Problems in Degenerate Diffusion

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Proefschrift

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to my parents
who, each in their own way,
have greatly contributed to this book

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Preface

Diffusion is the spread of substances by the natural movement of their particles. It is a process so basic to the material world and so ubiquitous that it touches every part of our daily lives. Our morning cup of tea, the delights of gastronomy, the scent of a neighbour, even the oxygen that we breathe, all these are brought to us by diffusion. It is not surprising that many mathematical models of physical and chemical processes contain diffusion as a fundamental element.

The classical way of incorporating diffusion into a mathematical model results in a strong spreading phenomenon. One of the consequences is that an increase in concentration at a certain point in space results, according to such a model, in an increase of the concentration everywhere else, and immediately. Although this increase may be very small, especially far away from the original change, it is always present. We often use the term 'infinite speed of propagation' to denote this property.

The word 'degenerate' in the title refers to a form of diffusion where the driving force behind the process vanishes, resulting in 'finite speed of propagation'. In many situations a model based on degenerate diffusion is more appropriate than a classical model. As an example, imagine an oil spill in the ground which is spreading under diffusion. A model based on degenerate diffusion will predict that the the patch of oil has a clear boundary outside of which there is no oil, while under classical diffusion small amounts oil would be present everywhere. Experiments clearly indicate that the predictions of the degenerate diffusion model are more accurate. Other examples are the flow of gas through rock and the spread of individuals of a biological population; degenerate diffusion plays a role in the elimination of noise from digitised images, and it has even been mentioned in the context of the spread of galactic civilisations.

In this thesis we focus on some mathematical problems that arise in degenerate diffusion models. In the Introduction we illustrate the existing theory with the Porous Medium Equation and derive the main model, which is concerned with the spread of chemical substances in the soil due to the flow of groundwater. We then give an overview of the rest of this thesis.

Chapter 1

Introduction

1.1 On degenerate diffusion

A typical diffusion process, for instance the diffusion of molecules of type A in a large quantity of other molecules, might be described by the equation

$$u_t = \operatorname{div}(D\nabla u) \quad \text{for} \quad x \in \Omega,\ t > 0. \tag{1.1}$$

The domain $\Omega \subset \mathbb{R}^N$ models the physical space that the molecules occupy, and $u(x, t)$ is the concentration of molecules of type A. Subscripts denote differentiation, the gradient operator ∇ is defined by

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right)^T = (u_{x_1}, \dots, u_{x_N})^T,$$

and the divergence operator by

$$\operatorname{div}(f_1, \dots, f_N)^T = \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}.$$

The diffusion coefficient D may depend on x, t, and any number of quantities including u and its derivatives, but its sign is always non-negative. In most physical situations the diffusion coefficient is bounded away from zero: $D(x, t, u, \cdots) \geq D_0 > 0$ for all $x, t, u, \dots$, and in that case (1.1) is called non-degenerate. 'Degenerate diffusion' refers to a diffusion process where the diffusion coefficient D is *not* bounded away from zero, but can vanish in parts of the domain[1].

For the interpretation of this property it is useful to go back to the modelling process that resulted in equation (1.1). We can write it as a conservation law

$$u_t + \operatorname{div} \boldsymbol{J} = 0 \tag{1.2}$$

[1]Sometimes the term 'degenerate diffusion' is also used for unbounded diffusion coefficients. We shall encounter such a situation in Section 1.11. In this introduction, however, we reserve the term 'degenerate diffusion' for a vanishing diffusion coefficient.

and a 'constitutive equation'

$$\boldsymbol{J} = -D\nabla u. \tag{1.3}$$

Equation (1.2) is called a conservation law because it is a differential form of the integral equation

$$\int_\Omega u(x,T)\,dx - \int_\Omega u(x,0)\,dx = -\int_0^T \int_{\partial\Omega} \boldsymbol{J}\cdot\nu,$$
$$\text{for every region } \Omega \subset \mathbb{R}^N \text{ and } T > 0.$$

Here $\partial\Omega$ denotes the boundary of the set Ω, with outward normal vector ν. This equation describes that $\int u$ is conserved; the only change in $\int u$ happens at the boundary of the domain of integration, and is described by $\boldsymbol{J}$, which is called the flux. Conservation laws, also called balance equations, lie at the basis of many physical models.

The constitutive equation (1.3) characterises the underlying physical model. When D does not depend on the gradient ∇u, equation (1.3) is often referred to as *Fick's Law*. Such a constitutive equation is typical for a diffusive process. Intuitively, Fick's Law states that molecules move in the direction of decreasing concentration: they spread out, away from each other.

Against the background of this model the vanishing diffusion coefficient represents a local disappearance of the driving force behind the spreading.

In many applications models incorporating degenerate diffusion arise in a natural way. We illustrate this with three examples.

1. *Spreading of biological populations.* Fick's law (1.3) is often employed to model the spread of a biological species, where $\boldsymbol{J}$ is the flux and u the concentration of individuals. Typically $D = D(x,t)$ is a parameter that models variations in the habitat, and this parameter can vanish in regions where the mobility of the species is effectively reduced to zero.

2. *Gas flow in porous rock [Mus37].* The mass density ρ of a gas flowing through a porous medium satisfies the conservation law

$$\phi\rho_t + \operatorname{div}(\rho\boldsymbol{v}) = 0$$

where ϕ is the porosity, i.e. the fraction of total volume that is available to the gas, and $\boldsymbol{v}$ is the velocity of the gas. Assuming *Darcy's Law*,

$$\boldsymbol{v} = -\frac{k}{\mu}\nabla p, \tag{1.4}$$

where μ is the viscosity, k the permeability, and p the pressure, and the *equation of state*

$$u = u_0 p^\gamma, \tag{1.5}$$

in which γ is a constant in (0, 1], we find by eliminating p and $\boldsymbol{v}$ the equation for ρ

$$\rho_t = c\,\Delta(\rho^m)$$

where $m = (\gamma + 1)/\gamma > 1$ and c is a positive constant. Note that if we define $\boldsymbol{J} = \rho\boldsymbol{v}$ in analogy with (1.2), then

$$\boldsymbol{J} = -mc\,\rho^{m-1}\nabla\rho,$$

which is a nonlinear version of Fick's Law (1.3). The diffusion coefficient $D = D(\rho) = mc\,\rho^{m-1}$ vanishes at the value $\rho = 0$.

3. *The interface between fresh and salt water in underground aquifers.* Consider a horizontal aquifer (a layer of porous material) of uniform thickness h, filled with a mixture of salt and fresh water. Under certain conditions we can suppose that the mixing zone between the salt and the fresh water is relatively thin. Since the salt water is the more dense, it will tend to lie underneath the fresh water, with a thin mixing zone—called the interface—separating the two fluids. De Josselin de Jong derived an equation for the movement of this interface [JdJ81]. In simplified form it reads

$$u_t = \operatorname{div}(u(h - u)\nabla u),$$

where the height u of the interface above the bottom of the aquifer takes values in $[0, h]$. This equation is said to have two-point degeneration: the diffusion coefficient $u(h - u)$ vanishes at the values $u = 0$ and $u = h$.

In the first example the degeneration is known beforehand, while in the other two examples the degeneration depends on the unknown function ρ or u. This latter case is more interesting and in the next section we shall briefly discuss some of the existing theory on this kind of degeneracy.

1.2 A primer on the Porous Medium Equation

The equation

$$u_t = \Delta u^m \quad \text{with} \quad m > 1, \tag{1.6}$$

is often called the 'porous medium equation' or, mostly in the Russian literature, the 'filtration equation'. Equation (1.6) is an example of (1.1) with $D = D(u) = mu^{m-1}$. Historically and mathematically this equation stands at the basis of the degenerate diffusion theory: many of the properties which are now known to be typical for problems with degenerate diffusion were first proved for equation (1.6). In this respect (1.6) serves as a model for a broad class of equations.

There are a number of review articles [Pel81, Aro86, Kal87, Váz92a] that together give an excellent overview of the existing literature. Here we will be content to explain some basic features of this equation which can serve as stepping-stones towards the work that follows.

The Barenblatt-Pattle solution. A famous explicit solution of (1.6) was found independently by Barenblatt [Bar52] and Pattle [Pat59]:

$$u_{\mathrm{BP}}(x,t) = t^{-\alpha}\left[\gamma - \frac{\alpha(m-1)}{2Nm}\frac{|x|^2}{t^{2\alpha/N}}\right]_+^{1/(m-1)}. \tag{1.7}$$

The number α equals $(m-1+2/N)^{-1}$, N is the space dimension, and $[\cdot]_+$ stands for $\max\{\cdot,0\}$. The parameter $\gamma > 0$ can be chosen freely, and characterises the mass of the solution $\int_{\mathbb{R}^N} u_{\mathrm{BP}}(x,t)\,dx$, which is conserved in time. Since equation (1.6) is autonomous in space and time, a translated copy of u_{BP} is again a solution of (1.6). Note that u_{BP} is a *self-similar solution*: there exist constants α, $\beta \in \mathbb{R}$ and a function f such that u_{BP} can be written in the form

$$u_{\mathrm{BP}}(x,t) = t^{-\alpha} f\left(\frac{|x|}{t^\beta}\right).$$

For such a solution equation (1.6) reduces to an ordinary differential equation, greatly simplifying the analysis. In many cases self-similar solutions are known to give the characteristic asymptotic behaviour for general solutions, and u_{BP} is no exception: as $t \to \infty$, solutions of (1.6) with finite mass $\left(\int_{\mathbb{R}^N} u(x,t)\,dx < \infty\right)$ converge to an appropriately scaled and translated version of u_{BP} [KK73, FK80, Váz83]. We shall see other examples of behaviour characterised by special solutions in Sections 1.5, 1.6 and 1.7.

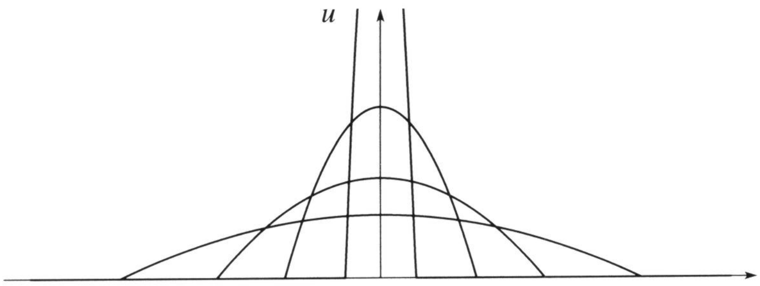


Figure 1.1: The Barenblatt-Pattle solution. The solution is drawn here for $N = 1$, $m = 2$, and $t = 0.1,\ 2,\ 10,\ 40$.

Lack of smoothing and weak solutions. A characteristic property of uniformly parabolic equations is a *smoothing effect:* if we solve the equation

$$u_t = \Delta u \tag{1.8}$$

in a domain $Q_T = \Omega \times (0, T]$ with initial and boundary data on the parabolic boundary $\Omega \times \{0\} \cup \partial\Omega \times (0, T]$, then the solution u is differentiable any number of times in the interior of Q_T. This phenomenon is called the smoothing effect, and it is a consequence of the specific form of the equation.

The Barenblatt-Pattle solution u_{BP} of equation (1.6) is smooth on the positivity set $\mathcal{P} = \{(x, t) \in Q_T : u_{\text{BP}}(x, t) > 0\}$. On this set, the diffusion coefficient $D(u) = mu^{m-1}$ in (1.6) is strictly positive, and the smoothing properties of (1.6) are similar to those of (1.8). At the boundary of the set $\mathcal{P}$, where $u_{\text{BP}} = 0$, the smoothness of u_{BP} is limited, and this is a consequence of the vanishing diffusion coefficient. This is what we call 'lack of smoothing'.

Although from a physical point of view we would like to admit u_{BP} as a solution of equation (1.6), the function u_{BP} does not have the regularity which is implicitly assumed in the formulation (1.6), i.e. twice differentiable in x and once in t. In order to admit such functions as solutions we need to broaden the definition of a solution. As an illustration, consider the following problem:

$$u_t = \Delta u^m \qquad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+ \tag{1.9a}$$

$$u(x, 0) = u_0(x) \qquad \text{for} \quad x \in \mathbb{R}^N. \tag{1.9b}$$

The corresponding definition of a 'weak solution' would be

Definition 1.1 — *A non-negative function $u \in L^\infty(Q_T)$ is a weak solution of (1.9) with initial data u_0 if*

$$-\int_{Q_T} (u\varphi_t + u^m \Delta\varphi) = \int_{\mathbb{R}^N} u_0\varphi(0) \tag{1.10}$$

for all $\varphi \in C_c^{2,1}(\overline{Q_T})$ *such that* $\varphi(T) = 0$.

$C^{2,1}(\overline{Q_T})$ is the set of functions on $\overline{Q_T}$ that are twice continuously differentiable in x and once in t. $C_c^{2,1}(\overline{Q_T})$ is the set of functions in $C^{2,1}(\overline{Q_T})$ whose support is compact in Q_T. For a function χ defined on Q_T we shall often need the value of that function on the time-section $t = \tau$; we denote this by $\chi(\tau)$, i.e.

$$\chi(\tau)(x) \stackrel{\text{def}}{=} \chi(x, t).$$

Therefore the integral on the right-hand side of (1.10) is a different way of writing

$$\int_{\mathbb{R}^N} u_0(x)\varphi(x, 0)\, dx.$$

Equation (1.10) is obtained by multiplying equation (1.9a) by φ and integrating by parts. Consequently any classical ($C^{2,1}$) solution of (1.9a) is also a weak solution: by exchanging the classical notion of a solution for Definition 1.1, we extend the set of solutions.

Existence and uniqueness. The comparison principle for uniformly parabolic equations can be generalised to equations of type (1.6): if of two solutions the initial and boundary values are ordered, then the solutions are also ordered in the interior of the domain. This result immediately implies uniqueness and generally plays an important role throughout the theory of (1.6). As an example we give an outline of the existence proof in [OKYL58] (see also the description given in [Ole63]).

For given non-negative initial data u_0, define the approximate solutions u^ε for $\varepsilon > 0$ as the solutions of the problem

$$u_t = \Delta u^m \qquad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+ \tag{1.11a}$$

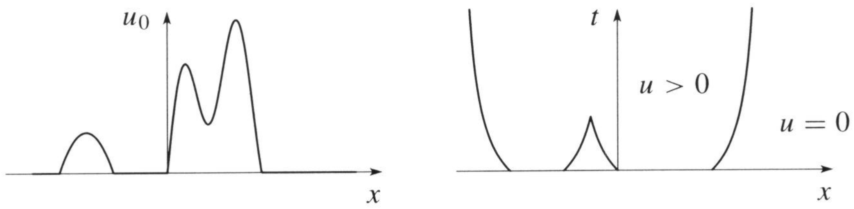
$$u(x, 0) = u_0(x) + \varepsilon \qquad \text{for} \quad x \in \mathbb{R}^N. \tag{1.11b}$$

Since $u \equiv \varepsilon$ is a solution of (1.11a), we have the a priori inequality $u^\varepsilon \geq \varepsilon$ by the comparison principle. By changing the function u^m in (1.11a) for values $u < \varepsilon$ we can render equation (1.11a) uniformly parabolic and apply classical results to obtain the existence of the solution u^ε [Fri64, LSU68]. This procedure yields a sequence of solutions u^ε such that $u^\varepsilon \geq \varepsilon$ and therefore every u^ε satisfies (1.11a). We then pass to the limit $\varepsilon \to 0$ and find a solution of (1.9).

Other methods of proving existence include monotonicity methods [Lio69] and semigroup theory [BC81].

Since weak solutions are defined in terms of test functions φ, the comparison principle is proved by choosing appropriate test functions. We will come back to this in Section 1.4 and Chapter 2.

Interfaces. The Barenblatt-Pattle solution demonstrates the existence of *interfaces*: curves or surfaces in the x, t-plane that separate the regions $\{u > 0\}$ and $\{u = 0\}$. In general, if the initial datum u_0 is non-negative but not strictly positive, as in Figure 1.2(a), then the corresponding solution of (1.6) will have interfaces. These are drawn schematically in Figure 1.2(b).



(a) The initial datum (b) The x, t-plane

Figure 1.2: How interfaces develop from the initial datum

This behaviour is essentially different from the case of the linear heat equation (1.8). If we solve equation (1.8) with an initial datum as in Figure 1.2(a), then the resulting solution will be strictly positive everywhere. This distinctive behaviour of equation (1.6) is a consequence of the vanishing diffusion coefficient.

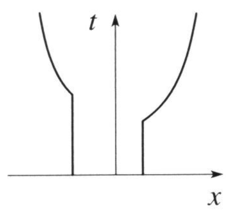


Figure 1.3: Waiting times

Interfaces in the case of equation (1.6) always move outwards, into the region $\{u = 0\}$. They can remain stationary for a finite time, but only initially, i.e. starting at $t = 0$: once the interface starts moving it never stops. If there is a stationary period $[0, t^*]$ then t^* is called a *waiting time* [Kal72]. An extensive research effort has resulted in a complete determination of the regularity of the interface: in each of the time slots $[0, t^*)$ and (t^*, ∞) the location of the interface is an analytic function of time, while the junction at t^* is at least Lipschitz continuous [HK86, AV87, Ang88].

1.3 The adsorption-convection model

A large part of this thesis, Chapters 2-5, is devoted to a mathematical analysis of a model that arises in the transport of a chemical substance by groundwater flow. Since an understanding of the physical background can assist in the interpretation of the mathematical results, we briefly sketch the derivation of this model. More detailed derivations, especially emphasising the underlying assumptions, can be found in [Bea72, Bea79, BV87, ZR94].

We investigate the spread of a chemical substance, or contaminant, in the soil. We suppose that the soil is saturated with water, and that this water moves with a known velocity, or more precisely *discharge* $\boldsymbol{q}$.[2] We suppose that the medium is rigid and the water incompressible, hence

$$\operatorname{div} \boldsymbol{q} = 0.$$

The chemical contaminant can be found in two states, or phases: dissolved in the water or adsorbed on the soil surface. We introduce C as the scaled dissolved concentration of the contaminant and S as the scaled adsorbed concentration. The total flux of contaminant is then given by the sum of convective flux and diffusive/dispersive flux, i.e.

$$\boldsymbol{J} = \boldsymbol{q}C - \boldsymbol{D}\nabla C.$$

where the tensor $\boldsymbol{D}$ combines the effects of molecular diffusion and mechanical dispersion ([Bea79], Chapter 7, or [BV87], Chapter 6)[3]. Since the total scaled

[2]In hydrology it is common to use the discharge rather than the vaguely defined 'flow' or 'velocity'. The discharge is the volume of water that passes through a given cross-section, per unit area of this cross-section and per unit of time, and is expressed in the same physical units as velocity. The relation between the discharge and what one could call 'average water velocity' involves the porosity. As an example to illustrate this, consider a (hypothetical) medium with a porosity close to zero, i.e. relatively little pore space. For such a medium the ratio between average velocity and discharge will be large, since of all the volume only a small fraction is occupied by the moving water particles which contribute to the discharge. On the other hand, if the porosity is close to one, then average velocity and discharge will not differ by much.

[3]A common expression for the dispersion tensor is

$$\boldsymbol{D} = (\alpha_L - \alpha_T)\frac{\boldsymbol{q} \otimes \boldsymbol{q}}{|\boldsymbol{q}|} + \alpha_T\,|\boldsymbol{q}|\,\boldsymbol{I},$$

which is equivalent to

$$\boldsymbol{D}\boldsymbol{q} = \alpha_L\,|\boldsymbol{q}|\,\boldsymbol{q} \quad \text{and} \quad \boldsymbol{D}\boldsymbol{r} = \alpha_T\,|\boldsymbol{q}|\,\boldsymbol{r} \text{ if } \boldsymbol{r} \perp \boldsymbol{q}.$$

Here α_L and α_T are the scaled dispersion lengths in the longitudinal and transversal direction with respect to the flow $\boldsymbol{q}$.

quantity of contaminant present in a unit volume is given by $C+S$, the equation of mass balance for the contaminant reads

$$(C + S)_t + \operatorname{div}(\boldsymbol{q}C - \boldsymbol{D}\nabla C) = 0. \tag{1.12}$$

We model the interaction between the dissolved and the adsorbed form of the contaminant by the first-order differential equation

$$S_t = k\mathcal{F}(S, C), \tag{1.13}$$

where the rate parameter k and the rate function $\mathcal{F}$ are usually determined experimentally. It is common to assume that the equilibrium condition $\mathcal{F}(S, C) = 0$ is uniquely solvable in S for given values of C, thus defining a function Ψ such that

$$\mathcal{F}(S, C) = 0 \iff S = \Psi(C). \tag{1.14}$$

Since the function Ψ is determined by experiments at constant temperature, it is generally referred to as an *isotherm*.

The model described above is referred to in the literature as the *non-equilibrium adsorption* model. Its name arises from the assumption, which is implicitly present in (1.13), that the changes in solute concentration C due to adsorption take place on the same time scale as those due to water flow and dispersion. Alternatively, one can adopt the assumption of *equilibrium adsorption* by supposing that the adsorption kinetics are significantly faster than the flow kinetics—which formally corresponds to[4] $k = \infty$—and that therefore S and C can be assumed to be constantly coupled by $S = \Psi(C)$. In this case the system (1.12)-(1.13) reduces to

$$\big(C + \Psi(C)\big)_t + \operatorname{div}(\boldsymbol{q}C - \boldsymbol{D}\nabla C) = 0. \tag{1.15}$$

During the last ten years a generalisation of these two models has received attention in the literature in which the porous matrix is assumed to have different types of adsorption sites characterised by different rate parameters k and possibly different rate functions $\mathcal{F}$ (see [ZR94], p. 23–24 for a survey). Typically, one type of adsorption site is assumed to satisfy the equilibrium assumption, while at other sites the adsorption/desorption process is slow with respect to the hydrodynamical processes. Such models, which we shall refer to as *multiple-rate models*, appear to predict field tests with significantly higher accuracy than the single-rate model.

[4]cf. Section 1.8

A different situation leading to the same mathematical formulation is that of dual porosity. In this situation there are spatial variations in the porosity which give rise to large fluctuations in the permeability (often of several orders of magnitude). One could say that such a soil is physically heterogeneous instead of chemically heterogeneous. The difference in permeability implies that the flow is concentrated in the highly permeable regions, and the adsorption sites in the low-porosity regions are only accessible to the flow via molecular diffusion. This results in a decreased effective adsorption rate for the sites in the low-porosity regions.

To accommodate these differences, we shall assume throughout that the total adsorbed concentration is given by

$$S = S_I + S_{II}.$$

We suppose that the reaction rate is high at adsorption sites of type I (equilibrium adsorption), and low at those of type II. This yields for S_I

$$S_I = \Psi_I(C),$$

and for S_{II}

$$S_{IIt} = k_{II}\mathcal{F}_{II}(S_{II}, C).$$

For $S_I \equiv 0$ we regain the non-equilibrium adsorption model and for $S_{II} \equiv 0$ the equilibrium adsorption model.

Remark 1.1 While experimental determination of an isotherm is relatively straightforward, the rate function $\mathcal{F}$ itself and the rate parameter k are extremely difficult to measure accurately. Because of this difficulty it is common practice to assume certain specific forms for $\mathcal{F}$, such as

$$\mathcal{F}(C, S) = \Psi(C) - S \tag{1.16}$$

or

$$\mathcal{F}(C, S) = C - \Psi^{-1}(S). \tag{1.17}$$

If Ψ is monotonically increasing—which is the common situation in practice—then these forms satisfy the additional monotonicity properties

$$\frac{\partial \mathcal{F}}{\partial C} \geq 0 \quad \text{and} \quad \frac{\partial \mathcal{F}}{\partial S} \leq 0.$$

We will show in Chapter 2 that such monotonicity implies uniqueness of solutions of an associated problem as well as a comparison principle. •

Various choices are made in the literature for the isotherms Ψ_I and Ψ_{II}. We refer to [DK92a] for the derivation of the most important types, and to [EHB76] for a comparison of the predictions of models based on different isotherms. We shall only mention the two most common categories (the classification is from [GSH74]):

1. Isotherms of Langmuir type:

   $$\Psi \text{ is concave near } C = 0 \text{ and } \Psi'(0+) < \infty$$

   with the generic example $\Psi(C) = \dfrac{\kappa_1 C}{1 + \kappa_2 C}$, where κ_1 and κ_2 are positive constants;

2. Isotherms of Freundlich type:

   $$\Psi \text{ is concave near } C = 0 \text{ and } \Psi'(0+) = \infty$$

   for which a typical example is $\Psi(C) = \kappa C^p$ for some $0 < p < 1$ and $\kappa > 0$.

In order to simplify the notation we write

$$u = C, \quad v = S_{II}, \quad \varphi(u) = \Psi_I(C), \quad \beta(u) = u + \varphi(u)$$

and thus obtain

$$\big(\beta(u) + v\big)_t + \operatorname{div}(\boldsymbol{q}u - \boldsymbol{D}\nabla u) = 0 \tag{1.18a}$$

$$v_t = k\mathcal{F}(u, v). \tag{1.18b}$$

Note that if $\boldsymbol{D} = D$ is a scalar, then by scaling time with a factor k and space with a factor $(kD)^{1/2}$ we can render (1.18) in the form

$$\beta(u)_t + v_t + \operatorname{div}(\boldsymbol{q}u - \nabla u) = 0 \tag{1.19a}$$

$$v_t = \mathcal{F}(u, v), \tag{1.19b}$$

where $\boldsymbol{q}(kD)^{-1/2}$ has been replaced by $\boldsymbol{q}$. The equilibrium adsorption case $v \equiv 0$ can be written in the form

$$\beta(u)_t + \operatorname{div}(\boldsymbol{q}u - \nabla u) = 0. \tag{1.20}$$

Problems (1.19) and (1.20) will be the starting point for our investigations. With this derivation in mind we shall sometimes call (1.19) the non-equilibrium problem and (1.20) the equilibrium problem.

Remark 1.2 The assumption that the discharge field $\boldsymbol{q}$ is known beforehand is not without importance. It implies that the distribution of the contaminant does not influence the water flow. For the modelling of the spread of tracer elements, like pesticides and herbicides, concentrations are very low and the contaminants do not interfere with the driving forces. When concentrations are high enough to cause a noticeable density difference, like in the case of salt and fresh water (a difference of $\pm 2.5\%$) this can result in a considerable force acting on the fluid. In such cases the flow can not be determined beforehand but becomes part of the problem.

In supposing that $\boldsymbol{q}$ is a given entity we also dismiss a host of problems that arise in practical situations. Due to small-scale variations in the soil properties there is a large degree of uncertainty in the measured values of (especially) the permeability. Consequently the actual discharge field is extremely difficult to determine to any reasonable accuracy. Nonetheless we feel that the mathematical investigation of these models can contribute to the general understanding of the problem of ground water contamination. •

Remark 1.3 Isotherms of Freundlich type lead to degeneracy in equations (1.19) and (1.20). If for instance

$$\Psi_I(C) = \kappa C^p, \quad \text{with} \quad 0 < p < 1,$$

then $\beta(u) = u + \kappa u^p$. If we set $w = \beta(u)$ and write $\phi = \beta^{-1}$ then w satisfies

$$(w + v)_t + \operatorname{div}\big(\boldsymbol{q}\phi(w) - \phi'(w)\boldsymbol{D}\nabla w\big) = 0.$$

This equation is degenerate since $\phi'(0^+) = 1/\beta'(0^+) = 0$. With this remark in mind we shall call the function β degenerate whenever β' is unbounded.

A form of degeneracy that has not yet been mentioned occurs when Ψ_{II} is of Freundlich type. If $\mathcal{F}$ is given by (1.16) we find

$$\frac{\partial \mathcal{F}}{\partial C}(0^+, 0^+) = \Psi_{II}(0^+) = \infty. \tag{1.21}$$

Note that if $\mathcal{F}$ is given by (1.17) then $\partial\mathcal{F}/\partial C$ is globally bounded. We will show in Section 1.9 that degeneracy of type (1.21) can give rise to interfaces, much like degeneracy of β. •

To conclude this introduction to the physical background of the problems that we investigate we note that many other models lead to similar equations. The reaction in a permeable catalyst particle is a typical example from the large field of chemical engineering. This leads to a system of equations closely related to (1.19) where u models the concentration of a mobile reactant and v the concentration of a reactant that is fixed to the particle [HHP96b, Ari75]. Other well-known examples are found in the theory of combustion ([BE89], Chapter 4, or [Maj81, DS95]).

In Chapters 2, 3, 4, and 5 of this thesis we will study various aspects of problems (1.19) and (1.20), such as the existence and uniqueness of general solutions, the existence of special solutions, the stability properties of these solutions, and the existence of interfaces. In the next part of this Introduction, sections 1.4–1.9, we will give an overview of these results and their relationship to previous work. Sections 1.10 and 1.11, corresponding to Chapters 6 and 7, are devoted to interfaces for an inhomogeneous version of (1.6) and self-similar solutions for (1.6) with $0 < m < 1$.

1.4 Existence and uniqueness for system (1.19)

For results concerning existence for initial-boundary value problems related to equation (1.20) the reader is referred to [AL83, BT84, DK87a, Gil77, Gil89, GP76, Gon89]; the papers cited above also give some results on uniqueness, and some more general results are found in [ACP82, BKP85, Ott95a, Ott95b]. In Appendix 2.A we give an example of a uniqueness result for equation (1.20) on an unbounded domain under weak conditions on the regularity of the solution. In Chapter 4 we give our own proof of existence of a solution for a radial version of equation (1.20), since the subsequent developments make explicit use of the approximating sequence.

For system (1.19) the literature is much less extensive. We mention the detailed treatment of Knabner [Kna91], as well as [DK87b, DS95, HHP96b, HHP96a]. None of these studies, however, cover the simultaneous degeneration of $\mathcal{F}$ and β (i.e., the situation when both $\mathcal{F}$ and β are non-Lipschitz continuous in u). In Chapter 2 we therefore present some well-posedness results that explicitly allow for this case.

The major issue in well-posedness of system (1.19) is the question of uniqueness. Formally, if $\mathcal{F}$ satisfies the monotonicity assumptions

$$\frac{\partial \mathcal{F}}{\partial u} \geq 0 \quad \text{and} \quad \frac{\partial \mathcal{F}}{\partial v} \leq 0, \tag{1.22}$$

then the system satisfies a comparison principle, which implies uniqueness. This can be seen in two ways.

The first comes from existing theory of parabolic systems. We can write the system in the form

$$\beta(u)_t + \operatorname{div}(\boldsymbol{q}u - \boldsymbol{D}\nabla u) = g_1(u, v) \tag{1.23a}$$

$$v_t = g_2(u, v) \tag{1.23b}$$

with $g_1(u, v) = -g_2(u, v) = -\mathcal{F}(u, v)$. Since $\partial g_1/\partial v \geq 0$ and $\partial g_2/\partial u \geq 0$, the right-hand side is a quasi-monotone reaction term (one also finds the qualification 'the system is cooperative'). If the left-hand side were uniformly parabolic, the result would follow from standard theory [Pao92].

The second way is more technical but allows for a rigorous equivalent, which is given in Section 2.2. Subtract equation (1.23a) for (u_2, v_2) from the same for (u_1, v_1), multiply by $H(u_1 - u_2)$, and integrate in space. Here H is the Heaviside function (with $H(0) = 0$). Formally

$$\frac{d}{dt}\int [\beta(u_1) - \beta(u_2)]_+ = \int \big(\beta(u_1) - \beta(u_2)\big)_t H(u_1 - u_2) \tag{1.24}$$

and

$$\int \operatorname{div}\big(\boldsymbol{q}(u_1 - u_2) - \boldsymbol{D}\nabla(u_1 - u_2)\big) H(u_1 - u_2) \geq 0,$$

and therefore we find

$$\frac{d}{dt}\int [\beta(u_1) - \beta(u_2)]_+ \leq \int \big(g_1(u_1, v_1) - g_1(u_2, v_2)\big) H(u_1 - u_2).$$

By repeating the argument for equation (1.23b) and adding the two we find

$$\begin{aligned}\frac{d}{dt}\int \big([\beta(u_1) - \beta(u_2)]_+ + [v_1 - v_2]_+\big) &\leq \\ \leq \int \Big(\big(g_1(u_1, v_1) - g_1(u_2, v_2)\big) H(u_1 - u_2) \\ + \big(g_2(u_1, v_1) - g_2(u_2, v_2)\big) H(v_1 - v_2)\Big) \\ = \int \big(\mathcal{F}(u_1, v_1) - \mathcal{F}(u_2, v_2)\big)\big(H(v_1 - v_2) - H(u_1 - u_2)\big).\end{aligned}$$

From the monotonicity assumption (1.22) it follows that the right-hand side of this expression is non-positive. Therefore

$$\begin{aligned}\int \big([\beta(u_1(t)) - \beta(u_2(t))]_+ + [v_1(t) - v_2(t)]_+\big) \\ \leq \int \big([\beta(u_1(0)) - \beta(u_2(0))]_+ + [v_1(0) - v_2(0)]_+\big)\end{aligned}$$

for all $t \geq 0$, provided the term on the right-hand side is finite.

This result has two immediate consequences:

1. A comparison principle: if $u_{01} \geq u_{02}$ and $v_{01} \geq v_{02}$, then the corresponding solutions satisfy $u_1 \geq u_2$ and $v_1 \geq v_2$ for all x and t.

2. A contraction in L^1: if

$$|\beta(u_{01}) - \beta(u_{02})| + |v_{01} - v_{02}| \in L^1,$$

then

$$\int \big(|\beta(u_1(t)) - \beta(u_2(t))| + |v_1(t) - v_2(t)|\big) \leq \int \big(|\beta(u_{01}) - \beta(u_{02})| + |v_{01} - v_{02}|\big)$$

for all $t \geq 0$.

The reasoning given above is formal because equation (1.24) can only be justified under additional assumptions on the regularity of the solution. This well-known problem arises in many situations in degenerate diffusion and has been approached in the past in different ways:

- By assuming additional regularity. If $\beta(u)_t \in L^1_{loc}(Q)$—supposing we are solving system (1.19) on a domain $Q = \Omega \times (0, T]$—then the argument given above can be made rigorous. During the last few years this condition has been weakened to conditions of the form

$$\beta(u) \in BV\big(0, T; L^1(\Omega)\big), \tag{1.25}$$

  i.e. $\beta(u)_t$ is a Radon measure ([Yin90] or [Pad95], Theorem II.3.1).

  This approach has two disadvantages: first, condition (1.25) does not seem to have any sensible physical interpretation; second, the existence of solutions satisfying (1.25) can often only be proved under additional restrictions on the regularity of the initial data ([Pad95], Chapter II.4, or [GMT96], Proposition 3.4). For less regular data there may exist no solution with the required regularity.

- By adopting a different definition of a solution. A number of authors [Bam77, BF91, DT94, Iva95, MT] (see also [Bre73], Chapter III) consider weak solutions that are obtained as the pointwise limit of solutions of approximate problems. The comparison principle that holds for the approximate problems then transfers to the limit, implying uniqueness.

Recently two novel approaches have appeared on the scene, both based on the variable-doubling method that was originally introduced by Kružkov [Kru70] in the theory of hyperbolic conservation laws. The first approach consists in proving that any solution implicitly satisfies an entropy condition [Car94, GMT94, Ott95a, Ott95b, KO96] which is related to the condition that is used in the original setup of Kružkov (see also [GMT96], Section 3.4). An alternative approach originated in the works of Plouvier and Gagneux [Plo95, PDG96, Urr96a, Urr96b] and introduces an equivalent formulation of the problem in terms of renormalised solutions. For these solutions the entropy condition can be bypassed.

Knabner and Otto have given a proof of L^1-contraction and uniqueness for this system, following the earlier work of Otto [KO96]. In Chapter 2 we do the same, but for a specific case under less general conditions on the coefficients. This has the advantage of allowing a simpler presentation while still providing the basis for later results.

For system (1.19) in one space dimension with $\boldsymbol{q} = 1$, i.e.

$$\beta(u)_t + v_t + u_x - u_{xx} = 0 \tag{1.26a}$$

$$v_t = \mathcal{F}(u, v) \tag{1.26b}$$

on the spatial domain $\mathbb{R}$, we prove the following theorem.

Theorem 1.2 — *Let $\mathcal{F}$ be continuous and satisfy the monotonicity condition (1.22), let $\mathcal{F}(0,0) = 0$, and let $\beta \in C\big([0,\infty)\big) \cap C^2\big((0,\infty)\big)$ such that $\beta' \geq b_0 > 0$. For every $u_0, v_0 \in L^\infty(\mathbb{R})$ with $u_0, v_0 \geq 0$, system (1.26) has exactly one solution with initial data (u_0, v_0).*

See Section 2.1 for the definition of a solution. In Chapter 2 we also prove a number of properties of solutions, like a comparison principle and the conservation of mass.

1.5 Large-time behaviour in one space dimension

Chapters 3 and 4 of this thesis are concerned with the large-time behaviour of solutions of equation (1.20) and system (1.19). In this section we give some background information and explain the interdependencies between different cases.

As an introduction to the results we examine Burgers' equation without diffusion,

$$u_t + \tfrac{1}{2}(u^2)_x = 0$$

which also can be written

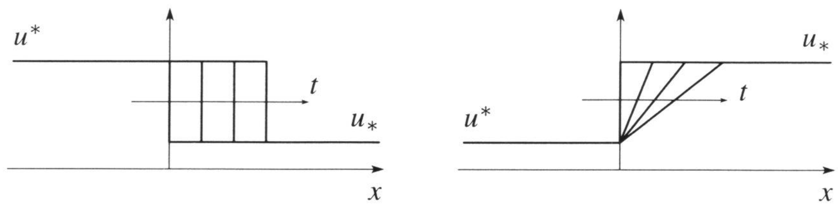
$$(\sqrt{u})_t + \tfrac{1}{2}u_x = 0,$$

by replacing u^2 by u. The latter form shows more clearly the relationship with equation (1.20).

It is well known that the entropy solution of the Riemann problem

$$\begin{aligned} &u_t + \tfrac{1}{2}(u^2)_x = 0 && \text{on } \mathbb{R} \times \mathbb{R}^+ \\ &u(x,0) = u^* && \text{for } x < 0 \\ &u(x,0) = u_* && \text{for } x > 0. \end{aligned}$$

depends qualitatively on the sign of $u^* - u_*$. If $u^* > u_*$, then the solution is a travelling (shock) wave; if $u^* < u_*$, then the entropy solution is a rarefaction wave, i.e. $u(x,t) = w(\eta)$ with $\eta = x/t$. Note that this sign condition is connected with the convexity of the function $u \mapsto u^2$; with a concave function the conclusion is reversed.



(a) $u^* > u_*$ (b) $u^* < u_*$

Figure 1.4: The two special solutions of Burgers' equation

The important fact for our purposes is that these two types of solutions are stable with respect to perturbations. In fact, any solution resembling initially (in some well-defined sense) one of these two 'fundamental' solutions converges to it as $t \to \infty$. Thus we can turn the argument around: given 'any' initial datum u_0 with limits u^* and u_* at minus and plus infinity, the corresponding solution converges either to a travelling wave (if $u^* > u_*$) or to a rarefaction wave ($u^* < u_*$).

For the problem of equation (1.20) in one space dimension,[5]

$$\beta(u)_t + u_x - u_{xx} = 0, \tag{1.27}$$

Van Duijn and De Graaf [DG87] proved similar statements to those about Burgers' equation: the large-time behaviour of solutions of (1.27) is either of travelling-wave type or of rarefaction-wave type, depending on the signs of $u^* - u_*$ and β''. In Section 1.6 and Chapter 3 we extend the travelling wave behaviour to system (1.19).[6] The rarefaction wave situation introduces an additional difficulty, and we will comment on this in Section 1.8.

There is a strong correspondence between the mathematical results on large-time behaviour and experimental practice. For instance, column experiments may show travelling wave profiles that are constant in time to a high degree of accuracy, demonstrating that despite inhomogeneities and boundary effects the travelling wave behaviour exhibits itself very strongly. This observation is mirrored by the mathematical result—convergence to travelling waves—that singles out the behaviour that will dominate, despite disturbances.

1.6 Large-time behaviour for system (1.19)

We place ourselves in one space dimension with constant coefficients, for which system (1.19) can be written as

$$\beta(u)_t + v_t + u_x - u_{xx} = 0 \tag{1.26a}$$

$$v_t = \mathcal{F}(u, v), \tag{1.26b}$$

[5]In one space dimension the flow field $\boldsymbol{q}$ and the diffusion/dispersion tensor $\boldsymbol{D}$ both reduce to scalars. Under the incompressibility condition $\operatorname{div} \boldsymbol{q} = 0$ the flow becomes constant, and if the medium is homogeneous this implies that $\boldsymbol{D}$ is constant, too (see footnote 3 on page 8). Equation (1.27) follows by scaling.

[6]A priori the signs of $u^* - u_*$ and $v^* - v_*$ give rise to four distinct possibilities for the ordering at infinity. However, the limit values at plus and minus infinity must satisfy the condition of chemical equilibrium $\mathcal{F}(u, v) = 0$. Together with the assumption of monotonicity M (page 47), this reduces the number of alternatives to two.

after an appropriate scaling (see also footnote 5). Being interested in travelling waves we consider these equations on the spatial domain $\mathbb{R}$. For the example that we give in this introduction we adopt a specific form for $\mathcal{F}$ (see also Remark 1.1):

$$\mathcal{F}(u, v) = \psi(u) - v.$$

The function ψ is continuous and strictly increasing.

