On the Overflow Process from a Finite Markovian Queue

Erik A. van Doorn
Centre for Mathematics and Computer Science, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

Received 3 May 1983
Revised 21 February 1984; 7 May 1984

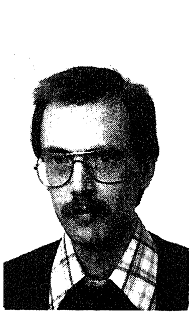
We determine the distribution of the time between overflows for a single server Markovian queueing system with finite waiting room and state-dependent service and arrival rates. The result is subsequently used to analyse a GI/M/∞ system where the arrival process is the overflow process from the M/M/s/r queue.

Keywords: Overflow Process, Markovian Queue, Infinite-Server Queue, Teletraffic Theory, Peakedness Factor.

1. Introduction

Consider a queueing system $\mathscr{Q}$ with s servers and r waiting places, where $0 < s < \infty$ and $0 \leqslant r < \infty$. If a customer arrives to find $s + r$ customers in the system, he departs never to return, and he is then said to have overflowed. Otherwise he enters the system and, depending on whether there are free servers or not, is served immediately or occupies a free waiting place until his turn to be served comes up. Our interest centers on the point process of overflowing customers which will be denoted by $(\mathscr{Q})_{\text{overflow}}$ and called the overflow process from the system $\mathscr{Q}$.

The study of overflow processes is of importance in teletraffic theory, since telephone systems usually provide for alternative routes for calls that are blocked on a specific trunk group. In this context Palm [36] studies the loss system GI/M/1/0 and shows that the overflow process is a renewal process. Further, he relates the Laplace–Stieltjes transform of the interoverflow time distribution to that of the interarrival time distribution. Palm also observes that the overflow process from a GI/M/s/0 loss system, where $s > 1$, may be conceived as the overflow process from a $(\text{GI/M/}s-1/0)_{\text{overflow}}$/M/1/0 system, so that his analysis actually pertains to GI/M/s/0 for all $s > 0$. We refer to Khintchine [22], Takàcs [44,45], Beneš [4], Riordan [41], Pearce and Potter [37], Wallin [48] and Potter [38] for treatments of Palm's theory and its ramifications. Several of these authors, including Palm, give detailed results for the overflow process from the system M/M/s/0 (see Descloux [11] for related results). As an aside we remark that the essentials of Palm's analysis can be traced back to Vaulot [47].



Erik A. van Doorn was born in Zeist, The Netherlands, in 1949. He received the M.S. degree in Mathematics from Eindhoven University of Technology in 1974 and the Ph.D. degree in Technical Sciences from Twente University of Technology in 1979. From 1980 to 1982 he was with the Dr. Neher Laboratories of the Netherlands Postal and Telecommunications Services. Since 1982 he has been with the Centre for Mathematics and Computer Science. His research interests are in the areas of queueing and teletraffic theory, stochastic processes, orthogonal polynomials and graph theory.

North-Holland
Performance Evaluation 4 (1984) 233–240

Determination of the overflow process from the system GI/M/s/r, when r, the number of waiting positions, is positive, is more difficult. For, although the overflow process is still renewal, an iterative argument as when $r = 0$ is no longer valid. The case $s = 1$, $0 \leqslant r < \infty$ was treated by Cinlar and Disney [6], while for arbitrary s and r only recently De Smit [13] and McNickle [30] have derived an explicit expression for the Laplace–Stieltjes transform of the interoverflow time distribution.

An even more complicated situation arises when one assumes non-exponential service time distributions, since then the overflow process is not in general a renewal process. The only available results are those of Halfin [14] who studies the overflow process from a GI/G/1/0 loss system.

One can generalize in another direction, however, without losing the renewal property of the overflow process. Namely, the overflow process from a GI/$M_{(n)}$/s/r queue, the index (n) indicating state-dependent rates, is renewal as observed by Descloux [12], who also develops procedures for determining the moments of the interoverflow time distribution.

The renewal property is also preserved in the model with which this paper is concerned. Concretely, we will analyse the overflow process from a Markovian queueing system with one server and a finite waiting room of size $r \geq 0$, for which the arrival and service rates may depend on the number of customers in the system. The queueing system is referred to as $M_{(n)}/M_{(n)}/1/r$. Evidently, with appropriate interpretation of the service rates this model encompasses any Markovian delay and loss system $M_{(n)}/M_{(n)}/s/r$ where $s > 1$.

