordinates (i, z_i) by straight line segments, then the points where such a line segment meets the x-axis will be called a node of the solution. We can now cite the following classical result [14].

LEMMA 3 (Sturm's separation theorem for difference equations). *For any system of equations* (13) *where* $b_n>0$, *the nodes of any two linearly independent solutions separate each other.*

Suppose $a_n + 1 < -b_n < 0$ for $n > N \ge 0$ and let two arbitrary numbers $\hat{z}_N > \hat{z}_{N-1} \ge 0$ determine a solution $\{\hat{z}_n\}_{-1}^{\infty}$ of (13). Then we have by induction

$$\hat{z}_n - \hat{z}_{n-1} = -(a_n+1)(\hat{z}_{n-1} - \hat{z}_{n-2}) - (a_n + b_n + 1)\hat{z}_{n-2} > 0$$

for $n>N$. Lemma 3 now implies that any solution $\{z_n\}$ of (13) has at most one node in the interval $[N-1, \infty)$. Hence, also noting that $z_n z_{n-2} \le 0$ if $z_{n-1}=0$, we can state the following lemma, which is also essentially contained in [9].

LEMMA 4. *If $a_n+b_n+1<0$ and $b_n>0$ for $n>N$, then any nontrivial solution $\{z_n\}$ of* (13) *for which $z_{m-1}z_m\le 0$ for some $m\ge N$ has the property that* $\operatorname{sign}(z_{m+k})=\operatorname{sign}(z_m)$ *if $z_m\ne 0$, and* $=\operatorname{sign}(-z_{m-1})$ *if $z_m=0$, for all $k>0$.*

Back to our orthogonal system (1) we let x be any real number and define the quantities y_n as in (15). Further, we let $\{\chi_0,\chi_1,\cdots\}$ be any sequence of positive numbers and define

$$z_{-1}=0,\ z_0=1 \quad\text{and}\quad z_n=(-1)^n(\chi_1\chi_2\cdots\chi_n)^{-1}y_n,\qquad n>0. \tag{26}$$

Finally, we let b_1 be positive,

$$a_n=-(c_n-x)/\chi_n \quad\text{and}\quad b_{n+1}=\lambda_{n+1}/(\chi_n\chi_{n+1}),\qquad n>0. \tag{27}$$

Then $\{z_n\}$ satisfies the recurrence relation (13) with $b_n>0$, so that the second condition in Lemma 4 is satisfied for $n>0$. In terms of c_n, λ_n, χ_n and x, the first condition in this lemma reads

$$c_n-\frac{\lambda_n}{\chi_{n-1}}-\chi_n>x, \tag{28}$$

provided $n>1$. Supposing (28) to be valid for $n>0$, we can choose $b_1>0$ so small that $a_n+b_n+1<0$ for $n>0$. Hence, Lemma 4 applies and we have $\operatorname{sign}((-1)^k y_k)=\operatorname{sign}(z_k)$ $=\operatorname{sign}(z_0)=1$, since $z_{-1}z_0=0$. Thus, by the basic oscillation theorem, $x\le\xi_1$. A trivial argument subsequently leads to our next theorem.

THEOREM 7. *For any sequence of positive numbers $\{\chi_0,\chi_1,\cdots\}$, one has*

$$\inf_{n\ge 1}\left\{c_n-\frac{\lambda_n}{\chi_{n-1}}-\chi_n\right\}\le\xi_1. \tag{29}$$

Remark. This theorem may also be obtained via the identification (11) for $P_n(x)$. Namely, the zeros $x_{n1},x_{n2},\cdots,x_{nn}$ of $P_n(x)$ are the eigenvalues of A_n; and therefore, also of the matrix $\Phi_n^{-1}A_n\Phi_n$, where $\Phi_n=\operatorname{diag}(\phi_1,\phi_2,\cdots,\phi_n)$ and $\phi_i>0$. With Gershgorin's theorem (see [12, p. 146]), one may subsequently prove that

$$x_{n1}\ge\min_{i\le n}\left\{c_i-\frac{\phi_{i-1}}{\phi_i}\sqrt{\lambda_i}-\frac{\phi_{i+1}}{\phi_i}\sqrt{\lambda_{i+1}}\right\}, \tag{30}$$

where $\phi_0=1$, say. Taking $\{\phi_i\}$ such that $\phi_{i+1}=\chi_i\phi_i/\sqrt{\lambda_{i+1}}$ and letting n tend to infinity yields (29).