We seek bounded travelling waves that tend to limit values (u^*, v^*) and (u_*, v_*) at plus and minus infinity:

$$\begin{aligned} (f, g) &\to (u^*, v^*) \quad \text{as} \quad \eta \to -\infty \quad \text{(upstream concentration)} \\ (f, g) &\to (u_*, v_*) \quad \text{as} \quad \eta \to \infty \quad \text{(downstream concentration).} \end{aligned}$$

Here $\eta = x - ct$ is the travelling wave coordinate. A necessary condition for convergence at infinity is the condition

$$v^* = \psi(u^*) \quad \text{and} \quad v_* = \psi(u_*),$$

which describes the assumption that (u_*, v_*) and (u^*, v^*) are states of chemical equilibrium. We assume that $u^* > u_*$ and $v^* > v_*$, and for convenience we set $u_* = v_* = 0$.

Van Duijn and Knabner [DK91] extensively studied travelling waves for system (1.26) and characterised the conditions under which they are to be found.

Theorem 1.3 ([DK91, DK92b]) — *If the 'total isotherm' $\chi(u) = \beta(u) + \psi(u)$ satisfies*[7]

$$\chi(s) > \frac{\chi(u^*)}{u^*} s \quad \textit{for all } 0 < s < u^*. \tag{1.29}$$

then there exists a travelling wave solution $(U, V)(x, t) = (f, g)(\eta)$ of system (1.26) with limits $(0, 0)$ and (u^*, v^*) at plus and minus infinity. The functions f and g are strictly decreasing.

Again the convexity condition (1.29) and the signs of $u^* - u_*$ and $v^* - v_*$ are linked. If we invert the sign of (1.29), then travelling waves exist if we

[7]In fact the total isotherm is $\varphi(u) + \psi(u)$, where $\beta(u) = u + \varphi(u)$ (see Section 1.3 and [DK91, DK92b]); however, for condition (1.29), as well as for other uses below, the added linear term makes no difference. We use this definition of χ to avoid introducing the additional function φ into the formulation.

invert either the direction of the convection (i.e. the sign of the term u_x) or the values at plus and minus infinity.

In Chapter 3 we use the technique of Osher and Ralston [OR82] to prove that these travelling waves are stable with respect to perturbations:

Theorem 1.4 — *Let (u_0, v_0) satisfy*

$$0 \le u_0 \le u^* \quad \text{and} \quad 0 \le v_0 \le v^*,$$

and

$$\int_{\mathbb{R}} \Big(|\beta(u_0) - \beta(f)| + |v_0 - g| \Big) < \infty, \tag{1.30}$$

where $(U, V)(x, t) = (f, g)(x - ct)$ is the travelling wave given by Theorem 1.3. Let (u, v) be the corresponding solution. Then there exists a translation of (U, V), again denoted (U, V), such that

$$\int_{\mathbb{R}} \Big(|\beta(u(t) - \beta(U(t))| + |v(t) - V(t)| \Big) \to 0$$

as $t \to \infty$.

Remark 1.4 An important question in the study of convergence of general solutions towards travelling waves is the following: given the initial data (u_0, v_0), to which translation of the travelling wave (U, V) does the solution converge? In the course of the proof of Theorem 1.4 it is shown that it follows from hypothesis (1.30) that there is exactly one translation of (f, g) that satisfies

$$\int_{\mathbb{R}} \Big(\beta(u_0) - \beta(f) + v_0 - g \Big) = 0. \tag{1.31}$$

Condition (1.31) characterises in a unique way the travelling wave to which the solution converges. •

Remark 1.5 A common method of proving convergence to travelling waves consists of linearising the problem around the travelling wave solution. Estimates for the decay of solutions of this linear problem are obtained by a spectral analysis, and convert into convergence estimates for the solution of the nonlinear problem in a neighbourhood of the travelling wave solution [Pel69, Eva72a, Eva72b, Sat76, CLS, Log]. This technique can only be applied when the nonlinear problem admits a meaningful linearisation. In the case of degenerate diffusion problems, for instance, the linearised problem is not well-posed in a reasonable sense. By contrast, the technique of Osher and Ralston contains no linearisation and can be applied to situations of classical, degenerate, and singular diffusion alike (see for instance [COR93]). Another advantage of this method is that it provides an explicit domain of attraction of the travelling wave, something which is usually not the case when proving stability by linearisation. •

1.7 Injection from a well: stability of self-similar solutions

Sections 1.5 and 1.6 were concerned with the large-time behaviour in one space dimension of solutions of equation (1.20) and system (1.19). Here we consider a situation in two dimensions where the large-time behaviour for equation (1.20) is given by a self-similar solution.

One of the methods of cleaning a polluted soil *in situ*, without removing it, is to pump water through it, possibly containing surface reactants or bacteria. A common technique is to drill wells in the soil, and inject water through some wells and recover it from others. Here we investigate the model problem of one injection well in a homogeneous two-dimensional medium.

The flow domain is modelled by the set $\Omega_\varepsilon = \mathbb{R}^2 \setminus \{r \le \varepsilon\}$, $\varepsilon > 0$, and the boundary $r = \varepsilon$ corresponds to the surface of the well. The flow field $\boldsymbol{q}$ is

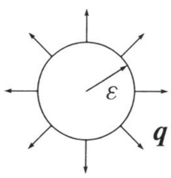


Figure 1.5: The injection well

a two-dimensional radial field caused by injection into the soil:

$$\boldsymbol{q}(r) = \frac{\lambda}{r}\boldsymbol{e}_r,$$

in which $r = |x|$ and λ is the Peclet number characterising the ratio of the pump velocity and the diffusion coefficient.

We assume a radially symmetric initial condition u_0 and expect it to give rise to radial symmetry in the solution. With the postulate $u = u(r, t)$ equation (1.20) reads

$$\beta(u)_t + \frac{1}{r}(\lambda u - r u_r)_r = 0$$

or

$$\beta(u)_t + \frac{\lambda - 1}{r} u_r - u_{rr} = 0. \tag{1.32}$$

We seek solutions of (1.32) on the spatial domain $\{\varepsilon < r < \infty\}$. The region $\{r \leq \varepsilon\}$ models the well by which water is injected, and we obtain the boundary condition on $r = \varepsilon$ by assuming continuity of the flux at the boundary:

$$\frac{\lambda}{\varepsilon} u(\varepsilon, t) - u_r(\varepsilon, t) = \frac{\lambda}{\varepsilon} u_e \quad \text{for } t > 0, \tag{1.33}$$

where u_e is the concentration of contaminant in the injected fluid. We study two cases:

$$\text{Contamination event:} \quad u_e = 1,\ u_0 \equiv 0 \tag{1.34}$$

$$\text{Remedial event:} \quad u_e = 0,\ u_0 \equiv 1. \tag{1.35}$$

Self-similar solutions

Equation (1.32) admits self-similar solutions of the kind

$$u(r, t) = f\Big(\frac{r}{\sqrt{t}}\Big),$$

where the function $f = f(\eta)$ satisfies the ordinary differential equation

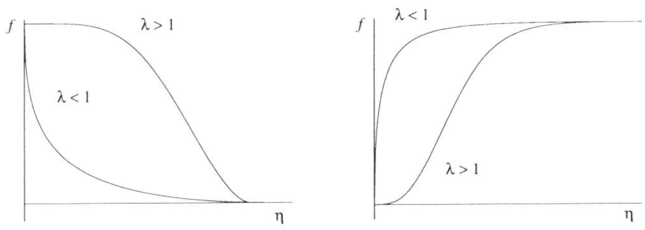
$$\tfrac{1}{2}\eta^2\{\beta(f)\}' + (\eta f' - \lambda f)' = 0 \quad \text{for} \quad 0 < \eta < \infty. \tag{1.36}$$

Here primes denote differentiation with respect to the variable η. The two events described above correspond to boundary conditions on the solution f of (1.36):

$$\text{Contamination event:} \quad f(0) = 1,\ f(\infty) = 0 \tag{1.37}$$

$$\text{Remedial event:} \quad f(0) = 0,\ f(\infty) = 1. \tag{1.38}$$

In section 4.3 phase plane techniques and approximation arguments are used to prove the existence and uniqueness of solutions of (1.36) with boundary conditions (1.37) and (1.38). The qualitative behaviour of these solutions shows a marked difference between the cases $0 < \lambda < 1$ and $\lambda > 1$, as can be expected from the sign of the singular convection term $(\lambda - 1)r^{-1}u_r$ in (1.32). This is shown in Figure 1.6.



(a) $f(0) = 1,\ f(\infty) = 0$ (b) $f(0) = 0,\ f(\infty) = 1$

Figure 1.6: The self-similar solutions of equation (1.20), with nonlinearity $\beta(u) = u + \sqrt{u}$.

Stability of self-similar solutions

Our aim is to prove that general solutions of (1.32) with boundary conditions (1.33) converge to self-similar solutions as either $\varepsilon \to 0$ or $t \to \infty$, showing that in some sense the self-similar solutions are the generic behaviour.

In order to compare solutions u^ε of (1.32) for different values of ε we transform the equation to a fixed domain by introducing the following variables:

$$x = \frac{r}{\varepsilon}, \qquad \tau = \frac{t}{\varepsilon^2}, \qquad \text{and} \quad w(x,\tau) = u^\varepsilon(r,t). \tag{1.39}$$

In the variables w, x, and τ, equation (1.32) with boundary condition (1.33) reads

$$\beta(w)_\tau + \frac{\lambda - 1}{x} w_x - w_{xx} = 0 \qquad 1 < x < \infty,\ \tau > 0 \tag{1.40a}$$

$$w_x(1,\tau) = \lambda\big(w(1,\tau) - u_e\big) \qquad \tau > 0 \tag{1.40b}$$

Using this formulation the two limit processes

$$\varepsilon \to 0, \quad t \text{ fixed}$$

and

$$t \to \infty, \quad \varepsilon \text{ fixed}$$

both correspond to $\tau \to \infty$. Remark that since $rt^{-1/2} = x\tau^{-1/2}$, the transformation (1.39) leaves self-similar solutions of (1.32) invariant, and if f is a solution of (1.36) then the function $w(x,\tau) = f(x/\sqrt{\tau})$ is a (self-similar) solution of (1.40a). Combining these two facts we conclude that to prove the convergence (in some sense) of a solution $u(r,t)$ to a self-similar solution $f(r/\sqrt{t})$

when either $\varepsilon \to 0$ or $t \to \infty$ it is sufficient to prove that the corresponding function $w(x, \tau)$ converges to $f(x/\sqrt{\tau})$ when τ tends to infinity.

We prove the following theorem.

Theorem 1.5 — *Let $w = w(x, \tau)$ be the solution of*

$$\begin{cases} \beta(w)_\tau + \dfrac{\lambda - 1}{x} w_x - w_{xx} = 0 & 1 < x < \infty,\ \tau > 0 \\ w_x(1, \tau) = \lambda\big(w(1, \tau) - u_e\big) & \tau > 0 \\ w(x, 0) = u_0 \in \mathbb{R} & x > 1 \end{cases}$$

where the pair (u_e, u_0) is either $(1, 0)$ or $(0, 1)$. Let f be the solution of (1.36) with corresponding values at $\eta = 0$ and at $\eta = \infty$. Then

$$\sup_{1<x<\infty} \big|w(x, \tau) - f(x/\sqrt{\tau})\big| \le C\tau^{-m}$$

for some constant C where m is given by

$$m = \begin{cases} \lambda/3 & \text{if } \lambda < 1 \\ 1/3 & \text{if } \lambda \ge 1. \end{cases}$$

By the remarks above we conclude

Corollary 1.6 — *Let the pair (u_e, u_0) be either $(1, 0)$ or $(0, 1)$, and let u^ε be the solution of*

$$\begin{cases} \beta(u^\varepsilon)_t + \dfrac{\lambda - 1}{r} u_r^\varepsilon - u_{rr}^\varepsilon = 0 & \varepsilon < r < \infty,\ t > 0 \\ u_r^\varepsilon(\varepsilon, t) = \lambda\big(u^\varepsilon(\varepsilon, t) - u_e\big) & t > 0 \\ u^\varepsilon(r, 0) = u_0 & r > \varepsilon, \end{cases}$$

and let f be the solution of (1.36) with corresponding values at $\eta = 0$ and at $\eta = \infty$. Then

$$\sup_{\varepsilon<r<\infty} \big|u^\varepsilon(r, t) - f(r/\sqrt{t})\big| \le C\left(\frac{\varepsilon^2}{t}\right)^m,$$

where C and m are as in Theorem 1.5.

Remark 1.6 For the limit process $t \to \infty$ it is possible to extend Theorem 1.5 to the case of non-constant initial data, and we do so in Chapter 4. For instance, instead of the condition $u_0 \equiv 0$ in (1.34) one can allow $u_0 \in L^\infty(\varepsilon, \infty)$ subject to the condition

$$\int_\varepsilon^\infty r\beta(u_0)\,dr < \infty. \tag{1.41}$$

Since $\beta(u) = C + \Psi_I(C)$, condition (1.41) can be interpreted as stating that the total amount of contaminant initially present should be finite. An analogous extension can be done for (1.35).

It is not difficult to see why such an extension is not possible when we let ε tend to zero while keeping t fixed. For fixed finite time, the concentration profile is strongly determined by the initial condition, especially for small times. We can therefore not expect a limit behaviour that is given by f, and therefore is independent of the initial distribution. •

1.8 The 'fast reaction limit' and large-time behaviour

We can organise the results on large-time behaviour that are mentioned in the Sections above in Table 1.1. In this diagram we take the function β and the

	(1.20)	(1.19)
1-d, const $\boldsymbol{q}$ $u^* > u_*$	TW [DG87]	TW [Sec. 1.6]
1-d, const $\boldsymbol{q}$ $u^* < u_*$	RW [DG87]	$*$
2-d, radial	SSS [Sec. 1.7]	$*$

Table 1.1: The different types of large-time behaviour: Travelling Waves, Rarefaction Waves, and Self-Similar Solutions (i.e. functions of the variable $\eta = x/\sqrt{t}$).

total isotherm $\chi = \beta + \psi$ to be concave as in the previous sections. In this section we will briefly discuss the two entries marked by '$*$'. As an introduction we first comment on what is often called the 'fast reaction limit'.

The parameter k in equation (1.13) governs the rate of the adsorptiondesorption reaction, the exchange between the adsorbed chemical and the dissolved chemical. If we do not scale out this parameter then system (1.19) becomes

$$\bigl(\beta(u) + v\bigr)_t + \operatorname{div}(\boldsymbol{q}u - \nabla u) = 0 \tag{1.42a}$$

$$v_t = k\mathcal{F}(u, v). \tag{1.42b}$$

The fast reaction limit is the limit $k \to \infty$, corresponding to an adsorption/desorption process whose dynamics are substantially faster than those of the convection and diffusion processes.

Equation (1.42b) when written in the form

$$\mathcal{F}(u, v) = \frac{1}{k} v_t$$

suggests that in the limit $k = \infty$ the functions u and v satisfy $\mathcal{F}(u, v) \equiv 0$. We can add to the credibility of this formal result, and gain some insight in the process, by considering the two-dimensional ordinary differential equation

$$\beta(u)_t = -k\mathcal{F}(u, v)$$
$$v_t = k\mathcal{F}(u, v),$$

which is essentially (1.42) with the non-local terms left out. If $\mathcal{F}(u, v) = \psi(u) - v$, for instance, where ψ is an increasing function, then the phase plane is similar to Figure 1.7. For this dynamical system the parameter k simply

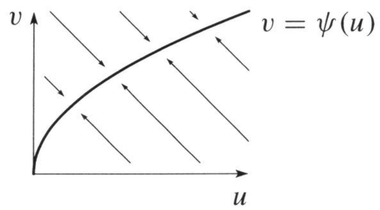


Figure 1.7: A typical phase plane

determines how fast the orbit converges to the equilibrium set $\{v = \psi(u)\}$. Obviously, in system (1.42) the 'pull' towards the set $\{v = \psi(u)\}$ will be stronger when k is larger. Consequently we expect that in the limit we have $v \equiv \psi(u)$ and the single equation for u

$$\chi(u)_t + u_x - u_{xx} = 0,$$

where $\chi(u) = \beta(u) + \psi(u)$ as in Section 1.6.

This formal result has been made rigorous by Knabner [Kna91] and Hilhorst, Van der Hout, and L. A. Peletier [HHP96a]. Knabner considers very general systems with weak hypotheses on the coefficients, and proves a convergence result in L^p-spaces over Q_T. Hilhorst et al. consider more specific systems and boundary and initial data and prove a stronger result, uniform convergence on compact subsets of Q_T.

With these results in mind we now take a look at the limit case $t \to \infty$ for a simple system of type (1.19) with constant reaction rate coefficient k,

$$\beta(u)_t + v_t + u_x - u_{xx} = 0 \tag{1.43a}$$

$$v_t = \psi(u) - v \tag{1.43b}$$

with rarefaction-wave conditions at infinity:

$$u^* < u_* \quad \text{and} \quad v^* < v_*.$$

This is one of the entries in Table 1.1 that is marked with a star. Supposing that the limit behaviour of the solution is of rarefaction-wave type, i.e.

$$u(x,t) \sim f(x/t) \quad \text{and} \quad v(x,t) \sim g(x/t),$$

then it makes sense to set

$$u_\lambda(x,t) = u(\lambda x, \lambda t) \quad \text{and} \quad v_\lambda(x,t) = v(\lambda x, \lambda t). \tag{1.44}$$

Under this transformation the system becomes

$$\beta(u_\lambda)_t + v_{\lambda t} + u_{\lambda x} - \frac{1}{\lambda} u_{\lambda xx} = 0$$

$$v_{\lambda t} = \lambda\big(\psi(u_\lambda) - v_\lambda\big).$$

The limit process $t \to \infty$ has transformed into $\lambda \to \infty$, and the limit profile is given by $\lim_{\lambda\to\infty}(u_\lambda, v_\lambda)|_{t=1}$.

When $\lambda \to \infty$ we see two degeneracies appearing in these equations. The first, that of the parameter multiplying $u_{\lambda xx}$, is a well-known one—the vanishing of the diffusion term makes the solution converge to a solution (in some sense) of the rest of the equation. This is the basis of the theorem proved by Van Duijn and De Graaf. The second degeneration is new, and comparable to the limit $k \to \infty$ that we discussed above.

We now give a sketch of how these features could be combined to give a result on the large-time behaviour of solutions of system (1.43). We start by splitting the limit behaviour: let $\nu, k > 0$ and consider the system

$$\beta(u_\nu^k)_t + v_{\nu t}^k + u_{\nu x}^k - \nu u_{\nu xx}^k = 0$$

$$v_{\nu t}^k = k\big(\psi(u_\nu^k) - v_\nu^k\big).$$

and the intermediate equation

$$\chi(u_\nu)_t + u_{\nu x} - \nu u_{\nu xx} = 0$$

where we again set $\chi(u) = \beta(u) + \psi(u)$. We now need two results:

1. *Convergence as* $k \to \infty$. For every $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon) > 0$ and $\nu_0 = \nu_0(\varepsilon) > 0$ such that

$$\left\| u_\nu^k(1) - u_\nu(1) \right\|_{L^\infty(\mathbb{R})} + \left\| v_\nu^k(1) - \psi\left(u_\nu^k(1)\right) \right\|_{L^\infty(\mathbb{R})} \le \varepsilon$$

   for all $k \ge k_0$ and $0 < \nu \le \nu_0$.

2. *Convergence as* $\nu \to 0$. For every $\varepsilon > 0$ there exists $\mu = \mu(\varepsilon) > 0$ such that

$$\left\| u_\nu(1) - u_{\mathrm{RW}}(1) \right\|_{L^\infty(\mathbb{R})} \le \varepsilon$$

   for all $0 < \nu \le \mu$. Here u_{RW} is the rarefaction wave solution of

$$\chi(u)_t + u_x = 0$$

   with limits u^* and u_* and minus and plus infinity. (Note that u_{RW} is invariant under the scaling (1.44)).

By combining these two results we find $\lambda_0 = \lambda_0(\varepsilon) > 0$ such that

$$\left\| u_{1/\lambda}^\lambda(1) - u_{\mathrm{RW}}(1) \right\|_{L^\infty(\mathbb{R})} \le \varepsilon$$

if $\lambda \ge \lambda_0$, and using the uniform continuity of ψ,

$$\left\| v_{1/\lambda}^\lambda(1) - \psi\left(u_{\mathrm{RW}}(1)\right) \right\|_{L^\infty(\mathbb{R})} \le \varepsilon.$$

This tells us that u_{RW} is the limit profile for u in system (1.19), and that v converges to $\psi(u_{\mathrm{RW}})$.

Of the two hypothetical convergence results mentioned above the second is in fact an alternative formulation of the theorem proved by Van Duijn and De Graaf. The first result, however, we could not find in the literature in the form in which it stands here. The major obstacle seems to be the convergence that is uniform in ν, without a lower bound on ν. It does seem possible to render a theorem due to Knabner ([Kna91], Th. 5.9) independent of ν, but some additional work would be necessary to obtain the result in the form given above.

For the second starred entry in Table 1.1 the situation is similar. A transformation of the type (1.44) leads to a fast reaction limit in the adsorption/desorption equation. In this case, however, the diffusion coefficient does not vanish, but remains constant under the transformation. By a similar combination of limit processes we obtain a similar (formal) result: as $t \to \infty$, system (1.19) effectively reduces to an equation of type (1.20), and the large-time behaviour of the two solutions is therefore essentially the same.

1.9 Existence of interfaces for system (1.19)

If we cast equation (1.27) in the form

$$w_t + \varphi(w)_x - \varphi(w)_{xx} = 0 \tag{1.45}$$

with $\varphi = \beta^{-1}$ then we expect that if $\varphi(u)$ resembles the function u^m in some sense we should find interfaces for equation (1.45), just like for equation (1.6). This has been investigated by a number of authors [OKYL58, Kal73, Pel74, Gil88] (see also the survey articles mentioned on page 4), and the final answer is that interfaces appear if and only if

$$\int_0 \frac{\varphi'(s)}{s}\, ds < \infty,$$

which is equivalent to

$$\int_0 \frac{1}{\beta(s)}\, ds < \infty. \tag{1.46}$$

As an example of the practical meaning of this integrability condition, note that for the function $\beta(s) = s^p$ for some $p > 0$ condition (1.46) is equivalent to $0 < p < 1$. In this case $\varphi(s) = s^{1/p}$, and the ensuing condition $1/p > 1$ corresponds to $m > 1$ for equation (1.6).

When we supplement equation (1.20) with the equation for v to obtain system (1.19), we expect—under conditions on $\mathcal{F}$—that the property of existence of interfaces will persist. However, the travelling waves investigated by Van Duijn and Knabner [DK92b, DK91] demonstrate that interfaces can also appear in the case of degeneration of the function $\mathcal{F}$, even if β is non-degenerate, i.e. $\int 1/\beta(s)\,ds = \infty$. We will first discuss the results of [DK92b, DK91], then describe our own work on general solutions.

Let (f, g) be the travelling wave given by Theorem 1.3. Again we assume $u_* = v_* = 0$ and set

$$L = \sup\{\eta \in \mathbb{R} : f(\eta) > 0\} = \sup\{\eta \in \mathbb{R} : g(\eta) > 0\}.$$

The first equality is a definition, the second is proved by Van Duijn and Knabner. We collect as a theorem a number of results given in [DK91, DK92b]. We recall that the total isotherm χ is given by $\chi(u) = \beta(u) + \psi(u)$.

Theorem 1.7 ([DK91, DK92b]) — *Same hypotheses as Theorem 1.3.*

1. $L < \infty \implies 1/\chi$ *is integrable near* 0;
2. $1/\beta$ *integrable near* $0 \implies L < \infty$;
3. *If* $\mathcal{F}(u, v) = k(\psi(u) - v)$ *for some* $k > 0$, *then*

   $$L < \infty \iff s \mapsto \frac{1}{X^{1/2}(s) + \psi(s)} \quad \textit{is integrable near } 0,$$

   where $X(s) = \int_0^s \chi(\sigma)\,d\sigma$;
4. *If* $\mathcal{F}(u, v) = k\big(u - \psi^{-1}(v)\big)$ *for some* $k > 0$, *then*

   $$L < \infty \iff 1/\beta \quad \textit{is integrable near } 0.$$

Remark 1.7 As an explicit example consider the case

$$\beta(u) = u^p \quad \text{and} \quad \mathcal{F}(u, v) = u^q - v,$$

with $p, q > 0$. Following part 3 of Theorem 1.7, we then have

$$L < \infty \iff p < 1 \quad \text{or} \quad q < 1.$$

•

Remark 1.8 The last two parts of this theorem serve as an example to show that it is not possible to close the gap between parts 1 and 2 without more specific knowledge about the form of the function $\mathcal{F}$. •

Clearly not only degeneration of β, the classical case, but also degeneration of $\mathcal{F}$ can lead to the existence of interfaces in travelling waves. In Chapter 5 we investigate interfaces of general solutions of the system of equations

$$\beta(u)_t + v_t - \operatorname{div} \boldsymbol{A}(x, t, u, \nabla u) + B(x, t, u, \nabla u) = 0 \tag{1.47a}$$

$$v_t = \mathcal{F}(x, t, u, v), \tag{1.47b}$$

on some domain $Q \subset \mathbb{R}^N \times \mathbb{R}^+$. The structural conditions on the different nonlinearities are given later. They are chosen as to include the case of system (1.19).

We mentioned above the extensive literature on interfaces for single equations. For systems of equations the results are less abundant. We mention the articles by Díaz and Stakgold [DS95] who considered problem (1.47) with a reaction term of the form $\mathcal{F}(u, v) = g(u)h(v)$, and Hilhorst, Van der Hout, and L. A. Peletier [HHP96a] who extended this to more general reaction terms. Both works consider one-dimensional autonomous situations with constant initial and boundary data. A much more general result is given in [Kna91], for system (1.19) with general convection and diffusion terms, but in one space dimension with additional restrictions on the time derivative. This latter restriction was necessary to apply a comparison principle, as is explained in Section 1.4, and with the recent results on uniqueness this restriction is probably unnecessary.

Most of the earlier results, including the three papers on systems mentioned above, have been obtained by comparison with travelling waves or other special solutions. The method that we use here, often called 'the energy method for free boundary problems', is different, and in fact does not require that the system satisfy a comparison principle. It was originally introduced by Antontsev [Ant81], rendered in a mathematically rigorous form by Díaz and Véron [DV85], and later extended and applied by these and several other authors, amongst whom Bernis [Ber96] and Shmarev [ADS95]. In [AD91] this method is applied to a different system of equations, arising in two-phase flow in porous media. We refer to [AD] for a good overview of the existing literature on this method.

The method has two principal features. On the one hand, it is a local method: it operates in subsets of the domain without taking into account global information like boundary conditions or boundedness of the domain. On the other hand, it has a very general setting, allowing to consider, for instance, problems in any space dimension, (x, t)-dependence of the different terms of the equations, and anisotropic diffusion. It is worth noting that the method is essentially qualitative, in the sense that it does not provide, in general, quantitative estimates of the evolution of the support.

In Chapter 5 we give the details of the hypotheses, theorems, and proofs. Here we shall merely try to give a flavour of the results that we obtain. Let us mention that Section 5.4 contains a formal outline of the method and is meant to serve as an introduction.

Two important conditions on the nonlinearities are the following:

$$m_0 u^{p+1} \leq \Phi(u) \leq m_1 u^{p+1} \quad \text{for} \quad u \geq 0, \tag{1.48}$$

where $\Phi(u) = \int_0^u s\beta'(s)\,ds$ and $p \in (0, 1]$, and

$$B(x, t, u, \nabla u) = \boldsymbol{q}(x, t) \cdot \nabla\beta(u)$$

with $\boldsymbol{q} \in L^\infty(Q; \mathbb{R}^N)$ satisfying $\operatorname{div} \boldsymbol{q} = 0$ in the sense of distributions on Q. Condition (1.48) characterises the degeneracy ($p < 1$) or non-degeneracy ($p = 1$) of the nonlinearity β. The condition on B is more of a technical nature, and signifies that the term B, when interpreted as a convection term, corresponds to a divergence-free and uniformly bounded flow. We assume of $\boldsymbol{A}$ that

$$\boldsymbol{A}(\cdot, \cdot, \cdot, \xi) \cdot \xi \geq m_2 |\xi|^2 \quad \text{and} \quad |\boldsymbol{A}(\cdot, \cdot, \cdot, \xi)| \leq m_3 |\xi| \quad \text{for all} \quad \xi \in \mathbb{R}^N$$

The theorems that we formulate depend crucially on some hypotheses on $\mathcal{F}$. The first is[8]

I_1 There exists a number $0 < \bar{v} \leq \infty$ and a non-negative function $\theta : [0, \bar{v}) \to \mathbb{R}$ such that

$$\big(u - \theta(v)\big)\, \mathcal{F}(\cdot, \cdot, u, v) \geq 0$$

for all $u \geq 0$ and $0 \leq v < \bar{v}$. If $\bar{v} < \infty$ then we set $\theta(v) = \infty$ for all $v \geq \bar{v}$.

Although this may not be clear at first sight, hypothesis I_1 is a very natural one in view of the underlying model. If the rate function $\mathcal{F}$ has one of the forms mentioned in Remark 1.1 on page 10, then

$$\big(u - \psi^{-1}(v)\big)\, \mathcal{F}(\cdot, \cdot, u, v) \geq 0$$

[8]Throughout this work we use sans-serif letters for important hypotheses, and we try to choose the letters in an intuitive way. Here 'I' stands for 'Interfaces'.

for all $u \geq 0$ and all $v \geq 0$ for which $\psi^{-1}(v)$ is defined. Obviously ψ^{-1} is an ideal candidate for the function θ of hypothesis I_1.

The first property that we prove is often called Finite Speed of Propagation (FSP). We give an exact definition in Chapter 5; in the meantime it is a good approximation to interpret FSP as 'the existence of interfaces'.

Theorem 1.8 — *Let hypothesis I_1 be satisfied. If $p < 1$ then system (1.47) has property* FSP.

We already mentioned on page 7 the existence of *waiting-times*: the interface of equation (1.6) for $m > 1$ can remain stationary over a time interval $[0, t^*]$, in which case the time $t^* > 0$ is called a waiting-time. For this to happen it is necessary that the initial data u_0 satisfy a certain 'flatness condition' near the interface, i.e. if $u_0(x) = 0$ for $x \geq x_0$, then $u_0(x)$ may not grow too quickly for $x < x_0$.

The occurrence of waiting-times has been thoroughly investigated by means of comparison arguments. The energy method that we apply supplies a different means of determining the existence of waiting-times (we will call this property WT).

Theorem 1.9 — *Let hypothesis I_1 be satisfied and suppose that $B = 0$. If $p < 1$ then system (1.47) has property* WT *(see Chapter 5 for the accompanying flatness condition).*

Remark 1.9 In one dimension, the equation with convection

$$\beta(u)_t + \beta(u)_x - u_{xx} = 0$$

can be reduced to an equation without convection

$$\beta(u)_t - u_{\xi\xi} = 0$$

by defining the new space variable $\xi = x - t$. If a solution of the second equation has a waiting-time, then the corresponding interface of the solution of the first has an initial speed of 1. This shows that the restriction to the no-convection case is a natural one. •

Both theorems stated above are concerned with degeneracy in β: $p < 1$ in (1.47). One could interpret I_1 as a hypothesis that guarantees that equation (1.47b) does not interfere with the interface properties of equation (1.47a).

We now state a theorem which also proves existence of interfaces when for instance $p = 1$ but $\mathcal{F}$ is degenerate. Introduce the hypotheses

I_2 $0 \leq \mathcal{F}(\cdot, \cdot, u, 0) \leq k_1 u^p$ for all $u \geq 0$;

I_3 $k_2 u^\gamma \leq \mathcal{F}(\cdot, \cdot, u, 0) \leq k_3 u^\gamma$ for all $u \geq 0$.

Here the exponent p is the same as above (i.e., given by (1.48)) and the exponent γ is free to be chosen in $(0, 1)$. The k_i are positive constants.

Theorem 1.10 — *Let either of the following conditions be satisfied:*

$$\mathsf{I}_2 \text{ with } p < 1 \qquad \text{or} \qquad \mathsf{I}_3 \text{ with } \gamma < 1.$$

Then

1. *System (1.47) has the property* FSP;
2. *If $B = 0$, then system (1.47) also has the property* WT.

Again the flatness condition is given in Chapter 5.

Under hypothesis I_3, Theorem 1.10 allows for $p = 1$, $\gamma < 1$, in which case the FSP and WT properties are consequence of the degeneracy in $\mathcal{F}$.

We now describe the topics of Chapters 6 and 7. Both are concerned with different, although related, problems.

1.10 Blow-up of interfaces

The Barenblatt-Pattle solution (1.7) of equation (1.6) has interfaces at $|x|^2 = 2\gamma Nm/(\alpha(m-1))\, t^{2\alpha/N}$, and although they move outwards as time increases, they clearly remain finite for all finite time. A similar property holds true for general solutions u of the initial value problem associated with (1.6). If the initial value $u(\cdot, 0)$ is bounded and has compact support, then we can choose an appropriately shifted and scaled version of the Barenblatt-Pattle solution u_{BP}, such that $u_{\mathrm{BP}}(\cdot, 0) \geq u(\cdot, 0)$ on $\mathbb{R}$. This ordering then persists for all time, and the finiteness of the interfaces of u_{BP} carries over to the function u. The interfaces may (and do) move outwards, but at every finite time they are bounded.

We wish to investigate the situation for a slightly modified version of (1.6), namely

$$\rho(x) u_t = \Delta u^m \qquad x \in \mathbb{R}^N, \quad t > 0. \tag{1.49}$$

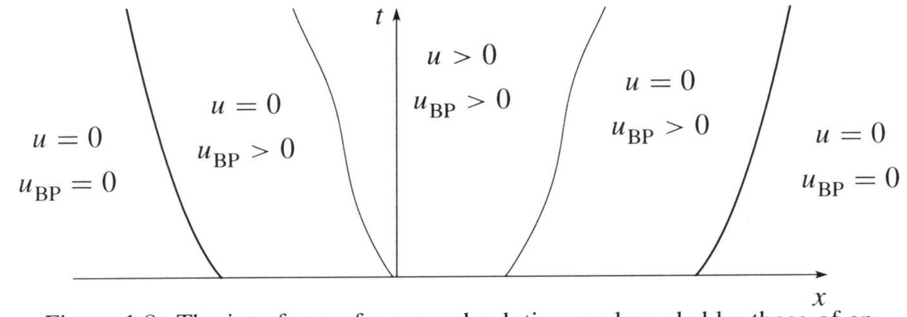


Figure 1.8: The interfaces of a general solution are bounded by those of an appropriately shifted and scaled Barenblatt-Pattle solution.

This equation (with $m = 6$) has been proposed as a model for heat transport in a gas through radiation rather than conduction. In this model u represents the temperature and ρ the mass density of the gas [ZR66].

Kamin and Kersner [KK93] investigated the behaviour of the interfaces of solutions of (1.49) when $N \geq 3$. They showed that if ρ satisfies $\rho \in L^1(\mathbb{R}^N)$, then for any initial datum u_0 with compact support there is a time $T = T(u_0)$ such that the corresponding solution $u = u(x, t)$ of (1.49) has non-compact spatial support for all $t > T$. In other words, the interface runs off to infinity in finite time. We call this phenomenon interface blow-up, and inspired by this result we investigated it further.

A detail that we should mention is that in one and two space dimensions the quantity $\int_{\mathbb{R}^N} \rho u(t)\, dx$ is conserved. If $\int \rho < \infty$ then we speak of a medium of finite mass, and we show in Theorem 6.2 that $u \to \bar{u}$, uniformly on compact sets, where $\bar{u}$ is the average of the initial data u_0:

$$\bar{u} \stackrel{\text{def}}{=} \frac{\displaystyle\int_{\mathbb{R}^N} \rho(x) u_0(x)\, dx}{\displaystyle\int_{\mathbb{R}^N} \rho(x)\, dx}.$$

This is in contrast to the case of constant ρ, in which solutions with finite initial mass tend to zero uniformly on the domain. In space dimensions three and higher the mass $\int \rho u(t)$ is not conserved, and it is by obtaining a contradiction to this fact that Kamin and Kersner prove the result mentioned above.

A first result on the interfaces [Pel94], valid for all $N \geq 1$, gives a sufficient condition on non-blow-up, as well as a nearly-explicit bound for the interface. The result is based on an explicit supersolution—essentially a generalisation of the Barenblatt-Pattle solution—that supplies a bound on the interfaces of the solution itself.

We suppose that ρ is radially symmetric and non-increasing, and that u is the solution of (1.49) with initial data u_0. The function

$$v(x,t) = \left(\frac{1}{f(t)} \left[1 - \frac{|x|^2}{g(t)^2} \right]_+ \right)^{1/(m-1)},$$

is a supersolution of equation (1.49) if f and g satisfy the inequalities

$$f'(t) \le 2mN \frac{1}{\rho(x) g^2(t)} \tag{1.50a}$$

$$g'(t) \ge \frac{2m}{m-1} \frac{1}{\rho(x) f(t) g(t)}, \tag{1.50b}$$

for all x and t such that $0 \le x \le g(t)$. We show in Chapter 6 that functions f and g can be found that satisfy (1.50) and that for any initial datum u_0 with compact support we can also choose them such that $v(x, 0)$ lies above u_0. It follows from the comparison principle that $0 \le u \le v$, and therefore the support of u must then be bounded by the interfaces of v, i.e. the curves $\{|x| = g(t)\}$. A possible choice for the function g is given by

$$\int_{g(0)}^{g(t)} r\rho(r)\, dr = \frac{2mt}{m-1}.$$

We summarise this in the

Theorem 1.11 — *Let $N \ge 1$ and suppose that ρ is radially symmetric and non-increasing in $r = |x|$. Let a and b be numbers such that*

$$0 \le u_0 \le \left(a \left[1 - \frac{r^2}{b^2} \right]_+ \right)^{1/(m-1)}$$

Then the support of the solution u to (1.49) with initial datum u_0 satisfies

$$\operatorname{supp} u(t) \subset \{x \in \mathbb{R}^N : r \le g(t)\},$$

where g is given by

$$\int_{b}^{g(t)} r\rho(r)\, dr = \frac{2mt}{m-1}.$$

It immediately follows that

Corollary 1.12 — *Under the conditions of Theorem 1.11, finite-time blow-up can not occur if*

$$\int_0^\infty r\rho(r)\,dr = \infty.$$

Remark 1.10 The conditions on ρ can be relaxed, and this is done in Chapter 6. However, some important questions remain when ρ is strongly non-radial. •

Written for radially symmetric densities ρ, the sufficient condition for blow-up introduced by Kamin and Kersner [KK93] reads

$$\int_0^\infty r^{N-1}\rho(r)\,dr < \infty. \tag{1.51}$$

This result is valid in space dimensions $N \geq 3$, and in this case there is obviously a gap between this result and Corollary 1.12. But in one space dimension, Corollary 1.12 turns out to be sharp:

Theorem 1.13 — *Let $N = 1$ and suppose that*

$$\int_{-\infty}^\infty |x|\,\rho(x)\,dx < \infty.$$

If $u_0 \geq 0$ and $u_0 \not\equiv 0$, then the outer bounds of $\operatorname{supp} u(t)$ *will tend to infinity in finite time.*

This follows from a simple formal argument which we make rigorous in Chapter 6. By multiplying (1.49) by x and integrating over $(0, \infty) \times (0, \tau)$, we find

$$\begin{aligned}
\int_0^\infty x\rho(x)u(x,\tau)\,dx - \int_0^\infty x\rho(x)u_0(x)\,dx &= \int_0^\tau\int_0^\infty x(u^m)_{xx}\,dxdt \\
&= -\int_0^\tau\int_0^\infty (u^m)_x\,dxdt + \int_0^\tau \left[x(u^m)_x\right]_0^\infty dt \\
&= \int_0^\tau \left[x(u^m)_x - u^m\right]_0^\infty dt
\end{aligned}$$

If we suppose that the interface is bounded for $0 \leq t \leq \tau$, then the boundary terms at infinity vanish, and we are left with

$$\int_0^\infty x\rho(x)u(x,\tau)\,dx = \int_0^\infty x\rho(x)u_0(x)\,dx + \int_0^\tau u^m(0,t)\,dt.$$

Since $\int |x|\,\rho(x)\,dx < \infty$ implies $\int \rho(x)\,dx < \infty$, we have $u \to \bar{u} > 0$ as $t \to \infty$ (see page 35), the last term in this equation will tend to infinity while the left-hand side is bounded by

$$\left(\max_{\mathbb{R}} u_0\right) \int_0^\infty x\rho(x)\,dx.$$

We find a contradiction to our assumption that for all $\tau > 0$ interfaces were bounded in space on $[0, \tau)$.