The purpose of this paper is twofold. First, in Section 2, we will show that the overflow process from an $M_{(n)}/M_{(n)}/1/r$ system is a renewal process of hyperexponential type and we derive an expression for the Laplace–Stieltjes transform of the interoverflow time distribution. Then we will exhibit that this knowledge may advantageously be used to examine Markovian queueing systems where an overflow process from one queue is the arrival process to another. One such system, to wit $(M/M/s/r)_{\text{overflow}}/M/\infty$, will be studied in detail in Section 3.

2. The overflow process from the $M_{(n)}/M_{(n)}/1/r$ queue

Let the system $M_{(n)}/M_{(n)}/1/r$ have arrival rate λ_n and service rate μ_n when there are n customers in the system. Denoting by $T_0 = 0$, T_1, T_2, ... the successive moments at which customers overflow, the overflow process $\{T_0, T_1, T_2, \ldots\}$ is obviously a renewal process. The distribution of the time between overflows is given in the next theorem, where it is convenient to let $K = r + 1$.

Theorem. *The interoverflow time distribution F(t) corresponding to an* $M_{(n)}/M_{(n)}/1/K-1$ *queue* $(1 \leqslant K < \infty)$ *with state-dependent arrival and service rates* λ_n *and* μ_n, *respectively, is a mixture of* $K+1$ *distinct exponential distributions. The Laplace–Stieltjes transform* $\phi(z)$ *of F(t) is given by*

$$\phi(z) \equiv \int_0^\infty \exp\{-zt\}\,\mathrm{d}F(t) = Q_K(-z)/Q_{K+1}(-z), \quad z \geqslant 0, \tag{1}$$

where Q_k *and* Q_{K+1} *are polynomials of degree K and* $K+1$, *respectively, defined by the recurrence relations*

$$\begin{aligned} &Q_{-1}(x) = 0, \quad Q_0(x) = 1 \\ &\lambda_n Q_{n+1}(x) = (\lambda_n + \mu_n - x)Q_n(x) - \mu_n Q_{n-1}(x), \quad n = 0, 1, \ldots, K. \end{aligned} \tag{2}$$

Finally, the intensity ν *of the overflow process is given by*

$$\nu \equiv \left\{\int_0^\infty t\,\mathrm{d}F(t)\right\}^{-1} = \lambda_K \pi_K \left\{\sum_{n=0}^{K} \pi_n\right\}^{-1}, \tag{3}$$

where

$$\pi_0 = 1 \quad \text{and} \quad \pi_n = \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n}, \quad n \geqslant 1. \tag{4}$$

Proof. Consider a birth–death process $\{X(t)\}$ with state space $S = \{0, 1, \ldots, K, K+1\}$, birth rate λ_n in state n ($n = 0, 1, \ldots, K$), and death rates μ_n in state n ($n = 1, 2, \ldots, K$) and 0 in state $K+1$, so that $K+1$ is an absorbing state for $\{X(t)\}$. Clearly, $F(t)$ equals the distribution of the time until absorption of the process $\{X(t)\}$ when the initial state is K. So the overflow process is (intrinsically) a renewal process of phase type (cf. Neuts [35]). The more specific characterization of the theorem is obtained if we interpret $F(t)$ as the first passage time distribution from state K into state $K+1$ of $\{X(t)\}$. It then follows from a result of Karlin and McGregor [18,19] (see also Keilson [20]) that the Laplace–Stieltjes transform of $F(t)$ is given by (1) and (2). Now writing $R_{-1}(x) = 0$, $R_0(x) = 1$ and

$$R_{n+1}(x) = (-1)^n \lambda_0 \lambda_1 \ldots \lambda_n Q_{n+1}(x), \quad n = 0, 1, \ldots, K, \tag{5}$$

we see that the polynomials $R_n(x)$ satisfy a three term recurrence formula of the form