Various consequences of Theorem 7 suggest themselves; e.g., one could take $\chi_n=1$ for all n, or, $\chi_0=1$ and $\chi_n=\lambda_{n+1}$ ($n>0$), the latter result being implicit in Maki [11]. We will explicitly state as a corollary the case $\chi_0=1$ and $\chi_n=\sqrt{\lambda_{n+1}}$ ($n>0$), since this result improves directly upon Lemma 3 of Nevai [15, p. 21].

COROLLARY 7.1.

$$\inf_{n\ge 1}\left\{c_n-\sqrt{\lambda_n}-\sqrt{\lambda_{n+1}}\right\}\le\xi_1.$$

By choosing $\chi_0=1$ and $\chi_n=\lambda_{n+1}/(c_{n+1}-\phi_{n+1})$ ($n>0$), where $\phi_n<c_n$ ($n>1$), we obtain the following useful, alternative formulation of Theorem 7.

THEOREM 7′. *For any sequence $\{\phi_1,\phi_2,\cdots\}$, with $\phi_1\le c_1$ and $\phi_n<c_n$ ($n>1$), one has*

$$\inf_{n\ge 1}\{\phi_n-\lambda_{n+1}/(c_{n+1}-\phi_{n+1})\}\le\xi_1. \tag{31}$$

Thus formulated, Theorem 7 is seen to improve upon a result of Léopold [10], specified for the present context, which amounts to (31) with a fixed value ϕ ($\leq c_n$ for all n) for all ϕ_n.

As a final lower bound for ξ_1, we mention a theorem of Chihara. Actually, Chihara gives the corresponding result for σ, but his argument applies equally well here (cf. [2], [4] and [6, Thm. IV.3.3]).

THEOREM 8 (Chihara). *For any chain sequence* $\{\beta_n\}_{n=1}^{\infty}$, *one has*

$$\inf_{n\geq 1} \frac{1}{2}\left\{c_n + c_{n+1} - \left((c_{n+1}-c_n)^2 + 4\lambda_{n+1}/\beta_n\right)^{1/2}\right\} \leq \xi_1. \tag{32}$$

Remark. $\{\beta_n\}_{n=1}^{\infty}$ is a *chain sequence* if there exists a sequence $\{g_k\}_{k=0}^{\infty}$ with $0\leq g_0<1$ and $0<g_k<1$ ($k>0$), such that $\beta_n=(1-g_{n-1})g_n$; $\{g_k\}$ is called a *parameter sequence* for $\{\beta_n\}$. For instance, $\{\frac{1}{4}\}$ is a chain sequence for which $\{\frac{1}{2}\}$ is a parameter sequence.

Remark. Theorems 7 and 8 are in a sense best possible since equality may be obtained in (29) and (32). To this end, one should take $\beta_n=\alpha_n(\xi_1)\equiv\lambda_{n+1}/((c_{n+1}-\xi_1)(c_n-\xi_1))$ (which is a chain sequence according to [6, Thm. IV.2.1]) in (32) and $\chi_n=(c_n-\xi_1)(1-g_{n-1})$, with $\{g_k\}$ a parameter sequence for $\{\alpha_n(\xi_1)\}$, in (29). Thus we have actually obtained new characterizations for the true interval of orthogonality.

Using an argument similar to that for Theorem 7 or, alternatively, exploiting Theorems 1 and 7, one easily produces the following general lower bound for σ.

THEOREM 9. *For any sequence of positive numbers* $\{\chi_0, \chi_1, \cdots\}$, *one has*

$$\liminf_{n\to\infty}\left\{c_n - \frac{\lambda_n}{\chi_{n-1}} - \chi_n\right\} \leq \sigma. \tag{33}$$

We will explicitly state as a corollary of Theorem 9 the case where $\chi_n=\sqrt{\lambda_{n+1}}$ for $n>0$.

COROLLARY 9.1.

$$\liminf_{n\to\infty}\left\{c_n - \sqrt{\lambda_n} - \sqrt{\lambda_{n+1}}\right\} \leq \sigma.$$

The latter result has been given by Wouk [23, Thm. 8(f)], while it is a slight generalization of a result of Chihara [2, p. 704]; see also Nevai [15, p. 22].

In this context we remark that the proof and subsequent formulation of another one of Wouk's results [23, Thm. 8(h)] contains an error. The corrected version of this theorem is an easy consequence of the above corollary.

For completeness' sake we finally mention the analogue to Theorem 8, Chihara's lower bound for σ.

THEOREM 10 (Chihara [2], [4], see also [6, Thm. IV.3.3]). *For any chain sequence* $\{\beta_n\}$

$$\liminf_{n\to\infty} \frac{1}{2}\left\{c_n + c_{n+1} - \left((c_{n+1}-c_n)^2 + 4\lambda_{n+1}/\beta_n\right)^{1/2}\right\} \leq \sigma. \tag{34}$$

Remark. It can be shown that the left-hand sides of (33) and (34) can be made arbitrarily close to σ by a suitable choice of $\{\chi_n\}$ and $\{\beta_n\}$, respectively.