Remark 1.11 Recently the asymptotic behaviour of the solution and the interface near the blow-up time have been studied by Galaktionov and King [GK]. •

Two space dimensions

A simple trick allows us to extend this complete characterisation of the occurrence of blow-up from one dimension to two dimensions with radial symmetry. If ρ and u satisfy (1.49) with both ρ and u being radially symmetric then by setting $s = \log r = \log |x|$ the functions

$$\hat{u}(s) = u(r) \quad \text{and} \quad \hat{\rho}(s) = r^2\rho(r)$$

satisfy the one-dimensional equation

$$\hat{\rho}(s)\hat{u}_t = (\hat{u}^m)_{ss} \quad \text{for} \quad s \in \mathbb{R} \quad \text{and} \quad t > 0.$$

This gives us the result

Corollary 1.14 — *Suppose that the function $r \mapsto r^2\rho(r)$ is non-increasing on $[0, \infty)$, and suppose that the initial datum u_0 is radially symmetric and has compact support. Then interface blow-up occurs if and only if*

$$\int_1^\infty r^2 \log r\rho(r)\,dr < \infty.$$

Remark 1.12 Using the comparison principle we can extend this result to non-radial solutions u by comparing a general solution with radially symmetric solutions that lie above and below it at $t = 0$. •

A question still unanswered is whether it is at all possible to characterise occurrence of blow-up entirely in terms of the density function ρ. In one dimension the answer is obviously yes, at least for non-increasing densities. For $N = 2$ with radially symmetric density Corollary 1.14 gives a complete characterisation, but for different types of densities one expects different results. For instance, if ρ only depends on the first coordinate x_1, then by comparison with the one-dimensional case we would expect instances of blow-up if

$$\int_{-\infty}^{\infty} |x_1|\, \rho(x_1)\, dx_1 < \infty.$$

Remark 1.13 In Chapter 6 we consider a slightly more general situation, in which the nonlinearity u^m is replaced by a general function $A(u)$ which is allowed to degenerate in two points (scaled to 0 and 1). This situation arises commonly in the modelling of two-phase flow in porous media, e.g. oil and water in porous rock ([AS79], equation (2.91)) or salt and fresh water in aquifers [JdJ81]. •

1.11 Fast diffusion

The equation known as the 'fast diffusion equation',

$$u_t = \Delta u^m \quad \text{for} \quad m < 1, \tag{1.52}$$

arises in a number of areas of application such as plasma physics, gas kinetics, and semiconductors. This equation has a life of its own and we shall briefly enter into it.

Herrero and Pierre [HP85] proved that the Cauchy Problem in $\mathbb{R}^N$ for (1.52) is well-posed for all $0 < m < 1$ with initial data in $L^1_{loc}(\mathbb{R}^N)$. This situation is different from the heat equation $m = 1$ for which conditions on the behaviour of the initial data are necessary to ensure uniqueness of the solution. In the parameter range $-1 < m \leq 0$ for $N = 1$ an interesting result was proved in [RV90]. For any $u_0 \in L^1(\mathbb{R})$ and $f, g \in L^\infty_{loc}(0, \infty)$, $f, g \geq 0$, there exists a unique solution of (1.52) with initial datum u_0 and satisfying the flux conditions at infinity

$$\lim_{x\to\infty} u^{m-1} u_x(x,t) = -f(t), \qquad \lim_{x\to-\infty} u^{m-1} u_x(x,t) = g(t).$$

Such flux conditions could be interpreted as inhomogeneous Neumann boundary conditions at infinity.

We will be interested in the long-term behaviour of solutions with finite mass and therefore concentrate on initial data $u_0 \in L^1(\mathbb{R}^N)$. For $(N-2)_+/N < m < 1$, i.e. $(N-2)/N < m < 1$ for $N \geq 2$ and $0 < m < 1$ for $N = 1$, it is known [FK80] that the asymptotic behaviour is given by the Barenblatt-Pattle solution (1.7). For $N \geq 3$ the range $0 < m < (N-2)/N$ is not covered. It is shown by a formal argument in [Pel81] and rigorously in [BC81] that in this range any solution with finite mass extinguishes in finite time due to a non-zero flux at infinity.

In Chapter 7 we prove the existence and additional properties of self-similar solutions of (1.52) in the parameter range $0 < m < (N-2)/N$. These solutions have finite mass and therefore also extinguish in finite time. Galaktionov and L. A. Peletier have proved that general solutions of the Cauchy Problem follow this self-similar behaviour prior to extinction [GP96a].

We seek solutions that are of the self-similar structure

$$u(x,t) = (T-t)^{\alpha} f(\eta) \quad \text{where} \quad \eta = |x|(T-t)^{-\beta}, \tag{1.53}$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are constants that need to be determined. Such solutions were also considered by Philip [Phi94] and in more detail by King [Kin93b], who gave a formal motivation for the existence of such solutions, and for the convergence of solutions with arbitrary initial distributions to these self-similar profiles. With Theorems 1.15 and 1.16 we provide a rigorous proof of King's conjectures concerning existence and uniqueness of self-similar solutions and some of their properties.

Substituting expression (1.53) into (1.52), we find that if we choose

$$\alpha(1-m) + 2\beta = 1, \tag{1.54}$$

then f satisfies the equation

$$\eta^{1-N}(\eta^{N-1} f^{m-1} f')' - \beta\eta f' + \alpha f = 0 \quad \text{for} \quad \eta > 0. \tag{1.55}$$

Symmetry and smoothness require that

$$f' = 0 \quad \text{at} \quad \eta = 0. \tag{1.56}$$

The restriction that f represent a solution of (1.52) of finite mass translates into the condition

$$\int_0^{\infty} \eta^{N-1} f(\eta)\, d\eta < \infty. \tag{1.57}$$

One can show that (1.57), when combined with (1.55), is equivalent with the statement that the flux $F(\eta) = \eta^{N-1} f^{m-1} f'(\eta)$ has a finite (negative) limit at infinity. This statement is equivalent to the assertion that

$$f(\eta) \asymp \eta^{-(N-2)/m} \quad \text{as} \quad \eta \to \infty, \tag{1.58}$$

where the notation $a(t) \asymp b(t)$ signifies

$$\lim_{t\to\infty} \frac{a(t)}{b(t)} \quad \text{exists and is positive.}$$

To conclude our preliminary remarks about equation (1.55), note that the scaling

$$\bar{f}(\eta) = \gamma^{-2/(1-m)} f(\eta/\gamma) \quad \text{for} \quad \gamma > 0 \tag{1.59}$$

leaves the equation as well as both boundary conditions invariant. We therefore always assume $f(0) = 1$.

Therefore the problem we shall study is: Find $f : [0, \infty) \to \mathbb{R}$, positive and smooth, and parameters $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$\eta^{1-N}(\eta^{N-1} f^{m-1} f')' - \beta\eta f' + \alpha f = 0, \quad f > 0 \qquad \text{for } \eta > 0 \tag{1.60a}$$

$$f'(0) = 0 \quad \text{and} \quad f(0) = 1 \tag{1.60b}$$

$$f(\eta) \asymp \eta^{-(N-2)/m} \qquad \text{as } \eta \to \infty \tag{1.60c}$$

$$\alpha(1-m) + 2\beta = 1. \tag{1.60d}$$

The relation (1.54) between the two parameters introduced by the Ansatz (1.53) arises from the requirement that f satisfy an equation involving only η. In situations where the problem under consideration satisfies a conservation law (e.g. conservation of mass), this law supplies a second condition on α and β, thus fixing the parameters. In this case we speak of *self-similar solutions of the first kind.* Since we seek solutions that do not conserve mass, there is no second condition on α and β for Problem (P). This extra degree of freedom gives it the character of a nonlinear eigenvalue problem: the parameter α (or β) is to be determined together with the solution function f. The function f is then called a *self-similar solution of the second kind* [Bar79].

The main results are summarised in the following two theorems. The first one gives existence and uniqueness for Problem (P).

Theorem 1.15 — *For every $N > 2$ and $0 < m < (N-2)/N$, Problem (P) has exactly one solution (f, α, β). Moreover,*

$$0 < \alpha < \frac{N-2}{(1-m)N-2}, \tag{1.61}$$

and β is given by (1.54).

This theorem implies that for every value of m in the given range, there exists exactly one self-similar solution of equation (1.52) of the form (1.53).

The second result concerns the behaviour of the eigenvalues α and β, as given by Theorem 1.15 and equation (1.54), when we vary the parameter m. We indicate the dependence of α and β on m by writing $\alpha(m)$ and $\beta(m)$. Let $m_0 = (N-2)/(N+2)$. We prove the following assertions:

Theorem 1.16 —

1. *$\alpha(m)$ and $\beta(m)$ depend continuously on m;*
2. *$\beta(m_0) = 0$; if $m < m_0$ then $\beta(m) > 0$, and if $m > m_0$ then $\beta(m) < 0$;*
3. *When $m \uparrow (N-2)/N$, then $\alpha(m) \to \infty$ and $\beta(m) \to -\infty$;*
4. *When $m \downarrow 0$, then $\alpha(m) \to 0$ and $\beta(m) \to \frac{1}{2}$.*

Theorem 1.16 can be interpreted in the following way. The parameter α determines the decay rate of the maximum of the solution. When m approaches zero, $\alpha(m)$ tends to zero, implying that the decay of the solution near $t = T$ is very slow. On the other hand, when m tends to $(N-2)/N$, $\alpha(m)$ tends to infinity, signifying a very fast decay rate. The parameter β determines the spread of the profile. When $\beta < 0$, the profile of the solution spreads out as t approaches T, while for $\beta > 0$ the profile shrinks, all mass concentrating in the origin. Because $\beta(m_0) = 0$, the solution u for $m = m_0$ is separable, consisting of a fixed profile multiplied by the factor $(T-t)^{(N+2)/4}$. This situation is very similar to the one considered by Berryman and Holland in [BH80].

1.12 Comments and miscellaneous references

In the preceding sections many references have been mentioned in relation to the subjects treated in this thesis. Others are slightly less related but deserve to be mentioned nonetheless.

Related models

Van Kooten wrote a thesis [vK95] on an unusual model for contaminant transport by a known discharge field. Contaminant particles are convected along streamlines and diffuse in perpendicular directions. This allows for a relatively efficient numerical approximation.

Two-phase immiscible and incompressible flow in porous media (for instance, oil and water in underground reservoirs) is an important source of degenerate diffusion problems. The origin of the degeneracy lies in the (experimental) observation that when the concentration of a phase approaches its minimal value, the resistance of the medium to flow of that phase becomes unbounded. Such models show the same kind of two-point degeneration as the fresh-salt water model derived by De Josselin de Jong (see page 3). See [AS79] or ([Bea72], Chapter 9) for a physical exposition, or [KL84, ADB85, GMT96] for well-posedness results.

Large-time behaviour for (1.20)

In models of two-phase flow in porous media, especially in oil-water systems, equations of type (1.20) arise with convex-concave nonlinearities (Buckley-Leverett). Recently the long-term behaviour of the solutions of these equations has been shown to exist of combinations of travelling waves and rarefaction waves [BGH].

Section 1.5 does not cover the case $u^* = u_*$, which is more subtle. It was investigated in [GDD94] for one space dimension, and in [DDG94] for two space dimensions. The analysis shows an interesting unfolding of cases, depending on the nonlinearity β. Closely related questions, also in several space dimensions, were considered by Escobedo and others in [EVZ93].

Travelling waves for (1.19)

Besides the work by Van Duijn and Knabner on travelling waves for the non-equilibrium model (1.19) many other authors have constructed travelling wave solutions. Most of these results are for linear models (see the introduction of [Zee90] for an overview). By contrast the nonlinear theory is much less developed. Van der Zee [Zee90, Zee91] introduces approximate travelling wave solutions, by disregarding a second-order derivative. On these solutions Logan and Ledder perform a perturbation analysis [LL95]. Also the article [DKZ93] should be mentioned, in which the authors apply the results of [DK91] to a specific combination of isotherms.

Well-injection

Goncerzewicz [Gon92] considered a problem related to that of section 1.7 and Chapter 4 in which water is not injected into the medium but extracted from it by maintaining a low water level in the well. This results in Dirichlet boundary conditions at $r = \varepsilon$. The resulting equations are similar and some of the existence and uniqueness results overlap.

Chapter 2

Well-posedness for system (1.19)

In this chapter we set the stage for later investigations of Problem (1.19). The problem we treat is a Cauchy Problem associated with (1.19) in one dimension,

$$\beta(u)_t + v_t + u_x - u_{xx} = 0 \tag{2.1a}$$
$$v_t = \mathcal{F}(u, v) \tag{2.1b}$$

on $Q_T = \mathbb{R} \times (0, T]$ with $0 < T < \infty$ with the initial condition

$$(u, v) = (u_0, v_0)$$

at $t = 0$. For the length of this chapter we shall call this Problem (P).

2.1 Formulation of the problem

Definition 2.1 — *A solution of the Cauchy Problem (P) is a pair of functions (u, v) such that*

1. $u \in C(Q_T) \cap L^\infty(Q_T)$ *with* $u_x \in L^2_{loc}(Q_T)$;
2. $v \in C([0, T]; L^\infty(\mathbb{R}))$;
3. u *and* v *satisfy the equations*

$$-\int_{Q_T} \big((\beta(u) + v)\varphi_t + u(\varphi_{xx} + \varphi_x)\big) = \int_{\mathbb{R}} (\beta(u_0) + v_0)\varphi(0) \tag{2.2}$$

and

$$-\int_{Q_T} v\varphi_t = \int_{\mathbb{R}} v_0\varphi(0) + \int_{Q_T} \mathcal{F}(u, v)\varphi \tag{2.3}$$

for every $\varphi \in C_c^{2,1}(\overline{Q_T})$ *such that* $\varphi(T) = 0$.

Sub- and supersolutions of Problem (P) satisfy parts 1 and 2, as well as (2.2) and (2.3) with the equality signs replaced by $\leq$ for subsolutions and $\geq$ for supersolutions.

Remark 2.1 The regularity assumptions on u and v deserve some explanation. It is possible to define solutions of degenerate diffusion problems with less a priori regularity (see e.g. [AL83, Ott95a, Ott95b]), whilst retaining the well-posedness of the problem. A posteriori one can show that the solutions are in fact more regular than assumed in the Definition. In the case here at hand, regularity does not depend on the coupling between the equations; each equation produces its own regularity. In fact, any pair of functions $(u, v) \in L^\infty(Q_T)^2$ that satisfies equations (2.2) and (2.3) automatically has the regularity of parts 1 and 2 of this definition. Before we show why, let us note that this is only true for solutions, not for sub- and supersolutions. In order to prove a comparison principle for sub- and supersolutions (Theorem 2.2) we need to assume a priori the additional regularity.

The boundedness of $\mathcal{F}$ when u and v are bounded automatically implies $v \in C([0, T]; L^\infty(\mathbb{R}))$. The a priori regularity of u follows from equation (2.1a) in two steps. The first is a uniqueness result for the equation

$$\beta(u)_t + u_x - u_{xx} = g \tag{2.4}$$

with initial data $u_0 \in L^\infty(\mathbb{R})$. Solutions of this problem are defined similar to (2.2). Here the right-hand side $g \in L^\infty(Q_T)$ is assumed to be given. Uniqueness of a solution $u \in L^\infty(Q_T)$ of this problem is established by extending results from [BKP85] (Theorem C of the Appendix of that reference; see also Appendix 2.A for a different proof).

The second step consists in remarking that because of this uniqueness any solution of (2.4) is the pointwise limit of regular solutions u_n of regularised problems. In such a case well-known results [DiB83, Sac83, Zie82] imply that solutions are continuous; also, a bound of the type

$$\|u_{nx}\|_{L^2((0,T)\times\Omega)} \leq C$$

is readily obtained from which the regularity $u_x \in L^2_{loc}(Q_T)$ follows. •

Remark 2.2 It will be useful to note that because of the continuity in time an equivalent formulation of part 3 of Definition 2.1 reads

$$\int_{\mathbb{R}} \big(\beta(u(t)) + v(t)\big)\varphi(t) - \int_{Q_t} \big((\beta(u) + v)\varphi_t + u(\varphi_x + \varphi_{xx})\big)$$
$$= \int_{\mathbb{R}} (\beta(u_0) + v_0)\varphi(0) \quad (2.5)$$

and

$$\int_{\mathbb{R}} v(t)\varphi(t) - \int_{Q_t} v\varphi_t = \int_{\mathbb{R}} v_0\varphi(0) + \int_{Q_t} \mathcal{F}(u, v)\varphi \quad (2.6)$$

for all $0 < t \le T$ and $\varphi \in C_c^{2,1}(\overline{Q_t})$. •

We now state the conditions on the nonlinearities β and $\mathcal{F}$.

B_1 $\beta \in C([0,\infty)) \cap C^2((0,\infty))$, $\beta(0) = 0$, $\beta' \ge b_0 > 0$, and β' and β'' are bounded on compact sets away from the origin;

OSL $\mathcal{F} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is continuous, $\mathcal{F}(0,0) = 0$, and $\mathcal{F}$ satisfies the one-sided Lipschitz conditions

$$\frac{\mathcal{F}(u_1, v) - \mathcal{F}(u_2, v)}{u_1 - u_2} \ge -L_u \quad \text{for all} \quad u_1, u_2, v \ge 0$$

and

$$\frac{\mathcal{F}(u, v_1) - \mathcal{F}(u, v_2)}{v_1 - v_2} \le L_v \quad \text{for all} \quad u, v_1, v_2 \ge 0$$

for some $L_u, L_v \ge 0$.

M $\mathcal{F}$ satisfies OSL with $L_u = L_v = 0$.

Hypothesis B_1 will always be tacitly assumed; the other two will be mentioned explicitly where necessary. By '$\mathcal{F}$ is monotone' we shall mean hypothesis M.

Remark 2.3 Condition M is a very natural one from the point of view of the physical model described in Section 1.3. The function $\mathcal{F}$ models the adsorption rate, the amount of chemical moving from dissolved to adsorbed phase per units of time and space. The natural situation would be that this rate is an increasing function of the concentration of dissolved chemical, and a decreasing function of the concentration of adsorbed chemical. In fact it would be difficult to imagine a chemical mechanism that did otherwise. However in many other models, for instance in mathematical biology, non-monotone behaviour of zero-order terms is not uncommon. •

2.2 Uniqueness for monotone functions $\mathcal{F}$

Throughout this section we assume that $\mathcal{F}$ satisfies hypothesis M. Define for every $\lambda > 0$ the weight function

$$\omega_\lambda(x) = e^{-\lambda\sqrt{1+x^2}} \quad \text{for} \quad x \in \mathbb{R}.$$

Theorem 2.2 — *Let (u_1, v_1) be a subsolution and (u_2, v_2) a supersolution of Problem (P) with corresponding initial data (u_{01}, v_{01}) and (u_{02}, v_{02}). Then there exists a constant $C > 0$ that does not depend on λ such that*

$$\begin{aligned} &\int_{\mathbb{R}} \omega_\lambda\big([\beta(u_1(t)) - \beta(u_2(t))]_+ + [v_1(t) - v_2(t)]_+\big) \\ &\qquad \leq e^{C(\lambda+\lambda^2)t} \int_{\mathbb{R}} \omega_\lambda\big([\beta(u_{01}) - \beta(u_{02})]_+ + [v_{01} - v_{02}]_+\big) \qquad (2.7) \end{aligned}$$

for all $0 \leq t \leq T$.

Theorem 2.2 simultaneously provides three distinct results:

1. A *comparison principle:*

   if $u_{01} \geq u_{02}$ and $v_{01} \geq v_{02}$, then $u_1 \geq u_2$ and $v_1 \geq v_2$ on Q_T;

2. A *contraction in $L^1(\mathbb{R})$*: if $\beta(u_{01}) - \beta(u_{02}) \in L^1(\mathbb{R})$ and $v_{01} - v_{02} \in L^1(\mathbb{R})$ then

   $$\begin{aligned} &\|\beta(u_1(t)) - \beta(u_2(t))\|_{L^1(\mathbb{R})} + \|v_1(t) - v_2(t)\|_{L^1(\mathbb{R})} \\ &\qquad \leq \|\beta(u_{01}) - \beta(u_{02})\|_{L^1(\mathbb{R})} + \|v_{01} - v_{02}\|_{L^1(\mathbb{R})}, \qquad (2.8) \end{aligned}$$

   for all $0 \leq t \leq T$. This follows from the limit $\lambda \to 0$ in (2.7);

3. *Continuous dependence on initial data* in a weighted space $L^1_{\omega_\lambda}(\mathbb{R})$ induced by the norm

$$\|u\|_{L^1_{\omega_\lambda}(\mathbb{R})} = \int_{\mathbb{R}} \omega_\lambda \, |u|$$

of the form

$$\begin{aligned}&\|\beta(u_1(t)) - \beta(u_2(t))\|_{L^1_{\omega_\lambda}(\mathbb{R})} + \|v_1(t) - v_2(t)\|_{L^1_{\omega_\lambda}(\mathbb{R})} \\ &\quad \leq e^{C(\lambda+\lambda^2)t}\big(\|\beta(u_{01}) - \beta(u_{02})\|_{L^1_{\omega_\lambda}(\mathbb{R})} + \|v_{01} - v_{02}\|_{L^1_{\omega_\lambda}(\mathbb{R})}\big),\end{aligned}$$

for all $0 \leq t \leq T$.

While the contraction in $L^1(\mathbb{R})$ only has a sensible definition for initial data with difference in $L^1(\mathbb{R})^2$, the continuous dependence result in $L^1_{\omega_\lambda}(\mathbb{R})^2$ holds for all bounded initial data. We might as well mention a fourth property that we prove in Corollary 2.3:

4. *Conservation of mass:* if

$$\int_{\mathbb{R}} \big(|\beta(u_{01}) - \beta(u_{02})| + |v_{01} - v_{02}|\big) < \infty,$$

then

$$\int_{\mathbb{R}} \big(\beta(u_1) - \beta(u_2) + v_1 - v_2\big) \quad \text{is constant in time.} \tag{2.9}$$

Remark 2.4 In physical terms, the quantity $\beta(u) + v$ corresponds to the total amount of chemical contaminant that is present in a unit volume. Hence the interpretation ‘Conservation of mass’ for (2.9). Although properties 2 and 3 clearly are related to the physical mass $\beta(u)+v$, their exact interpretation is not obvious. Let us note that the contraction in L^1 will play an important role in the proof of convergence of general solutions to travelling waves (Chapter 3).

•

Proof of Theorem 2.2. We will apply a theorem due to Otto [Ott95a, Ott95b] that gives an estimate for single equations. Although strictly speaking the hypotheses of [Ott95a, Ott95b] exclude application to the situation at hand, it is easily verified in the proof that these restrictions are non-essential. We shall briefly comment on this.

We shall apply the theorem proved by Otto to the equation

$$\beta(u)_t + u_x - u_{xx} = g$$

on a domain $(a, b) \times (0, T)$ for some $a < b$, with non-homogeneous Dirichlet boundary data on $\{a, b\} \times (0, T)$. Clearly both u_1 and u_2 satisfy such an equation with different zero-order terms g and boundary data. Note that for u_i, $g = -\mathcal{F}(u_i, v_i) \in L^\infty((a, b) \times (0, T))$.

There are two main differences between our situation and the hypotheses of Otto's theorem. First, Otto disallows zero-order terms that depend explicitly on x and t. It is readily verified that the proof holds unchanged for zero-order terms that are in L^∞. Second, Otto demands Dirichlet data that are constant in time. For test functions with support inside $(a, b) \times [0, T]$ this is not essential either.

We now proceed with the proof. We set $\tilde{u} = u_1 - u_2$, $\tilde{v} = v_1 - v_2$, $\tilde{\beta} = \beta(u_1) - \beta(u_2)$, and $\tilde{\mathcal{F}} = \mathcal{F}(u_1, v_1) - \mathcal{F}(u_2, v_2)$. Let χ be any non-negative element of $C_c^\infty(\mathbb{R})$, and choose $a, b \in \mathbb{R}$ such that $\operatorname{supp} \chi \subset (a, b)$. Define H to be the Heaviside function with $H(0) = 0$. When we apply Otto's theorem to the solutions u_1 and u_2 with the test function $\gamma(x, \tau) = H(t - \tau)\chi(x)$ we obtain

$$\int_{\mathbb{R}} \chi \tilde{\beta}_+(t) - \int_{\mathbb{R}} \chi \tilde{\beta}_+(0) \le -\int_{Q_t} \chi_x H(\tilde{u})(\tilde{u}_x - \tilde{u}) - \int_{Q_t} \chi \tilde{\mathcal{F}} H(\tilde{u}).$$

Here $Q_t = \mathbb{R} \times (0, t]$. Now since

$$-\int_{Q_t} \chi_x H(\tilde{u})\tilde{u}_x = -\int_{Q_t} \chi_x (\tilde{u}_+)_x = \int_{Q_t} \chi_{xx} \tilde{u}_+,$$

we obtain

$$\int_{\mathbb{R}} \chi \tilde{\beta}_+(t) - \int_{\mathbb{R}} \chi \tilde{\beta}_+(0) \le \int_{Q_t} \tilde{u}_+(\chi_x + \chi_{xx}) - \int_{Q_t} \chi \tilde{\mathcal{F}} H(\tilde{u}).$$

Now let χ converge to the function ω_λ, for instance by setting $\chi = \omega_\lambda \eta_n$ where η_n is a cut-off function. By observing that there exists a constant $c > 0$ independent of λ such that

$$|w_{\lambda x}|, |w_{\lambda xx}| \le c(\lambda + \lambda^2)\omega_\lambda \quad \text{on} \quad \mathbb{R},$$

we find that

$$\int_{\mathbb{R}} \omega_\lambda \tilde{\beta}_+(t) - \int_{\mathbb{R}} \omega_\lambda \tilde{\beta}_+(0) \le 2c(\lambda + \lambda^2) \int_{Q_t} \omega_\lambda \tilde{u}_+ - \int_{Q_t} \omega_\lambda \tilde{\mathcal{F}} H(\tilde{u}).$$

Next we consider equation (2.3). When we multiply the difference of equations (2.3) for v_1 and v_2 by $H(\tilde{v})\omega_\lambda$ and integrate we obtain

$$\int_{\mathbb{R}} \omega_\lambda \tilde{v}_+(t) - \int_{\mathbb{R}} \omega_\lambda \tilde{v}_+(0) \leq \int_{Q_t} \omega_\lambda \tilde{\mathcal{F}} H(\tilde{v}),$$

and by adding this inequality to the previous one,

$$\int_{\mathbb{R}} \omega_\lambda (\tilde{\beta}_+(t) + \tilde{v}_+(t)) - \int_{\mathbb{R}} \omega_\lambda (\tilde{\beta}_+(0) + \tilde{v}_+(0)) \leq$$
$$\leq \int_{Q_t} \omega_\lambda \tilde{\mathcal{F}}(H(\tilde{v}) - H(\tilde{u})) + 2c(\lambda + \lambda^2) \int_{Q_t} \omega_\lambda \tilde{u}_+. \quad (2.10)$$

Set $h = H(\tilde{v}) - H(\tilde{u})$. Clearly h takes values in the set $\{-1, 0, 1\}$. At points $(x, t) \in Q_t$ where $h(x, t) = 1$ we have $v_1 > v_2$ and $u_1 \leq u_2$; therefore, by hypothesis M,

$$\mathcal{F}(u_1, v_1) \leq \mathcal{F}(u_2, v_2)$$

and $\tilde{\mathcal{F}} h \leq 0$. Similarly, where $h = -1$ we also have $\tilde{\mathcal{F}} h \leq 0$. This implies that the first term on the right-hand side in (2.10) is non-positive. Since $\beta' \geq b_0 > 0$, we can estimate the last term in (2.10) by

$$\frac{2c}{b_0}(\lambda + \lambda^2) \int_{Q_t} \omega_\lambda \tilde{\beta}_+.$$

Thus we can use Gronwall's Lemma to conclude that

$$\int_{\mathbb{R}} \omega_\lambda (\tilde{\beta}_+(t) + \tilde{v}_+(t)) \leq e^{C(\lambda+\lambda^2)t} \int_{\mathbb{R}} \omega_\lambda (\tilde{\beta}_+(0) + \tilde{v}_+(0)),$$

where $C = 2c/b_0$, as asserted. This concludes the proof of Theorem 2.2.

•

Corollary 2.3 (Conservation of mass) — *If*

$$\int_{\mathbb{R}} \big(|\beta(u_{01}) - \beta(u_{02})| + |v_{01} - v_{02}|\big) < \infty,$$

then the integral

$$\int_{\mathbb{R}} \big(\beta(u_1) - \beta(u_2) + v_1 - v_2\big)$$

is constant in time.

Proof. Subtract equations (2.5) for u_1, u_2 and a test function $\chi \in C_c^\infty(\mathbb{R})$:

$$\int_{\mathbb{R}} \chi\big(\beta(u_1(t)) - \beta(u_2(t)) + v_1(t) - v_2(t)\big)$$
$$= \int_{\mathbb{R}} \chi\big(\beta(u_{01}) - \beta(u_{02}) + v_{01} - v_{02}\big) + \int_0^t \int_{\mathbb{R}} (u_1 - u_2)(\chi_x + \chi_{xx}).$$

Since $\beta' \geq b_0 > 0$, the function $u_1 - u_2 \in L^1(\mathbb{R} \times (0, t))$; the result then follows from letting χ converge towards the function 1. •

For the proof of existence of a solution of the Cauchy Problem (P) (Theorem 2.7) we shall need a comparison theorem for the Cauchy-Dirichlet Problem on bounded sets. We state here the definition of a solution and the comparison theorem. The proof of this theorem is a slight perturbation of that of Theorem 2.2.

Let $\Omega \subset \mathbb{R}$ be a bounded open interval. We consider the problem

$$\beta(u)_t + v_t + u_x - u_{xx} = 0$$
$$v_t = \mathcal{F}(u, v)$$

on $Q_T^\Omega = \Omega \times (0, T]$ with the boundary condition

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, T]$$

and the initial condition

$$(u, v) = (u_0, v_0)$$

at $t = 0$. We call this Problem (CD).

Definition 2.4 — *A solution of the Cauchy-Dirichlet Problem (CD) is a pair of functions (u, v) such that*

1. $u \in C(Q_T^\Omega) \cap L^2(0, T; H^1(\Omega))$;
2. $v \in C([0, T]; L^\infty(\Omega))$;
3. *u and v satisfy the equations*

$$-\int_{Q_T^\Omega} \big((\beta(u) + v)\varphi_t + u(\varphi_{xx} + \varphi_x)\big) = \int_\Omega (\beta(u_0) + v_0)\varphi(0) \quad (2.11)$$

and

$$-\int_{Q_T^\Omega} v\varphi_t = \int_\Omega v_0\varphi(0) + \int_{Q_T^\Omega} \mathcal{F}(u, v)\varphi \tag{2.12}$$

for every $\varphi \in C^{2,1}(\overline{Q_T^\Omega})$ *such that* $\varphi(T) = 0$ *and* $\varphi = 0$ *on* $\partial\Omega \times (0, T]$, *and the boundary condition*

$$u = 0 \quad on \quad \partial\Omega \times (0, T]. \tag{2.13}$$

Sub- and supersolutions of Problem (CD) *satisfy parts 1 and 2, as well as (2.11), (2.12), and (2.13), with the equality signs replaced by* $\leq$ *for subsolutions and* $\geq$ *for supersolutions.*

Note that we do not incorporate the boundary condition (2.13) into the function space in part 1. This allows us to consider sub- and supersolutions with non-zero boundary values.

We recall that we assume hypothesis M to be satisfied.

Theorem 2.5 — *Let* (u_1, v_1) *be a subsolution and* (u_2, v_2) *a supersolution of Problem* (CD) *with corresponding initial data* (u_{01}, v_{01}) *and* (u_{02}, v_{02}). *Then*

$$\int_\Omega \big([\beta(u_1(t)) - \beta(u_2(t))]_+ + [v_1(t) - v_2(t)]_+\big) \leq \int_\Omega \big(\beta(u_{01}) - \beta(u_{02})]_+ + [v_{01} - v_{02}]_+\big) \tag{2.14}$$

for all $0 \leq t \leq T$.

Proof. The proof follows the same lines as the proof of Theorem 2.2. A non-obvious difference is that we apply the theorem of Otto to the test function $\gamma(x, \tau) = H(t - \tau)$, a function that is constant in space. The restriction to test functions with compact support does not hold for the case of zero Dirichlet data. •

2.3 Uniqueness for non-monotone $\mathcal{F}$

For completeness we mention here a generalisation of Theorem 2.2 to the case of non-monotone rate functions $\mathcal{F}$. Under hypothesis OSL a comparison principle does not hold in general—as shown by examples in the next section—but the continuous dependence in $L^1_{\omega_\lambda}$ and the contraction in L^1 still hold in weakened form:

Theorem 2.6 — *Let hypothesis* OSL *be satisfied and let* (u_1, v_1) *be a subsolution and* (u_2, v_2) *a supersolution of Problem* (P) *with corresponding initial data* (u_{01}, v_{01}) *and* (u_{02}, v_{02}). *Then*

$$\int_{\mathbb{R}} \omega_\lambda \big(|\beta(u_1(t)) - \beta(u_2(t))| + |v_1(t) - v_2(t)| \big)$$
$$\leq e^{(L+C(\lambda+\lambda^2))t} \int_{\mathbb{R}} \omega_\lambda \big(|\beta(u_{01}) - \beta(u_{02})| + |v_{01} - v_{02}| \big) \quad (2.15)$$

for all $0 \leq t \leq T$. *Here* $L = 2\max\{L_u/b_0, L_v\}$.

Proof. We enter the arguments of the proof of Theorem 2.2 at inequality (2.10). Under hypothesis M the first term on the right-hand side is non-positive and can be discarded. Under hypothesis OSL this term yields a positive contribution and can be estimated by

$$\int_{Q_t} \omega_\lambda \big(L_u \, |\tilde{u}| + L_v \, |\tilde{v}| \big),$$

where s_- is defined as $[-s]_+$. By interchanging (u_1, v_1) and (u_2, v_2) and adding we obtain

$$\int_{\mathbb{R}} \omega_\lambda \big(|\tilde{\beta}(t)| + |\tilde{v}(t)| \big) - \int_{\mathbb{R}} \omega_\lambda \big(|\tilde{\beta}(0)| + |\tilde{v}(0)| \big)$$
$$\leq 2\max(L_u/b_0, L_v) \int_{Q_t} \omega_\lambda \big(|\tilde{\beta}| + |\tilde{v}| \big) + 2c(\lambda + \lambda^2) \int_{Q_t} \omega_\lambda \, |\tilde{u}| \, .$$

Estimate (2.15) follows by applying Gronwall's Lemma to this inequality.

•

2.4 Counterexamples

To motivate the hypotheses that we have set on $\mathcal{F}$ in order to prove uniqueness and a comparison principle, we briefly discuss a number of counterexamples. See also the book by Pao [Pao92], section 1.6.

As an example to show the necessity of M for a comparison principle, consider the problem

$$\begin{aligned} &u_t + v_t - u_{xx} = 0 && 0 < x < 1, t > 0 \\ &v_t = -u - v && 0 < x < 1, t > 0 \\ &u(0,t) = u(1,t) = 0 && t > 0 \\ &u(x,0) = v(x,0) = 0 && 0 < x < 1. \end{aligned} \tag{2.16}$$

Clearly $u_1 \equiv v_1 \equiv 0$ is a solution to this problem. If we construct a second solution (u_2, v_2) by solving this problem with initial data $v_2(\cdot, 0) \equiv 0$ and $u_2(\cdot, 0) > 0$ then by equation (2.16) solution v_2 instantly becomes negative everywhere on $(0, 1)$. This shows that a comparison principle can not hold for this system. This idea can be extended to show that for (2.1) for smooth rate functions $\mathcal{F}$ a comparison principle can not hold if $\mathcal{F}$ violates the monotonicity assumption M at any value of (u, v).

When M is relaxed to OSL, the comparison principle is lost but uniqueness still holds. When the rate function $\mathcal{F}$ also violates the one-sided Lipschitz conditions then non-uniqueness can occur.

This is well known in the case of ordinary differential equations: if we seek a solution y of the problem

$$y' = y^p, \quad \text{with} \quad y(0) = 0, \tag{2.17}$$

then for $p \geq 1$ there is a unique solution $y \equiv 0$. For $0 < p < 1$, however, there is an additional solution

$$y(t) = \big((1-p)t\big)^{1/(1-p)},$$

and since $y'(0) = 0$ the functions

$$y_c(t) = \begin{cases} 0 & 0 \leq t \leq c \\ y(t-c) & t > c \end{cases}$$

for any $c > 0$ are also solutions of problem (2.17). In this case, the loss of Lipschitz continuity of the nonlinearity clearly results in non-uniqueness.

For a second example we extend these ideas to partial differential equations. Consider

$$u_t = u_{xx} + u^p \qquad t > 0,\ x \in \Omega \subset \mathbb{R} \tag{2.18}$$
$$u(x,0) = 0 \qquad x \in \Omega$$

Here we write $u^p = |u|^{p-1}u$.

On $\Omega = \mathbb{R}$ we can immediately identify the solution $u(x,t) \equiv y(t)$. We can also find other non-zero solutions, for instance the self-similar solution

$$u(x,t) = t^{1/(1-p)} f(\eta), \qquad \eta = \frac{x}{\sqrt{t}},$$

where f satisfies the equation

$$-f'' + \frac{1}{1-p} f - \frac{1}{2}\eta f' = f^p \quad \text{on} \quad \mathbb{R}.$$

We are free to choose $f(0)$ and $f'(0)$, and here we choose $f(0) = 0$ and $f'(0)$ positive but not too large. The resulting function f is plotted in Figure 2.1.

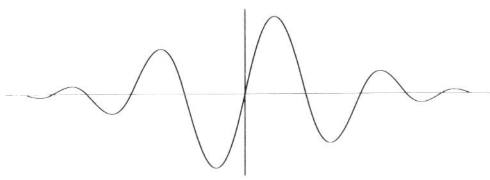


Figure 2.1: The function f.

We use this self-similar solution to construct a non-zero solution to (2.18) on the bounded domain $\Omega = (0,1)$ with homogeneous Dirichlet boundary data $u(0,t) = u(1,t) = 0$. Denote the first positive zero of f by η_1. If we define the function

$$\underline{u}(x,t) = \begin{cases} t^{1/(1-p)} f\left(\dfrac{x}{\sqrt{t}}\right) & \text{for} \quad 0 \le x \le \eta_1\sqrt{t} \\ 0 & \text{for} \quad \eta_1\sqrt{t} < x \le 1 \end{cases}$$

then for $0 \le t \le \eta_1^{-2}$ the function $\underline{u}$ is a subsolution (in distributional sense) of equation (2.18). By applying the technique of sub- and supersolutions to $\underline{u}$ and y we prove that there is at least one solution u such that $\underline{u} \le u \le y$ on $[0,1] \times [0, \eta_1^{-2}]$. Clearly u can not be identically equal to zero and this establishes non-uniqueness on a bounded domain.

2.5 Existence for monotone $\mathcal{F}$

Theorem 2.7 — *Suppose $\mathcal{F}$ satisfies condition* M *and let* $(u_0, v_0) \in L^\infty(\mathbb{R})^2$, $u_0, v_0 \geq 0$. *Then there exists a solution* (u, v) *of Problem* (P) *as defined in Definition 2.1.*

Proof. Step 1: approximation on a bounded domain. Define for $n \in \mathbb{N}$ the spatial domain $\Omega_n = (-n, n)$ and set $Q_n = \Omega_n \times (0, T]$. We consider the approximate problem (P_n)

$$\begin{aligned}
&\beta(u)_t + v_t + u_x - u_{xx} = 0 && x \in \Omega_n,\ 0 < t \leq T && (2.19)\\
&v_t = \mathcal{F}(u, v) && x \in \Omega_n,\ 0 < t \leq T && (2.20)\\
&u = 0 && x \in \{-n, n\},\ 0 < t \leq T &&\\
&(u, v) = (u, v) && \text{at} \quad t = 0. &&
\end{aligned}$$

Solutions to this problem are defined as in Definition 2.4.

We first introduce an a priori bound. Define the functions $\bar{u}, \bar{v} : [0, T] \to \mathbb{R}$ by

$$\begin{aligned}
\beta(\bar{u})' &= -\mathcal{F}(\bar{u}, \bar{v}), \qquad & \bar{u}(0) = \bar{u}_0 = \max_{\mathbb{R}} u_0,\\
\bar{v}' &= \mathcal{F}(\bar{u}, \bar{v}), \qquad & \bar{v}(0) = \bar{v}_0 = \max_{\mathbb{R}} v_0.
\end{aligned}$$

Since $(\beta(\bar{u})+\bar{v})' = 0$, both $\beta(\bar{u}(t))$ and $\bar{v}(t)$ are uniformly bounded by $\beta(\bar{u}_0)+\bar{v}_0$. From the lower bound on β' in B_1 it follows that there exists $M > 0$ such that $\bar{u}(t) \leq M$ and $\bar{v}(t) \leq M$ for all $0 \leq t \leq T$.