$$R_{n+1}(x) = (x - a_n) R_n(x) - b_n R_{n-1}(x), \quad n \geqslant 0, \tag{6}$$

with $b_0 = 0$ and $b_n = \lambda_{n-1}\mu_n > 0$ ($n > 0$), so that they constitute part of an orthogonal system with respect to a positive definite moment functional [7, Theorem I.4.4]. This implies that the zeros of $R_n(x)$ (and hence of $Q_n(x)$) are real and distinct [7, Theorem I.5.2]. Further, since $a_n = \lambda_n + \mu_n$, the parameters a_n and b_n satisfy a criterion due to Stieltjes [7, p.47] implying that the zeros of $Q_n(x)$ are positive. A further appeal to the theory of orthogonal polynomials [7, p.29] yields that the partial fraction decomposition

$$\frac{Q_K(-z)}{Q_{K+1}(-z)} = \sum_{n=1}^{K+1} \frac{\omega_n z_n}{z + z_n}, \tag{7}$$

where the z_n ($n = 1, 2, \ldots, K+1$) are the (positive) zeros of Q_{K+1}, has

$$\omega_n = -\frac{1}{z_n} \frac{Q_K(z_n)}{Q'_{K+1}(z_n)} > 0. \tag{8}$$

Also, by (7) and the fact that $Q_n(0) = 1$, we have that $\sum_{n=1}^{K+1} \omega_n = 1$, as it should be. So

$$F(t) = \sum_{n=1}^{K+1} \omega_n (1 - \exp\{-z_n t\}), \quad t \geqslant 0, \tag{9}$$

a hyperexponential distribution of order $K+1$ with distinct parameters for the components.

Finally, since $\nu = -1/\phi'(0)$, we obtain from (1),

$$\nu^{-1} = Q'_K(0) - Q'_{K+1}(0). \tag{10}$$

Result (3) now follows readily from the recurrence relations (2). □

Remarks. (1) The fact that the first passage time distribution from state K into state $K+1$ of $\{X(t)\}$ is hyperexponential of order $K+1$ was shown earlier by Keilson [21], who used a different argument.

(2) The result (3) follows also from the observation that the intensity of the overflow process equals the arrival rate in state K times the stationary probability that there are K customers in the system.

(3) Substitution of $\lambda_n = \lambda$ and $\mu_n = n\mu$ ($n = 0, 1, \ldots, K$) leads to results which are easily seen to coincide with those of Palm [36] and others on the overflow process from an M/M/K/0 system.

(4) Various sources give procedures for and numerical experience with the problem of determining the zeros of the polynomial Q_{K+1}. We mention Kuczura [27] for the M/M/K/0, case, and Machihara [28,29] for the $M_{(n)}/M_{(n)}/1/K-1$ case in general.

3. The system $(M/M/s/r)_{\text{overflow}}/M/\infty$

We consider an M/M/s/r queue (s servers, r waiting places) with arrival rate λ and service rate μ per server, and let $\alpha = \lambda/\mu$. The overflow process from this queue is offered to an infinite server system also with service rate μ per server, and we are interested in the stationary distribution $\{p(i), i = 0, 1, \dots\}$ of the number of busy servers in the secondary system.

This model is of importance in a teletraffic context, where it is customary to characterize a stream of calls by the trunk occupancy distribution it induces on an infinite trunk group. For $r = 0$ the model is a classical one [23,49] and of basic interest in the analysis and design of public telephone trunk networks. The presence of waiting positions is a more modern development which occurs for instance in mobile communication systems (the context which incited this study) and private-line networks.

The system $(M/M/s/r)_{\text{overflow}}/M/\infty$ has been the subject of a paper by Rath and Sheng [40] who describe an approximative procedure for determining the distribution of the number of busy servers in the secondary system. Exact analyses of the model have been performed by Basharin [1], Herzog and Kühn [15] and Kokotushkin (see [2]). In [1] and [15] algorithmic solutions are given to the problem of determining the moments of the stationary busy-server distribution, the variance of this distribution being explicitly determined by Herzog and Kühn. Both these analyses are based on the equilibrium equations for the joint probabilities $p(i, j)$ of having i customers in the $M/M/s/r$ system and j busy servers in the infinite server system. Kokotushkin's analysis has yielded explicit expressions for the moments of the busy-server distribution, which are cited in [2]. His approach is apparently based on the concept of 'Markov chain flows', which is identical to Kosten's [24] concept of 'Markov driven flows' (MDF's). Indeed, Kosten's [24] results for the system $MDF/M/\infty$ can be used to reproduce Kokotushkin's results. We will show, however, that the simplest way to derive explicit expressions for the binomial moments

$$B_k = \sum_{i=k}^{\infty} \binom{i}{k} p(i), \quad k = 1, 2, \dots \tag{11}$$

is to exploit the overflow theorem of the previous section and standard results for the $GI/M/\infty$ system. Before elaborating on this approach we remark that it does not seem possible to obtain the explicit results of this section by applying the techniques of Ramaswami and Neuts [39] for the system $PH/G/\infty$.