5. Bounds on spread and centre. As mentioned in the introduction, we can straightforwardly produce lower (upper) bounds for η_1 (or τ) on the basis of upper (lower) bounds for ξ_1 (or σ) by considering the polynomials $\bar{P}_n(x)=(-1)^nP_n(-x)$

which are determined by the recurrence formula (1) via the parameters $\bar{c}_n = -c_n$ and $\bar{\lambda}_n = \lambda_n$, and thus have $[-\eta_1, -\xi_1]$ ($[-\tau, -\sigma]$) as their true (limit) interval of orthogonality. Then various upper (lower) bounds on the spread of the true (or limit) interval of orthogonality may be obtained by combining upper (lower) bounds for ξ_1 (or σ) with lower (upper) bounds for η_1 (or τ). Similarly, we should combine upper (lower) bounds for ξ_1 (or σ) with upper (lower) bounds for η_1 (or τ) to obtain upper (lower) bounds on the centre of the true (or limit) interval of orthogonality. We will not pursue this approach in any detail except that we show how known results on the spread of the true interval of orthogonality may be reproduced in this way. Also, we show that additional information on the centre of the true (or limit) interval of orthogonality may be obtained by exploiting Stieltjes' criterion (5).

Let us first note that as a consequence of Corollary 4.3 and its dual result for η_1, we have the following theorem, which is essentially due to Mirsky [13], who states it in a finite eigenvalue context (the term *spread* is taken from Mirsky).

THEOREM 11.

$$\eta_1 - \xi_1 > \left((c_{n+1} - c_n)^2 + 4\lambda_{n+1}\right)^{1/2}, \qquad n = 1, 2, \cdots.$$

This is the simplest result combining parameters c_n and λ_n. A bound involving only c_n's, which is not necessarily worse than Theorem 11, is

$$\eta_1 - \xi_1 > c_n - c_m, \qquad n, m = 1, 2, \cdots, \tag{35}$$

which follows from Corollary 4.1. However, Theorem 11 does improve upon a result involving only λ_n's which, together with (35), was given already by Shohat [18], [19], viz.,

$$\eta_1 - \xi_1 > 2\sqrt{\lambda_n}, \qquad n = 2, 3, \cdots. \tag{36}$$

But then, the latter inequality can be sharpened in another direction on the basis of (19) (with $\chi_n \equiv 1$) as follows.

THEOREM 12. *For any two integers $k > 0$ and $M \geq 0$, one has*

$$\eta_1 - \xi_1 > \frac{4}{k+1} \sum_{m=M+1}^{M+k} \sqrt{\lambda_m}. \tag{37}$$

In particular, it follows that $\eta_1 - \xi_1 \geq 4\sqrt{\lambda}$ when $\lambda_m \to \lambda$ as $m \to \infty$.

So much for the spread.

Regarding the centre of the true interval of orthogonality, let us assume $\eta_1 < \infty$. Then, by Stieltjes' criterion (in dual form), we have

$$-c_n = -\eta_1 + \gamma_{2n-2} + \gamma_{2n-1}, \qquad \lambda_{n+1} = \gamma_{2n-1}\gamma_{2n}$$

for $n > 0$, where $\gamma_0 = 0$ and $\gamma_n > 0$ for $n > 0$. For convenience, we define $\gamma_{-1} = 1$. By (29) we then get

$$\inf_{n \geq 1} \left\{ \eta_1 - \gamma_{2n-2} - \gamma_{2n-1} - \frac{\gamma_{2n-3}\gamma_{2n-2}}{\chi_{n-1}} - \chi_n \right\} \leq \xi_1. \tag{38}$$

Subsequently, substituting $\chi_n = \gamma_{2n-1}$ for $n \geq 0$ yields

$$-\eta_1 + 2\inf\{c_n\} \leq \xi_1. \tag{39}$$

Combining this inequality and its dual result, we obtain the next theorem.

THEOREM 13. *If* $\xi_1 > -\infty$ *or* $\eta_1 < \infty$, *then*

$$\inf\{c_n\} \le \frac{1}{2}(\xi_1 + \eta_1) \le \sup\{c_n\}. \tag{40}$$

Similarly, we obtain the corresponding result for the centre of the limit interval of orthogonality.