We prove the existence of a solution to Problem (P_n) by the Schauder fixed point theorem. Define the convex set

$$X = \{u \in L^2(Q_n) : 0 \leq u(x, t) \leq \bar{u}(t) \text{ on } Q_n\}.$$

We introduce the operator $\mathcal{T}_1 : X \to L^2(Q_n)$, where $\mathcal{T}_1 u$ is defined as the solution $\hat{v}$ of the ordinary differential equation

$$\hat{v}_t = \mathcal{F}(u, \hat{v}) \quad \text{on} \quad Q_n \tag{2.21}$$

with initial condition $\hat{v}(\cdot, 0) = v_0$. By hypothesis M the function $\bar{v}$ is an upper bound for $\hat{v}$: if for any $x \in Q_n$, $0 \leq t \leq T$ we have $\hat{v}(x, t) = \bar{v}$, then

$$\hat{v}_t(x, t) = \mathcal{F}(u(x, t), \bar{v}) \leq \mathcal{F}_n(\bar{u}, \bar{v}) = 0.$$

The possibility that $\hat{v}_t(x,t) = 0$ is ruled out by local uniqueness since $\mathcal{F}$ is one-sided Lipschitz continuous in its second argument. By a similar argument the constant 0 is a lower bound for $\hat{v}$. As a result, $\mathcal{T}_1 u$ belongs to the set

$$Y = \{v \in L^2(Q_n) : 0 \le v(x,t) \le \bar{v}(t) \text{ on } Q_n\}.$$

The operator $\mathcal{T}_2 : Y \to L^2(Q_n)$ is defined in the following way: $\hat{u} = \mathcal{T}_2 v$ is the solution of

$$\beta(\hat{u})_t + \hat{u}_x - \hat{u}_{xx} = -\mathcal{F}(\hat{u}, v) \quad \text{on} \quad Q_n, \tag{2.22}$$

subject to the boundary condition $\hat{u} = 0$ and the initial condition $\hat{u} = u_0$. The functions $u = 0$ and $u = \bar{u}$ are sub- and supersolutions: for $\bar{u}$ this follows from

$$\beta(\bar{u})_t = -\mathcal{F}(\bar{u}, \bar{v}) \ge -\mathcal{F}(\bar{u}, v)$$

since $v \le \bar{v}$ on Q_n, and for $u = 0$ the argument is similar. As a result the composite operator $\mathcal{T} = \mathcal{T}_2 \circ \mathcal{T}_1$ maps X into X.

Next we prove that the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ are continuous in the L^2-norm. For the length of this proof, let $\|\cdot\|$ denote the norm of $L^2(Q_n)$. We first consider $\mathcal{T}_1$. Let $u^m, u \in X$, $\|u^m - u\| \to 0$ as $m \to \infty$, and set $v^m = \mathcal{T}_1 u^m$, $v = \mathcal{T}_1 u$. We need to prove that $\|v^m - v\| \to 0$. By multiplying the difference of equations (2.21) with $w^m = v^m - v$ we find

$$\sup_{0\le t\le T} \frac{1}{2} \int_{\Omega_n} (w^m)^2 \le \sup_{0\le t\le T} \int_0^t \int_{\Omega_n} \big(\mathcal{F}(u^m, v^m) - \mathcal{F}(u^m, v)\big) w^m + \\ \big\|\mathcal{F}(u^m, v) - \mathcal{F}(u, v)\big\| \, \big\|w^m\big\| . \tag{2.23}$$

The first term on the right-hand side is non-positive by hypothesis M and we find the estimate

$$\big\|w^m\big\| \le C \big\|\mathcal{F}(u^m, v) - \mathcal{F}(u, v)\big\| .$$

Since $\mathcal{F}$ is continuous, there is a subsequence along which $\mathcal{F}(u^m, v) - \mathcal{F}(u, v) \to 0$ almost everywhere in Q_n, and consequently $\|w^m\| \to 0$ along that sequence. By the uniqueness of the limit the whole sequence converges to zero.

For the continuity of $\mathcal{T}_2$ we take analogously a sequence $v^m, v \in Y$, such that $\|v^m - v\| \to 0$, and define in the same way $u^m = \mathcal{T}_2 v^m$ and $u = \mathcal{T}_2 v$, and $z^m = u^m - u$. By the results of Sacks [Sac83] the sequence u^m is uniformly continuous and we can therefore extract a subsequence along which u^m and v^m converge pointwise almost everywhere in Q_n. We can then pass to the limit in the equation

$$-\int_0^T \int_{\Omega_n} \big(\beta(u^m)\varphi_t + u^m(\varphi_x + \varphi_{xx})\big) = \int_{\Omega_n} \beta(u_0)\varphi(0) + \int_0^T \int_{\Omega_n} \mathcal{F}(u^m, v^m)\varphi$$

for every $\varphi \in C_c^\infty([0, T] \times \Omega_n)$. The function $u = \mathcal{T}_2 v$ is the unique solution of the limit equation which proves that the pointwise limit of u^m is equal to u. Again the uniqueness of the limit implies that the whole sequence converges.

The uniform continuity of u^m also implies that $\mathcal{T}$ is compact. It then follows by the Schauder fixed point theorem that there exists a $u \in X$ such that $\mathcal{T}u = u$, implying that the pair (u, v) with $v = \mathcal{T}_1 u$ solves Problem (P_n).

Step 2: the limit $n \to \infty$. We denote the solutions of (P_n) obtained in this way by (u_n, v_n) and we let n tend to infinity. By the comparison principle on bounded domains (Theorem 2.5) u_n and v_n form increasing sequences. We pass to the limit in equations (2.2) and (2.3).

The regularity assumptions of Definition (2.1) are satisfied for v by the uniform boundedness of $v_t = \mathcal{F}(u, v)$ and for u by the uniform continuity of the functions u^m due to Sacks and Remark 2.1. •

Remark 2.5 If the problem is semilinear, i.e. in the case of equations (2.1) if β is linear, then it is possible to construct an operator for which one seeks a fixed point by inverting the principal part. The resulting operator is compact and one finds an existence result for small time [CL95]. •

Appendix 2.A Uniqueness proof for equation (2.4)

For sake of completeness we give here a uniqueness proof for the equation

$$\beta(u)_t + u_x - u_{xx} = g \tag{2.4}$$

on Q_T under weak conditions on the regularity of the solution. We assume $g \in L^\infty(Q_T)$ and the initial data $u_0 \in L^\infty(\mathbb{R})$ to be given. We consider a solution to be a function $u \in L^\infty(Q_T)$ such that

$$-\int_{Q_T} \big(\beta(u)\varphi_t + u(\varphi_x + \varphi_{xx})\big) = \int_{\mathbb{R}} \beta(u_0)\varphi(0) + \int_{Q_T} g\varphi \tag{2.24}$$

for every $\varphi \in C_c^{2,1}(\overline{Q_T})$ such that $\varphi(0) = 0$.

Theorem 2.8 — *Let u_1 and u_2 be two solutions of the problem described above with identical initial data and right-hand side. Then $u_1 \equiv u_2$.*

Note that the uniqueness theorem proved by Otto demands that $u_x \in L^2(Q_T)$. Theorem 2.8 demonstrates that this regularity is automatic for solutions defined in the weaker sense of (2.24).

Proof. Set $\tilde{u} = u_1 - u_2$. For a given test function φ we have

$$\int_{Q_T} \tilde{u}(B\varphi_t + \varphi_x + \varphi_{xx}) = 0. \tag{2.25}$$

Here B is defined by

$$B = \begin{cases} \dfrac{\beta(u_1) - \beta(u_2)}{u_1 - u_2} & \text{if} \quad u_1 \neq u_2 \\ b_0 & \text{otherwise.} \end{cases}$$

Note that since $\beta' \geq b_0$ by B_1, this definition implies that $B \geq b_0 > 0$ a.e. on Q_T. We remark here that since $\tilde{u} \in L^\infty(Q_T)$ and $\tilde{u}B \in L^\infty(Q_T)$, by approximation any function $\varphi \in C^{2,1}(\overline{Q_T})$ is admissible in (2.25) if $\varphi(T) = 0$ and $\varphi_x, \varphi_{xx}, \varphi_t \in L^1(Q_T)$.

Let B_n be a smooth and bounded approximation of B such that $B \geq B_n \geq b_0$ and $B_n \to B$ a.e. on Q_T. Choose $\chi \in C_c^\infty(Q_T)$, and define the sequence of test functions ζ_n by

$$\begin{aligned} &B_n\zeta_{nt} + \zeta_{nx} + \zeta_{nxx} = \chi && \text{on} \quad Q_T \\ &\zeta_n(T) = 0 && \text{on} \quad \mathbb{R}. \end{aligned} \tag{2.26}$$

Lemma 2.9 — *The functions ζ_n have the following properties:*

1. *There exists a constant M_1 independent of n such that*

$$|\zeta_n| \leq M_1(|x| + 1)e^{-|x|} \quad \textit{for all} \quad (x,t) \in Q_T;$$

2. *There exist constants M_n depending on n such that*

$$|\zeta_{nt}|,\ |\zeta_{nx}|,\ |\zeta_{nxx}| \leq M_n(|x| + 1)e^{-|x|} \quad \textit{for all} \quad (x,t) \in Q_T;$$

3. *There exists a constant M_2 independent of n such that*

$$\int_{Q_T} B_n\zeta_{nt}^2 \leq M_2.$$

4. *The integral*

$$\int_0^T \int_{|x|>r} \sqrt{B_n}\, |\zeta_{nt}|$$

tends to zero as r tends to infinity, uniformly in n.

We conclude the proof of the Theorem and give the proof of Lemma 2.9 afterwards. Take $\varphi = \zeta_n$ in equation (2.25):

$$\begin{aligned}
\left|\int_{Q_T} \tilde u \chi\right| &= \left|\int_{Q_T} \tilde u (B - B_n)\zeta_{nt}\right| \\
&\le \int_{Q_T} \left|\tilde u \frac{B - B_n}{\sqrt{B_n}}\right| \sqrt{B_n}\, |\zeta_{nt}| \\
&\le \frac{1}{\sqrt{b_0}} \|\tilde u B\|_{L^\infty(Q_T)} \int_0^T \int_{|x|>r} \sqrt{B_n}\, |\zeta_{nt}| \\
&\qquad + \left(\int_0^T \int_{|x|\le r} \tilde u^2 \frac{(B - B_n)^2}{B_n}\right)^{1/2} \left\|\sqrt{B_n}\zeta_{nt}\right\|_{L^2(Q_T)}.
\end{aligned}$$

The first term is small uniformly in n when $r \to \infty$ by part 4 of this Lemma, and for fixed r the second term tends to zero as $n \to \infty$ by the pointwise convergence of B_n to B. Consequently

$$\int_{Q_T} \tilde u \chi = 0$$

for every $\chi \in C_c^\infty(Q_T)$. •

Proof of Lemma 2.9. First note that by comparing ζ_n with functions that are constant in space it follows that ζ_n is bounded by a constant that only depends on b_0, χ, and T.

Part 1 follows from the observation that if $R > 0$ is such that supp $\chi \subset (-R, R)$, then the function $\zeta(x) = M_1 e^{-x}$ is a solution for equation (2.26) on $x > R$, and the function $\zeta(x) = -M_1 x e^x$ a supersolution on $x < -R$ (if $R > 3/2$). By choosing M_1 such that ζ and ζ_n are ordered at $x = \pm R$, the result follows.

Part 2 follows directly from part 1 by the classical Bernšteĭn estimates. The dependence on n of the constant M_n arises from the non-boundedness of B_n and its derivatives.

Part 3 is also more or less standard. By multiplying with ζ_{nt} and integrating by parts we find

$$\int_t^T \int_{\mathbb{R}} B_n \zeta_{nt}^2 + \tfrac{1}{2} \int_{\mathbb{R}} \zeta_{nx}^2(t) = \int_t^T \int_{\mathbb{R}} \chi \zeta_{nt} - \int_t^T \int_{\mathbb{R}} \zeta_{nx} \zeta_{nt}$$

and therefore

$$\begin{aligned}
&\left\|\sqrt{B_n}\zeta_{nt}\right\|^2_{L^2(Q_T)} + \sup_{0\le t\le T} \tfrac{1}{2} \|\zeta_{nx}(t)\|^2_{L^2(\mathbb{R})} \\
&\qquad\qquad \le C \|\zeta_{nt}\|_{L^2(Q_T)} + \frac{b_0}{2} \|\zeta_{nt}\|^2_{L^2(Q_T)} + \frac{1}{2b_0} \|\zeta_{nx}\|^2_{L^2(Q_T)}
\end{aligned}$$

Since $B_n \geq b_0$ and

$$\|\zeta_{nx}\|^2_{L^2(Q_T)} \leq T \sup_{0\leq t\leq T} \|\zeta_{nx}(t)\|^2_{L^2(\mathbb{R})},$$

the boundedness of $\|\sqrt{B_n}\zeta_{nt}\|_{L^2(Q_T)}$ and $\|\zeta_{nx}\|_{L^2(Q_T)}$ follows if $T \leq b_0/2$. If this is not the case, we divide Q_T into time slices and repeat the procedure.

Part 4 seems to be the novel part of this Lemma. Its proof breaks up into three steps.

An estimate of ζ_{nx} in $L^1(Q_T)$. By comparing ζ_n with a translated copy of itself it follows that $\zeta_{nx}(x,t) \leq 0$ for $x \geq R$. Therefore

$$\int_0^T \int_r^\infty |\zeta_{nx}| \leq T \sup_{0\leq t\leq T} \zeta_n(r,t) \leq T M_1 r e^{-r} \quad \text{for} \quad r > R$$

by part 1. A similar statement holds for $r < -R$.

An estimate of ζ_{nx} in $L^p(Q_T)$. Pick $p \in (2,\infty)$ and multiply equation (2.26) by $|\zeta_{nx}|^{p-2}\zeta_{nt}$:

$$\int_{Q_T} B_n |\zeta_{nx}|^{p-2} \zeta_{nt}^2 + \frac{1}{p(p-1)} \sup_{0\leq t\leq T} \int_{\mathbb{R}} |\zeta_{nx}(t)|^p$$
$$\leq \frac{b_0}{2} \int_{Q_T} |\zeta_{nx}|^{p-2} \zeta_{nt}^2 + \frac{1}{b_0} \int_{Q_T} |\zeta_{nx}|^p + C\left(\int_{Q_T} |\zeta_{nx}|^p\right)^{(p-2)/p}$$

where the constant C does not depend on n. By a similar reasoning with T as for part 3 we show the norm $\|\zeta_{nx}\|_{L^p(Q_T)}$ to be bounded independently of n.

A local estimate of ζ_{nt} in L^2. Pick a cut-off function $\eta \in C_c^\infty(\mathbb{R})$ which satisfies $\eta(x) \equiv 1$ for $|x| \leq 1$ and $\eta(x) \equiv 0$ for $|x| \geq 2$. Choose $r > R+2$. If we repeat the same argument as under part 3, now by multiplying with $\eta^2(x-r)\zeta_{nt}(x,t)$ instead of just ζ_{nt}, we find,

$$\int_t^T \int_{\mathbb{R}} B_n \eta^2 \zeta_{nt}^2 + \tfrac{1}{2} \int_{\mathbb{R}} \eta^2 \zeta_{nx}^2(t) = -2 \int_t^T \int_{\mathbb{R}} \zeta_{nx}\zeta_{nt}\eta_x\eta - \int_t^T \int_{\mathbb{R}} \eta^2 \zeta_{nx}\zeta_{nt}$$

and therefore

$$\int_{Q_T} B_n \eta^2 \zeta_{nt}^2 \leq C \int_{Q_T} \eta^2 \zeta_{nx}^2$$

where C does not depend on r or n. From the bounds on ζ_{nx} in L^1 and L^p we find by interpolation

$$\|\zeta_{nx}\|_{L^2((r,\infty)\times(0,T))} \leq C r^\theta e^{-\theta r},$$

where $\theta = (p-2)/(2p-2) \in (0,1)$. Therefore

$$\int_0^T \int_{r-1}^{r+1} \sqrt{B_n}\,|\zeta_{nt}| \leq \sqrt{2}\left(\int_0^T \int_{r-1}^{r+1} B_n \zeta_{nt}^2\right)^{1/2} \leq C r^\theta e^{-\theta r}.$$

This proves the result. •

Chapter 3

Convergence to travelling waves

In this chapter we prove the convergence of general solutions of the system

$$\beta(u)_t + v_t + u_x - u_{xx} = 0 \tag{3.1a}$$

$$v_t = \mathcal{F}(u, v) \tag{3.1b}$$

to travelling wave solutions. Collecting all necessary regularity assumptions (those needed to ensure the existence and uniqueness of a solution of this system, and the hypotheses of Theorem 1.3), we assume that β and $\mathcal{F}$ satisfy

$$\beta \in C([0, \infty)) \cap C^2((0, \infty)) \tag{3.2}$$

$$\mathcal{F} \in C([0, \infty) \times [0, \infty)) \cap \mathrm{Lip}((0, \infty) \times (0, \infty)) \tag{3.3}$$

$$\mathcal{F}(0, 0) = 0, \quad \mathcal{F}_u \geq 0, \quad \text{and} \quad \mathcal{F}_v \leq 0 \quad \text{a.e. on} \quad (0, \infty) \times (0, \infty) \tag{3.4}$$

We make use of the results of Van Duijn and Knabner [DK91, DK92b] on travelling wave solutions of (3.1). The theorem that these authors prove supposes that associated to $\mathcal{F}$ there is an isotherm ψ:

> There exists a strictly increasing function $\psi \in C([0, \infty))$ such that
>
> $$\mathcal{F}(u, v) \gtrless 0 \iff \psi(u) \gtrless v \tag{3.5}$$
>
> for all $u, v \geq 0$.

Note that by conditions (3.3)-(3.4), a function $\psi \in C([0, \infty))$ satisfying (3.5) always exists; the new element is the strict monotonicity.

The most important condition for the existence of a travelling wave is

$$\chi(s) > \frac{\chi(u^*)}{u^*} s \quad \text{for all} \quad 0 \leq s \leq u^*,$$

where the total isotherm χ is given by $\chi(s) = \beta(s) + \psi(s)$. We commented on the interpretation of this condition on page 19.

This chapter has been submitted as D. Hilhorst and M. A. Peletier, *Convergence to Travelling Waves in a Reaction-Diffusion System Arising in Contaminant Transport.*

Theorem 3.1 ([DK91, DK92b]) — *Under the given hypotheses there exists a travelling wave solution* $(U, V)(x, t) = (f, g)(x - ct)$ *of system (3.1) with limits* $(0, 0)$ *and* (u^*, v^*) *at plus and minus infinity. The functions* f *and* g *are strictly decreasing.*

Remark 3.1 A travelling wave (f, g) satisfies the system of ordinary differential equations

$$-c\beta(f)' - cg' + f' - f'' = 0$$
$$-cg' = \mathcal{F}(f, g).$$

If β and $\mathcal{F}$ satisfy the regularity assumptions (3.2) and (3.3) above, then $f, g \in C^1(\mathbb{R})$. We use this regularity in the proof of Theorem 3.2. •

We prove the following convergence result:

Theorem 3.2 — *Suppose that* $\mathcal{F}$ *is strictly monotone, i.e. the inequalities in (3.4) can be replaced by strict inequalities almost everywhere. Let* $(u_0, v_0) \in L^\infty(\mathbb{R})^2$ *satisfy*

$$0 \le u_0 \le u^* \quad \textit{and} \quad 0 \le v_0 \le v^*,$$

and

$$\int_{\mathbb{R}} \big(|\beta(u_0) - \beta(f)| + |v_0 - g|\big) < \infty, \tag{3.6}$$

where $(U, V)(x, t) = (f, g)(x - ct)$ *is the travelling wave given by Theorem 3.1. Let* (u, v) *be the corresponding solution of (3.1). Then there exists a translation of* (U, V)*, again denoted* (U, V)*, such that*

$$\int_{\mathbb{R}} \big(|\beta(u(t)) - \beta(U(t))| + |v(t) - V(t)|\big) \to 0$$

as $t \to \infty$.

The proof is based on the ideas of Osher and Ralston [OR82] who proved a similar result for a single convection-diffusion equation (see also [BH86, HH91, COR93, Zha93]). The main difference with the previous articles lies in the generalisation from an equation to a system.

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3.1 Construction of a semigroup

By Theorems 2.7 and 2.2 we can define a semigroup operator that maps the initial data (u_0, v_0) to the corresponding solution $(u(t), v(t))$. For future convenience we consider pairs of the form $(\beta(u_0), v_0)$ instead of (u_0, v_0) and define $S(t)(\beta(u_0), v_0)$ as the solution $(\beta(u(t)), v(t))$ corresponding to the initial data $(\beta(u_0), v_0)$. The domain of definition of the semigroup $S(t)$ is the set

$$\mathcal{L} = \left\{z_0 = (\beta(u_0), v_0) \in L^\infty(\mathbb{R})^2 : u_0, v_0 \geq 0\right\}.$$

We introduce the norm on $L^1(\mathbb{R})^2$ for elements $z = (\beta(u), v) \in \mathcal{L}$:

$$\|z\|_{L^1(\mathbb{R})^2} = \int_{\mathbb{R}} \big(|\beta(u)| + |v|\big),$$

and a partial ordering $\geq$ on $\mathcal{L}$:

$$z_1 \geq z_2 \quad \text{if} \quad u_1 \geq u_2 \text{ and } v_1 \geq v_2 \text{ on } \mathbb{R}.$$

Using this notation the semigroup $S(t)$ has the following properties:

1. $S(t)$ preserves L^1, i.e.

$$\|S(t)z_1 - S(t)z_2\|_{L^1(\mathbb{R})^2} \leq \|z_1 - z_2\|_{L^1(\mathbb{R})^2} \tag{3.7}$$

   for each $t > 0$ and $z_1, z_2 \in \mathcal{L}$ such that $z_1 - z_2 \in L^1(\mathbb{R})^2$.

2. $S(t)$ preserves order, i.e. if $z_1, z_2 \in \mathcal{L}$ and $z_1 \geq z_2$, then $S(t)z_1 \geq S(t)z_2$ for all $t > 0$.

These properties are simply reformulations of Theorem 2.2. As a by-product of the proof of Theorem 2.2 we obtain the following extra information:

Proposition 3.3 — *For $z_1, z_2 \in \mathcal{L}$, $z_1 - z_2 \in L^1(\mathbb{R})^2$, set $(\beta(u_i(t)), v_i(t)) = S(t)z_i$. If the set*

$$\{(x, \tau) \in \mathbb{R} \times (0, t) : (u_1 - u_2)(v_1 - v_2)(x, \tau) < 0\}$$

has non-zero measure, then

$$\|S(t)z_1 - S(t)z_2\|_{L^1(\mathbb{R})^2} < \|z_1 - z_2\|_{L^1(\mathbb{R})^2}.$$

Proof. The proof follows from an inspection of inequality (2.10). By Theorem 2.2 the integral $\int_{Q_t} \tilde{u}_+$ is finite, and therefore we can set $\lambda = 0$ in (2.10). This leads to

$$\int_{\mathbb{R}} \big(\tilde{\beta}_+(t) + \tilde{v}_+(t)\big) - \int_{\mathbb{R}} \big(\tilde{\beta}_+(0) + \tilde{v}_+(0)\big)$$
$$\leq \int_{Q_t} \big(\mathcal{F}(u_1, v_1) - \mathcal{F}(u_2, v_2)\big)\big(H(\tilde{v}) - H(\tilde{u})\big).$$

By hypothesis (3.4) the integral on the right-hand side is strictly negative. •

On a region $\Omega \times (0, T)$ on which a solution (u, v) is bounded away from zero, assumptions (3.2) and (3.3) imply that equation (3.1a) is uniformly parabolic, and therefore by classical regularity theory u_t and u_{xx} are functions and equation (3.1a) is satisfied almost everywhere. We use this to prove a Strong Comparison Principle:

Proposition 3.4 — *Let (u_1, v_1) and (u_2, v_2) both satisfy equations (3.1a) and (3.1b) on a domain $\Omega \times (0, T)$ and let u_1 and u_2 be bounded away from zero. Suppose that $|u_{2t}|$ is bounded on $\Omega \times (0, T)$. If $u_1 \geq u_2$ and $v_1 \geq v_2$, then*

$$\text{either} \quad u_1 \equiv u_2 \quad \text{or} \quad u_1 > u_2$$

on $\Omega \times (0, T)$.

Note that we do not explicitly impose any regularity on u_{1t} or $\beta(u_1)_t$, other than follows from the uniform parabolicity.

Proof. The strict positiveness of u_1 and u_2 and (3.2) imply that β'' is bounded on the values of u_1 and u_2. Then we have after setting $w = u_1 - u_2$,

$$\beta'(u_1)w_t - w_{xx} + w_x + cw = -\mathcal{F}(u_1, v_1) + \mathcal{F}(u_2, v_2).$$

where

$$c = \frac{\beta'(u_1) - \beta'(u_2)}{u_1 - u_2} u_{2t}$$

is a bounded function of x and t. Using the monotonicity of $\mathcal{F}$ in v we have

$$-\mathcal{F}(u_1, v_1) + \mathcal{F}(u_2, v_2) \geq -\frac{\mathcal{F}(u_1, v_1) - \mathcal{F}(u_2, v_1)}{u_1 - u_2} w,$$

and the result then follows from standard parabolic theory. •

3.2 Proof of Theorem 3.2

By the hypothesis

$$\int_{\mathbb{R}} \big(|\beta(u_0) - \beta(f)| + |v_0 - g|\big) < \infty, \tag{3.8}$$

we can assume without loss of generality that

$$\int_{\mathbb{R}} \big(\beta(u_0) - \beta(f) + v_0 - g\big) = 0. \tag{3.9}$$

This follows from the observation that if a locally integrable function $\varphi : \mathbb{R} \to \mathbb{R}$ has finite limits at plus and minus infinity, then

$$\begin{aligned} \int_{\mathbb{R}} \big(\varphi(x+h) - \varphi(x)\big)\, dx &= \lim_{R\to\infty} \left\{ \int_R^{R+h} \varphi - \int_{-R}^{-R+h} \varphi \right\} \\ &= (\varphi(\infty) - \varphi(-\infty))h. \end{aligned}$$

Condition (3.9) pins down the travelling wave that (u, v) will converge to. We define the translation operator τ_h, for $h \in \mathbb{R}$, by

$$(\tau_h y)(x) = y(x-h) \quad \text{for all} \quad x \in \mathbb{R}$$

for any function y on $\mathbb{R}$.

In order to compare the general solution of Problem (1.60) with the travelling wave solution we introduce a change of variables: we set $\eta = x - ct$ where c is the wave speed of the travelling wave (U, V) and consider the solution as a function of (η, t) instead of (x, t). This amounts to considering instead of $S(t)$ the semigroup

$$\Sigma(t) = \tau_{-ct} \circ S(t).$$

The properties of $S(t)$ discussed above are passed unchanged to $\Sigma(t)$, and by construction $\zeta = (\beta(f), g)$ is a fixed point of $\Sigma(t)$.

First we prove stability in a special case:

Proposition 3.5 — *Suppose that z_0 lies between two travelling waves, i.e. there exist $h_1, h_2 \in \mathbb{R}$ such that*

$$\tau_{h_1}\zeta \le z_0 \le \tau_{h_2}\zeta.$$

Then $\|z(t) - \zeta\|_{L^1(\mathbb{R})^2} \to 0$ as $t \to \infty$.

Proof. The proof consists of six steps. For the length of this proof set $z(t) = \Sigma(t)z_0$.

Step 1 *The set* $\{z(t) - \zeta\}_{t>0}$ *is compact in* $L^1(\mathbb{R})^2$.

By (3.7),

$$\|\tau_h z(t) - z(t)\|_{L^1(\mathbb{R})^2} \leq \|\tau_h z_0 - z_0\|_{L^1(\mathbb{R})^2}, \qquad (3.10)$$

for all $h \in \mathbb{R}$. This implies that $\|\tau_h z(t) - z(t)\|_{L^1(\mathbb{R})^2} \to 0$ uniformly in t as $h \to 0$. By the comparison principle the fact that initially the solution z lies between travelling waves implies that the same holds for all positive time t. Therefore $\beta(u) - \beta(f)$ and $v - g$ have tails at plus and minus infinity that are integrable uniformly in t. We can then apply for instance Corollary IV.26 of [Bre83] or Theorem IV.8.20 of [DS58] to conclude that $\{\beta(u(t)) - \beta(f)\}_{t>0}$ and $\{v(t) - g\}_{t>0}$ both are compact in $L^1(\mathbb{R})$.

Let the ω-limit set ω be defined as

$$\omega = \{y \in \zeta + L^1(\mathbb{R})^2 : \exists t_n \to \infty, z(t_n) \to y \text{ in } L^1(\mathbb{R})^2\}.$$

By Step 1, $\omega \neq \emptyset$.

Remark 3.2 By the results of Sacks [Sac83] solutions of (3.1a) have a modulus of continuity in space and time that does not depend on t for large t. Consequently if $y = (\beta(y_u), y_v) \in \omega$, then y_u is necessarily uniformly continuous in space, and the first component of $\Sigma(t)y$ is uniformly continuous in space and time. The second component of $\Sigma(t)y$ is Lipschitz continuous in time by equation (3.1b). Finally, it follows from Corollary 2.3 that

$$\int_{\mathbb{R}} (\beta(y_u) + y_v) = 0. \qquad (3.11)$$

•

We introduce a class of Lyapunov functionals V_h, for $h \in \mathbb{R}$,

$$V_h(y) = \|y - \tau_h \zeta\|_{L^1(\mathbb{R})^2} \quad \text{for all} \quad y \in \zeta + L^1(\mathbb{R})^2.$$

Step 2 *The functional* V_h *is constant on* ω.

This is a classical result in the theory of dynamical systems. By (3.7) the functional V_h decreases along the trajectory $z(t)$. If $y_1, y_2 \in \omega$, then we can find a sequence $t_n \to \infty$ such that $z(t_{2n}) - y_1 \to 0$ and $z(t_{2n+1}) - y_2 \to 0$ in $L^1(\mathbb{R})^2$. It follows that $V_h(y_1) = V_h(y_2)$.

Step 3 *If $y \in \omega$, then $\Sigma(t)y \in \omega$ for all $t > 0$.*
Again this is classical: if the sequence $t_n \to \infty$ is such that $z(t_n) - y \to 0$, then

$$\begin{aligned}\|z(t_n + t) - \Sigma(t)y\|_{L^1(\mathbb{R})^2} &= \|\Sigma(t)z(t_n) - \Sigma(t)y\|_{L^1(\mathbb{R})^2} \\ &\le \|z(t_n) - y\|_{L^1(\mathbb{R})^2}\end{aligned}$$

and this last term tends to zero by hypothesis.
Step 4 *If $y \in \omega$, $V_0(y) > 0$, then there exists $h \in \mathbb{R}$ such that*

$$V_h(\Sigma(t)y) < V_h(y) \quad \textit{for all time} \quad t > 0.$$

Set $y = (\beta(u), v)$. When we compare u with $\tau_h f$ for different values of h, there are two possibilities:

1. There exists $h \in \mathbb{R}$ such that $u \equiv \tau_h f$ on $\mathbb{R}$;
2. There exists $h \in \mathbb{R}$ and $I = [\eta_1, \eta_2] \subset \mathbb{R}$ such that $u - \tau_h f$ is of two signs on I and $u, f > 0$ on I.

In case 1 the possibility $v \equiv \tau_h g$ implies $h = 0$ by the mass restriction (3.11). Therefore $V_0(y) = 0$ which is ruled out by hypothesis. From $v \not\equiv \tau_h g$ it follows that by choosing h close to 0 we can obtain that the set

$$\{\eta \in \mathbb{R} : (u - \tau_h f)(v - \tau_h g) < 0\}$$

has non-zero measure. We then conclude by the continuity in time and Proposition 3.3.

In case 2 we obtain the result by the strong maximum principle. Denote by $\beta(u(t))$ and $v(t)$ the first and second components of $\Sigma(t)y$, so that $(\beta(u(0)), v(0)) = y$. We define the initial conditions (on $\mathbb{R}$)

$$\check{u}_0 = \max\{u(0), \tau_h f\} \quad \text{and} \quad \check{v}_0 = \max\{v(0), \tau_h g\},$$

and we set

$$\check{u} = \max\{u(t), \tau_h f\} \quad \text{and} \quad \check{v} = \max\{v(t), \tau_h g\}.$$

Let $(\beta(\bar{u}(t)), \bar{v}(t)) = \Sigma(t)(\beta(\check{u}_0), \check{v}_0)$. By the comparison principle we have that $u, \tau_h f \le \bar{u}$ and $v, \tau_h g \le \bar{v}$. Since by construction $\tau_h f \not\equiv \check{u}_0$ on I, we deduce from the strong comparison principle (Proposition 3.4) that $u, \tau_h f < \bar{u}$

on $I \times (0, 1]$. Here we use the fact that f is continuously differentiable so that the travelling wave has a bounded time derivative. Then

$$\check{u}(t) \leq \bar{u}(t) \quad \text{and} \quad \check{u}(t) \not\equiv \bar{u}(t) \quad \text{on} \quad I \times (0, 1].$$

If we construct analogous functions $\hat{u}$, $\underline{u}$, $\hat{v}$, and $\underline{v}$, where max is replaced by min, then we have for $0 < t < 1$,

$$\begin{aligned}
&\int_{\mathbb{R}} \Big(|\beta(u(t)) - \beta(\tau_h f)| + |v(t) - \tau_h g| \Big) \\
&\qquad = \int_{\mathbb{R}} \big(\beta(\check{u}(t)) - \beta(\hat{u}(t)) + \check{v}(t) - \hat{v}(t)\big) \\
&\qquad < \int_{\mathbb{R}} \big(\beta(\bar{u}(t)) - \beta(\underline{u}(t)) + \bar{v}(t) - \underline{v}(t)\big) \\
&\qquad \leq \int_{\mathbb{R}} \big(\beta(\check{u}_0) - \beta(\hat{u}_0) + \check{v}_0 - \hat{v}_0\big) \\
&\qquad = \int_{\mathbb{R}} \big(|\beta(u(0)) - \beta(\tau_h f)| + |v(0) - \tau_h g|\big).
\end{aligned}$$

This implies that $V_h(\Sigma(t)y) < V_h(y)$ for all $t > 0$.

Step 5 *Conclusion.*

We combine these building blocks in the following manner: by Step 1, the ω-limit set ω is not empty. By Step 3, it consists of trajectories, and by Step 2 every functional V_h is constant along these trajectories. By Step 4, this implies that $V_0(\omega) = 0$. Therefore $\omega = \{\zeta\}$. •

Theorem 3.2 is a simple consequence of Proposition 3.5 and the property of contraction in L^1:

Proof of Theorem 3.2. By (3.8) we can approximate z_0 by functions z_{0n} such that z_{0n} lies between two travelling waves, $\|z_{0n} - z_0\|_{L^1(\mathbb{R})^2} \leq 1/n$, and

$$\int_{\mathbb{R}} \big(\beta(u_{0n}) - \beta(u_0) + v_{0n} - v_0\big) = 0.$$

By applying Proposition 3.5 to the sequence of functions $z_n(t) = \Sigma(t)z_{0n}$ it follows that $\|z_n(t) - \zeta\|_{L^1(\mathbb{R})^2} \to 0$ as $t \to \infty$ for all n. Then

$$\begin{aligned}
\|z(t) - \zeta\|_{L^1(\mathbb{R})^2} &\leq \|z(t) - z_n(t)\|_{L^1(\mathbb{R})^2} + \|z_n(t) - \zeta\|_{L^1(\mathbb{R})^2} \\
&\leq \|z_0 - z_{0n}\|_{L^1(\mathbb{R})^2} + \|z_n(t) - \zeta\|_{L^1(\mathbb{R})^2},
\end{aligned}$$

and by choosing n and t large enough these two norms can be made as small as necessary. •

Chapter 4

Well injection: stability of self-similar solutions

4.1 Introduction

In this chapter we consider the asymptotic behaviour as $t \to \infty$ and as $\varepsilon \downarrow 0$ of radial solutions of the equation

$$\beta(u)_t + \operatorname{div}\left(u\frac{\lambda}{|x|}\boldsymbol{e}_r - \nabla u\right) = 0 \qquad \text{for } (x,t) \in \Omega_\varepsilon \times \mathbb{R}^+, \tag{4.1}$$

satisfying the boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\lambda}{\varepsilon}(u_e - u) \qquad \text{for } (x,t) \in \partial\Omega_\varepsilon \times \mathbb{R}^+. \tag{4.2}$$

The set Ω_ε is the outer domain $\{x \in \mathbb{R}^2 : |x| > \varepsilon\}$, $\boldsymbol{e}_r$ is the unit vector in the radial direction, $\lambda > 0$ and u_e are given constants, and $\beta : [0,\infty) \to [0,\infty)$ is a function to be specified later.

Since we only consider radial solutions, we seek a function $u = u(r,t)$ that satisfies

$$\beta(u)_t + \frac{\lambda - 1}{r}u_r - u_{rr} = 0 \qquad \varepsilon < r < \infty,\ t > 0 \tag{4.3a}$$

$$u_r = \frac{\lambda}{\varepsilon}(u - u_e) \qquad r = \varepsilon,\ t > 0 \tag{4.3b}$$

$$u(r,0) = u_0(r) \qquad r > \varepsilon, \tag{4.3c}$$

where $u_0 : [\varepsilon,\infty) \to [0,\infty)$ denotes the radially symmetric initial distribution. Without loss of generality we may consider the cases

$$\text{Contamination process:} \qquad u_e = 1,\ u_0(\infty) = 0, \tag{4.4}$$

$$\text{Remediation process:} \qquad u_e = 0,\ u_0(\infty) = 1, \tag{4.5}$$

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where we suppose that $u_0(\infty) := \lim_{r\to\infty} u_0(r)$ exists. Furthermore we suppose that u_0 satisfies

A_1 $u_0 \in C^{0,1}([\varepsilon, \infty))$, $0 \le u_0 \le 1$, and $ru_0'(r)$ is uniformly bounded on (ε, ∞).

About the function β we assume (cf. [DK92b]):

B_1 $\beta \in C^3(0, \infty) \cap C([0, \infty))$;

B_2 $\beta(0) = 0$, $\beta'(s) > 0$ and $\beta''(s) \le 0$ for $s > 0$.

A Cauchy-Dirichlet problem for equation (4.3a) with $\lambda > 1$ was studied by Goncerzewicz [Gon92], generalising results by Gilding [Gil89] and Díaz and Kersner [DK87a] who considered general convection-diffusion equations in $\mathbb{R}^1$. Following these authors we introduce weak solutions in the following sense. Let T be some fixed positive number, which eventually will tend to infinity, and consider the half strip $S_T^\varepsilon = \{(r, t) : \varepsilon < r < \infty,\ 0 < t < T\}$.

Definition 4.1 — *A non-negative function $u : \overline{S_T^\varepsilon} \to \mathbb{R}$ is called a weak solution of Problem (4.3) if*

1. *$u \in C(\overline{S_T^\varepsilon})$ and u has a bounded weak derivative u_r in S_T^ε;*
2. *for every test function $\varphi \in H^1(S_T^\varepsilon)$ that vanishes for large r and at $t = T$,*

$$\int_{S_T^\varepsilon} \left\{\beta(u)\varphi_t r + (\lambda u - r u_r)\varphi_r\right\} dr dt + \int_\varepsilon^\infty \beta(u_0(r))\varphi(r, 0)\, r dr + \lambda u_e \int_0^T \varphi(\varepsilon, t)\, dt = 0. \quad (4.6)$$

If u satisfies (4.6) with the equality replaced by $\ge$ ($\le$) and with $\varphi \ge 0$ in S_T^ε then we call u a sub(super)solution.

Hypotheses B_1-B_2 and A_1 ensure the existence of a unique weak solution u which is smooth in the set $\{(r, t) \in S_T^\varepsilon : u(r, t) > 0\}$. This is proved in Section 4.2.

Remark 4.1 Observe that when (4.3a) is interpreted as a convection-diffusion equation in $\mathbb{R}^1$, the sign of $\lambda - 1$ determines the direction of the convection: when $\lambda < 1$ it is directed towards the origin, and when $\lambda > 1$ away from the origin. This distinction will turn out to be important when studying the asymptotic behaviour as $\varepsilon \downarrow 0$. •

Our aim is to show that under certain conditions, solutions of Problem (4.3) converge to self-similar solutions when either $\varepsilon \downarrow 0$ or $t \to \infty$. The combination of these two limit processes is explained by the following transformation:

$$\xi = \frac{r}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon^2}$$

under which Problem (4.3) becomes

$$\beta(u)_\tau + \frac{\lambda - 1}{\xi} u_\xi - u_{\xi\xi} = 0 \qquad \xi > 1,\ \tau > 0 \qquad (4.7a)$$

$$u_\xi = \lambda(u - u_e) \qquad \xi = 1,\ \tau > 0 \qquad (4.7b)$$

$$u(\xi, 0) = u_0(\varepsilon\xi) \qquad \xi > 1. \qquad (4.7c)$$

Obviously the behaviour of solutions of Problem (4.7) for large τ is strongly linked to that of solutions of Problem (4.3) for either $\varepsilon \downarrow 0$ or $t \to \infty$.

A scaling argument leads us to investigate self-similar solutions of equation (4.3a) of the form

$$u(r, t) = f\left(\frac{r}{\sqrt{t}}\right) = f(\eta),$$

satisfying the equation

$$\tfrac{1}{2}\eta^2\{\beta(f)\}' + (\eta f' - \lambda f)' = 0, \qquad (4.8)$$

where primes denote differentiation with respect to η. Since these self-similar solutions are expected to arise in the limit $\varepsilon \downarrow 0$, we solve equation (4.8) in the domain $0 < \eta < \infty$ with the combinations (4.4) and (4.2) as boundary conditions:

$$\text{Contamination process:} \quad f(0) = 1,\ f(\infty) = 0, \qquad (4.9)$$

$$\text{Remediation process:} \quad f(0) = 0,\ f(\infty) = 1. \qquad (4.10)$$

Note that $\eta = r/\sqrt{t} = \xi/\sqrt{\tau}$, and therefore the self-similar solution satisfies both equation (4.3a) and equation (4.7a).