In concurrence with previous notation we let $F(t)$ denote the interoverflow time distribution of the $M/M/s/r$ queue and $\phi(z)$ its Laplace–Stieltjes transform; also, ν^{-1} will denote the mean interoverflow time. The classical results of Takács [43,45] and Cohen [8] for the system $GI/M/\infty$ then state that

$$B_k = \frac{\nu}{k\mu} \prod_{j=1}^{k-1} \kappa_j, \quad k = 1, 2, \dots, \tag{12}$$

where the empty product is interpreted as unity and

$$\kappa_j = \frac{\phi(j\mu)}{1 - \phi(j\mu)}, \quad j = 1, 2, \dots . \tag{13}$$

Application of our theorem with $K = s + r$ and

$$\lambda_n = \lambda \quad \text{and} \quad \mu_n = \min\{n, s\}\mu, \quad n = 0, 1, \dots, s + r, \tag{14}$$

now yields

$$\nu = \frac{\lambda \alpha^{s+r}}{s! s^r} \left\{ \sum_{n=0}^{s-1} \frac{\alpha^n}{n!} + \frac{\alpha^s}{s!} \frac{1 - (\alpha/s)^{r+1}}{1 - \alpha/s} \right\}^{-1} \tag{15}$$

and

$$\phi(z) = Q_{s+r}(-z)/Q_{s+r+1}(-z), \quad z \geqslant 0, \tag{16}$$

where the Q_i are determined by (2) and (14). According to Karlin and McGregor [17] we have

$$Q_n(\mu x) = c_n(x, \alpha), \quad n = 0, 1, \dots, s, \tag{17}$$

and, with $\xi(x) \equiv \xi(x, \alpha, s) = \frac{1}{2}(\alpha s)^{-1/2}(s + \alpha - x)$,

$$Q_{s+n}(\mu x) = (s/\alpha)^{n/2}\left\{c_s(x, \alpha)U_n(\xi(x)) - (s/\alpha)^{1/2}c_{s-1}(x, \alpha)U_{n-1}(\xi(x))\right\},$$
$$n = 0, 1, \ldots, r+1. \tag{18}$$

Here the c_n are Charlier polynomials with parameter α, defined by the recurrence relation

$$\begin{aligned} &c_{-1}(x, \alpha) = 0, \quad c_0(x, \alpha) = 1 \\ &(n + \alpha - x)c_n(x, \alpha) = nc_{n-1}(x, \alpha) + \alpha c_{n+1}(x, \alpha), \quad n \geq 1, \end{aligned} \tag{19}$$

and the U_n Chebysev polynomials of the second kind, recurrently defined by

$$\begin{aligned} &U_{-1}(x) = 0, \quad U_0(x) = 1 \\ &2xU_n(x) = U_{n-1}(x) + U_{n+1}(x), \quad n \geqslant 1 \end{aligned} \tag{20}$$

(cf. [7]). Writing

$$v_n(x) \equiv v_n(x, \alpha, s) = (s/\alpha)^{n/2}U_n(\xi(-x)), \quad n \geq 0, \tag{21}$$

and suppressing the parameter α in c_n, we readily arrive at

$$\phi(j\mu) = \frac{\alpha c_s(-j)v_r(j) - sc_{s-1}(-j)v_{r-1}(j)}{\alpha c_s(-j)v_{r+1}(j) - sc_{s-1}(-j)v_r(j)}, \quad j \geqslant 1. \tag{22}$$

Now exploiting another recurrence relation for Charlier polynomials, viz.,

$$c_n(x+1, \alpha) - c_n(x, \alpha) = -(n/\alpha)c_{n-1}(x, \alpha), \quad n \geq 0 \tag{23}$$