THEOREM 14. *If* $\sigma > -\infty$ *or* $\tau < \infty$, *then*

$$\lim_{n\to\infty} \inf\{c_n\} \le \frac{1}{2}(\sigma + \tau) \le \lim_{n\to\infty} \sup\{c_n\}. \tag{41}$$

Appendix. A second order oscillation theorem. In this appendix, we shall assume $\xi_1 > -\infty$. We define

$$Q_n(x) = P_n(x)/P_n(\xi_1), \qquad n = 0, 1, \cdots, \tag{A1}$$

where $\{P_n\}$ is given by (1), and wish to study the behaviour of the sequence $\mathbf{Q}(x) = \{Q_0(x), Q_1(x), \cdots\}$. To this end, we define the polynomials $P_n^*(x)$, $n = 0, 1, \cdots$, by

$$P_n^*(x) = P_{n+1}(\xi_1)(Q_{n+1}(x) - Q_n(x))/(x - \xi_1), \tag{A2}$$

i.e., $\{P_n^*\}$ is the set of *kernel polynomials* with parameter ξ_1 which is associated with our original system $\{P_n\}$ (see [6, §I.7]). These kernel polynomials form an orthogonal system. The zeros of $P_n^*(x)$ will be denoted by x_{nk}^*, $k = 1, 2, \cdots, n$, and in an obvious manner we define the numbers ξ_k^* and η_k^*, $k = 0, 1, \cdots$. The following lemma holds.

LEMMA A1. *For all* $k > 0$, *one has* $\xi_k^* = \xi_{k+1}$ *and* $\eta_k^* = \eta_k$.

Proof. There is a separation theorem saying that

$$x_{nk} < x_{nk}^* < x_{n+1,k+1} \tag{A3}$$

[6, Thm. I.7.2], whence the second statement holds.

Regarding ξ_k^* we can only conclude from (A3) that

$$\xi_k \le \xi_k^* \le \xi_{k+1}, \qquad k = 1, 2, \cdots. \tag{A4}$$

However, there exists a distribution $d\psi(x)$ with respect to which the polynomials P_n are orthogonal whose support contains the points ξ_k, $k = 1, 2, \cdots$, but no other points smaller than σ [6, Thm. II.4.5]. The polynomials P_n^* are then orthogonal with respect to the distribution $d\psi^*(x) = (x - \xi_1)d\psi(x)$ [21, Thm. 3.1.4]. Assuming that $d\psi^*$ is the only distribution with respect to which the P_n^* are orthogonal, we subsequently obtain from [6, Thm. II.4.5] that $\xi_k^* = \xi_{k+1}$ $(k = 1, 2, \cdots)$.

Now suppose that $d\psi^*$ is not uniquely determined by $\{P_n^*\}$. We see from (A4) that $\xi_1^* \le \xi_2$. But $\xi_1^* < \xi_2$ would be contradictory to the fact that the support of $d\psi^*$ contains at least one point in $(-\infty, \xi_1^*]$ (see [6, Thm. II.4.4(i)]). Consequently, $\xi_1^* = \xi_2$. Invoking [3, Thm. 5], we conclude that $d\psi^*$ is the unique distribution corresponding to $\{P_n^*\}$ whose support is contained in $[\xi_2, \infty)$, and that $\xi_k^* = \xi_{k+1}$ for $k > 1$ too. □

The following second order oscillation theorem is the main result of this appendix.

THEOREM A1. *The polynomials* Q_n *defined by* (A1) *and* (1) *satisfy*

$$S(\mathbf{Q}(x)) = S(\mathbf{\Delta}_Q(x)) = k \tag{A5}$$

iff $\xi_k < x \le \xi_{k+1}$ $(k = 0, 1, \cdots)$. *Here* $\mathbf{Q}(x) = \{Q_0(x), Q_1(x), \cdots\}$ *and* $\mathbf{\Delta}_Q(x) = \{Q_0(x), Q_1(x) - Q_0(x), Q_2(x) - Q_1(x), \cdots\}$.

Proof. The fact that $S(Q(x))=k$ iff $\xi_k<x\leq\xi_{k+1}$ is a restatement of the basic oscillation theorem. The second part follows by application of the basic oscillation theorem to the polynomials P_n^* and observing that, by Corollary 4.1, $Q_0(x)(Q_1(x)-Q_0(x))<0$ when $x>\xi_1$. □

When $\eta_1<\infty$ a similar theorem may be obtained for the polynomials

$$R_n(x)=P_n(x)/P_n(\eta_1), \qquad n=0,1,\cdots. \tag{A6}$$

In closing, we remark that a finite version of Theorem A1 is stated in [7] in the context of birth-death processes. Indeed, the results of this paper apply to these stochastic processes as is shown in [8].

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