The boundary value problems (4.8)-(4.9) and (4.8)-(4.10) are studied in Section 4.3. In Section 4.4 we prove the main results of this paper. They concern the asymptotic behaviour of weak solutions of Problem I. We shall need an additional hypothesis on u_0 and β in order to prove these results:

A_2 $$\left| \int_\varepsilon^\infty r\big\{\beta(u_0(r)) - \beta(u_0(\infty))\big\}\, dr \right| < \infty;$$

Hypothesis A_2 can be interpreted physically as stating that the perturbation $u_0 - u_0(\infty)$ of the constant state $u_0(\infty)$ has finite mass. We will show in Section 4.2 that A_2 implies that

$$\left| \int_{\varepsilon}^{\infty} r\bigl\{\beta(u(r,t)) - \beta(u_0(\infty))\bigr\}\, dr \right| < \infty$$

for all $t > 0$.

The double degeneration of (4.8) with (4.10)—the degeneration of $\beta(f)$ at $f = 0$ and the degeneration of the equation at $\eta = 0$, which coincide—forces us to assume a technical hypothesis in order to prove the result for the remediation process:

B_3 $\quad \lim\limits_{s\downarrow 0} \dfrac{s\beta''(s)}{\beta'(s)} = p - 1$ where $0 < p \leq 1$.

Note that in the case of a Freundlich isotherm the condition B_3 is satisfied with $0 < p < 1$, and in the case of a Langmuir isotherm with $p = 1$.

The precise asymptotic statements are:

Theorem 4.2 — *Let hypotheses* B_1-B_2 *and* A_1-A_2 *be satisfied. Further let* u *be the solution of Problem (4.3) with* $u_e = 1$ *and* $u_0(\infty) = 0$ *(contamination process), and let* f *denote the solution of (4.8) and (4.9).*

1. *If* ε *is fixed, then*

$$\sup_{\varepsilon \leq r < \infty} |u(r,t) - f(r/\sqrt{t})| = O(t^{-k}) \quad as \quad t \to \infty;$$

2. *If* $u_0 \equiv 0$*, then for fixed* $t > 0$

$$\sup_{\varepsilon \leq r < \infty} |u(r,t) - f(r/\sqrt{t})| = O\Bigl(\frac{\varepsilon^2}{t}\Bigr)^k, \quad as \quad \varepsilon \to 0.$$

Here the exponent k *is given by*

$$k = \begin{cases} \lambda/3 & for\ \lambda < 1, \\ 1/3 & for\ \lambda \geq 1. \end{cases}$$

Theorem 4.3 — *Let hypotheses* B_1-B_3 *and* A_1-A_2 *be satisfied. Further let* u *be the solution of Problem (4.3) with* $u_e = 0$ *and* $u_0(\infty) = 1$ *(remediation process), and let* f *denote the solution of (4.8) and (4.10). Then the conclusions are the same as those of Theorem 4.2 (with* $u_0 \equiv 1$ *in part (b)).*

Remark 4.2 The restriction to constant u_0 when ε is varied is a natural one. Since the influence of changes in ε on the solution is small at a fixed time and away from the well, it is necessary for convergence to self-similar solutions that the initial behaviour of the general solution corresponds to the initial behaviour of the self-similar solution. In practical terms, this means u_0 has to be constant. Observe that when u_0 is constant, the two limit processes $\varepsilon \downarrow 0$ and $t \to \infty$ are truly equivalent. •

Remark 4.3 As a by-product of the proof of Theorems 4.2 and 4.3 we obtain a pointwise estimate of u. In the contamination case the self-similar solution is a subsolution for the general solution, which implies the following inequality:

$$0 \leq 1 - u(r,t) \leq 1 - f(r/\sqrt{t}) \quad \text{for all} \quad r > \varepsilon,\ t > 0.$$

The behaviour of $1 - f(\eta)$ near $\eta = 0$ is shown to be proportional to η^λ (Proposition 4.14), and therefore for fixed $r > \varepsilon$

$$1 - u(r,t) = O(t^{-\lambda/2}) \quad \text{as} \quad t \to \infty.$$

In the same way an estimate follows for the remediation case:

$$0 \leq u(r,t) \leq f(r/\sqrt{t}) \quad \text{for all} \quad r > \varepsilon,\ t > 0.$$

Here the behaviour of f (Proposition 4.16) translates in a similar way to the behaviour of $u(r,t)$ for fixed r as t tends to infinity. •

4.2 Weak solutions: existence and uniqueness

We present here the existence and uniqueness results for weak solutions of Problem I. Most of these results are obtained by a straightforward generalisation of the work of Díaz and Kersner [DK87a], Gilding [Gil89], and Goncerzewicz [Gon92]. In those cases we omit the details and only give the appropriate references. However, special attention has to be given to the flux boundary condition at $r = \varepsilon$.

As is usual we obtain weak solutions as limits of approximating positive classical solutions. Since these approximations are used later on in this paper to prove the asymptotic results, we describe the procedure in some detail in the existence proof below.

Theorem 4.4 — *Let hypotheses* B_1-B_2 *and* A_1 *be satisfied. Then Problem (4.3) has a unique weak solution.*

Proof. To show existence, we slightly alter the initial and boundary conditions in Problem I, ensuring that the corresponding solution remains strictly positive. This is achieved by considering approximating sequences $\{u_{0n}\}_{n\geq 1}$ and $\{u_{en}\}_{n\geq 1}$, satisfying

$$u_{0n} \in C^{\infty}([\varepsilon, n]);$$

$$u_{0n} \downarrow u_0 \quad \text{as } n \to \infty, \text{ uniformly on bounded subsets of } [\varepsilon, \infty);$$

$$\frac{1}{n} \leq u_{0n} \leq 1 \quad \text{on} \quad [\varepsilon, n]; \tag{4.11}$$

$$\sup_{\varepsilon\leq r\leq n} |ru_{0n}{}'(r)| \leq \sup_{\varepsilon\leq r<\infty} |ru_0'(r)|; \tag{4.12}$$

$$u_{0n}(r) = \delta_n \quad \text{for } n-1 \leq r \leq n; \tag{4.13}$$

$$u_{0n}{}'(\varepsilon) = 0,$$

and

$$u_{en}(t) = u_e - (u_e - u_{0n}(\varepsilon))e^{-nt} \quad \text{for} \quad 0 \leq t \leq T,$$

where the constants δ_n are chosen in $[1/n, 1]$. Note that the compatibility condition

$$u_{en}(0) = u_{0n}(\varepsilon)$$

is satisfied. In Problem (4.3) we now replace u_0 by u_{0n} and u_e by u_{en}. This yields the approximate problems

$$\beta(u)_t + \frac{\lambda - 1}{r}u_r - u_{rr} = 0 \qquad \text{in } S_T^{\varepsilon,n} = (\varepsilon, n) \times (0, T], \tag{4.14a}$$

$$u_r = \frac{\lambda}{\varepsilon}(u - u_{en}(t)) \qquad \text{at } r = \varepsilon,\ t \in (0, T], \tag{4.14b}$$

$$u = \delta_n \qquad \text{at } r = n,\ t \in (0, T], \tag{4.14c}$$

$$u = u_{0n} \qquad \text{at } t = 0,\ r \in [\varepsilon, n], \tag{4.14d}$$

for $n \geq 1$. Existence and uniqueness of solutions to this problem are classical and can be found in e.g. [LSU68], page 491. The solutions obtained have the regularity $C^{\infty}(S_T^{\varepsilon,n}) \cap C^{2+\alpha,1+\alpha/2}(\overline{S_T^{\varepsilon,n}})$.

In order to obtain an estimate on the spatial derivative of the solutions u_n, we derive an equation for the flux

$$F_n = \lambda u_n - r u_{nr}.$$

The functions u_n satisfy the equation

$$\beta'(u_n)u_{nt} + \tfrac{1}{r}F_{nr} = 0. \qquad (4.15)$$

Differentiating this equation with respect to r yields for F_n the uniformly parabolic equation

$$\beta'(u_n)F_{nt} = F_{nrr} - F_{nr}\left\{\frac{\lambda+1}{r} + \frac{\beta''(u_n)}{\beta'(u_n)}u_{nr}\right\} \quad \text{in} \quad S_T^{\varepsilon,n}.$$

Using (4.15) once more we find the boundary conditions

$$\begin{aligned} F_n &= \lambda u_{en}, & r &= \varepsilon; \\ F_{nr} &= 0, & r &= n. \end{aligned}$$

Hypothesis A_1 and properties (4.11) and (4.12) of the functions u_{0n} imply that F_n is bounded uniformly in n at $t = 0$. By the maximum principle, the same then holds for F_n on the whole of $\overline{S_T^{\varepsilon,n}}$. Therefore

$$\sup_{\overline{S_T^{\varepsilon,n}}} |u_{nr}| \le L \qquad (4.16)$$

for some $L > 0$ that is independent of n.

Next we investigate the regularity in time. We first consider the behaviour of u_n at the boundary $r = \varepsilon$.

Lemma 4.5 — *There exists a positive constant c independent of n such that*

$$|u_n(\varepsilon, t_2) - u_n(\varepsilon, t_1)| \le c|t_2 - t_1|^{\frac{1}{2}}$$

for all $0 \le t_1 \le t_2 \le T$.

Proof. We shall only prove the inequality $u_n(\varepsilon, t_2) - u_n(\varepsilon, t_1) \ge -c(t_2 - t_1)^{1/2}$ for $t_2 \ge t_1$; the opposite inequality follows along the same lines.

We first consider an auxiliary problem: find $z : [0,\infty) \times [0,\infty) \to \mathbb{R}$ that satisfies

$$z_t = z_{xx} \quad \text{for all} \quad (x,t) \in (0,\infty) \times (0,\infty),$$

along with initial condition $z(\cdot, 0) = 0$ and boundary condition $z_x(0, \cdot) = 1$. This problem has a unique solution which is of the form

$$z(x, t) = \sqrt{t}\, f\Big(\frac{x}{\sqrt{t}}\Big).$$

It is not difficult to verify that f is negative on $[0, \infty)$, has a finite limit $f(0)$, and satisfies $f'' < 0$ on $(0, \infty)$.

We now construct a comparison function for equation (4.14a) that is based on the function z. For $0 < b \leq \min\{\beta'(s) : 0 \leq s \leq 1\}$ to be chosen later, define

$$v(r, t) = u_n(\varepsilon, t_1) + z\Big(b^{-1/2}(r - \varepsilon), t - t_1\Big) - L(r - \varepsilon) - m(t - t_1),$$

on the set $\varepsilon \leq r \leq n$, $t_1 \leq t \leq T$, where $m = |\lambda - 1| \Big(b^{-1/2} + L\Big)/(b\varepsilon)$. It then follows that

$$\begin{aligned} \beta(v)_t - v_{rr} + \frac{\lambda - 1}{r}\, v_r &= \beta'(v) \left\{ v_t - \frac{1}{\beta'(v)} v_{rr} + \frac{\lambda - 1}{r\beta'(v)}\, v_r \right\} \\ &\leq \beta'(v) \left\{ v_t - \frac{1}{b}\, v_{rr} + \frac{|\lambda - 1|}{b\varepsilon}\, |v_r| \right\} \\ &\leq 0, \end{aligned} \tag{4.17}$$

where we have used the fact that $v_{rr} < 0$ in the first inequality. In (4.17) we have changed the nonlinearity β outside the range of u_n such that $\beta(v)$ is well-defined and $0 < b < \beta'(s) < \infty$ for all $s \in \mathbb{R}$. This is necessary because v may not be positive everywhere on its domain.

We prove that $u_n \geq v$, which implies the assertion. It follows from (4.17) that the minimum of $u_n - v$ is assumed on the parabolic boundary of the set $\{\varepsilon < r < n,\ t_1 < t \leq T\}$. The bound (4.16) ensures that $u_n \geq v$ at $t = t_1$, and since $u_n(n, t) = 1 \geq u_n(\varepsilon, t_1)$ the same holds on the right boundary $\{r = n,\ t_1 < t \leq T\}$. Therefore a negative minimum of $u_n - v$ can only be assumed on the boundary $r = \varepsilon$, where

$$\begin{aligned} (u_n - v)_r &= \frac{\lambda}{\varepsilon}(u_n - u_{en}(t)) - b^{-1/2} z_r + L \\ &\leq \frac{\lambda}{\varepsilon} + L - b^{-1/2}. \end{aligned}$$

Choosing b sufficiently small we therefore obtain $(u_n - v)_r < 0$ on the boundary $r = \varepsilon$, and conclude that $u_n \geq v$ on $\{\varepsilon \leq r \leq n,\ t_1 \leq t \leq T\}$. •

The regularity result of Gilding [Gil76] then yields that

$$||u_n||_{C^{0+1,0+1/2}(\overline{S_T^{\varepsilon,n}})} \leq C \quad \text{for all } n.$$

This suffices to apply the Arzelà-Ascoli Theorem and conclude that there exists a subsequence that converges uniformly on compact subsets of $\overline{S_T^{\varepsilon}}$. By a familiar argument (see e.g. Oleinik [Ole63], p. 361) the limit function u can be shown to be a weak solution of Problem (4.3). This concludes the proof of existence. •

The uniqueness follows directly from a comparison principle:

Proposition 4.6 (Comparison Principle) — *Let u^1 be a subsolution and u^2 be a supersolution of Problem (4.3), with initial values u_0^1 and u_0^2, and boundary conditions at $r = \varepsilon$:*

$$u_r^1 \geq \frac{\lambda}{\varepsilon}(u^1 - u_e^1) \quad \textit{and} \quad u_r^2 \leq \frac{\lambda}{\varepsilon}(u^2 - u_e^2).$$

If $u_0^1 \leq u_0^2$ on $[\varepsilon, \infty)$ and $u_e^1 \leq u_e^2$, then $u^1 \leq u^2$ on S_T^{ε}.

The proof of Proposition 4.6 is a simple extension of the proof in Goncerzewicz [Gon], and follows the ideas of Díaz and Kersner [DK87a]. This completes the proof of Theorem 4.4 •

We conclude this section with a property of solutions of Problem (4.3) that is crucial for the large-time behaviour.

Proposition 4.7 (Mass Conservation) — *Let u be a solution of Problem (4.3). Then*

$$\int_{\varepsilon}^{\infty} \{\beta(u(r,t)) - \beta(u_0(\infty))\}\, r dr = \int_{\varepsilon}^{\infty} \{\beta(u_0(r)) - \beta(u_0(\infty))\}\, r dr + \lambda t (u_e - u_0(\infty)).$$

This can be interpreted as stating that the only increase of 'mass'—in the case of the model described in the introduction, this would be mass of contaminant—comes from the injection at the boundary. The proof of this statement is analogous to the proof of mass conservation for the porous media equation (1.6) [Gil77].

4.3 Limit profiles

In order to obtain solutions of (4.8) subject to boundary conditions (4.9), (4.10) we consider the slightly more general problem

$$\mathrm{P}(a,b)\begin{cases}\frac{1}{2}\eta^2\{\beta(f)\}' + (\eta f' - \lambda f)' = 0, & 0 < \eta < \infty \\ f(0) = a, \quad f(\infty) = b, \end{cases} \tag{4.18}$$

for any $a, b \in [0, 1]$. We first prove existence and uniqueness of solutions of $\mathrm{P}(a, b)$ and then enter more deeply into the specific cases $\mathrm{P}(0, 1)$ and $\mathrm{P}(1, 0)$. Some of the proofs will only be sketched; the reader can find comprehensive and detailed studies of Problem $\mathrm{P}(1, 0)$ in [DK94] and of Problem $\mathrm{P}(0, 1)$ in [Pel93].

Existence and uniqueness

Because of the possible degeneration of the equation when $f = 0$, we must again define the notion of a solution of this problem. For convenience we set

$$F(\eta) = \eta f'(\eta) - \lambda f(\eta), \quad \eta > 0.$$

Definition 4.8 — *Let a, $b \in [0, 1]$. A function $f : [0, \infty) \to [0, 1]$ is called a solution of Problem $\mathrm{P}(a, b)$ if F and $\beta(f)$ are locally absolutely continuous on $[0, \infty)$, and satisfy*

$$F' + \frac{1}{2}\eta^2\{\beta(f)\}' = 0 \quad \text{a.e. on } (0, \infty);$$

and

$$f(0) = a \quad \text{and} \quad f(\infty) = b.$$

We can directly deduce from this definition

Proposition 4.9 — *Let f be a solution of Problem $\mathrm{P}(a, b)$ and let $\mathcal{P}$ be the positivity set $\{\eta > 0 : f(\eta) > 0\}$. Then*

1. *$f \in C^1((0, \infty)) \cap C^\infty(\mathcal{P})$;*
2. *f is monotone, and $f' \neq 0$ on $\mathcal{P}$ unless $a = b$;*
3. *$F(\eta) \to -\lambda b$ as $\eta \to \infty$.*

Proof. Parts 1 and 3 are proven in [DK94], and part 2 follows from a local uniqueness argument as in [AP71, AP74]. •

About the positivity set $\mathcal{P}$ we remark that

- if $a = 0$ and $b > 0$, then $\mathcal{P} = (0, \infty)$ [Pel93];
- if $a > 0$ and $b = 0$, then we distinguish two cases: if $1/\beta(s)$ is integrable at $s = 0$, then $\mathcal{P} = [0, L)$ for some $L > 0$; otherwise $\mathcal{P} = [0, \infty)$ [DK94].

When $\mathcal{P}$ is unbounded, we have an a priori estimate of the rate of convergence at infinity:

Proposition 4.10 — *Let f be a solution of Problem $P(a, b)$. Then there exist positive constants η_0, C, and K, such that*

$$|b - f(\eta)| \le C \int_{\frac{1}{\lambda}\eta^\lambda}^{\infty} e^{-K\sigma^{2/\lambda}} d\sigma \tag{4.19}$$

for all $\eta > \eta_0$.

The proof is given in [Pel93] and uses a lower bound of $\beta'(s)$ near $s = b$. Note that (4.19) implies that

$$\begin{aligned} |b - f(\eta)| &\le C \int_{\frac{1}{\lambda}\eta^\lambda}^{\infty} e^{-K\sigma^{2/\lambda}} d\sigma \\ &\le C \int_{\frac{1}{\lambda}\eta^\lambda}^{\infty} \frac{2K}{\lambda} \sigma^{2/\lambda - 1} e^{-K\sigma^{2/\lambda}} d\sigma \\ &= Ce^{-K\lambda^{-2/\lambda}\eta^2}, \end{aligned}$$

if η is large enough.

We have the following comparison principle.

Proposition 4.11 (Comparison Principle) — *Let f_i, for $i = 1, 2$, be solutions of $P(a_i, b_i)$ with $a_i, b_i \in [0, 1]$. If $a_1 \le a_2$ and $b_1 \le b_2$ then $f_1 \le f_2$ on $[0, \infty)$.*

Proof. Denote the positivity sets of the functions f_1 and f_2 by $\mathcal{P}_1$ and $\mathcal{P}_2$. Suppose that the difference $v = f_1 - f_2$ assumes a positive maximum at $\eta_0 \in (0, \infty)$. Then $f_1(\eta_0) > 0$ so that $\eta_0 \in \mathcal{P}_1$, which implies that f_1 is twice continuously differentiable in η_0.

- If $\eta_0 \in \mathcal{P}_2$ then f_2 also is twice differentiable in η_0, and then the result follows from subtracting the equations for f_1 and f_2 at $\eta = \eta_0$.

- If $\eta_0 \in \mathbb{R}^+ \setminus \mathcal{P}_2$ then $f_2'(\eta_0) = 0$, which implies $f_1'(\eta_0) = 0$. From Proposition 4.9 it follows that this only is possible when $a_1 = b_1$ and f_1 is constant on $\mathcal{P}_1$. Because $f_1(\eta_0) > 0$ we have $a_1 = b_1 > 0$, and the boundary conditions then imply that f_2 is not monotone. This contradicts Proposition 4.9.

•

Corollary 4.12 — *For every $a, b \in [0, 1]$, Problem $P(a, b)$ has at most one solution.*

Proposition 4.13 — *For every $a, b \in [0, 1]$, Problem $P(a, b)$ has a solution (which is unique by Corollary 4.12).*

Proof. With the change of variables $s = \frac{1}{\lambda}\eta^\lambda$ and $g(s) = f(\eta)$, Problem $\mathrm{P}(a, b)$ can be written as

$$\mathrm{P}_g(a, b) \begin{cases} g'' + \mu s^\alpha \{\beta(g)\}' = 0 & \text{for} \quad 0 < s < \infty \\ g(0) = a, \quad g(\infty) = b \end{cases} \tag{4.20}$$

where $' = d/ds$ and the constants α and μ are given by

$$\alpha = \frac{2}{\lambda} - 1 \quad \text{and} \quad \mu = \tfrac{1}{2}\lambda^\alpha.$$

A solution of $\mathrm{P}_g(a, b)$ is defined in a sense similar to the case of Problem $\mathrm{P}(a, b)$, and it can easily be verified that the two problems are equivalent.

If both a and b are positive, then by the Comparison Principle any solution of Problem $\mathrm{P}(a, b)$ will take values between a and b. Therefore the problem is non-degenerate and the existence of a solution to the boundary value problem $\mathrm{P}(a, b)$ can be shown by a shooting argument: if h is the solution of (4.20) with initial conditions $h(0) = a$ and $h'(0) = A$, then $\lim_{s\to\infty} h(s)$ exists for all $A > 0$, depends continuously on A, and tends to zero or infinity when $A \to 0$ or $A \to \infty$. This implies that there exists an A such that the limit is equal to b. The details of this argument can be found in [Pel93] and a similar argument is used by Gilding and L. A. Peletier ([GP77], p. 532). In the rest of this proof we will suppose that $a = 0$ and $b > 0$, and merely assert that the other case, $b = 0$ and $a > 0$, can be handled in an analogous way.

A solution of $\mathrm{P}_g(0, b)$ is constructed as the limit as $\varepsilon \downarrow 0$ of solutions of $\mathrm{P}_g(\varepsilon, b)$. For $\varepsilon > 0$ the solution of $\mathrm{P}_g(\varepsilon, b)$ is defined and unique, and by the Comparison Principle the sequence $\{g_\varepsilon\}$ depends monotonically on ε. We now show that the pointwise limit of this sequence, denoted by g, is a solution of Problem $\mathrm{P}_g(a, b)$. By twice integrating the equation in $\mathrm{P}_g(\varepsilon, b)$ we find the following integral identity for g_ε:

$$g_\varepsilon(s) = b - \mu \int_s^\infty \big[(1+\alpha)\sigma - \alpha s\big]\sigma^{\alpha-1}\big\{\beta(b) - \beta(g_\varepsilon(\sigma))\big\}\, d\sigma \qquad (4.21)$$

for all $s \in [0, \infty)$. The finiteness of the integral follows from the exponential convergence proved in Proposition 4.10. Since $g_\varepsilon \downarrow g$ as $\varepsilon \to 0$, and therefore $(\beta(b) - \beta(g_\varepsilon)) \uparrow (\beta(b) - \beta(g))$ on $[0, \infty)$, we can apply the monotone convergence theorem to the integral in (4.21) to conclude that it converges; the positivity of the left-hand side implies that the limit is finite. This results in the same integral equality for the limit function g:

$$g(s) = b - \mu \int_s^\infty \big[(1+\alpha)\sigma - \alpha s\big]\sigma^{\alpha-1}\big\{\beta(b) - \beta(g(\sigma))\big\}\, d\sigma \qquad (4.22)$$

for all $s \in [0, \infty)$. Starting with (4.22) and differentiating twice we can show that g is a solution to Problem $\mathrm{P}_g(0, b)$. This implies that the corresponding function f is a solution of Problem $\mathrm{P}(0, b)$. •

Behaviour near zero

In the proofs of Section 4.4 we need an estimate of the behaviour of the similarity solution near the origin. We restrict ourselves to the cases $\mathrm{P}(0, 1)$ and $\mathrm{P}(1, 0)$.

Proposition 4.14 — *Let f be the solution of Problem $P(1, 0)$. Then*

$$\lim_{\eta \downarrow 0} \eta^{1-\lambda} f'(\eta) \quad \textit{exists in} \quad (-\infty, 0).$$

Proof. Writing equation (4.18) in the form

$$\frac{f''}{f'} = \frac{\lambda - 1}{\eta} - \tfrac{1}{2}\eta\beta'(f(\eta)),$$

we obtain for arbitrary $\eta, \eta_0 \in \mathcal{P}$,

$$\eta^{1-\lambda} f'(\eta) = \eta_0^{1-\lambda} f'(\eta_0)\, e^{-\frac{1}{2}\int_{\eta_0}^{\eta} y\beta'(f(y))dy}.$$

Letting $\eta \downarrow 0$ yields the result. •

For P(0, 1) the analysis is more involved because the degeneracy of the nonlinearity and the geometric degeneracy coincide at $\eta = 0$. We encounter these two elements when describing the behaviour of solutions. In order to be able to make definite statements we must assume the extra hypothesis on β

$$\mathsf{B}_3 \qquad \lim_{f\downarrow 0} \frac{f\beta''(f)}{\beta'(f)} = p - 1 \quad \text{for some constant} \quad p \in (0, 1].$$

This condition expresses that for small data β behaves essentially as a power with exponent p.

For a nonlinearity β in the form of $\beta(f) = cf^p$, equation (4.18) has certain scaling properties that allow us to transform it into an autonomous one, and then apply a phase plane analysis. This analysis, which contains a complete classification of the behaviour of solutions near the origin, is given in [Pel93]. Here we summarise the results.

Proposition 4.15 — *Let f be the solution of* P(0, 1)*, where $\beta(f) = cf^p$ for some $p \in (0, 1)$ and $c > 0$, and let μ be given, as in Proposition 4.13, by $\mu = \frac{1}{2}\lambda^{\frac{2}{\lambda}-1}$.*

1. *If $\lambda < 2/(1-p)$, then the limit*

$$\lim_{\eta\downarrow 0} \frac{f(\eta)}{\eta^\lambda} \quad \textit{exists in } (0, \infty);$$

2. *If $\lambda = 2/(1-p)$, then*

$$\lim_{\eta\downarrow 0} \frac{f^{1-p}(\eta)}{\eta^2 \log \eta} = -2c\mu p \lambda^{1-p};$$

3. *If $\lambda > 2/(1-p)$, then*

$$\lim_{\eta\downarrow 0} \frac{f(\eta)}{\eta^{\frac{2}{1-p}}} = \frac{A}{\lambda^k},$$

in which $k = 2/(\lambda(1-p))$ and

$$A = \left(\frac{c\mu p}{1-k}\right)^{\frac{1}{1-p}}.$$

For more general nonlinearities β the analysis is more involved, and the results less precise. We find

Proposition 4.16 — *Let f be the solution of Problem $P(0,1)$, where β satisfies* B_1-B_3. *Then*

1. *If* $\lambda < \dfrac{2}{1-p}$, *then* $\lim_{\eta\downarrow 0} \eta^{-\lambda} f(\eta)$ *exists in* $(0,\infty)$;

2. *If* $\lambda \geq \dfrac{2}{1-p}$, *then* $\lim_{\eta\downarrow 0} \eta^{2}\beta'(f(\eta)) = 2\left(\lambda - \dfrac{2}{1-p}\right)$.

The number $2/(1-p)$ *should be replaced by* ∞ *when* $p = 1$.

Proof. Introducing the variables

$$\gamma(s) = \frac{sg'(s)}{g(s)} \quad \text{and} \quad \delta(s) = \mu s^{\alpha+1}\beta'(g(s))$$

in equation (4.20), we find that they satisfy the system of equations

$$\begin{aligned} s\gamma' &= \gamma(1-\gamma-\delta) \\ s\delta' &= \delta(\alpha+1+\zeta(s)\gamma), \end{aligned}$$

where

$$\zeta(s) = \frac{g(s)\beta''(g(s))}{\beta'(g(s))}.$$

By B_3 and the boundary condition $g(0) = 0$ we observe that $\zeta(s) \to p-1$ as $s \downarrow 0$. Consequently this system is asymptotically autonomous in the sense of Thieme [Thi92] as $s \downarrow 0$ (or if $\sigma = \log s$, as $\sigma \to -\infty$). We wish to apply a theorem of Poincare-Bendixson type (Theorem 1.6 of the same reference) to conclude that (γ, δ) tends to an equilibrium of the 'limit' system

$$\begin{aligned} s\gamma' &= \gamma(1-\gamma-\delta) \\ s\delta' &= \delta(\alpha+1+(p-1)\gamma), \end{aligned} \tag{4.23}$$

as $s \downarrow 0$. According to [Thi92], the only remaining condition to be verified is that the orbit under consideration is bounded as $s \downarrow 0$.

To show that this is the case, remark that the concaveness of g implies that

$$g(s) \geq sg'(s) \quad \text{for all} \quad s > 0,$$

which gives $0 \leq \gamma \leq 1$ for all $s > 0$. Since δ is positive, the orbit (γ, δ) can only be unbounded in the positive δ-direction. To force a contradiction, suppose that there exists a sequence $s_n \downarrow 0$ such that $\delta(s_n) \to \infty$ and such that

$\delta'(s_n) < 0$ and $|\delta'(s_n)/\gamma'(s_n)| \to \infty$. Since $\alpha+1 > 0$ and $\zeta(s) \to p-1$ when $s \downarrow 0$, there exists an $\varepsilon > 0$ such that δ' is positive when $\gamma < \varepsilon$. It therefore follows that $\gamma(s_n) \geq \varepsilon$. On the other hand, we can write

$$\frac{\delta'}{\gamma'} = \frac{\delta(\alpha+1+\zeta(s)\gamma)}{\gamma(1-\gamma-\delta)}$$

and if $\gamma \geq \varepsilon$ then the right-hand side of this expression is bounded from above and below when δ is large. This contradicts the assumption that $|\delta'(s_n)/\gamma'(s_n)| \to \infty$ as $n \to \infty$, and we conclude that the orbit (γ, δ) is bounded and therefore tends to an equilibrium of the limit system (4.23).

For the analysis of the equilibrium points of (4.23) it is convenient to introduce

Definition 4.17 — *Let $\varphi \in C^1(0,\delta)$ for some $\delta > 0$. Then*

$$\nu(\varphi) \stackrel{\text{def}}{=} \lim_{x\downarrow 0} \frac{x\varphi'(x)}{\varphi(x)} \quad \textit{(provided this limit exists)}$$

is called the index of φ.

If φ is a power of its argument, $\nu(\varphi)$ simply is the exponent. One can derive some properties of ν which extend this correspondence: if φ and ψ are such that $\nu(\varphi)$ and $\nu(\psi)$ are defined, then

1. $\nu(\varphi\psi) = \nu(\varphi) + \nu(\psi)$;
2. $\nu(\varphi \circ \psi) = \nu(\varphi)\nu(\psi)$ provided $\psi(0) = 0$;
3. $\nu(\varphi^\alpha) = \alpha\nu(\varphi)$ for all $\alpha \in \mathbb{R}$;
4. $\nu(\varphi) > -1 \implies \varphi \in L^1(0,\delta)$.

Besides, by de l'Hôpital's rule, the existence of $\nu(\varphi')$ implies that $\nu(\varphi)$ exists and that

5. $\nu(\varphi) = 1 + \nu(\varphi')$.

Note that with this notation assumption B_3 can be written as $\nu(\beta') = p - 1$.

The system (4.23) has the equilibria $e_0 = (0,0)$ and $e_1 = (1,0)$, and if $\alpha + p < 0$ then the point

$$e_2 = \left(\frac{\alpha+1}{1-p}, -\frac{\alpha+p}{1-p}\right)$$

is also an equilibrium point. Of these equilibria the first, $(0,0)$, can be quickly ruled out: by definition $\nu(g) = \lim_{s\downarrow 0} \gamma(s)$, and by writing equation (4.20) as

$$\frac{sg''(s)}{g'(s)} = -\delta(s)$$

we see that $\nu(g') = -\lim_{s\downarrow 0} \delta(s)$. Consequently $(\gamma,\delta) \to (0,0)$ implies on one hand $\nu(g') = 0$ and on the other hand $\nu(g) = 0$; this is incompatible by property 5 above. For the other two equilibria, we distinguish three cases:

- when $\alpha + p < 0$, the equilibrium $(1,0)$ is unstable (in backward time) and is therefore not admissible; it follows that $(\gamma,\delta) \to e_2$ as $s \downarrow 0$, and more specifically $\delta(s) \to -(\alpha+p)/(1-p)$;
- when $\alpha + p = 0$, $e_1 = e_2$ and therefore $\delta(s) \to 0$ as $s \downarrow 0$;
- when $\alpha + p > 0$, e_1 is the only admissible equilibrium and therefore $\nu(g) = 1$; using properties 1-3 we find that

  $$\nu\big(s^\alpha \beta'(g(s)) g'(s)\big) = \alpha + p - 1 > -1,$$

  which implies by (4.20) and property 4 that g'' is integrable; as a result, $g'(0+)$ is finite.

We can rearrange this information in the following form:

- When $\alpha + p > 0$, $\lim_{s\downarrow 0} g'(s)$ is finite;
- When $\alpha + p \le 0$, $\lim\limits_{s\downarrow 0} \mu s^{\alpha+1} \beta'(g(s)) = -\dfrac{\alpha+p}{1-p}$.

In terms of λ, f, and η, this is the statement of the theorem. •

4.4 The main results

This section is devoted to the proofs of Theorems 4.2 and 4.3. We shall discuss the proof in full for Theorem 4.2, and merely comment on the differences with Theorem 4.3.

Proof of Theorem 4.2. In order to compare the solution of the original problem with the self-similar solution we reformulate the problem in self-similar variables. If u is the solution of Problem (4.3), then define z by

$z(\eta, \tau) = u(r, t)$, where the independent variables are again linked through the relations

$$\eta = \frac{r}{\sqrt{t}} \quad \text{and} \quad \tau = \frac{t}{\varepsilon^2}.$$

The function z satisfies the equation

$$\tau\beta(z)_\tau - \tfrac{1}{2}\eta\beta(z)_\eta + \frac{\lambda - 1}{\eta} z_\eta - z_{\eta\eta} = 0 \quad \text{for} \quad \eta > 1/\sqrt{\tau},\ \tau > 0 \qquad (4.24)$$

and the boundary condition

$$z_\eta = \lambda\sqrt{\tau}(z - 1) \quad \text{for} \quad \eta = 1/\sqrt{\tau},\ \tau > 0.$$

The first step consists of an integral estimate, based on the conservation of mass.

Proposition 4.18 — *Under the conditions of 4.2, let $\Omega : \mathbb{R}^+ \to \mathbb{R}$ be defined by*

$$\Omega(\tau) = \int_{1/\sqrt{\tau}}^{\infty} \eta\big\{\beta(z(\eta, \tau)) - \beta(f(\eta))\big\}\, d\eta, \qquad \tau > 0,$$

and suppose that either ε is fixed or u_0 is constant. Then there is a constant χ such that

$$\Omega(\tau) \le \frac{\chi}{\tau} \quad \textit{for all } \tau > 0. \qquad (4.25)$$

If u_0 is constant, then χ does not depend on ε.

Proof. By integrating (4.8) and using boundary conditions (4.9) and Proposition 4.10 we find that

$$\int_{1/\sqrt{\tau}}^{\infty} \beta(f(\eta))\,\eta d\eta = \lambda - \int_0^{1/\sqrt{\tau}} \beta(f(\eta))\,\eta d\eta$$

The conservation of mass (Proposition 4.7) reads in the η, τ coordinates

$$\int_{1/\sqrt{\tau}}^{\infty} \beta(z)\,\eta d\eta = \frac{1}{\varepsilon^2\tau}\int_\varepsilon^\infty \beta(u_0(r))\, r dr + \lambda.$$

By combining these two we find that Ω is well-defined and that

$$\tau\Omega(\tau) = \frac{1}{\varepsilon^2}\int_\varepsilon^\infty \beta(u_0(r))\, r dr + \tau\int_0^{1/\sqrt{\tau}} \beta(f(\eta))\,\eta d\eta. \qquad (4.26)$$

The second term in (4.26) is bounded by $\beta(1)/2$. When ε is constant, the result follows immediately; when ε varies, but u_0 is equal to 0, the first term on the right-hand side vanishes and the remainder is bounded independent of ε. •

The interest of this integral estimate lies in the fact that z and f are ordered, and that therefore the argument of the integral is positive. Indeed, if v is the self-similar solution of equation (4.3a) corresponding to f, i.e. $v(r,t) = f(r/\sqrt{t})$, we can integrate equation (4.3a) from 0 to ε to obtain

$$\int_0^{\varepsilon} \beta(v)_t \, r dr + \left[\lambda v - r v_r\right]_0^{\varepsilon} = 0.$$

Now $v_t(r,t) = -\frac{1}{2} r t^{-3/2} f'(r/\sqrt{t}) > 0$ for all r and t and therefore we have $\lambda v(\varepsilon, t) - \varepsilon v_r(\varepsilon, t) \le \lambda$. By the Comparison Principle (Proposition 4.6) we then find that u lies above v on the whole of S_T^{ε}, which implies the same for z and f (on the appropriate domain).

Our aim is to convert an integral estimate related to (4.25) into a pointwise estimate. For this we need the next lemma (for an idea of the proof we refer to [Pel71]).

Lemma 4.19 — *Let ϕ be a non-negative continuous function on* $[0, \infty)$ *with lower Lipschitz constant L, i.e.*

$$\frac{\phi(x) - \phi(y)}{x - y} \ge -L \quad \text{for all } x, y \in [0, \infty),\ x \ne y.$$

Let $x_0 > 0$. If $\int_{x_0}^{\infty} x\phi(x)\,dx \le \alpha$, then

$$\sup_{x_0 \le x < \infty} \phi(x) \le \sqrt[3]{6L^2\alpha}.$$

We shall not apply this lemma directly to Ω, but to the integral

$$\int_{1/\sqrt{\tau}}^{\infty} \eta\{z(\eta,\tau) - f(\eta)\}\,d\eta.$$

For this integral we obtain an estimate similar to (4.25) by pointing out that, because β is concave and strictly increasing on its domain, the function $s \mapsto \beta(s) - \beta'(1)s$ is non-decreasing for $0 \le s \le 1$. This implies that $\beta(z(\eta,\tau)) - \beta(f(\eta)) \ge \beta'(1)(z(\eta,\tau) - f(\eta))$ and thus

$$\int_{1/\sqrt{\tau}}^{\infty} \eta\{z(\eta,\tau) - f(\eta)\}\,d\eta \le \frac{\chi}{\beta'(1)\tau}. \tag{4.27}$$

The crucial part in the application of Lemma 4.19 to estimate (4.27), with $\phi(\eta,\tau) = z(\eta,\tau) - f(\eta)$, is that we need to verify the lower Lipschitz continuity of ϕ with respect to the variable η. For general β, the function $\beta(z(\eta,\tau))$

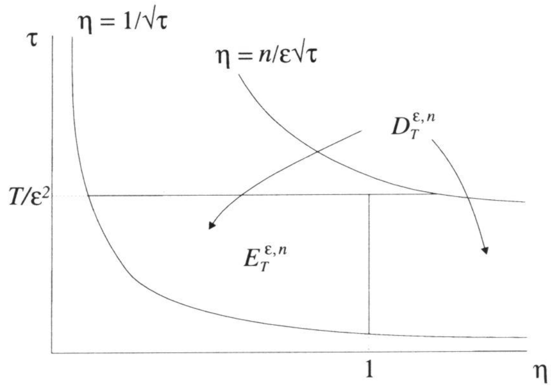

Figure 4.1: The domains $D_T^{\varepsilon,n}$ and $E_T^{\varepsilon,n}$.

need not be lower Lipschitz continuous with respect to η, and therefore we switched here from (4.25) to (4.27). From Proposition 4.9 we know that f is nonincreasing on $\mathbb{R}^+$, so the lower Lipschitz constant of ϕ only depends on z. We have

Proposition 4.20 — *If $0 < \lambda < 1$, then there exist positive constants ℓ and m independent of η and τ such that*

$$z_\eta(\eta, \tau) \geq -\ell\tau^{\frac{1-\lambda}{2}} - m \quad \textit{for all } \tau > 0 \textit{ and } \eta > 1/\sqrt{\tau}.$$

If $\lambda \geq 1$, then there exists a constant m independent of η and τ such that

$$z_\eta(\eta, \tau) \geq -m \quad \textit{for all } \tau > 0 \textit{ and } \eta > 1/\sqrt{\tau}.$$

If, for the moment, we consider this proposition proved, the conclusions of Theorem 4.2 follow by combining (4.27) and Proposition 4.20 and applying Lemma 4.19.

Proof of Proposition 4.20. Let $z_n(\eta, \tau) = u_n(r, t)$ where u_n denotes the regularised solution constructed in Section 4.2. The domain of definition of z_n is

$$D_T^{\varepsilon,n} = \left\{ (\eta, \tau) : \frac{1}{\sqrt{\tau}} < \eta < \frac{n}{\varepsilon\sqrt{\tau}}, 0 < \tau \leq \frac{T}{\varepsilon^2} \right\}$$

which is drawn in Figure 4.1. The first part of the proof is the following lemma.