(see, e.g., [16]) for $n = s$, we obtain, from (13) and (22),

$$\kappa_j = \frac{v_r(j) - \{1 - c_s(-j+1)/c_s(-j)\}v_{r-1}(j)}{v_{r+1}(j) - v_r(j) - \{1 - c_s(-j+1)/c_s(-j)\}(v_r(j) - v_{r-1}(j))}, \quad j \geq 1. \tag{24}$$

For completeness' sake we note that $v_n(j)$ may be written as

$$v_n(j) = (\gamma_2\gamma_1^{-n} - \gamma_1\gamma_2^{-n})/(\gamma_2 - \gamma_1), \tag{25}$$

where γ_1 and γ_2 are the roots of the equation

$$sx^2 - (\alpha + s + j)x + \alpha = 0. \tag{26}$$

For computational purposes, however, the recurrence relation

$$\begin{aligned} &v_{-1}(j) = 0, \quad v_0(j) = 1 \\ &(\alpha + s + j)v_n(j) = \alpha v_{n+1}(j) + sv_{n-1}(j), \quad n \geq 0, \end{aligned} \tag{27}$$

which follows from (20) and (21), is more useful. Similarly, an explicit expression for $c_s(-j)$ is given by

$$c_s(-j) = \begin{cases} 1 & j = 0, \\ \displaystyle\sum_{n=0}^{s} \binom{s}{n} \frac{(j+n-1)!}{(j-1)!}\alpha^{-n}, & j \geqslant 1 \end{cases} \tag{28}$$

(see, e.g., [16]), but for numerical work one had better use the recurrence formulas (19) and (23).

So (12), (15) and (24) give us expressions for the binomial moments B_k, which can be shown to agree with Kokotushkin's results as given in [2]. We remark that

$$c_n(-1, \alpha) = 1/E_n(\alpha), \quad n = 0, 1, \ldots, \tag{29}$$

where

$$E_n(\alpha) \equiv \frac{\alpha^n}{n!}\left\{\sum_{i=0}^{n} \frac{\alpha^i}{i!}\right\}^{-1} \tag{30}$$

is the Erlang loss function (see [16]). Thus we obtain for the variance V of the number of busy servers

$$V = 2B_2 + M - M^2 = M\left\{1 - M + \frac{v_r(1) - (1 - E_s)v_{r-1}(1)}{v_{r+1}(1) - v_r(1) - (1 - E_s)(v_r(1) - v_{r-1}(1))}\right\}, \tag{31}$$

where $E_s \equiv E_s(\alpha)$ and

$$M = B_1 = \nu/\mu \tag{32}$$

is the mean number of busy servers. The expression for V in (31) is equivalent to (but simpler than) formula (2.12) of Herzog and Kühn [15]. Substitution of $r = 0$ in (31) immediately yields the Molina–Nyquist result (see [49] or [10]).

The ratio V/M is called the peakedness factor of the overflow stream (cf. [46]). Kokotushkin's asymptotic formula for this peakedness factor (see [1,2]) is valid only for $\alpha < s$, but can easily be generalized as follows. By (15) and (32) we have

$$M \to \begin{cases} 0 & \text{if } \alpha < s \\ \alpha - s & \text{if } \alpha \geq s \end{cases} \quad \text{as } r \to \infty, \tag{33}$$

while from (25) we see that

$$v_{r-1}/v_r \to \gamma_1 \equiv (2s)^{-1}\left\{1 + s + \alpha - \sqrt{(1 + s + \alpha)^2 - 4\alpha s}\right\} \quad \text{as } r \to \infty. \tag{34}$$

Using these results in (31) gives us

$$V/M \to \tfrac{1}{2}\left\{1 + |s - \alpha| + \sqrt{(1 + s + \alpha)^2 - 4\alpha s}\right\} \quad \text{as } r \to \infty, \tag{35}$$

which incorporates Kokotushkin's result.

4. Concluding remarks

The quantities κ_j of (13) are also the basic elements in the expressions for the binomial moments of the stationary busy-server distribution in the system GI/M/s/0 (see, e.g., [45]). Hence, by substitution of (24) in these formulas we can generalize the results of Bech [3] and Brockmeyer [5] (see also Schehrer [42]), who analyze the system $(\mathrm{M/M}/s_1/r)_{\mathrm{overflow}}/\mathrm{M}/s_2/0$ for $r = 0$ on the basis of equilibrium equations.

A further generalization of the model is obtained when we assume that next to the overflow process an independent Poisson stream of customers arrives at the secondary system. The problem of finding the stationary busy-server distribution of the secondary system may then be tackled by observing that between arrivals from the overflow process the number of busy servers $Y(t)$ behaves as a birth–death process, so that, actually, $\{Y(t)\}$ is a 'Markovian regenerative process' [9] or a 'piecewise Markov process' [26], the latter setting being somewhat more general. Since, by our theorem, we have at our disposal the Laplace–Stieltjes transform of the interoverflow time distribution, techniques similar to those of Kuczura [25,27] may be employed to solve the problem.

In this context it is interesting to note that Morrison [31–34] studies similar models purely on the basis of equilibrium equations for the combined system of two queues, whose dimensions he substantially reduces. It may be shown, at least when one is interested in the stationary busy-server distribution for the system $\mathrm{M} + (\mathrm{M/M}/s_1/0)_{\mathrm{overflow}}/\mathrm{M}/s_2/0$, that Morrison's approach requires approximately the same amount of numerical work as Kuczura's method.

Acknowledgement

The author wishes to thank André Roosma and a referee for helpful comments.

References

[1] G.P. Basharin, On analytical and numerical methods of switching system investigation, Proc. 6th Internat. Teletraffic Congress, Munich (1970) paper 231.

[2] G.P. Basharin, V.A. Kokotushkin and V.A. Naumov, The method of equivalent substitutions for calculating fragments of communication networks for a digital computer I, Engrg. Cybernetics 17 (1979) 66–73.

[3] N.I. Bech, Metode till beregning af spaerring i alternativ trunking-og gradingsystemer, Teleteknik 5 (1954) 435–448 (in Danish).

[4] V.E. Beneš, Transition probabilities for telephone traffic, Bell Syst. Tech. J. 39 (1960) 1297–1320.

[5] E. Brockmeyer, Det simple overflowproblem i telefontrafikteorien, Teleteknik 5 (1954) 361–374 (in Danish).

[6] E. Cinlar and R.L. Disney, Stream of overflows from a finite queue, Oper. Res. 15 (1967) 131–134.

[7] T.S. Chihara, An Introduction to Orthogonal Polynomials (Gordon & Breach, New York, 1978).

[8] J.W. Cohen, The full availability group of trunks with an arbitrary distribution of the inter-arrival times and a negative exponential holding time distribution, Simon Stevin 31 (1957) 169–181.

[9] J.W. Cohen, The Single Server Queue (North-Holland, Amsterdam, rev. ed., 1982).

[10] R.B. Cooper, Introduction to Queueing Theory (Edward Arnold, London, 2nd ed., 1981).

[11] A. Descloux, On overflow processes of trunk groups with Poisson inputs and exponential service times, Bell Syst. Tech. J. 42 (1963) 383–397.

[12] A. Descloux, On Markovian servers with recurrent input, Proc. 6th Internat. Teletraffic Congress, Munich (1970) paper 331.

[13] J.H.A. de Smit, The overflow process of the multi-server queue with exponential service times and finite waiting room, Memorandum Nr. 408, Department of Applied Mathematics, Twente University of Technology, Enschede, The Netherlands, 1982.

[14] S. Halfin, Distribution of the interoverflow time for the GI/G/1 loss system, Math. Oper. Res. 6 (1981) 563–570.

[15] U. Herzog and P. Kühn, Comparison of some multiqueue models with overflow and load-sharing strategies for data transmission and computer systems, in: J. Fox, ed., Proc. Symp. on Computer-Communications Networks and Teletraffic, pp. 449–472 (Polytechnic Press, Brooklyn, NY, 1972).

[16] D.L. Jagerman, Some properties of the Erlang loss function, Bell Syst. Tech. J. 53 (1974) 525–551.

[17] S. Karlin and J.L. McGregor, Many server queueing processes with Poisson input and exponential service times, Pacific J. Math. 8 (1958) 87–118.

[18] S. Karlin and J.L. McGregor, A characterization of birth and death processes, Proc. Nat. Acad. Sci.-U.S.A. 45 (1959) 375–379.

[19] S. Karlin and J.L. McGregor, Coincidence properties of birth and death processes, Pacific J. Math. 9 (1959) 1109–1140.

[20] J. Keilson, A review of transient behaviour in regular diffusion and birth–death processes, J. Appl. Probab. 1 (1964) 247–266.