Lemma 4.21 — *There exists a positive constant C, depending on β, λ and u_0, such that*

$$|z_{n\eta}| \leq \frac{C}{\eta} \quad \text{on } D_T^{\varepsilon,n}.$$

The constant C does not depend on n or T, and if u_0 is constant, then it does not depend on ε, either.

Proof of Lemma 4.21. We use again the flux F_n introduced in the proof of Theorem 4.4. First note that

$$F_n(r,t) = \lambda z_n(\eta,\tau) - \eta z_{n\eta}(\eta,\tau)$$

on the relevant domains. The the estimate follows from the observation that both F_n and z_n are bounded uniformly in n. If $u_0 = 0$, then by choosing $\delta_n = 1/n$ in (4.13) the constant C can be chosen independently of ε as well.

•

The remainder of the proof is based on the application of the maximum principle for parabolic equations to certain flux-type functions, depending on the value of λ. We distinguish two cases.

CASE I. $0 < \lambda \leq 2$. We truncate the unbounded domain $D_T^{\varepsilon,n}$ by considering

$$E_T^{\varepsilon,n} = D_T^{\varepsilon,n} \cap \{\eta < 1\}.$$

We assume that $n > \sqrt{T}$, so that the domain $E_T^{\varepsilon,n}$ is as is shown in Figure 4.1. On $E_T^{\varepsilon,n}$ we define the modified flux function

$$F_n = F_n(\eta,\tau) = \eta^{1-\lambda}\left[z_{n\eta}(\eta,\tau) + \tfrac{1}{2}\eta\beta(z_n(\eta,\tau))\right].$$

Using equation (4.24) we find that F_n satisfies

$$\tau\beta' F_{nt} - F_{n\eta\eta} - bF_{n\eta} - cF_n = d \quad \text{on } E_T^{\varepsilon,n}, \tag{4.28}$$

where the coefficients b, c, and d are given by

$$b(\eta,\tau) = -\frac{1-\lambda}{\eta} + \tfrac{1}{2}\eta\beta' - \frac{\beta''}{\beta'}z_{n\eta}, \quad c(\eta,\tau) = \tfrac{1}{2}(\lambda-2)\left\{\beta' - \frac{\beta''}{\beta'}\beta(z_n)\right\},$$

$$d(\eta,\tau) = \tfrac{1}{2}(\lambda-2)\,\eta^{2-\lambda}\frac{\beta''}{\beta'}\beta(z_n)^2.$$

Here we note β' and β'' for $\beta'(z_n)$ and $\beta''(z_n)$. Due to the regularisation, the coefficients in (4.28) are all smooth and bounded on $E_T^{\varepsilon,n}$. Note that $c \leq 0$ and $d \geq 0$, and that therefore F_n is a supersolution for the equation

$$\tau\beta' G_\tau - G_{\eta\eta} - bG_\eta - cG = 0 \quad \text{on } E_T^{\varepsilon,n}.$$

By the maximum principle (see e.g. [PW67]), a non-positive minimum of F_n on $E_T^{\varepsilon,n}$ must be assumed on its parabolic boundary, i.e. $\Gamma_1 \cup \Gamma_2$.

On Γ_1, given by $\eta = 1/\sqrt{\tau}$, we use the boundary condition and find

$$\begin{aligned} F_n(1/\sqrt{\tau}, \tau) &= \tau^{(\lambda-1)/2}\{\lambda\sqrt{\tau}(z_n(1/\sqrt{\tau}, \tau) - 1) + \tfrac{1}{2}\beta(z_n)/\sqrt{\tau}\} \\ &\geq -\lambda\tau^{\lambda/2}(1 - f(1/\sqrt{\tau})) \\ &\geq A, \end{aligned}$$

in which A is the (negative) limit value from Proposition 4.14. On Γ_2, where $\eta = 1$, we have $F_n = z_{n\eta}(1, \tau) + \frac{1}{2}\beta(z_n(\eta, 1)) \geq -C - \frac{1}{2}\beta(1)$ by Lemma 4.21. This implies that $F_n \geq -\ell := \min\{A, -C - \frac{1}{2}\beta(1)\}$ on $E_T^{\varepsilon,n}$ where $\ell > 0$, and therefore we have $z_{n\eta} \geq -\ell\eta^{\lambda-1} - \frac{1}{2}\beta(1)$ on $E_T^{\varepsilon,n}$ for all $n > \sqrt{T}$. When we combine this with Lemma 4.21 we obtain the required result.

CASE II. The case $\lambda > 2$ demands a different modified flux function:

$$F_n = F_n(\eta, \tau) = \eta^{1-\lambda}\big[z_{n\eta}(\eta, \tau) - \tfrac{1}{2}\lambda\eta(\beta(1) - \beta(z_n(\eta, \tau)))\big],$$

which satisfies

$$\begin{aligned} \tau\beta' F_{nt} = F_{n\eta\eta} + bF_{n\eta} &+ c\big\{F_n + \tfrac{1}{2}\lambda\eta^{2-\lambda}[\beta(1) - \beta(z_n)]\big\} \\ &+ d\big\{F_n + (\lambda - 1)\eta^{2-\lambda}[\beta(1) - \beta(z_n)]\big\} \quad \text{on} \quad E_T^{\varepsilon,n}, \end{aligned} \tag{4.29}$$

in which the coefficients b, c and d are given by

$$b(\eta, \tau) = -\frac{1-\lambda}{\eta} + \tfrac{1}{2}\eta\beta' - \frac{\beta''}{\beta'}z_{n\eta}, \quad c(\eta, \tau) = \tfrac{1}{2}\lambda(\lambda - 2)\frac{\beta''}{\beta'}\big[\beta(1) - \beta(z_n)\big],$$
$$d(\eta, \tau) = -\tfrac{1}{2}\lambda\beta'.$$

Now define the function ω by

$$\begin{aligned} \omega(F, \eta, \tau) \stackrel{\text{def}}{=} c\big\{F &+ \tfrac{1}{2}\lambda\eta^{2-\lambda}[\beta(1) - \beta(z_n)]\big\} \\ &+ d\big\{F + (\lambda - 1)\eta^{2-\lambda}[\beta(1) - \beta(z_n)]\big\}, \end{aligned} \tag{4.30}$$

and remark that $c, d < 0$. We claim that the function $\eta \mapsto \eta^{2-\lambda}(\beta(1) - \beta(z_n))$ is bounded on $\mathbb{R}^+$ by a constant independent of τ and n: on one hand,

$$\begin{aligned} 0 \leq \eta^{2-\lambda}\big[\beta(1) - \beta(z_n)\big] &\leq \eta^{2-\lambda}\big[\beta(1) - \beta(f(\eta))\big] \\ &\leq 2\beta'(1)\frac{A}{\lambda}\eta^2, \end{aligned}$$

if η is small enough, in which A is again the limit value from Proposition 4.14. On the other hand,

$$0 \leq \eta^{2-\lambda}\big[\beta(1) - \beta(z_n)\big] \leq \beta(1)\eta^{2-\lambda}.$$

The combination of the first for small η and the second for large η yields the uniform bound. Therefore, by choosing $F_0 \in \mathbb{R}$, $F_0 < A$ negative and large enough, $\omega(F_0, \eta, \tau)$ can be made positive for all η and τ. This implies that the constant F_0 is a subsolution for equation (4.29), and by following the same line of reasoning as for case I, we can conclude that $F_n \geq F_0$ on $E_T^{\varepsilon,n}$, for all $n > \sqrt{T}$. The required result is then obtained in a similar fashion. •

This concludes the proof of Theorem 4.2.

The proof of Theorem 4.3 follows the same lines, with some alterations. First, the ordering of the solution u and the self-similar solution f is reversed, which gives rise to the definition

$$\Omega(\tau) = \int_{1/\sqrt{\tau}}^{\infty} \eta\big\{\beta(f(\eta)) - \beta(z(\eta, \tau))\big\}\, d\eta, \quad \tau > 0.$$

The assertion of Proposition 4.18, however, holds unchanged, as does its proof. Second, if we denote $f(\eta) - z(\eta, \tau)$ by $\phi(\eta, \tau)$, the application of Lemma 4.19 requires an estimate of ϕ_η from below. From Section 4.3 we know that f is strictly increasing on $\mathbb{R}^+$. For an upper bound on z_η, we have

Proposition 4.22 — *Let z be the solution of Problem (4.3) with $u_e = 0$ and $u_0(\infty) = 1$, transported to the η, τ-plane. If $0 < \lambda < 1$ then there exist positive constants ℓ and m independent of η and τ such that*

$$z_\eta(\eta, \tau) \leq \ell\tau^{\frac{1-\lambda}{2}} + m \quad \text{for all } \tau > 0 \text{ and } \eta > 1/\sqrt{\tau};$$

if $\lambda \geq 1$ then there exists a constant m independent of η and τ such that

$$z_\eta(\eta, \tau) \leq m \quad \text{for all } \tau > 0 \text{ and } \eta > 1/\sqrt{\tau}.$$

The proof of Proposition 4.22 follows the same lines as that of Proposition 4.20, and we shall only mention the flux function that is used:

$$F_n = \eta^{1-\gamma}\left[z_{n\eta} + \tfrac{1}{2}\eta\beta(z_n)\right]$$

where $\gamma = \min\{\lambda, 2\}$. The result is then again reached by combination of Propositions 4.18 and 4.22 and Lemma 4.19. •

Chapter 5

Interface behaviour for system (1.19)

5.1 Introduction

In this chapter we study the support evolution properties of solutions of the system of equations

$$\beta(u)_t + v_t - \operatorname{div} \boldsymbol{A}(u, \nabla u) + B(u, \nabla u) = 0 \tag{5.1a}$$

$$v_t = \mathcal{F}(u, v). \tag{5.1b}$$

As explained in the Introduction, a travelling wave analysis of (5.1) shows that solutions with interfaces can exist when either β or the rate function $\mathcal{F}$ (or both) is degenerate. This phenomenon is often called *finite speed of propagation.* In this chapter we wish to investigate finite speed of propagation and the existence of waiting-times for general solutions of (5.1).

We start by giving the definitions of the properties of system (5.1) that we will prove in this chapter:

Definition 5.1 (Finite speed of propagation, FSP) — *If (u, v) is a solution such that $u(\cdot, 0)$ and $v(\cdot, 0)$ both vanish on a ball $B(x_0, \rho_0)$, then there exists an instant $t_0 > 0$ and a continuous function $\rho : [0, t_0] \to \mathbb{R}$, $\rho(0) = \rho_0$, such that $u(\cdot, t)$ and $v(\cdot, t)$ vanish almost everywhere in $B(x_0, \rho(t))$ for all $t \leq t_0$.*

Finite speed of propagation, as defined by the Definition above, is depicted in Figure 5.1. If the initial data resemble the bottom graph, then there is a region in the x, t-plane (the cone $|x| \leq \rho(t)$) in which $u \equiv v \equiv 0$. The thin line indicates the real interface, which generally will lie further away.

We speak of a waiting time if we have a condition that guarantees that ρ is constant over a finite time interval:

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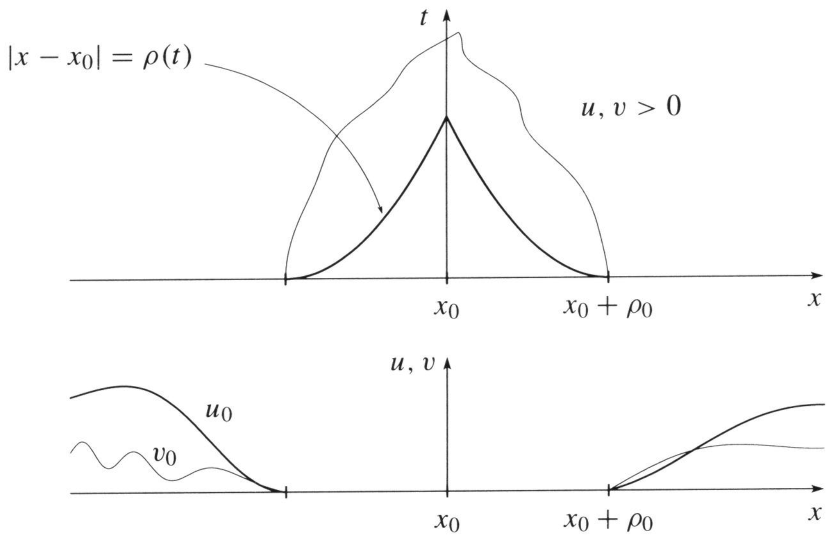

Figure 5.1: Finite speed of propagation

Definition 5.2 (Waiting times, WT) — *If (u, v) is a solution such that $u(\cdot, 0)$ and $v(\cdot, 0)$ both vanish on a ball $B(x_0, \rho_0)$ and satisfy a flatness condition in an annulus $B(x_0, \rho_1) \setminus \overline{B(x_0, \rho_0)}$, then there exists an instant $t^* > 0$ such that $u(\cdot, t)$ and $v(\cdot, t)$ vanish almost everywhere in $B(x_0, \rho_0)$ for all $t \in [0, t^*]$.*

Before we continue two remarks are due. First, we have not yet defined the notion of a solution; we postpone this to Section 5.5. Second, the flatness condition that the definition of the WT property refers to is unspecified; the precise condition depends on the assumptions that we make about the components of (5.1). This will become clear in what follows.

5.2 Statement of results

Let us state our hypotheses. We consider the problem

$$\beta(u)_t + v_t - \operatorname{div} \boldsymbol{A}(x, t, u, \nabla u) + \boldsymbol{q} \cdot \nabla\beta(u) = 0 \qquad (x, t) \in Q \qquad (5.2a)$$
$$v_t = \mathcal{F}(x, t, u, v) \qquad (x, t) \in Q \qquad (5.2b)$$
$$(u, v) = (u_0, v_0) \qquad \text{at } t = 0 \qquad (5.2c)$$

on a domain $Q = \Omega \times (0, T]$, where Ω is a bounded domain in $\mathbb{R}^N$. Solutions to this problem are defined in Definition 5.6. The following hypotheses shall hold throughout this chapter, even when not stated explicitly:

1. $u_0, v_0 \in L^\infty(\Omega)$;

2. β is continuous, and $\boldsymbol{A}$ and $\mathcal{F}$ are Carathéodory functions;[1]

3. $m_0 u^{p+1} \le \Phi(u) \le m_1 u^{p+1}$ for all $u \ge 0$, where the function Φ is given by

$$\Phi(u) = \int_0^u s\beta'(s)ds. \tag{5.3}$$

4. $\boldsymbol{A}(\cdot,\cdot,\cdot,\xi)\cdot\xi \ge m_2 |\xi|^2, \quad \xi \in \mathbb{R}^N$;

5. $|\boldsymbol{A}(\cdot,\cdot,\cdot,\xi)| \le m_3 |\xi|, \quad \xi \in \mathbb{R}^N$;

6. $\boldsymbol{q} \in L^\infty(Q;\mathbb{R}^N)$ and $\operatorname{div}\boldsymbol{q} = 0$.

7. $\mathcal{F}$ is Lipschitz continuous in the second variable:

$$|\mathcal{F}(x,t,u,v_1) - \mathcal{F}(x,t,u,v_2)| \le L\,|v_1 - v_2| \tag{5.4}$$

for all $u, v_1, v_2 \in \mathbb{R}$, and $(x,t) \in Q$.

Here $0 < p \le 1$ and the numbers m_i are positive constants. In view of Properties FSP and WT defined previously we fix once and for all $x_0 \in \Omega$ and $\rho_0 > 0$ such that $B_{\rho_0} = B(x_0,\rho_0) \subset \Omega$ and $u_0 = v_0 = 0$ on B_{ρ_0}. In addition, for Property WT we assume that $\rho_1 > \rho_0$ and $B_{\rho_1} \subset \Omega$. We shall use the notation 'B_ρ' for $B(x_0,\rho)$.

We shall use the following hypotheses in the formulation of the different theorems (I for Interfaces):

I_1 There exists a number $0 < \bar{v} \le \infty$ and a non-negative function $\psi : [0,\bar{v}) \to \mathbb{R}$ such that

$$(u - \psi(v))\,\mathcal{F}(x,t,u,v) \ge 0$$

for all $u \ge 0$, $0 \le v < \bar{v}$, $x \in B_{\bar{\rho}}$ and for all $t > 0$. Here $\bar{\rho} > \rho_0$ and, if appropriate, $\bar{\rho} > \rho_1$. If $\bar{v} < \infty$ then we set $\psi(v) = \infty$ for all $v \ge \bar{v}$.

I_2 $0 \le \mathcal{F}(\cdot,\cdot,u,0) \le k_1 u^p$ for all $u \ge 0$;

[1]A Carathéodory function $f(x,t,u)$ is continuous in u for almost every fixed (x,t) and measurable in (x,t) for every fixed u. This guarantees that $f(x,t,u(x,t))$ is measurable if $u(x,t)$ is measurable.

I_3 $k_2 u^\gamma \le \mathcal{F}(\cdot, \cdot, u, 0) \le k_3 u^\gamma$ for all $u \ge 0$.

Here the exponent p is the same as above and the exponent γ is free to be chosen in $(0, 1)$. The k_i are positive constants. Whenever possible we shall omit the variables x, t in expressions of the type $\boldsymbol{A}(x, t, u, \nabla u)$.

Although at first sight hypothesis I_1 may seem far-fetched, it arises in a natural way in the derivation of the model underlying (5.2), as explained in the Introduction.

We shall now state our main results. The first one extends a known result for the 'porous medium equation' (1.6): if $p < 1$, then under a weak condition on $\mathcal{F}$ system (5.2) has property FSP. Besides, an advection term of the form $\boldsymbol{q} \cdot \nabla\beta(u)$ does not change this property:

Theorem 5.3 — *Let hypothesis I_1 be satisfied. If $p < 1$ then Problem (5.2) has property* FSP.

For the theorem on waiting times we introduce an auxiliary function:

$$\Psi(s) = \int_0^s \psi(\sigma)\, d\sigma$$

where ψ is given by I_1. If $s \ge \bar{v}$, then $\Psi(s)$ is taken equal to infinity.

Theorem 5.4 — *Let hypothesis I_1 be satisfied and suppose that $\boldsymbol{q} = 0$. If $p < 1$ then Problem (5.2) has property* WT. *The accompanying flatness condition reads*

$$\begin{cases} \text{There exists a constant } C > 0 \text{ such that} \\ \displaystyle\int_{B_\rho} \Phi(u_0) + \int_{B_\rho} \Psi(v_0) \le C(\rho - \rho_0)_+^{1/(1-\delta)} \quad \text{for all} \quad 0 < \rho < \rho_1. \end{cases}$$

Here δ is given by (5.14).

It was already known from a travelling wave analysis ([DK91]) that if the function $\mathcal{F}$ satisfies a certain kind of degeneracy, then finite speed of propagation can occur even for regular (i.e. Lipschitz continuous) nonlinearities β. The following theorem makes this statement precise.

Theorem 5.5 — *Let either of the following conditions be satisfied:*

I_2 with $p < 1$ or I_3 with $\gamma < 1$.

Set $\eta = p$ for I_2 and $\eta = \gamma$ for I_3. Then,

1. *Problem (5.2) has the property* FSP;

2. *if $q = 0$, then Problem (5.2) also has the property* WT *under the assumption of the flatness condition*

$$\begin{cases} \textit{There exists a constant } C > 0 \textit{ such that} \\ \displaystyle\int_{B(x_0,\rho)} u_0^{\eta+1} + \int_{B(x_0,\rho)} v_0^{\frac{\eta+1}{\eta}} \le C(\rho-\rho_0)_+^{1/(1-\delta)} \\ \qquad\qquad\qquad\qquad \textit{for all} \quad 0 < \rho < \rho_1. \end{cases}$$

Here δ is given by (5.14) for hypothesis I_2 and by (5.29) for hypothesis I_3.

Remark 5.1 For a quick interpretation of Theorems 5.3, 5.4, and 5.5, consider the case $\beta(u) = u^p$ and $\mathcal{F}(\cdot, \cdot, u, 0) = u^q$ with $p, q > 0$. We then prove the FSP and WT properties for the following range of parameters:

$$\begin{array}{lll} p = 1, & 0 < q < 1 & \text{(by means of } \mathsf{I}_3) \\ p < 1, & p \le q < \infty & \text{(by means of } \mathsf{I}_2) \\ p < 1, & 0 < q < 1 & \text{(by means of } \mathsf{I}_3) \end{array} \Bigg\} \Rightarrow p < 1, \quad 0 < q < \infty.$$

•

5.3 A comparison with the method of travelling waves

In many cases results of the type of Theorem 5.3 (finite speed of propagation) are proved by comparing the solution with travelling waves. This allows the often detailed information that can be obtained on travelling waves to be transferred to general solutions.

For this method to apply it is however necessary that the problem is autonomous, i.e. invariant under translations. Although the study of such equations and systems can lead to valuable insight, many applications explicitly require results that remain valid when this spatial invariance condition is relaxed. A typical example is the model of transport of chemical substances through a porous medium that is derived in Section 1.3. Even if the medium itself is supposed homogeneous, together with its characteristics such as the functions β and $\mathcal{F}$, then the dispersion coefficient $\boldsymbol{D}$ will generally depend on space and time, because it depends on the discharge field $\boldsymbol{q}$ (see footnote 3 on page 8). Note that such a dispersion coefficient does not only bring space- and time-dependence into the problem, but also anisotropy.

The price to be paid for the gain in generality, however, is obvious; for instance, while for the travelling wave solutions (necessarily in homogeneous media) that were examined in [DK91] a very precise characterisation could be given of the occurrence and non-occurrence of bounded supports, in the general case we must make do with the one-sided results of Theorems 5.3, 5.4 and 5.5.

For the travelling wave method it is also necessary that the problem satisfy a comparison principle. It is not difficult to see that if $\boldsymbol{q} = 0$ and Φ is an increasing function, system (5.1) satisfies a comparison principle if and only if $\mathcal{F}$ is increasing in u and decreasing in v. A comparison principle is a very important tool, not only in proving existence and uniqueness, but also in proving finite speed of propagation (property FSP) or its converse, by comparing the solution with travelling waves. Generically the information about the travelling waves immediately carries over to the full problem.

Therefore it is important to note that the conditions that we set on $\mathcal{F}$ allow for non-monotonicity. Indeed, they could be said to imply a form of 'weak monotonicity': a function $\mathcal{F}$ that is increasing in u and decreasing in v automatically satisfies condition I_1 as well as the requirement

$$\mathcal{F}(u, 0) \geq 0 \quad \text{for all} \quad u > 0$$

which is part of I_2 and I_3.

As a second point of difference we should note that in general a travelling wave comparison method can not prove the existence of waiting times, since travelling waves mostly have non-zero speed.

5.4 An outline of the method

As was mentioned in the Introduction, an important aspect of this Energy Method is its applicability to very general equations and systems. The other side of the coin is that proofs tend to be very technical and obscure the underlying ideas. We therefore start with an introduction to the method, aimed at conveying philosophy rather than mathematically correct statements. After this introduction we shall prove Theorems 5.3, 5.4, and 5.5 in full rigour.

We shall discuss the method applied to a simplified version of (5.2):

$$u_t^p + v_t - \Delta u = 0 \tag{5.5a}$$

$$v_t = \mathcal{F}(u, v) \tag{5.5b}$$

The result we seek is that of Theorem 5.3, i.e. if $p < 1$ and $\mathcal{F}$ satisfies I_1, then system (5.5) has property FSP.

The key idea is to derive an ordinary differential inequality from this system of partial differential equations and to conclude by means of the study of this inequality. Following the definition of property FSP we assume that the initial data u_0 and v_0 both vanish in the ball $B_{\rho_0} = B(x_0, \rho_0)$. We multiply the first equation of (5.5) by the solution u and integrate by parts on a ball B_ρ centered in x_0 with radius $\rho < \rho_0$. We obtain

$$\frac{p}{p+1}\frac{d}{dt}\int_{B_\rho} u^{p+1} + \int_{B_\rho} |\nabla u|^2 = \int_{\partial B_\rho} u \nabla u \cdot \nu - \int_{B_\rho} u \mathcal{F}(u, v). \tag{5.6}$$

By integrating over $(0, t)$ for some $0 < t < T$,

$$\frac{p}{p+1}\int_{B_\rho} u(t)^{p+1} + \int_0^t \int_{B_\rho} |\nabla u|^2 = \int_0^t \int_{\partial B_\rho} u \nabla u \cdot \nu - \int_0^t \int_{B_\rho} u \mathcal{F}(u, v). \tag{5.7}$$

We now define the non-negative functions b and E, which represent generalised *energies* (whence the term 'energy method'):

$$b(\rho, t) = \sup_{0 \le \tau \le t} \int_{B_\rho} u(\tau)^{p+1} \quad \text{and} \quad E(\rho, t) = \int_0^t \int_{B_\rho} |\nabla u|^2 .$$

Both are non-decreasing with respect to ρ; using

$$\frac{\partial E}{\partial \rho}(\rho, t) = \int_0^t \int_{\partial B_\rho} |\nabla u|^2 \text{ for } \rho < \rho_0 \tag{5.8}$$

it follows from the Hölder inequality that

$$\int_0^t \int_{\partial B_\rho} u \nabla u \cdot \nu \le \Big(\int_0^t \int_{\partial B_\rho} u^2\Big)^{\frac{1}{2}} \Big(\frac{\partial E}{\partial \rho}\Big)^{\frac{1}{2}}. \tag{5.9}$$

Using this in (5.7) we find that

$$\frac{p}{p+1} b(\rho, t) + E(\rho, t) \le \Big(\int_0^t \int_{\partial B_\rho} u^2\Big)^{\frac{1}{2}} \Big(\frac{\partial E}{\partial \rho}\Big)^{\frac{1}{2}} - \int_0^t \int_{B_\rho} u \mathcal{F}(u, v). \tag{5.10}$$

The next step consists in using an interpolation-trace inequality (see Appendix 5.B) to derive the estimate

$$\left(\int_0^t \int_{\partial B_\rho} u^2\right)^{\frac{1}{2}} \leq C t^{\frac{1-\theta}{2}} K(T)(E+b)^\kappa,$$

where

$$\kappa = \theta/2 + (1-\theta)/(p+1), \tag{5.11}$$

$$K(T) = \max(1, T^{\frac{\theta}{2}}) \max\left(1, b(T, \rho_1)^{\frac{\theta(1-p)}{2(p+1)}}\right),$$

and

$$\theta = \frac{N(1-p)+p+1}{N(1-p)+2p+2}. \tag{5.12}$$

It is important to remark here that $\kappa > 1/2$ if and only if $p < 1$. Therefore the arguments that follow can only be executed if $p < 1$, since they require that $\kappa > 1/2$. This is the point in the reasoning where the degeneracy of the nonlinearity is essential. Applying Young's inequality we obtain from (5.9) that

$$\begin{aligned}\left(\int_0^t \int_{\partial B_\rho} u^2\right)^{\frac{1}{2}} \left(\frac{\partial E}{\partial \rho}\right)^{\frac{1}{2}} &\leq C t^{\frac{1-\theta}{2}} K(T)(E+b)^\kappa \left(\frac{\partial E}{\partial \rho}\right)^{\frac{1}{2}} \\ &\leq \varepsilon(E+b) + C_\varepsilon (C t^{\frac{1-\theta}{2}} K(T))^{\frac{1}{1-\kappa}} \left(\frac{\partial E}{\partial \rho}\right)^{\frac{1}{\delta}}\end{aligned} \tag{5.13}$$

where $0 < \varepsilon < p/(p+1)$ and

$$\delta = 2(1-\kappa) = \frac{3p+1+N(1-p)}{2p+2+N(1-p)}. \tag{5.14}$$

By combining (5.10) and (5.13) we obtain

$$b + E \leq C_1 t^{\frac{1-\theta}{\delta}} \left(\frac{\partial E}{\partial \rho}\right)^{\frac{1}{\delta}} - C_2 \int_0^t \int_{B_\rho} u\mathcal{F}(u, v), \tag{5.15}$$

where C_1 and C_2 collect all the different constants.

In the rigorous proofs we shall show how to handle the second term on the right-hand side in (5.15), depending on the assumptions on $\mathcal{F}$. For the moment we assume that it is non-positive, allowing us to find an ordinary differential inequality for the function E:

$$E(\rho, t) \leq C_1 t^{\frac{1-\theta}{\delta}} \left(\frac{\partial E}{\partial \rho}\right)^{\frac{1}{\delta}} (\rho, t), \tag{5.16}$$

which holds for all $0 < \rho < \rho_0$ and $0 < t < T$. For such ordinary differential inequalities it is not difficult to prove the following Lemma, which is a special case of Lemma 5.8, part 2:

Lemma — *Let $\nu > 0$ and $0 < \delta < 1$, and let $\psi \in C([0, T] \times [0, \rho_0])$ be a non-negative function such that*

$$\psi^\delta(t, \rho) \leq K t^\nu \frac{\partial \psi}{\partial \rho}(t, \rho)$$

for almost every $\rho \in (0, \rho_0)$ and for all $t \in [0, T]$. Then there exists a time $t^* \leq T$ and a continuous function $r : [0, t^*] \to \mathbb{R}$ with $r(0) = \rho_0$ such that

$$\psi(\rho, t) = 0 \quad \textit{for all } \rho \textit{ and } t \textit{ such that} \quad \rho \leq r(t).$$

We conclude from this Lemma that in the region $\rho \leq r(t)$ the function $E(t, \rho)$, and therefore also the function $b(t, \rho)$, is equal to zero. Therefore system (5.5) possesses property FSP.

As we said above, this is not more than an intuitive outline of a general method which can be proved in full rigour. Many steps are only formal, and the treatment of the function $\mathcal{F}$ has been completely neglected. In Sections 5.5, 5.6, and 5.7 we shall give the details which are necessary to render the ideas presented above rigorous.

5.5 Proof of Theorem 5.3

Even for the simple system (5.5) the proof of Theorem 5.3 given above is not complete. The steps that were omitted were:

1. the definition of a solution of the problem;
2. the justification of equations (5.6) and (5.8) and estimate (5.9). This depends strongly on the choice of the function space to which a solution (u, v) should belong;

3. the treatment of the term $\displaystyle\int_0^t \int_{B_\rho} u\mathcal{F}(u, v)$.

Besides, in order to complete the proof for system (5.2), we need to consider the more general functions $\beta, \boldsymbol{A}, \boldsymbol{q}$, and $\mathcal{F}$ instead of their simple counterparts in (5.5). We shall discuss these points one by one.

The definition of a solution. This first omission is easily remedied:

Definition 5.6 — *A pair of measurable functions (u, v) defined in $Q = \Omega \times (0, T]$ is a weak solution of (5.2) with initial data (u_0, v_0) if the following conditions are satisfied:*

1. $u \in L^\infty(0, T; L^{p+1}(\Omega)) \cap L^2(0, T; H^1(\Omega))$ *and* $v \in L^\infty(Q)$;
2. $u \geq 0$ *and* $v \geq 0$;
3. $\boldsymbol{A}(\cdot, \cdot, u, \nabla u) \in L^1(Q; \mathbb{R}^N)$ *and* $\nabla\beta(u) \in L^1(Q)$;
4. $\liminf_{t\to 0} \Phi(u(\cdot, t)) = \Phi(u_0)$ *and* $\liminf_{t\to 0} v(\cdot, t) = v_0$ *in* $L^1(\Omega)$;
5. *for any* $\psi \in C^\infty([0, T]; C_c^\infty(\Omega))$ *we have*

$$\int_\Omega \Phi(u)\psi(T) - \int_Q \left\{\Phi(u)\psi_t - \boldsymbol{A}(u, \nabla u)\cdot\nabla\psi - \psi\boldsymbol{q}\cdot\nabla\beta(u)\right\} = \\ = \int_\Omega \Phi(u_0)\psi(0) - \int_Q \mathcal{F}(u, v)\psi \qquad (5.17)$$

and

$$\int_\Omega v\psi(T) - \int_Q v\psi_t = \int_\Omega v_0\psi(0) + \int_Q \mathcal{F}(u, v)\psi$$

where we have omitted the variable pair (x, t) for clarity.

We emphasise that we leave aside all questions of uniqueness, as well as existence under given boundary conditions. The arguments that follow only require the existence of a solution in the local sense of Definition 5.6.

Non-zero convection. In order to accommodate non-zero convection we shall use a domain of integration that is not the cylinder $B_\rho \times (0, T)$ but a truncated cone

$$K_{\rho,t} = \{(x, \tau) \in \Omega \times (0, t) : |x| < g(\rho, \tau)\}, \qquad (5.18)$$

where $g(\rho, \tau) := \rho - \alpha\tau$ and $\alpha > 0$ shall be fixed later. For a general function ψ, we introduce the notation

$$\int_{K_{\rho,t}} \psi \equiv \int_0^t \int_{B_g} \psi \equiv \int_0^t \int_{B_{g(\rho,\tau)}} \psi \, dx d\tau,$$

and similarly for the boundary integrals.

It can easily be verified that the following identity holds for smooth functions ζ:

$$\int_{B_{g(\rho,t)}} \frac{\partial \zeta}{\partial t}(x,t)\, dx = \frac{d}{dt}\int_{B_{g(\rho,t)}} \zeta(x,t)\, dx - g'(t)\int_{\partial B_{g(\rho,t)}} \zeta(x,t)\, ds. \quad (5.19)$$

By a truncation-regularisation scheme such as in [AD] or [DV85] we can combine this formula with equation (5.17) (for $\psi = u$) to obtain for all $0 < \rho < \rho_0$ and for all $0 < t < T$,

$$\int_{B_{g(\rho,t)}} \Phi(u(t)) + \int_0^t \int_{\partial B_g} (\boldsymbol{q} \cdot \nu + \alpha))\Phi(u) + \int_{K_{\rho,t}} \boldsymbol{A}(u, \nabla u) \cdot \nabla u =$$

$$= \int_0^t \int_{\partial B_g} u \boldsymbol{A}(u, \nabla u) \cdot \nu - \int_{K_{\rho,t}} u \mathcal{F}(u, v). \quad (5.20)$$

When ρ is allowed to take values in the interval (ρ_0, ρ_1), as is necessary for Property WT, the right-hand side of (5.20) contains the extra term $\int_{B_\rho} \Phi(u_0)$.

Equation (5.20) is the equivalent of (5.6) for general functions β, $\boldsymbol{A}$, and $\boldsymbol{q}$. Observe that by choosing $\alpha = \|\boldsymbol{q}\|_{L^\infty(Q)}$, the second term on the left-hand side becomes non-negative.

Remark 5.2 The use of a cone instead of a cylinder has a very simple physical interpretation. When $\alpha = \|\boldsymbol{q}\|_{L^\infty(Q)}$, the spatial boundary of the cone (i.e., $\partial B_{g(\rho,t)}$) moves inward with time with a velocity that is as least as large as the maximum velocity of the flow field. Therefore the convection term will not introduce any material from outside into the integration domain; the occurrence or non-occurrence of zero sets is then determined by the interplay between the time derivative, the diffusion term, and the function $\mathcal{F}$, as is the case when convection is absent. •

Justification of equation (5.8) and estimate (5.9). It follows from Fubini's theorem that for $u \in H^1(\Omega)$ and ρ_0 such that $B_{\rho_0} \subset \Omega$, the function

$$\rho \mapsto \int_{\partial B_\rho} |\nabla u|^2$$

is defined for almost every $\rho \in (0, \rho_0)$. Since the domain of integration is a cone, we now define the functions b and E in the following way:

$$b(\rho, t) = \sup_{0 \leq \tau \leq t} \int_{B_{g(\rho,\tau)}} u^{p+1} \quad \text{and} \quad E(\rho, t) = \int_{K_{\rho,t}} |\nabla u|^2 .$$

Definition 5.6 guarantees that these expressions are well-defined. It then follows that for almost every $\rho \in (0, \rho_0)$,

$$\frac{\partial E}{\partial \rho}(\rho, t) = \int_0^t \int_{\partial B_g} |\nabla u|^2 ,$$

and

$$\int_0^t \int_{\partial B_g} u \boldsymbol{A}(u, \nabla u) \cdot \nu \leq m_3 \Big(\int_0^t \int_{\partial B_g} u^2 \Big)^{\frac{1}{2}} \Big(\frac{\partial E}{\partial \rho} \Big)^{\frac{1}{2}}$$

We can then combine this with (5.20) to obtain

$$m_0 b(\rho, t) + m_2 E(\rho, t) \leq m_3 \Big(\int_0^t \int_{\partial B_g} u^2 \Big)^{\frac{1}{2}} \Big(\frac{\partial E}{\partial \rho} \Big)^{\frac{1}{2}} - \int_{K_{\rho,t}} u \mathcal{F}(u, v). \tag{5.21}$$

This inequality is the rigorous counterpart of (5.10).

Handling of the term $\int_{K_{\rho,t}} u\mathcal{F}(u, v)$. Let us now consider the last term in (5.21). Hypothesis I_1 ensures the existence of a function ψ. By multiplying the second equation in (5.2) by $\psi(v)$ and integrating we find that

$$\int_{B_{g(\rho,t)}} \Psi(v(t)) \leq \int_{K_{\rho,t}} \psi(v) \mathcal{F}(u, v),$$

for all $0 < \rho < \rho_0$. Note that since hypothesis I_1 allows the function $\psi(s)$ to assume the value ∞ for some values of s, the two integrals written above might both be infinite. Now we add this inequality to (5.21) to obtain

$$\begin{aligned} m_0 b(\rho, t) + \int_{B_{g(\rho,t)}} \Psi(v(t)) + m_2 E(\rho, t) \leq \\ \leq m_3 \Big(\int_0^t \int_{\partial B_g} u^2 \Big)^{\frac{1}{2}} \Big(\frac{\partial E}{\partial \rho} \Big)^{\frac{1}{2}} - \int_{K_{\rho,t}} (u - \psi(v)) \mathcal{F}(u, v). \end{aligned} \tag{5.22}$$

Hypothesis I_1 now ensures that the second term on the left-hand side is non-negative and the last term on the right-hand side non-positive. This also ensures that both sides of the inequality have finite values. We are left with

$$m_0 b(\rho, t) + m_2 E(\rho, t) \le m_3 \Big(\int_{K_{\rho,t}} u^2 \Big)^{\frac{1}{2}} \Big(\frac{\partial E}{\partial \rho} \Big)^{\frac{1}{2}}. \tag{5.23}$$

From here onwards the proof is the same as in the formal discussion, with the one exception that Lemma 5.8 should now be applied to a truncated cone instead of a cylinder.

5.6 Proof of Theorem 5.4

Throughout the previous section the variable ρ took values in $[0, \rho_0)$. For values outside of this range, i.e. for $\rho \in [\rho_0, \rho_1)$, essentially the same arguments hold. The main difference is that when integrating over the cylinder (that is, the passage from (5.6) to (5.7)) the terms at $t = 0$ do not necessarily vanish. The equivalent of inequality (5.22) then reads

$$\begin{aligned} m_0 b(\rho, t) + \int_{B_\rho} \Psi(v(t)) + m_2 E(\rho, t) \le \\ \le m_3 \Big(\int_0^t \int_{B_\rho} u^2 \Big)^{\frac{1}{2}} \Big(\frac{\partial E}{\partial \rho} \Big)^{\frac{1}{2}} + \int_{B_\rho} \Phi(u_0) + \int_{B_\rho} \Psi(v_0). \end{aligned} \tag{5.24}$$

The first term on the right-hand side is handled as before, and the proof of Theorem WT is concluded by the application of Lemma 5.8.

5.7 Proof of Theorem 5.5

The essential difference between Theorem 5.5 and the two other theorems lies in the assumptions on the nonlinearity $\mathcal{F}$. We shall therefore only discuss this part of the total argument.

The following Lemma combines some technical estimations.

Lemma 5.7 — *Let $\mathcal{F}$ satisfy I_2 or I_3. Then for all $\varepsilon > 0$ and $0 < t < T$,*

$$\begin{aligned} -\int_0^t \int_{B_g} u\mathcal{F}(u, v) \le -\int_0^t \int_{B_g} u\mathcal{F}(u, 0) + \varepsilon^{-1/\eta} C_1 t \int_{B_\rho} v_0^{\frac{\eta+1}{\eta}} \\ + \hat{C}(\varepsilon, t) \int_0^t \int_{B_g} u^{\eta+1}, \end{aligned} \tag{5.25}$$

where $\eta = p$ *for* I_2 *and* $\eta = \gamma$ *for* I_3, *and*

$$\hat{C}(\varepsilon, t) = L\varepsilon + \varepsilon^{-1/\eta} C_1 t \quad \textit{and} \quad C_1 = C_1(\eta, T, L) > 0.$$

Here ρ *takes values in* $(0, \rho_0)$ *for Property* FSP *and in* $(0, \rho_1)$ *for Property* WT.

We first continue the proof of Theorem 5.5 and prove this Lemma afterwards.

Let us first tackle the case of Property FSP under hypothesis I_2. In that case the integral $\int_{B_\rho} v_0^{\frac{p+1}{p}}$ is equal to zero by assumption. Fix $\eta = p$ Using the non-negativeness of the term $u\mathcal{F}(u, 0)$ stated in I_2 we obtain from (5.25)

$$-\int_0^t \int_{B_g} u\mathcal{F}(u, v) \le \hat{C} \int_0^t \int_{B_g} u^{p+1}. \tag{5.26}$$

Now, combining (5.21) and (5.26) we get

$$m_0 b(\rho, t) + m_2 E(\rho, t) \le \Big(\int_0^t \int_{B_g} u^2\Big)\Big(\frac{\partial E}{\partial \rho}\Big)^{\frac{1}{2}} + \hat{C} \int_0^t \int_{B_g} u^{p+1} \tag{5.27}$$

By choosing ε and t^* small enough,

$$\hat{C}(\epsilon, t)t \le \frac{1}{2} m_0 \quad \text{for all} \quad t \in [0, t^*].$$

Then (5.27) becomes

$$\frac{1}{2} m_0 b(\rho, t) + m_2 E(\rho, t) \le \Big(\int_0^t \int_{\partial B_g} u^2\Big)\Big(\frac{\partial E}{\partial \rho}\Big)^{\frac{1}{2}}, \tag{5.28}$$

and we conclude by a combination of Theorem 5.9 and Lemma 5.8, in the same way as in the proof of Theorem 5.3.

For Property WT we consider cylinders $B_\rho \times (0, t)$ where ρ now takes values in $(0, \rho_1)$, which introduces two extra terms in (5.28):

$$\frac{1}{2} m_0 b(\rho, t) + m_2 E(\rho, t) \le \Big(\int_0^t \int_{\partial B_\rho} u^2\Big)^{\frac{1}{2}}\Big(\frac{\partial E}{\partial \rho}\Big)^{\frac{1}{2}} + \\ + m_1 \int_{B_\rho} u_0^{p+1} + \varepsilon^{-1/\eta} e^{C_1 T} L t \int_{B_{\rho_0}} v_0^{\frac{\eta+1}{\eta}}.$$

The result follows in the same way as in the proof of Theorem 5.4.

When we trade hypothesis I_2 for hypothesis I_3 we introduce a new energy,

$$c(\rho, t) \stackrel{\text{def}}{=} \frac{1}{1+\gamma} \int_{K_{\rho,t}} u^{\gamma+1}.$$

Using Hypothesis I_3 and (5.25), inequality (5.21) becomes

$$\begin{aligned} &m_0 b(\rho, t) + m_2 E(\rho, t) + (k_2 - \hat{C}(\varepsilon, t)) c(\rho, t) \leq \\ &\leq m_3 \Big(\int_0^t \int_{\partial B_g} u^2 \Big)^{\frac{1}{2}} \Big(\frac{\partial E}{\partial \rho} \Big)^{\frac{1}{2}} + \varepsilon^{-1/\gamma} e^{C_1 T} L t \int_{B_\rho} v_0^{\frac{\gamma+1}{\gamma}}. \end{aligned}$$

For Property FSP the last term disappears, and we choose ε and t^* such that $\hat{C}(\varepsilon, t) \leq k_2/2$ for all $0 \leq t \leq t^*$. Then, applying Theorem 5.9 with parameter γ instead of p, we obtain

$$\Big(\int_0^t \int_{\partial B_g} u^2 \Big)^{\frac{1}{2}} \Big(\frac{\partial E}{\partial \rho} \Big)^{\frac{1}{2}} \leq \varepsilon (E + c) + \frac{C}{\varepsilon} (t^{\frac{1-\theta}{2}})^{\frac{1}{1-\kappa}} \Big(\frac{\partial E}{\partial \rho} \Big)^{\frac{1}{\delta}}$$

where κ, θ and δ all have the same values as in (5.11), (5.12) and (5.14) with p replaced by γ, and

$$K(T) = \max(1, T^{\frac{\theta}{2}}) \max \Big(1, c(T, \rho_1)^{\frac{\theta(1-\gamma)}{2(\gamma+1)}} \Big).$$

Notably, δ has the value

$$\delta = \frac{3\gamma + 1 + N(1-\gamma)}{2\gamma + 2 + N(1-\gamma)}, \tag{5.29}$$

and therefore $\delta < 1$ if and only if $\gamma < 1$. Property WT is handled analogously.

We conclude with the proof of Lemma 5.7. Multiply the second equation in (5.2) by v^{q-1}, where $q > 1$ will be fixed later. Integrating over $B_{g(\rho,t)}$ and using formula (5.19) we obtain

$$\frac{1}{q} \frac{d}{dt} \int_{B_g(\rho,t)} v^q \leq \int_{B_g(\rho,t)} v^{q-1} \mathcal{F}(u, v) \tag{5.30}$$

Now, if we write

$$\mathcal{F}(u, v) = \mathcal{F}(u, 0) + (\mathcal{F}(u, v) - \mathcal{F}(u, 0))$$

and use the assumption of Lipschitz continuity of $\mathcal{F}$ in v together with either I_2 or I_3—depending on which one is valid—we find

$$\begin{aligned}\int_{B_g(\rho,t)} v^{q-1}\mathcal{F}(u,v) &\le \int_{B_g(\rho,t)} v^{q-1}\mathcal{F}(u,0) + L\int_{B_g(\rho,t)} v^q \\ &\le C\int_{B_g(\rho,t)} v^{q-1}u^\eta + L\int_{B_g(\rho,t)} v^q\end{aligned}$$

and with Young's inequality with exponent q we obtain

$$\int_{B_g(\rho,t)} v^{q-1}\mathcal{F}(u,v) \le \left(L + C\frac{q-1}{q}\right)\int_{B_g(\rho,t)} v^q + \frac{C}{q}\int_{B_g(\rho,t)} u^{\eta q},$$

where

for I_2 we set $\eta = p$ and $C = k_1$, and

for I_3 we set $\eta = \gamma$ and $C = k_3$.

By Gronwall's Lemma it follows that (setting $C' = (q-1)C$)

$$\int_{B_{g(\rho,t)}} v^q(t) \le e^{C't}\int_{B_{\rho_0}} v_0^q + C\int_0^t e^{C'(t-\tau)}\int_{B_{g(\rho,\tau)}} u^{\eta q}(\tau)\,dx d\tau. \tag{5.31}$$

Using this estimate on v, we estimate the integral of $u\mathcal{F}(u,v)$ which appears in (5.21). Again using the decomposition $\mathcal{F}(u,v) = \mathcal{F}(u,0) + (\mathcal{F}(u,v) - \mathcal{F}(u,0))$, the Lipschitz continuity, and Young's inequality, we obtain for any $\epsilon > 0$,

$$\begin{aligned}-\int_0^t\int_{B_g} u\mathcal{F}(u,v) &\le -\int_0^t\int_{B_g} u\mathcal{F}(u,0) + L\int_0^t\int_{B_g} uv \\ &\le -\int_0^t\int_{B_g} u\mathcal{F}(u,0) + \varepsilon L\int_0^t\int_{B_g} u^{\eta+1} \\ &\qquad + \varepsilon^{-1/\eta}L\int_0^t\int_{B_g} v^{\frac{\eta+1}{\eta}},\end{aligned}$$

where we are writing again g for $g(\rho,\tau)$. Now if we use (5.31) with $q = (\eta+1)/\eta$ we obtain

$$\begin{aligned}-\int_0^t\int_{B_g} u\mathcal{F}(u,v) &\le -\int_0^t\int_{B_g} u\mathcal{F}(u,0) + \varepsilon L\int_0^t\int_{B_g} u^{\eta+1} \\ &\quad + \varepsilon^{-1/\eta}e^{C'T}Lt\int_{B_{\rho_0}} v_0^q + \varepsilon^{-1/\eta}e^{C'T}LCt\int_0^t\int_{B_g} u^{\eta+1},\end{aligned}$$

and by rearranging the different terms,

$$-\int_0^t \int_{B_g} u\mathcal{F}(u, v) \le -\int_0^t \int_{B_g} u\mathcal{F}(u, 0) + \varepsilon^{-1/\eta} e^{C'T} Lt \int_{B_{\rho_0}} v_0^q$$
$$+ L\left(\varepsilon + \varepsilon^{-1/\eta} Cte^{C'T}\right) \int_0^t \int_{B_g} u^{\eta+1}. \qquad (5.32)$$

This proves the Lemma.

Appendix 5.A A nonlinear ordinary differential inequality

Lemma 5.8 — *Let $\gamma \ge 0$ and $0 < \delta < 1$.*

1. *Let $\delta \ge 0$ and let $\mathcal{F} \in C([0, \rho_0 + \delta] \times [0, T])$ be a non-negative function such that*

$$\mathcal{F}^{\delta}(\rho, t) \le K t^{\gamma} \frac{\partial \mathcal{F}}{\partial \rho}(\rho, t) + \varepsilon(\rho - \rho_0)_+^{\frac{\delta}{1-\delta}}$$

for almost every $\rho \in [0, \rho_0 + \delta]$ and for all $t \in [0, T]$. Here ε is a non-negative number. If $\mathcal{F}$ is non-decreasing in both arguments, then there exists a time $0 < t^* \le T$ such that $\mathcal{F} = 0$ on $[0, \rho_0] \times [0, t^*]$.

2. *Let $\underline{K_{\rho_0,T}}$ be the cone defined in (5.18) with $\rho = \rho_0$ and $t = T$, and let $\mathcal{F} \in C(\overline{K_{\rho_0,T}})$ be a non-negative function such that*

$$\mathcal{F}^{\delta}(\rho, t) \le K t^{\gamma} \frac{\partial \mathcal{F}}{\partial \rho}(\rho, t) \qquad (5.33)$$

for all $t \in [0, T]$ and for almost all $\rho \in [0, g(\rho_0, t)]$. Then there exists a continuous function $r : [0, T] \to \mathbb{R}$ with $r(0) = \rho_0$ such that

$$\mathcal{F}(\rho, t) = 0 \quad \text{for all } \rho \text{ and } t \text{ such that} \quad \rho \le r(t).$$

Proof. For part 1 we consider an auxiliary function $z = z(\rho)$ that satisfies

$$z^{\delta}(\rho) = K(t^*)^{\eta} \frac{dz}{d\rho}(\rho) + \varepsilon(\rho - \rho_0)_+^{\frac{\delta}{1-\delta}} \qquad (5.34)$$

and

$$z(\rho_0 + \delta) \ge \phi(\rho_0 + \delta, t^*). \qquad (5.35)$$

Here $t^* > 0$ is still to be chosen. It is easily shown that the function

$$z(\rho) = A(\rho - \rho_0)_+^{\frac{\delta}{1-\delta}}$$

satisfies these two conditions if

$$A > \max\{\varepsilon^{\frac{1}{\delta}}, \phi(\rho_0 + \delta, t^*)\delta^{\frac{1}{1-\delta}}\}.$$

In that case t^* is deduced from (5.34):

$$A^\delta = K(t^*)^\eta \frac{A}{1-\delta} + \varepsilon.$$

The statement of the Lemma then follows from the monotonicity of $\mathcal{F}$ in t and ρ.

Part 2 is proved in the following way. Fix $0 < t \leq T$ and suppose that $\mathcal{F}$ is strictly positive on $(\underline{\rho}, g(\rho_0, t)]$. Remark that by (5.33) the function ϕ is non-decreasing in ρ, and therefore $\phi(\rho, t) > 0$ implies $\phi(\cdot, t) > 0$ on $(\rho, g(\rho_0, t)]$. Then for all $\underline{\rho} < \rho \leq g(\rho_0, t)$,

$$(1-\delta)K^{-1}t^{-\gamma} \leq \frac{\partial(\mathcal{F}^{1-\delta})}{\partial\rho}.$$

Now integrate over $(\underline{\rho}, g(\rho_0, t)]$:

$$(1-\delta)K^{-1}t^{-\gamma}(g(\rho_0, t) - \underline{\rho}) \leq \mathcal{F}^{1-\delta}(g(\rho_0, t), t) - \mathcal{F}^{1-\delta}(\underline{\rho}, t),$$

or equivalently,

$$\mathcal{F}^{1-\delta}(\underline{\rho}, t) \leq \mathcal{F}^{1-\delta}(g(\rho_0, t), t) - (1-\delta)K^{-1}t^{-\gamma}(g(\rho_0, t) - \underline{\rho}).$$

Clearly this implies a contradiction if

$$\underline{\rho} < g(\rho_0, t) - \frac{K}{(1-\delta)} t^\gamma \mathcal{F}^{1-\delta}(g(\rho_0, t), t) \tag{5.36}$$

Since $\mathcal{F}$ is continuous on the closed set $\overline{K_{\rho_0,T}}$, $\mathcal{F}^{1-\delta}$ is bounded by a constant $M > 0$ and as a consequence

$$\phi(\rho, t) = 0 \quad \text{if} \quad \rho \leq g(\rho_0, t) - \frac{KM}{(1-\delta)} t^\gamma.$$

•

Appendix 5.B An interpolation-trace inequality

As there exist some slightly different versions of interpolation-trace inequalities, we present here the one that is used in the present work. For further reference see [AD, LSU68]. We denote the norm of the space of Lebesgue integrable functions $L^p(X)$ on a measure space X by $\|\cdot\|_{p,X}$.

Theorem 5.9 — *Let Ω be a bounded domain in $\mathbb{R}^N$ with piecewise smooth boundary Γ and let $u \in W^{1,p}(\Omega)$, $1 < p < \infty$. The following inequality holds:*

$$\|u\|_{q,\Gamma} \leq C(\|\nabla u\|_{p,\Omega} + \|u\|_{\gamma,\Omega})^{\theta} \, \|u\|_{r,\Omega}^{1-\theta}$$

where

$$\theta = \frac{p}{q} \frac{qN - r(N-1)}{p(N+r) - Nr} \in (0, 1)$$

$$1 \leq \gamma < \infty$$

$$1 \leq r < \frac{Np}{N-p} \text{ and } 1 \leq q < \frac{p(N-1)}{N-p} \text{ if } N > p$$

$$1 \leq r, q < \infty \text{ if } p = N$$

$$1 \leq r, q \leq \infty \text{ if } p > N$$

and the constant C depends on Ω and the exponents.

Chapter 6

Blow-up of interfaces

6.1 Introduction

In this chapter we study some properties of solutions of the nonlinear diffusion equation

$$\rho(x)u_t = \Delta A(u) \qquad x \in \mathbb{R}^N,\ t > 0, \tag{6.1}$$

in one and two space dimensions. The nonlinearity A is such that $A' > 0$ on $(0, 1)$ and $A'(0) = A'(1) = 0$; the density function $\rho : \mathbb{R}^N \to (0, \infty)$ is supposed bounded and continuous, and we shall mostly be interested in the case where $\rho(x)$ tends to zero for large $|x|$.

Equations of type (6.1) arise in plasma physics [KR81, RK82], and in hydrology [JdJ81, Bea72, CHDHK89], and in order to set the ideas we shall briefly describe the hydrological model. In the interaction between fresh and salt water in underground aquifers, mixing of the two liquids occurs over length scales much smaller than the size of the aquifer, and in modelling this situation it is therefore generally assumed that a sharp interface separates the liquids. In a horizontal aquifer of even thickness, and under the assumption that the slope of the interface is not too large, the movement of the interface is governed by the equation [JdJ81, Bea72]

$$\varepsilon(x, y)\frac{\mu}{\gamma}\frac{\partial u}{\partial t} - \operatorname{div}\left(\kappa(x, y)\, u(1 - u)\frac{\nabla u}{1 + |\nabla u|^2}\right) = 0. \tag{6.2}$$

Here $u(x, y)$ represents the height of the interface, scaled to take values between zero and one. The constants μ and γ represent the viscosity and the density difference between the fluids, ε is the porosity, and κ is the permeability of the medium.

This chapter is to appear in Advances in Mathematical Sciences and Applications as M. Guedda, D. Hilhorst, & M. A. Peletier, *Disappearing Interfaces in Nonlinear Diffusion*. It also contains the main result of an earlier work [Pel94].

Since we shall mainly be interested in solutions u with relatively small gradients, we replace the quotient $\nabla u/(1+|\nabla u|^2)$ in (6.2) by ∇u. Furthermore, we shall mostly consider either one-dimensional or two-dimensional axially symmetric solutions. In the two-dimensional case with axial symmetry, equation (6.2) reduces to

$$\varepsilon(r)\frac{\mu}{\gamma}u_t - \frac{1}{r}\big(r\kappa(r)\,u(1-u)u_r\big)_r = 0 \tag{6.3}$$

where $r^2 = x^2 + y^2$ and subscripts denote differentiation. If we introduce a new space variable $\tilde{r}$, defined by

$$\log \tilde{r} \stackrel{\text{def}}{=} \int_1^r \frac{ds}{s\kappa(s)}$$

then (6.3) transforms into

$$\rho(\tilde{r})u_t - \frac{1}{\tilde{r}}\big(\tilde{r}u(1-u)u_{\tilde{r}}\big)_{\tilde{r}} = 0 \tag{6.4}$$

in which $\tilde{r}^2\rho(\tilde{r}) = (\mu/\gamma)\,r^2\varepsilon(r)\kappa(r)$. In one space dimension, the equation becomes

$$\rho(x)u_t - \big(u(1-u)u_x\big)_x = 0. \tag{6.5}$$

Both (6.4) and (6.5) are of the form (6.1).

We shall suppose that the degeneration of the nonlinearity A is such that at the values $u = 0$ and $u = 1$ interfaces can appear (we shall henceforth use the term 'interfaces' in the mathematical sense that is common in degenerate diffusion, instead of the physical sense used above). Such is the case for equations (6.4) and (6.5) above. Our main interest in this chapter lies in the behaviour of solutions of (6.1) and their interfaces for large time. This interest was fired by previous works by Kamin and Rosenau [KR81, RK82] on equation (6.1) with single degeneration ($A'(0) = 0$, $A'(s) > 0$ for all $s > 0$). Among other results they showed that as time tends to infinity the solution u converges uniformly on bounded sets to the weighted mean of the initial distribution u_0, i.e. $u \to \bar{u}$ where $\bar{u}$ is given by

$$\bar{u} \stackrel{\text{def}}{=} \frac{\int \rho(x)u_0(x)\,dx}{\int \rho(x)\,dx},$$

provided the numerator of this expression has a finite value. This extends a known result in the case of constant ρ, which states that a solution with finite initial mass decays to zero.

Recently an interesting result has been proved by Kamin and Kersner in [KK93]. They consider equation (6.1) in $\mathbb{R}^N$ with $N \geq 3$, again with single degeneration, and they proved that integrability of ρ on $\mathbb{R}^N$ ($\rho \in L^1(\mathbb{R}^N)$) implies that even if the initial distribution has compact support and therefore the solution also has compact support for small times, there is a time $0 < T < \infty$ such that for $t > T$ the support is no longer compact. This behaviour differs strongly from the case of constant ρ, in which the support of the solution is a compact set for all time $t > 0$. For the same equation a converse result has been proved in [Pel94]: in this paper the author exhibits an explicit supersolution that also has compact support for small time. In the case that ρ is radially symmetric and decreasing in r, the support of this supersolution remains bounded for all time if and only if $r\rho(r) \notin L^1(0, \infty)$. By means of the comparison principle this implies that if $r\rho(r) \notin L^1(0, \infty)$, then a solution of (6.1) with bounded initial support has a bounded support at all finite time.

In this chapter we shall be interested in the Cauchy problem for (6.1) in one and two space dimensions. This dimensional restriction is natural in the case of the hydrological model, and also the mathematical properties that we wish to examine are different for dimensions one and two on one hand and three and higher on the other. Since we will be interested in solutions with interfaces between the regions $\{u = 0\}$, $\{0 < u < 1\}$, and $\{u = 1\}$, we assume that

$$\begin{cases} A \in C^1([0,1]),\ A' > 0 \text{ on } (0,1),\ A(0) = A'(0) = A'(1) = 0, \\ \displaystyle\int_{0^+} \frac{A'(s)}{s}\, ds < \infty \quad \text{and} \quad \int^{1^-} \frac{A'(s)}{1-s}\, ds < \infty. \end{cases}$$

In addition, the density function ρ and the initial data u_0 should satisfy

$$\begin{aligned} &\rho \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ \ \rho > 0 \text{ on } \mathbb{R}^N; \\ &u_0 \in C(\mathbb{R}^N),\ \ 0 \leq u_0 \leq 1 \text{ on } \mathbb{R}^N. \end{aligned}$$

Throughout this chapter we shall suppose that these hypotheses are satisfied.

To our knowledge, existence and uniqueness for the Cauchy problem associated with (6.1) have not yet been proved in the literature. We therefore include these proofs in Appendix 6.A. The uniqueness is a consequence of the following Comparison Principle:

Theorem 6.1 — *Let N be equal to either one or two, and suppose that u_1 is a subsolution and u_2 a supersolution of Problem (P). If $\rho(u_{01} - u_{02})_+ \in L^1(\mathbb{R}^N)$, then $\rho(u_1 - u_2)_+(\cdot, t) \in L^1(\mathbb{R}^N)$ for all $t \geq 0$ and*

$$\int_{\mathbb{R}^N} \rho(u_1 - u_2)_+(\cdot, t) \leq \int_{\mathbb{R}^N} \rho(u_{01} - u_{02})_+$$

for all $t \geq 0$.

The definition of sub- and supersolutions is given in Appendix 6.A.

We prove the following theorems.

Theorem 6.2 (Large-time behaviour) — *Let N be equal to either one or two, and let u be the solution of (6.1) with initial data u_0. If $\rho u_0 \in L^1(\mathbb{R}^N)$, then*

$$u(t) \to \bar{u} \stackrel{\text{def}}{=} \frac{\displaystyle\int_{\mathbb{R}^N} \rho(x) u_0(x)\, dx}{\displaystyle\int_{\mathbb{R}^N} \rho(x)\, dx} \quad as \quad t \to \infty,$$

as $t \to \infty$, uniformly on compact subsets of $\mathbb{R}^N$.

Eidus has remarked in [Eid90] that a similar result holds in the case of a single degeneration in two space dimensions.

Let the support of a function f (supp f) be defined as the closure of the set $\{x : f(x) > 0\}$. A solution u of (6.1) for $N = 1$ is said to exhibit finite time blow-up if its support is bounded from above initially and there exists a time T such that supp $u(t)$ is unbounded from above for all time $t > T$. For the formulation of Theorem 6.3 we shall need an auxiliary density function σ defined by

$$\sigma(x) = \min_{0 \leq \xi \leq x} \rho(\xi),$$

the reason being that the function σ is monotonic while ρ need not be.

Theorem 6.3 (Blow-up in one dimension) — *Let u be a solution of (6.1) for $N = 1$, with non-zero initial data u_0, such that the support of u is bounded from above at time $t = 0$. Then the following implications hold:*

1. $\displaystyle\int_0^\infty x\rho(x)\, dx < \infty \implies$ *finite time blow-up;*

2. $\displaystyle\int_0^\infty x\sigma(x)\,dx = \infty \implies$ *no finite time blow-up.*

If ρ is not decreasing, the two conditions above leave a small gap. In the class of decreasing functions ρ, however, the characterisation is complete:

Corollary 6.4 — *Let the conditions of Theorem 6.3 be satisfied, and suppose in addition that ρ is non-increasing on $[K, \infty)$ for some $K > 0$. Then*

$$\textit{finite time blow-up} \iff \int_0^\infty x\rho(x)\,dx < \infty.$$

It follows from the inversion $\tilde{u} = 1 - u$ that similar statements hold for the interface at $u = 1$. Note that the behaviour of ρ and u_0 towards $-\infty$ has no influence on the (qualitative) behaviour of the upper boundary of the support. We can apply these statements once to $\{x > 0\}$ and once to $\{x < 0\}$ with independent results.

Using the Comparison Principle we can extend this result to a statement on a strip $\Omega = \mathbb{R} \times (-1, 1)$ with Neumann boundary conditions, with a density function ρ that does not depend on the vertical coordinate: $\rho(x, y) = \rho(x)$ on Ω. Consider the problem

$$\begin{aligned} &\rho u_t = \Delta A(u) && \text{in } Q_T = \Omega \times (0, T] \\ &\frac{\partial u}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0, T] \\ &u = u_0 && \text{at } t = 0. \end{aligned}$$

The following result easily follows from the Comparison Princixple:

Theorem 6.5 (Blow-up in a 2d strip) — *Let the initial condition u_0 be such that $u_0(x, y) = 1$ for small x and $u_0(x, y) = 0$ for large x. Let $\zeta_0(t)$ denote the interface between $\{u > 0\}$ and $\{u = 0\}$ at time t:*

$$\zeta_0(t) = \operatorname{supp} u(t) \cap \{(x, y) \in \Omega : u(x, y, t) = 0\}.$$

Then the following statements hold:

1. *If $\displaystyle\int_0^\infty x\rho(x)\,dx < \infty$ then the interface ζ_0 will run off to infinity in finite time;*

2. *If* $\int_0^\infty x\sigma(x)\,dx = \infty$ *then the interface* ζ_0 *will remain bounded for all finite time.*

A similar statement holds for the interface ζ_1 *between the sets* $\{u = 1\}$ *and* $\{u < 1\}$.

A different way of extrapolating the one-dimensional results is by considering the two-dimensional radially symmetric problem and transforming the ensuing (one-dimensional) equation to an equation of the form (6.1). In this case the auxiliary density function σ is different:

$$\sigma(r) = \min_{0 \le \xi \le r} \xi^2 \rho(\xi).$$

We prove the following result:

Theorem 6.6 (Blow-up in 2d, radially symmetric case) — *Let u be a solution of (6.1) with initial condition* u_0. *Suppose that both* ρ *and* u_0 *are radially symmetric, and that* $\operatorname{supp} u_0$ *is compact.*

1. *If* $\int_1^\infty \rho(r) r \log r\, dr < \infty$ *and* $0 \in \operatorname{Int}(\operatorname{supp} u_0)$, *then the support of u ceases to be compact in finite time;*

2. *If* $\int_1^\infty \sigma(r) \frac{\log r}{r}\, dr = \infty$, *then the support of u is compact for all time.*

Corollary 6.7 — *Suppose* u_0 *has compact support and* $0 \in \operatorname{Int}(\operatorname{supp} u_0)$. *If* $r \mapsto r^2 \rho(r)$ *is a decreasing function of r on a neighbourhood of* $+\infty$, *then the support of u becomes unbounded in finite time if and only if*

$$\int_1^\infty \rho(r) r \log r\, dr < \infty.$$

Remark 6.1 The proof of part (ii) of Theorem 6.3 is based on the construction of a supersolution. This construction can be done in all dimensions $N \ge 1$ [Pel94], leading to the following theorem:

Theorem 6.8 — *Let* $N \ge 1$ *and define* $\sigma(r) = \min\{\rho(x) : |x| \le r\}$ *for* $0 \le r < \infty$. *Suppose the solution u of Problem (P) has compact support initially. If*

$$\int_0^\infty r\sigma(r)\, dr = \infty$$

then $\operatorname{supp} u(t)$ *will be bounded for all time* $t \ge 0$.

There is an interesting gap between the statements of Theorem 6.8 for $N = 2$ and Corollary 6.7. Clearly, the condition $r\sigma(r) \notin L^1(0, \infty)$ is too weak in the case of radially symmetric densities. But if we take a density function $\rho = \rho(x, y)$ on $\mathbb{R}^2$ that is only a function of x, i.e. $\rho(x, y) = \rho(x)$, then in the same way as in Theorem 6.5 we can compare it with solutions of the one-dimensional problem. The result of this comparison is that for convenient initial distributions the blow-up of interfaces is *equivalent* with $x\rho(x) \in L^1(0, \infty)$, which implies that the condition $r\sigma(r) \notin L^1(0, \infty)$ is sharp. It is not clear what a general condition for blow-up of interfaces should be in a non-radially symmetric situation. •

Theorem 6.2 is proved in Section 6.2. The blow-up of interfaces in one space dimension (Theorem 6.3) is studied in Section 6.3, and in two space dimensions in Section 6.4 (Theorem 6.6).

6.2 Proof of Theorem 6.2

Theorem 6.2 was proved for the single-degeneration case in one dimension by Rosenau and Kamin [RK82]. We give here a completely different proof which also applies to the case studied by Rosenau and Kamin.

We shall use certain a priori estimates on the solution of Problem (P). The following Lemma is proved in Appendix 6.A:

Lemma 6.9 — *Let u be the solution of Problem (P) with initial function u_0, and set $v = A(u)$. Suppose that $\rho u_0 \in L^1(\mathbb{R}^N)$. Then the following statements hold.*

1. $\displaystyle\int_{\mathbb{R}^N} \rho u(\cdot, \tau) = \int_{\mathbb{R}^N} \rho u_0 \quad \textit{for all} \quad \tau \geq 0 \quad \textit{(conservation of mass);}$

2. $\displaystyle\int_{\mathbb{R}^N} \rho B(v(\cdot, \tau)) + \int_0^\tau \int_{\mathbb{R}^N} |\nabla v|^2 \leq \int_{\mathbb{R}^N} \rho B(v_0) \quad \textit{for all} \quad \tau \geq 0;$

3. $\displaystyle\int_{\mathbb{R}^N} |\nabla v|^2(\cdot, \tau) \leq \frac{c}{\tau} \quad \textit{for all} \quad \tau > 0.$

where $B(s) = \int_0^s \sigma\beta'(\sigma)d\sigma$ with $\beta = A^{-1}$ and $c > 0$ is a constant that does not depend on τ.

Remark 6.2 Estimates as given in Lemma 6.9 are well known in degenerate diffusion problems. The presence of the density function ρ does introduce a novel element, however: the conservation of mass is only true in this form in one and two space dimensions. In fact the main result of [KK93], valid for $N \geq 3$ (see the Introduction), is based on showing that conservation of mass does not hold. •

Proof of Theorem 6.2. It follows from the uniform continuity of the function v (this is a consequence of [DV94], as is shown in the proof of Theorem 6.13) that there exists a sequence $t_n \to \infty$ and a function $\bar{v} \in C(\mathbb{R}^N)$, $0 \leq \bar{v} \leq A(1)$, such that $v(t_n) \to \bar{v}$ as $n \to \infty$, uniformly on compact sets. Now let Ω be an arbitrary bounded set of $\mathbb{R}^N$. Then by Lemma 6.9, part 3,

$$\left\| v(t_n) - \frac{1}{|\Omega|}\int_\Omega v(t_n) \right\|_{L^2(\Omega)} \leq C\|\nabla v(t_n)\|_{L^2(\Omega)} \leq \frac{cC}{t_n},$$

where C is a constant that depends on Ω, so that

$$\bar{v} = \frac{1}{|\Omega|}\int_\Omega \bar{v}$$

for each bounded subset $\Omega \subset \mathbb{R}^N$. Therefore $\bar{v}$ is constant, and $u(t_n) = \beta(v(t_n)) \to \bar{u} \stackrel{\text{def}}{=} \beta(\bar{v})$ as $t_n \to \infty$, where $\beta \stackrel{\text{def}}{=} A^{-1}$. The value of $\bar{u}$ follows from the conservation of mass (part 1 of Lemma 6.9). The fact that this limit is uniquely defined implies the convergence of $u(t)$ as $t \to \infty$. This concludes the proof of Theorem 6.2. •

6.3 Proof of Theorem 6.3

The proof of Theorem 6.3 is based on the comparison principle. First we consider a special case.

Lemma 6.10 — *Let $u_0 \in L^1(\mathbb{R})$, $u_0 \not\equiv 0$, and suppose that the support of u_0 is bounded from above. If $\int_0^\infty x\rho(x)\,dx$ is finite, then there exists a time T after which the support of the solution u is unbounded from above.*

Proof. Define the upper interface function

$$\zeta(t) = \sup\{x \in \mathbb{R} : u(x,t) > 0\}.$$

For the purpose of contradiction we suppose that $\zeta(t) < \infty$ for all $t \in [0, \infty)$. Let the sequence of smooth functions χ_n be such that $\operatorname{supp} \chi_n$ is compact in $(0, \infty)$, χ_n and $|x\chi_n'(x)|$ are bounded uniformly in x and n, and finally $\chi_n \to 1$ and $\chi_n' \to 0$ pointwise on $(0, \infty)$. We substitute the test function

$$\psi(x) = \begin{cases} x\chi_n(x) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}$$

in equation (6.15). Then

$$\begin{aligned} \int_0^\infty x\rho(x)u(x,T)\chi_n(x)\,dx - \int_0^\infty x\rho(x)u_0(x)\chi_n(x)\,dx \\ = \int_0^T \int_0^\infty A(u)\{x\chi_n\}_{xx}\,dxdt \\ = -\int_0^T \int_0^\infty A(u)_x\{\chi_n + x\chi_n'\}\,dxdt. \end{aligned}$$

Note that the function $A(u)_x$ is well-defined by Lemma 6.9. Letting $n \to \infty$ and applying Lebesgue's dominated convergence theorem we deduce that

$$\begin{aligned} \int_0^\infty x\rho(x)u(x,T)\,dx - \int_0^\infty x\rho(x)u_0(x)\,dx &= -\int_0^T \int_0^{\zeta(t)} A(u)_x\,dxdt \\ &= \int_0^T A(u(0,t))\,dt. \end{aligned} \tag{6.6}$$

If $\rho \in L^1(\mathbb{R})$, then by Theorem 6.2, $u(0,t) \to \bar{u} > 0$ as $t \to \infty$. Since the left-hand side of (6.6) is bounded as $T \to \infty$, there exists a sequence $\{t_n\}$, $\lim_{n\to\infty} t_n = \infty$, such that $A(u(0,t_n)) \to 0$ as $n \to \infty$, implying a contradiction. On the other hand, if $\rho \notin L^1(\mathbb{R})$, then by Theorem 6.2 the function $u(\cdot,t)$ converges to zero pointwise on $\mathbb{R}$ as $t \to \infty$. By the dominated convergence theorem we conclude that the first integral in (6.6) tends to zero as $T \to \infty$. At some time T there will be a sign difference between the left and the right hand side of (6.6), again implying a contradiction. •

We now turn to the proof of Theorem 6.3. First consider the case in which $\int_0^\infty x\rho(x)\,dx < \infty$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function such that $\chi(x) = 1$ for all $x > 0$, $\chi(x) = 0$ for all $x < -1$ and $0 \leq \chi \leq 1$ on $\mathbb{R}$. Define $v_0(x) = u_0(x)\chi(x+d)$ for such a value of $d > 0$ that v_0 is not identically equal to zero. Then $v_0 \in L^1(\mathbb{R})$, and $\operatorname{supp} v_0$ is bounded from above. If we denote the solution of Problem (P) with initial data v_0 by v, then Lemma 6.10 implies that supp v will be unbounded from above in finite time. Since by the comparison principle $u \geq v$ on $\mathbb{R} \times \mathbb{R}^+$, the same holds for u.

Now assume that $\int_0^\infty x\sigma(x)\,dx = \infty$. In order to show that the support of u remains bounded for all time, we compare the solution u with a supersolution with bounded support. A similar supersolution was discussed in [Pel94].

Suppose for the time being that $u_0(x) = 0$ for all $x \geq 0$. Let the comparison function w be defined by

$$w(x,t) = \begin{cases} 1 & x \leq 0 \\ \eta^{-1}\left[a\left(1 - x^2/g(t)^2\right)\right] & 0 < x < g(t) \\ 0 & x \geq g(t), \end{cases}$$

where $\eta(s) = \int_0^s A'(\tau)/\tau\,d\tau$, $\eta(1) = a$, and $g : [0,\infty) \to [0,\infty)$ is a function to be specified later. By explicit calculation it follows that the following conditions are sufficient to guarantee that w is a weak supersolution in the sense of Definition 6.12:

$$\rho w_t \geq A(w)_{xx} \qquad \text{for } 0 < x < g(t),\ t > 0 \tag{6.7}$$

$$g'(t) \geq -\frac{1}{\rho(g(t))}\frac{\partial}{\partial x}\eta\big(w\big)(g(t),t) \qquad \text{for all } t > 0 \tag{6.8}$$

$$w(x,0) \geq u_0(x) \qquad \text{for all } x \in \mathbb{R}. \tag{6.9}$$

This follows from the following argument: if $P = \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : |x| < g(t)\}$ and $\Gamma = \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : |x| = g(t)\}$, then it follows from (6.15) that w is a supersolution if

$$-\int_P \{\rho w_t - A(w)_{xx}\}\psi + \int_\Gamma \{\rho w \nu_t - A(w)_x \nu_x\}\psi \leq 0 \tag{6.10}$$

for all appropriate test functions ψ. Here $\nu = (\nu_t, \nu_x)$ is the unit vector normal to P that points outward. If g is differentiable then $\nu_t = -g'(t)\nu_x$, and by conditions (6.7) and (6.8), condition (6.10) is met. The condition (6.9) is necessary to apply the comparison principle (Theorem 6.1).

Inequality (6.9) is satisfied due to our assumption that the support of u_0 is contained in $\{x \leq 0\}$. If we expand (6.7) we find

$$x^2\left\{\frac{\rho(x)g'(t)}{g(t)} - \frac{2a}{g(t)^2}\right\} \geq -A'(w) \quad \text{for} \quad 0 < x < g(t),\ t > 0. \tag{6.11}$$

The right-hand side is non-positive and therefore it is sufficient to require that g satisfy

$$g'(t) \geq \frac{2a}{g(t)\rho(x)} \quad \text{for all} \quad 0 < x < g(t),\ t > 0.$$

With the definition of σ in mind we define g by setting

$$g'(t) = \frac{2a}{g(t)\sigma(g(t))} \quad \text{for all} \quad t > 0 \tag{6.12a}$$

$$g(0) = 1. \tag{6.12b}$$

Since $\partial\eta(w)/\partial x$ takes the value $-2a/g(t)$ in $x = g(t)$, with this definition of g the function w also satisfies (6.8).

Now that the comparison function has been defined, we need to determine the behaviour of its interface $\{(x, t) : x = g(t)\}$. The solution g of the problem (6.12) is given by

$$\int_1^{g(t)} x\sigma(x)\, dx = 2at. \tag{6.13}$$

From the initial assumption $x\sigma(x) \notin L^1(0, \infty)$ it follows that $g(t)$ remains finite for all finite time t. By the comparison principle the same holds for u.

We can relax the condition on the support of u_0 by shifting the supersolution rightwards until the initial distributions u_0 and $w(\cdot, 0)$ are ordered. If w is shifted rightwards by a distance $d > 0$, then the ensuing condition on the behaviour of σ is $\int_d^\infty (x - d)\sigma(x)\, dx = \infty$; since

$$\int_d^\infty (x - d)\sigma(x)\, dx \geq \int_d^{2d} (x - d)\sigma(x)\, dx + \tfrac{1}{2} \int_{2d}^\infty x\sigma(x)\, dx = \infty,$$

this condition is satisfied. This concludes the proof of Theorem 6.3. •

Remark 6.3 If the condition $\int_0^\infty x\sigma(x)\, dx = \infty$ is satisfied, the proof of Theorem 6.3 not only shows that the support of u stays bounded for all time, but also gives a (more or less explicit) bound: $\operatorname{supp} u(t) \subset \{x \in \mathbb{R} : x \leq g(t)\}$, where the function g is given by (6.13). •

6.4 Radial symmetry in two dimensions

Theorem 6.6 is proved by comparison with radially symmetric solutions of the same problem. Let v be a radially symmetric solution of Problem (P). Then

$$\rho v_t = \frac{1}{r}\big(rA(v)_r\big)_r \quad \text{for } 0 < r < \infty,\ t > 0.$$

By the change of variables $s = \log r$ we find

$$\hat{\rho}(s)v_t = A(v)_{ss} \quad \text{for } -\infty < s < \infty,\ t > 0,$$

where $\hat{\rho}(s) = r^2\rho(r)$. Note that $\hat{\sigma}(s) := \min_{0\le\xi\le s}\hat{\rho}(\xi) = \sigma(r)$. Theorem 6.3 states that the behaviour of interfaces depends on the integrability of $s\hat{\rho}(s)$ and $s\hat{\sigma}(s)$ at infinity. This translates in the following way:

$$\int_0^\infty s\hat{\rho}(s)\,ds < \infty \iff \int_1^\infty \rho(r)r\log r\,dr < \infty$$

and

$$\int_0^\infty s\hat{\sigma}(s)\,ds = \infty \iff \int_1^\infty \sigma(r)\frac{\log r}{r}\,dr = \infty.$$

The statement of Theorem 6.6 then follows from Theorem 6.3. Note that the extra condition $0 \in \operatorname{Int}(\operatorname{supp} u_0)$ guarantees that we can find a subsolution with non-trivial support. •

The result of Theorem 6.6 is made possible by the existence of a scaling of the independent variable r ($s = \log r$) that maps the point $r = \infty$ to $s = \infty$ and gives the equation a one-dimensional form. This same scaling maps the point $r = 0$ to $s = -\infty$, which implies that by following exactly the same reasoning we can prove

Theorem 6.11 — *Let u be a solution of Problem (P) with initial condition u_0, let $\rho(x) = \rho(|x|)$, and suppose that $0 \notin \operatorname{supp} u_0$.*

1. *If* $\int_0^1 \rho(r)r\log r\,dr < \infty$, *then after finite time* $\operatorname{supp} u(t)$ *shall contain the point* $x = 0$;
2. *If* $\int_0^1 \sigma(r)\frac{\log r}{r}\,dr = \infty$, *then* $0 \notin \operatorname{supp} u(t)$ *for all time* $t > 0$.

Example. In [AG93] the authors describe a so-called *focusing solution* of the N-dimensional porous medium equation

$$u_t = \Delta u^m \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+. \tag{6.14}$$

The support of this solution contains a hole that shrinks as time increases, disappearing totally at some finite time t^*. The solution that they construct is radially symmetric and of self-similar form: if we set $t^* = 0$, and let v denote the (scaled) pressure associated with (6.14), $v = mu^{m-1}/(m-1)$, then the solution is given by

$$v(r,t) = r^{2-\alpha}\frac{\mathcal{F}(\eta)}{-\eta}, \qquad r > 0,\ t < 0,$$

where the self-similar variable η is given by $\eta = tr^{-\alpha}$. The function $\mathcal{F}$ and the exponent $\alpha \in (1,2)$ are obtained by solving the ensuing ordinary differential equation.