[21] J. Keilson, Log-concavity and log-convexity in passage time densities of diffusion and birth-death processes, J. Appl. Probab. 8 (1971) 391–398.

[22] A. Khintchine, Mathematical Methods in the Theory of Queueing (Griffin, London, 2nd English ed., 1969).

[23] L. Kosten, Über Sperrungswahrscheinlichkeiten bei Staffelschaltungen, Elektrische Nachrichten Technik 14 (1937) 5–12.

[24] L. Kosten, Approximate determination of congestion quantities by equivalent traffic methods, Delft Progr. Rept. 5 (1980) 227–252.

[25] A. Kuczura, Queues with mixed renewal and Poisson inputs, Bell Syst. Tech. J. 51 (1972) 1305–1326.

[26] A. Kuczura, Piecewise Markov processes, SIAM J. Appl. Math. 24 (1973) 169–181.

[27] A. Kuczura, Loss systems with mixed renewal and Poisson inputs, Oper. Res. 21 (1973) 787–795.

[28] F. Machihara, Transition probabilities of Markovian service system and their applications, Rev. Electrical Communication Labs. 29 (1981) 170–188.

[29] F. Machihara, On the property of eigenvalues of some infinitesimal generator, Oper. Res. Lett. 2 (1983) 123–126.

[30] D.C. McNickle, A note on congestion in overflow queues, Opsearch 19 (1982) 171–177.

[31] J.A. Morrison, Analysis of some overflow problems with queuing, Bell Syst. Tech. J. 59 (1980) 1427–1462.

[32] J.A. Morrison, Some traffic overflow problems with a large secondary queue, Bell Syst. Tech. J. 59 (1980) 1463–1482.

[33] J.A. Morrison, An overflow system in which queuing takes precedence, Bell Syst. Tech. J. 60 (1981) 1–12.

[34] J.A. Morrison and P.E. Wright, A traffic overflow system with a large primary queue, Bell Syst. Tech. J. 61 (1982) 1487–1517.

[35] M.F. Neuts, Renewal processes of phase type, Naval Res. Logist. Quart. 25 (1978) 445–454.

[36] C. Palm, Intensitätsschwankungen im Fernsprechverkehr, Ericsson Technics 44 (1943) 1–189.

[37] C. Pearce and R. Potter, Some formulae old and new for overflow traffic in telephony, Austral. Telecomm. Res. 11 (1977) 92–97.

[38] R.M. Potter, Explicit formulae for all overflow traffic moments of the Kosten and Brockmeyer systems with renewal input, Austral. Telecomm. Res. 13 (1980) 39–49.

[39] V. Ramaswami and M.F. Neuts, Some explicit formulas and computational methods for infinite-server queues with phase-type arrivals, J. Appl. Probab. 17 (1980) 498–514.

[40] J.H. Rath and D. Sheng, Approximations for overflows from queues with a finite waiting room, Oper. Res. 27 (1979) 1208–1216.

[41] J. Riordan, Stochastic Service Systems (Wiley, New York, 1962).

[42] R. Schehrer, Über die Momente höherer Ordnung von Überlaufverkehr hinter volkommen erreichbaren Bündeln, Wiss. Ber. AEG-Telefunken 50 (1977) 113–119.

[43] L. Takàcs, On the generalization of Erlang's formula, Acta Math. Acad. Sci. Hung. 7 (1956) 419–433.

[44] L. Takàcs, On the limiting distribution of the number of coincidences concerning telephone traffic. Ann. Math. Statist. 30 (1959) 134–142.

[45] L. Takàcs, Introduction to the Theory of Queues (Oxford University Press, New York, 1962).

[46] E.A. van Doorn, Some analytical aspects of the peakedness concept, Proc. 10th Internat. Teletraffic Congress, Montreal (1983) paper 4.4b–5.

[47] É. Vaulot, Sur l'application du calcul des probabilités à la théorie du trafic téléphonique, C.R. Acad. Sci. Paris 200 (1935) 1815–1818.

[48] J.F.E. Wallin, Overflow traffic from the viewpoint of renewal theory, Statist. Neerlandica 31 (1977) 171–178.

[49] R.I. Wilkinson, Theories for toll traffic engineering in the U.S.A., Bell Syst. Tech. J. 35 (1956) 421–514.