In the case $N = 2$ we can use this solution to construct an explicit example of disappearing interfaces. Again we perform the change of variables $s = \log r$, after which the solution u given by Aronson and Graveleau satisfies the equation

$$\hat{\rho}(s)u_t = (u^m)_{ss} \quad \text{on } \mathbb{R},$$

where $\hat{\rho}(s) = e^{2s}$. Initially—that is, at some finite time before $t = 0$—$\operatorname{supp} u = [-a, \infty)$, where a is a positive number. The transformation $s = \log r$ maps $r = 0$ to $s = -\infty$, and the closure of the hole in the support in the original variables therefore corresponds to a disappearing of the left interface, clearly in finite time. Given the results of this chapter, this also follows directly from the form of $\hat{\rho}$. The interest of this solution lies in the fact that the interface is given *explicitly*. The location of the interface is given by $r = c(-t)^{1/\alpha}$ in the original variables; in terms of s and t, the interface lies at

$$s = \frac{1}{\alpha}\log(-t) + c', \qquad t < 0.$$

Appendix 6.A Well-posedness and a priori estimates

This appendix is devoted to the proofs of existence and uniqueness of the solution of the Cauchy Problem

$$(P)\begin{cases} \rho(x)u_t = \Delta A(u) & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^N \end{cases}$$

in one and two space dimensions. We can write problem (P) in the equivalent form

$$(P_\beta)\begin{cases} \rho(x)\beta(v)_t = \Delta v & \text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ v(x,0) = A(u_0(x)) & \text{for } x \in \mathbb{R}^N \end{cases}$$

where $v = A(u)$ and $\beta = A^{-1}$.

We borrow the definition of a weak solution from [BKP85]. Set $Q = \mathbb{R}^N \times \mathbb{R}^+$, and $Q_T = \{(x,t) \in Q : t \le T\}$.

Definition 6.12 — *The function $u \in C(\bar{Q})$ is a weak solution of Problem (P) if*

1. $0 \le u \le 1$ *on* $\bar{Q}$;
2. u *satisfies the integral identity*

$$\int_\Omega \rho(x)u(x,t)\psi(x,t)\,dx - \int_\Omega \rho(x)u_0(x)\psi(x,0)\,dx$$
$$= \int_0^t \int_\Omega \{\rho u \psi_t + A(u)\Delta\psi\}\,dxd\tau - \int_0^t \int_{\partial\Omega} A(u)\frac{\partial \psi}{\partial \nu}\,dxd\tau \qquad (6.15)$$

for all smooth bounded domains $\Omega \subset \mathbb{R}^N$, for all non-negative functions $\psi \in C^{2,1}(\bar{\Omega} \times [0,T])$ that vanish on $\partial\Omega$ for all $t > 0$.

Weak sub- and supersolutions are defined similarly, after replacement in (6.15) of the equality sign by '$\le$' (for subsolutions) or '$\ge$' (for supersolutions).

We establish the following result.

Theorem 6.13 — *Let N be equal to either one or two. There exists a weak solution of Problem (P).*

Proof. We prove the theorem for $N = 2$, the extension to $N = 1$ being straightforward. We set $\Omega_n = \{x \in \mathbb{R}^2 : |x| < n\}$ and $Q_{nT} = \Omega_n \times (0,T)$ and we consider the problem

$$(P_n)\begin{cases} \rho_n\beta_n(v)_t = \Delta v & (x,t) \in Q_{nT} \\ \dfrac{\partial v}{\partial \nu} = 0 & (x,t) \in \partial\Omega_n \times (0,T) \\ v(x,0) = v_{0n}(x) & x \in \Omega_n \end{cases} \qquad (6.16)$$

in which

1. $\rho_n \in C^\infty(\Omega_n)$, $\rho_n > 0$, and $\rho_n \to \rho$ pointwise in $\mathbb{R}^2$;
2. $\beta_n \in C^\infty([0, A(1)])$, $\beta_n' \ge b_0 > 0$ on $[0, A(1)]$, $\beta_n \to \beta$ uniformly on $[0, A(1)]$, and $\beta_n' \to \beta'$ in $L^1(0, A(1))$;

3. $v_0 = A(u_0)$; $v_{0n} \in C^\infty(\Omega_n)$, $1/n \le v_{0n} \le A(1) - 1/n$, and $v_{0n} \to v_0$ almost everywhere on $\mathbb{R}^2$.

Problem (P_n) has a unique classical solution v_n [LSU68] and it follows from the comparison principle that $1/n \le v_n \le A(1) - 1/n$ on Q_{nT}.

We conclude from [DV94] that there exists a function $v \in C(\bar{Q})$ and a subsequence $\{v_{n_k}\}$ such that $v_{n_k} \to v$ uniformly on $\{|x| \le R\} \times [0, T]$ for all R. We deduce from a similar identity for v_n that v satisfies the integral identity

$$\int_\Omega \rho(x)\beta(v(x,t))\psi(x,t)\,dx - \int_\Omega \rho(x)u_0(x)\psi(x,0)\,dx = \int_0^t\int_\Omega \{\rho\beta(v)\psi_t + v\Delta\psi\}\,dxd\tau - \int_0^t\int_{\partial\Omega} v\frac{\partial\psi}{\partial\nu}\,dxd\tau$$

for all smooth bounded domains $\Omega \subset \mathbb{R}^2$, for all functions $\psi \in C^{2,1}(\bar{\Omega} \times [0, T])$ which vanish on $\partial\Omega$ and for all $t > 0$. The function $u = \beta(v)$ satisfies the assertion of the theorem. •

The proof of Theorem 6.1 that we give here is an adaptation of the proof of a similar property due to Bertsch, Kersner, and L. A. Peletier [BKP85]. It should be noted that although the techniques are similar, there is an interesting effect in the change from one or two spatial dimensions to three dimensions and higher. This is further explained in Remark 6.4.

Proof of Theorem 6.1. Again we only prove the theorem for $N = 2$; the extension to $N = 1$ is straightforward.

Define the functions $w = u_1 - u_2$ and $w_0 = u_{01} - u_{02}$. They satisfy

$$\int_\Omega \rho w(\cdot,t)\psi(\cdot,t) - \int_\Omega \rho w_0\psi(\cdot,0) \le \le \int_0^t\int_\Omega (w\rho\psi_t + (A(u_1) - A(u_2))\Delta\psi) - \int_0^t\int_{\partial\Omega} (A(u_1) - A(u_2))\psi_\nu \tag{6.17}$$

for all appropriate domains Ω and test functions ψ. For the length of this proof we adopt the notation $\psi_\nu = \partial\psi/\partial\nu$. Define $\Omega_n = \{x \in \mathbb{R}^2 : |x| < n\}$ and $Q_{nt} = \Omega_n \times (0, t]$, and the function

$$q(x,t) = \begin{cases} \dfrac{A(u_1) - A(u_2)}{u_1 - u_2} & \text{if } u_1 \ne u_2 \\ 0 & \text{if } u_1 = u_2 \end{cases}$$

Remark that $q \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^+)$, and that $\|q\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^+)} \le \|A'\|_{L^\infty(0,1)}$. We approximate q on Q_{nt} by functions q_n such that

$$n^{-2} \le q_n \le \|q\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^+)} + n^{-2} \text{ on } Q_{nt}; \tag{6.18}$$

$$\|(q_n - q)/\sqrt{q_n}\|_{L^2(Q_{nt})} \to 0 \text{ as } n \to \infty, \tag{6.19}$$

and introduce as test functions the solutions ψ_n of

$$\begin{cases} \rho\psi_t + q_n \Delta\psi = 0 & \text{in } Q_{nt} \\ \psi = 0 & \text{on } \partial\Omega_n \times [0, t] \\ \psi(x, t) = \chi(x) & \text{on } \Omega_n, \end{cases} \tag{6.20}$$

where χ is a fixed function that belongs to $C_c^\infty(\Omega_n)$ for n large enough and takes values in [0, 1]. The density ρ is bounded from below on Q_{nt}, so (6.20) has a unique solution $\psi_n \in C^{2,1}(\bar{Q}_{nt})$. By multiplying the equation for ψ with $\Delta\psi/\rho$ we find that

$$\int_0^t \int_{\Omega_n} q_n (\Delta\psi)^2 \le C \tag{6.21}$$

where C is a constant independent of n.

Using ψ_n as a test function in (6.17) we find that

$$\int_{\Omega_n} \rho\chi w(\cdot, t)\, dx - \int_{\Omega_n} \rho w_0 \psi_n(\cdot, 0)\, dx \le \int_0^t \int_{\Omega_n} (q - q_n)\Delta\psi_n - \int_0^t \int_{\partial\Omega_n} q w \psi_{n\nu}$$

Denote the two integrals on the right-hand side I_1 and I_2. We shall now show that both tend to zero as n tends to infinity. First consider I_1:

$$I_1^2 \le \int_0^t \int_{\Omega_n} \left| \frac{q - q_n}{\sqrt{q}_n} \right|^2 \int_0^t \int_{\Omega_n} q_n |\Delta\psi_n|^2$$

and the right-hand side of this expression tends to zero because of (6.21) and (6.19). To prove that I_2 tends to zero, we compare the function ψ_n with the solution z_n of

$$\begin{cases} \Delta z = 0 & r_0 < |x| < n \\ z = 0 & |x| = n \\ z = 1 & |x| = r_0 \end{cases}$$

where r_0 is such that $\operatorname{supp}\chi \subset \{|x| \le r_0\}$. The solution z_n of this problem is $z_n(x) = (\log n - \log|x|)/(\log n - \log r_0)$. Since both ψ_n and z_n are equal to zero on $|x| = n$, we have

$$0 \le -\psi_{n\nu} \le -z_{n_\nu} \quad \text{on } \partial\Omega_n.$$

Explicitly this implies that

$$|\psi_{n\nu}| \le \frac{1}{n(\log n - \log r_0)}. \tag{6.22}$$

We can then estimate I_2 by

$$|I_2| \le t\|A'\|_{L^\infty(0,1)} \frac{2\pi}{\log n - \log r_0}$$

and the right-hand side of this expression tends to zero as n tends to infinity.

Since by the comparison principle $0 \le \psi_n \le 1$ on Q_{nT}, we can deduce from (6.A) that

$$\int_{\mathbb{R}^2} \rho\chi w(\cdot, t)\, dx \le I_1 + I_2 + \int_{\Omega_n} \rho w_0 \psi_n(\cdot, 0)\, dx$$
$$\le I_1 + I_2 + \int_{\mathbb{R}^2} \rho w_{0+}\, dx. \tag{6.23}$$

The right-hand side of this expression is finite by the hypothesis of the Theorem. Passing to the limit in (6.23) yields

$$\int_{\mathbb{R}^2} \rho\chi w(\cdot, t)\, dx \le \int_{\mathbb{R}^2} \rho w_{0+}\, dx$$

for all $\chi \in C_c^\infty(\mathbb{R}^2)$ such that $0 \le \chi \le 1$. The theorem then follows immediately from this inequality by letting χ converge pointwise to the function $\operatorname{sgn}(w_+)$. •

Remark 6.4 The absence of a uniform lower bound for ρ introduces an interesting effect in the well-posedness of the Cauchy Problem for equation (6.1). If the proof of Theorem 6.1 is rewritten for spatial dimensions different from $N = 2$, the only important difference lies in the explicit function z_n. In one dimension, $z_n(x) = (n - x)/(n - r_0)$, so that $z_n'(n) = -1/(n - r_0)$ tends to zero as $n \to \infty$. This implies that I_2 tends to zero as $n \to \infty$, which is necessary to conclude. However, when $N \ge 3$, $z_n(r) = (r^{2-N} - n^{2-N})/(r_0^{2-N} - n^{2-N})$. In this case, $\int_{\partial\Omega_n} |z_n'|$ remains bounded away from zero, and without an additional assumption on the solution in fact uniqueness does not hold [KK93, Eid90, EK94]. •

Remark 6.5 The proof of the comparison principle still holds when the condition $A \in C^1([0, 1])$ is replaced by $A \in W^{1,\infty}(0, 1)$ and the condition $u_0 \in C(\mathbb{R}^N)$, $0 \le u_0 \le 1$ by $u_0 \in L^\infty(\mathbb{R}^N)$, $0 \le u_0 \le 1$ a.e. on $\mathbb{R}^N$. •

We conclude this appendix with the proof of Lemma 6.9.

Proof of Lemma 6.9. We first prove the second part of the Lemma. By Theorems 6.13 and 6.1 we can obtain v as the limit of functions v_n, which are defined for all $|x| < n$ and $0 \le t \le \tau$. First fix $R > 0$ and set $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. We multiply the differential equation in Problem (P_n) by v_n and integrate on $\{|x| < n\} \times (0, \tau)$:

$$\int_{B_R} \rho_n(x) B_n(v_n(x, \tau))\, dx + \int_0^\tau \int_{B_R} |\nabla v_n|^2$$
$$\le \int_{|x|<n} \rho_n(x) B_n(v_n(x, \tau))\, dx + \int_0^\tau \int_{|x|<n} |\nabla v_n|^2$$
$$= \int_{|x|<n} \rho_n(x) B_n(v_{0n})\, dx, \tag{6.24}$$

where $B_n(s) = \int_0^s \tau\beta_n'(\tau)d\tau$. The condition $\int \rho u_0 < \infty$ implies that the functions v_{0n} can be chosen such that $\int \rho_n B_n(v_{0n})$ is bounded independently of n. Since the function $B_n \circ \beta_n^{-1}$ is Lipschitz continuous with a Lipschitz constant L that does not depend on n, the last term in (6.24) is bounded as $n \to \infty$ and therefore we can extract a subsequence—without changing notation—such that ∇v_n converges weakly in $L^2(B_R \times (0,\tau))$. With the uniform convergence of v_n we can identify the limit as ∇v. Using the dominated convergence theorem and the weak convergence of ∇v_n we can pass to the limit in (6.24) to obtain

$$\int_{B_R} \rho(x)B(v(x,\tau))\,dx + \int_0^\tau \int_{B_R} |\nabla v|^2 \le \int_{\mathbb{R}^N} \rho(x)B(v_0)\,dx.$$

The result then follows from the monotone convergence theorem.

To prove part 1, consider a monotonic cut-off function $\eta \in C^\infty(\mathbb{R})$ such that $\eta = 1$ on $(-\infty, 1]$ and $\eta = 0$ on $[2,\infty)$. Take $\psi(x) = \eta(|x|/R)$ for some $R > 0$ as a test function in (6.15), giving

$$\int_{\mathbb{R}^N} \rho u(\cdot,\tau)\psi = \int_{\mathbb{R}^N} \rho u_0 \psi - \int_0^\tau \int_{\mathbb{R}^N} \nabla v \nabla \psi \tag{6.25}$$

where we have used the fact that $\nabla v \in L^2(\mathbb{R}^N \times (0,\tau))$ by part 2. We can estimate the last integral in (6.25) by

$$R^{N/2-1} \max_{\mathbb{R}} |\eta'| \left(\int_0^\tau \int_{R<|x|<2R} |\nabla v|^2 \right)^{1/2}$$

which tends to zero as $R \to \infty$. The result then follows from an application of the monotone convergence theorem.

To prove part 3, multiply by $t v_{nt}$ the equation satisfied by v_n and integrate:

$$\begin{aligned}\int_0^\tau \int_{|x|<n} t\rho_n\beta_n'(v_n)v_{nt}^2 &= -\frac{1}{2}\int_0^\tau \int_{|x|<n} t\frac{d}{dt}|\nabla v_n|^2 \\ &= \frac{1}{2}\int_0^\tau \int_{|x|<n} |\nabla v_n|^2 - \frac{\tau}{2}\int_{|x|<n} |\nabla v_n|^2(\cdot,\tau),\end{aligned}$$

or

$$\tau \int_{|x|<n} |\nabla v_n|^2(\cdot,\tau) \le \int_0^\tau \int_{|x|<n} |\nabla v_n|^2,$$

after which the result follows from the second part of the Lemma. •

Chapter 7

A self-similar solution in fast diffusion

7.1 Introduction

In this chapter we consider solutions of (1.6) in the case of fast diffusion, i.e. $0 < m < 1$. We shall write the equation as

$$u_t = \operatorname{div}(u^{-n}\nabla u). \tag{7.1}$$

For $n = 1$, equation (7.1) arises in the study of the expansion of a thermalised electron cloud [LH76], in gas kinetics as the central dynamical limit of Carleman's model of the Boltzman equation [Car57, KL80, KLR80, Kur73, McK76], and in ion exchange kinetics in cross-field convective diffusion of plasma [HP58]. In [Kin88] a model is described for the diffusion of impurities in silicon, in which equation (7.1) arises for values of n between 0 and 1.

We shall consider equation (7.1) in $\mathbb{R}^N$, for a spatial dimension N larger than two, subject to an initial condition

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \mathbb{R}^N, \tag{7.2}$$

where the initial distribution u_0 is non-negative and $u_0 \in L^1(\mathbb{R}^N)$.

When $0 < n < \frac{2}{N}$, it is well-known that solutions of the initial value problem (7.1), (7.2) are smooth and exist for all time (see e.g. [Pel81]). For values of $n \geq 1$ no solutions with finite initial mass exist [Váz92b]; in [BC81] it is proved that when $\frac{2}{N} < n < 1$, finite-mass solutions become identically equal to zero in finite time, due to a non-zero flux at infinity. We will be concerned with a special kind of such solutions, namely those which are of a self-similar form. In particular we take $N > 2$ and $\frac{2}{N} < n < 1$, and seek solutions u of (7.1) which vanish at a finite time T, and which are of the form

$$u(x,t) = (T-t)^{\alpha} f(\eta) \quad \text{where} \quad \eta = |x|(T-t)^{-\beta}, \tag{7.3}$$

This chapter has appeared as an article in Differential and Integral Equations [PZ95].

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are constants that need to be determined. Such solutions were also considered by Philip [Phi94] and in more detail by King [Kin93b], who gave a formal motivation for the existence of such solutions, and for the convergence of solutions with arbitrary initial distributions to these self-similar profiles. In this chapter we provide a rigorous proof of King's conjectures concerning existence and uniqueness of self-similar solutions and some of their properties. When $\beta = 0$, the solution u given in (7.3) is separable, and for this case Galaktionov and L. A. Peletier have proved convergence of general finite-mass solutions to the separable one [GP96a]. A similar statement on bounded domains can be found in [BH80].

The character of fast diffusion implies that at any time t at which a solution of (7.1) is not identically equal to zero, it is in fact strictly positive in $\mathbb{R}^N$ and smooth [HP85]. Hence, when looking for solutions of the form (7.3), it is no restriction to assume that $f(\eta)$ is positive and smooth for all $\eta \geq 0$.

Substituting expression (7.3) into (7.1), we find that if we choose

$$\alpha n + 2\beta = 1, \tag{7.4}$$

then f satisfies the equation

$$\eta^{1-N}(\eta^{N-1} f^{-n} f')' - \beta \eta f' + \alpha f = 0 \quad \text{for} \quad \eta > 0. \tag{7.5}$$

Symmetry and smoothness require that

$$f' = 0 \quad \text{at} \quad \eta = 0. \tag{7.6}$$

The restriction that f represent a solution of (7.1) of finite mass translates into the condition

$$\int_0^\infty \eta^{N-1} f(\eta)\, d\eta < \infty. \tag{7.7}$$

One can show that (7.7), when combined with (7.5), is equivalent with the statement that the flux $F(\eta) = \eta^{N-1} f^{-n} f'(\eta)$ has a finite (negative) limit at infinity. This statement is equivalent to the assertion that

$$f(\eta) \asymp \eta^{-(N-2)/(1-n)} \quad \text{as} \quad \eta \to \infty, \tag{7.8}$$

where the notation $a(t) \asymp b(t)$ signifies

$$\lim_{t\to\infty} \frac{a(t)}{b(t)} \quad \text{exists and is positive.}$$

To conclude our preliminary remarks about equation (7.5), note that the scaling

$$\bar{f}(\eta) = \gamma^{-2/n} f(\eta/\gamma) \quad \text{for} \quad \gamma > 0 \tag{7.9}$$

leaves the equation as well as both boundary conditions invariant. Throughout this chapter we therefore set $f(0) = 1$.

Therefore the problem to be studied in this chapter is: Find $f : [0, \infty) \to \mathbb{R}$, positive and smooth, and parameters $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$\eta^{1-N}(\eta^{N-1} f^{-n} f')' - \beta\eta f' + \alpha f = 0, \quad f > 0 \qquad \text{for } \eta > 0 \tag{7.10a}$$

$$f'(0) = 0 \quad \text{and} \quad f(0) = 1 \tag{7.10b}$$

$$f(\eta) \asymp \eta^{-(N-2)/(1-n)} \qquad \text{as } \eta \to \infty \tag{7.10c}$$

$$\alpha n + 2\beta = 1. \tag{7.10d}$$

The relation (7.4) between the two parameters introduced by the Ansatz (7.3) arises from the requirement that f satisfy an equation involving only η. In situations where the problem under consideration satisfies a conservation law (e.g. conservation of mass), this law supplies a second condition on α and β, thus fixing the parameters. In this case we speak of *self-similar solutions of the first kind.* Since we seek solutions that do not conserve mass, there is no second condition on α and β for Problem (7.10). This extra degree of freedom gives it the character of a nonlinear eigenvalue problem: the parameter α (or β) is to be determined together with the solution function f. The function f is then called a *self-similar solution of the second kind* [Bar79].

The main results of this chapter are summarised in the following two theorems. The first one gives existence and uniqueness for Problem (7.10).

Theorem 7.1 — *For every $N > 2$ and $\frac{2}{N} < n < 1$, Problem (7.10) has exactly one solution (f, α, β). Moreover,*

$$0 < \alpha < \frac{N-2}{nN-2}. \tag{7.11}$$

This theorem implies that for every value of n in the given range, there exists exactly one self-similar solution of equation (7.1) of the form (7.3).

The second result concerns the behaviour of the eigenvalues α and β, as given by Theorem 7.1 and equation (7.4), when we vary the parameter n. We indicate the dependence of α and β on n by writing $\alpha(n)$ and $\beta(n)$. Let $n_0 = 4/(N+2)$. We prove the following assertions:

Theorem 7.2 —

1. *$\alpha(n)$ and $\beta(n)$ depend continuously on n;*
2. *$\beta(n_0) = 0$; if $n > n_0$ then $\beta(n) > 0$, and if $n < n_0$ then $\beta(n) < 0$;*
3. *When $n \downarrow \frac{2}{N}$, then $\alpha(n) \to \infty$ and $\beta(n) \to -\infty$;*
4. *When $n \uparrow 1$, then $\alpha(n) \to 0$ and $\beta(n) \to \frac{1}{2}$.*

Theorem 7.2 can be interpreted in the following way. The parameter α determines the decay rate of the maximum of the solution. When n approaches one, $\alpha(n)$ tends to zero, implying that the decay of the solution near $t = T$ is very slow. On the other hand, when n tends to $\frac{2}{N}$, $\alpha(n)$ tends to infinity, signifying a very fast decay rate. The parameter β determines the spread of the profile. When $\beta < 0$, the profile of the solution spreads out as t approaches T, while for $\beta > 0$ the profile shrinks, all mass concentrating in the origin. Because $\beta(n_0) = 0$, the solution u for $n = n_0$ is separable, consisting of a fixed profile multiplied by the factor $(T-t)^{(N+2)/4}$. This situation is very similar to the one considered by Berryman and Holland in [BH80]. In Figure 7.1 the dependence of α and β on n is drawn for $N = 3$.

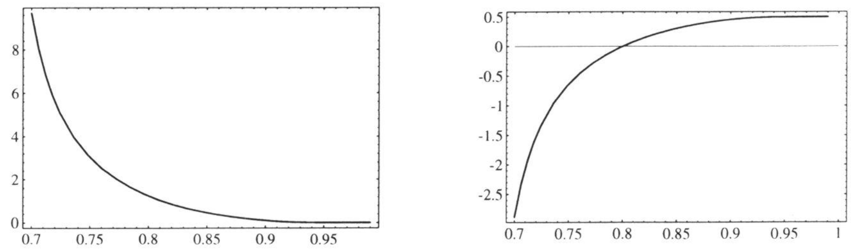

Figure 7.1: The dependence of α (left) and β on n, for $N = 3$ and $0.7 < n < 1$.

To prove these results we first consider in Section 2 an alternative formulation for Problem (7.10). In that section we also derive estimates and properties of solutions that will be used later. In Section 3 we prove the existence and uniqueness of solutions of Problem (7.10) (Theorem 7.1), and in Section 4 we prove Theorem 7.2.

Acknowledgement We wish to thank J. Hulshof for his valuable contribution.

7.2 Preliminaries

Inspired by the analysis of King [Kin93b] we first transform equation (7.5) into a first-order autonomous system. This is the key step in our approach because it allows for an analysis in the phase plane. In particular we concentrate on the first order equation which holds along integral curves in the phase plane.

Let $f \in C^2((0,\infty)) \cap C^1([0,\infty))$ be a positive solution of Problem (P). Then introduce the functions $t, z : (0,\infty) \to \mathbb{R}$, defined by

$$t(\eta) = \tfrac{1}{2}\log(2\eta^{-2}f^{-n}(\eta)) \quad \text{and} \quad z(\eta) = -1 - \frac{n}{2}\frac{\eta f'(\eta)}{f(\eta)}.$$

They are well-defined for all $\eta > 0$, and $\{(t(\eta), z(\eta)) : 0 < \eta < \infty\}$ is a continuously differentiable curve in the t, z-plane. Remark that this curve is invariant under the scaling (7.9). Along the curve we have for $z \neq 0$

$$\frac{dz}{dt} = \left(\frac{2}{n} - 2\right)z - \left(N + 2 - \frac{4}{n}\right) - \left(N - \frac{2}{n}\right)\frac{1}{z} + e^{-2t}\left(\lambda + \frac{1}{z}\right). \tag{7.12}$$

The boundary conditions (7.6) imply

$$t \to \infty \quad \text{and} \quad z \to -1 \qquad \text{as} \quad \eta \downarrow 0, \tag{7.13}$$

and (7.8) yields

$$t \to \infty \quad \text{and} \quad z \to L \stackrel{\text{def}}{=} \frac{nN - 2}{2(1-n)} \qquad \text{as} \quad \eta \to \infty. \tag{7.14}$$

To summarise, every solution f of Problem (P) can be represented as a continuously differentiable orbit in the t, z-plane that satisfies (7.12) and connects the points $(\infty, -1)$ and (∞, L).

For brevity we introduce the notation

$$a = \frac{2}{n} - 2 \quad \text{and} \quad \lambda = 2\beta,$$

and write equation (7.12) as

$$\frac{dz}{dt} = \frac{a}{z}(z - L)(z + 1) + e^{-2t}\left(\lambda + \frac{1}{z}\right). \tag{7.15}$$

A solution of equation (7.15) is locally unique, since for every $(t, z) \in \mathbb{R}^2$, either dz/dt or dt/dz depends on t and z in a Lipschitz continuous manner.

We can immediately use this formulation to restrict the admissible values of λ:

Lemma 7.3 — *Suppose there exists a* $\lambda \in \mathbb{R}$ *and a continuously differentiable orbit* γ *in the* t, z*-plane that satisfies (7.15) and connects the points* $(\infty, -1)$ *and* (∞, L)*. Then*

$$-\frac{1}{L} < \lambda < 1.$$

Proof. We argue by contradiction. First suppose $\lambda > 1$. By the continuity of γ there exists $(t_0, z_0) \in \gamma$ such that

$$z_0 = -\frac{1}{\lambda} \quad \text{and} \quad \frac{dz}{dt} \leq 0 \quad \text{in } (t_0, z_0),$$

which contradicts equation (7.15). If $\lambda = 1$, then the line $z = -1$ is a solution curve of (7.15); we will prove in Lemma 7.8 that for fixed values of λ, an orbit with behaviour (7.13) is unique. In a similar fashion one proves the lower bound: here the contradiction is also on the line $\{z = -1/\lambda\}$, but with the crossing in the other direction. •

Solution curves that satisfy (7.15) have a simple structure. This is the content of the following lemma.

Lemma 7.4 — *If* f *is a solution of Problem (7.10), and* γ *is the corresponding orbit in the* t, z*-plane, then* γ *intersects the* t*-axis exactly once. Furthermore, there exist functions* $z_+(t)$ *and* $z_-(t)$*, such that* $z_+ \geq 0$ *and* $z_- \leq 0$*, and that*

$$\gamma = \{(t, z) : z = z_+(t)\} \cup \{(t, z) : z = z_-(t)\}.$$

It follows immediately from the preceding remarks that the functions z_+ and z_- satisfy (7.15).

Proof. We can write the isocline $\Gamma = \{(t, z) : dz/dt = 0\}$ as the union of $\Gamma_+ = \{z = \phi_+(t)\}$ and $\Gamma_- = \{z = \phi_-(t)\}$, where the functions $\phi_\pm$ are given by

$$\phi_\pm(t) = -\frac{1-L}{2} + \frac{\lambda}{2a}e^{-2t} \pm \frac{1}{2}\sqrt{\left(1 - L + \frac{\lambda}{a}e^{-2t}\right)^2 + 4\left(L - \frac{1}{a}e^{-2t}\right)}.$$

The phase plane is drawn in Figure 7.2; we should remark that $\phi'_+ > 0$ and $\phi'_- < 0$, and that

$$\lim_{t\to\infty} \phi_+(t) = L \quad \text{and} \quad \lim_{t\to\infty} \phi_-(t) = -1.$$

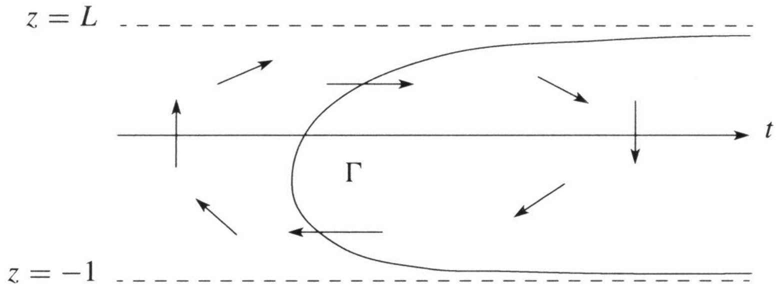

Figure 7.2: The phase plane

From the vector field in Figure 7.2 and the limiting behaviour (7.13) and (7.14) we can deduce that an orbit with more than one intersection with the t-axis has to intersect itself. This is ruled out by the local uniqueness. Any solution curve can therefore be split into two parts, one above the t-axis, and one below. Since dz/dt is finite whenever z is non-zero, the two parts can each be represented by a single-valued function of t, as in Figure 7.3. •

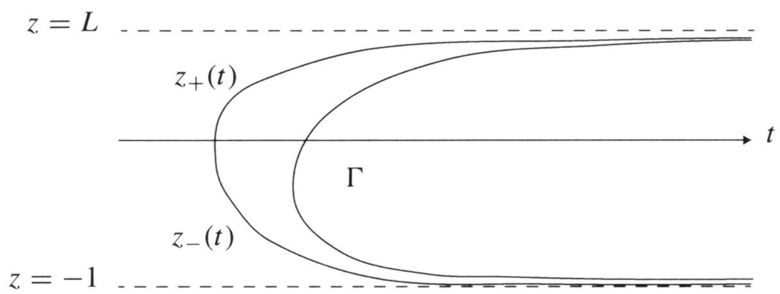

Figure 7.3: A typical solution

The following lemma describes how the functions z_+ and z_- approach their limits as t tends to infinity.

Lemma 7.5 —

1. Let z_+ satisfy equation (7.15) and the asymptotic behaviour $z_+(t) \to L$

as $t \to \infty$. *Then*

$$z_+(t) = L - Ae^{-2t} + x(t)$$

where

$$A = \frac{n}{2}\frac{1+\lambda L}{L+1-n} \quad \textit{and} \quad x(t) = O\left(e^{-4t}\right) \quad \textit{as} \quad t \to \infty.$$

2. *Let* z_- *satisfy equation (7.15) and the asymptotic behaviour* $z_-(t) \to -1$ *as* $t \to \infty$. *Then*

$$z_-(t) = -1 + Be^{-2t} + y(t)$$

where

$$B = \frac{1-\lambda}{N} \quad \textit{and} \quad y(t) = O\left(e^{-4t}\right) \quad \textit{as} \quad t \to \infty.$$

Proof. We only prove the first part; the proof of the second part is similar. First remark that the isocline $\phi_+(t)$ tends to L as $L - a^{-1}e^{-2t}$, and that therefore $L - z_+$ tends to zero at least as fast as $a^{-1}e^{-2t}$. Set

$$z_+(t) = L - Ae^{-2t} + x(t) \tag{7.16}$$

and define $q(t) = x(t)e^{2t}$. By the previous remark, q remains bounded as t tends to infinity. Using (7.16) in (7.15) we find the following equation for q:

$$\begin{aligned} q' &= \left(a\frac{L+1}{L} + 2\right)q + (aq - aA + 1)\frac{L - z_+}{z_+L}. \\ &= \kappa q + \mu(t). \end{aligned} \tag{7.17}$$

Remark that since $z_+(t) \to L$ as fast as e^{-2t}, $|\mu(t)| \leq Ce^{-2t}$ for some constant C. Equation (7.17) implies

$$q(t) = -e^{\kappa t}\int_t^\infty e^{-\kappa s}\mu(s)\,ds,$$

and thus

$$|q(t)| \leq \frac{C}{\kappa + 2}e^{-2t}.$$

•

We now are in a position to prove the equivalence of the two formulations that we have discussed so far.

Lemma 7.6 — *With every solution f of Problem (7.10) correspond functions z_+ and z_-, such that*

1. *z_+ and z_- are defined on $[T, \infty)$ for some $T \in \mathbb{R}$;*
2. *z_+ and z_- satisfy (7.15) on (T, ∞), and $z_+(T) = z_-(T) = 0$;*
3. *$z_+(t) \to L$ and $z_-(t) \to -1$ as $t \to \infty$.*

Conversely, every pair of continuously differentiable functions z_+ and z_- that satisfies the above conditions defines a solution f of Problem (7.10).

In what follows, we shall refer to z_+ as the *upper solution* and to z_- as the *lower solution* of equation (7.15).

Proof. The first assertion was shown in Lemma 7.4; we only need to prove the inverse case. First let us remark that we can choose a parametrisation $(\tilde{t}(\xi), \tilde{z}(\xi))$ of the union of the two curves $S = \{(t, z) : z = z_+(t)\} \cup \{(t, z) : z = z_-(t)\}$, in such a way that

$$\tilde{t}(0) = T \quad \text{and} \quad \tilde{z}(0) = 0; \tag{7.18}$$

$$\tilde{z}(\xi) = z_+(\tilde{t}(\xi)) \quad \text{if } \xi \geq 0, \text{ and } \quad \tilde{z}(\xi) = z_-(\tilde{t}(\xi)) \quad \text{if } \xi \leq 0; \tag{7.19}$$

$$\tilde{t}'(\xi) = \tilde{z}(\xi). \tag{7.20}$$

Indeed, with any point $(\tau, \zeta) \in S$ we associate the parameter value ξ as follows:

$$\xi = \begin{cases} \displaystyle\int_T^\tau \frac{ds}{z_+(s)} & \text{if } \zeta \geq 0 \\ \displaystyle\int_T^\tau \frac{ds}{z_-(s)} & \text{if } \zeta < 0. \end{cases} \tag{7.21}$$

From equation (7.15) we deduce that

$$\lim_{t \downarrow T} \frac{d}{dt} z_+^2(t) = \lim_{t \downarrow T} \frac{d}{dt} z_-^2(t) = -2aL + 2e^{-2T} \tag{7.22}$$

which implies that $1/z_+(t)$ and $1/z_-(t)$ are integrable near $t = T$. Therefore the integrals in (7.21) are well defined. Observation (7.22) also implies that the orbit thus obtained is continuously differentiable for all $\xi \in \mathbb{R}$.

We can then construct the solution f of Problem (7.10) by defining

$$\eta = e^{\xi} \quad \text{and} \quad f(\eta)^n = 2e^{-2\tilde{t}(\xi)}\eta^{-2}. \tag{7.23}$$

From differentiation of (7.23) it follows that f is a solution of equation (7.5). It remains to prove that boundary conditions (7.6) and (7.8) are satisfied. It follows from (7.23) that

$$\frac{d}{d\eta}(f^n)(\eta) = -4\eta^{-3}(1 + \tilde{z}(\xi))e^{-2\tilde{t}(\xi)}.$$

Using the limiting behaviour of z_- (Lemma 7.5) we find that $f'(0) = 0$. This proves (7.6). For the boundary condition at $\eta = \infty$, we calculate

$$\eta^{\frac{n(N-2)}{1-n}} f(\eta)^n = 2e^{2(L\xi - \tilde{t}(\xi))}.$$

The limiting behaviour of z_+ implies that

$$\frac{d}{d\xi}\left(L\xi - \tilde{t}(\xi)\right) = L - \tilde{z}(\xi) \leq 2Ae^{-2\tilde{t}(\xi)} \leq 2Ae^{-L\xi},$$

where the second inequality is true if ξ is large enough, and therefore

$$\lim_{\xi \to \infty} \left(L\xi - \tilde{t}(\xi)\right)$$

is finite. This concludes the proof. •

7.3 Existence and uniqueness

Problem (7.10) is a nonlinear eigenvalue problem: the number β is to be determined together with the solution function f. In this section we prove the following theorem:

Theorem 7.7 — *For every $N > 2$ and $\frac{2}{N} < n < 1$, Problem (7.10) has exactly one solution (f, α, β). Moreover,*

$$-\frac{1-n}{nN-2} < \beta < \frac{1}{2}. \tag{7.24}$$

Note that statement (7.24) is an immediate consequence of Lemma 7.3.

By Lemma 7.6, the assertion of Theorem 7.7 is equivalent to the existence and uniqueness of a number $\lambda \in \mathbb{R}$ and functions z_+ and z_- as described in the Lemma. The proof shall proceed as follows: for every $-1/L < \lambda < 1$ we show that there exist functions z_+ and z_-, solutions of (7.15), which have the prescribed behaviour at $t = \infty$. Both functions intersect the t-axis, but in general at different values of t. For exactly one value of λ, the two half-orbits connect in a continuous way, and therefore define a solution of Problem (7.10).

Lemma 7.8 — *For every $-1/L < \lambda < 1$, the following statements hold:*

1. *There exist unique solutions $z_+(t)$ and $z_-(t)$ of (7.15), defined for t large enough, such that*

$$z_+(t) \to L \quad \text{and} \quad z_-(t) \to -1$$

   as t tends to infinity;

2. *The solutions z_+ and z_- can be uniquely continued for decreasing t as long as they remain non-zero.*

Proof. If we choose t_0 sufficiently large, then $\phi_+(t_0) > 0$, and the part of the phase plane to the right of $t = t_0$ will have a structure as shown in Figure 7.4.

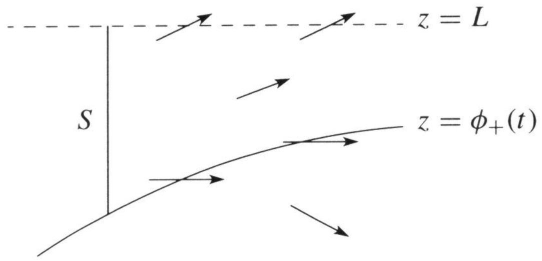

Figure 7.4: The existence proof

Let $\gamma_+(\tau, \zeta)$ denote the orbit in the t, z-plane that starts in (τ, ζ) and continues for increasing t. Define the set $S = \{(t, z) : t = t_0, \ \phi_+(t_0) \le z \le L\}$ and the subsets

$$S_1 = \{(t, z) \in S : \gamma_+(t, z) \text{ intersects the line } z = L\}$$
$$S_2 = \{(t, z) \in S : \gamma_+(t, z) \text{ intersects the curve } z = \phi_+(t)\}.$$

By a classical argument it can be shown that S_1 and S_2 are disjoint, and that both are non-empty and open relative to S. It follows that there exists an $s \in S \setminus (S_1 \cup S_2)$. The orbit $\gamma_+(s)$ then remains between $z = \phi_+(t)$ and $z = L$ for all $t > t_0$. Let z_+ be defined by

$$\gamma_+(s) = \{(t, z) : z = z_+(t),\ t \geq t_0\};$$

since $\phi_+(t) \to L$ as t tends to infinity, it follows that $z_+(t) \to L$ as $t \to \infty$ as well.

To prove the uniqueness of z_+, consider two solutions z_+ and $\bar{z}_+$, and suppose that $\bar{z}_+ > z_+$ on $t > t_0$ (local uniqueness does not permit that solution curves intersect). If we subtract the equations (7.15) for z_+ and $\bar{z}_+$ and integrate the result from $t_1 > t_0$ to $t_2 > t_1$, we find that

$$\left[\bar{z}_+^2 - z_+^2\right]_{t_1}^{t_2} \geq \tfrac{1}{2} a \int_{t_1}^{t_2} (\bar{z}_+^2 - z_+^2)\, dt,$$

provided t_1 is large enough. Letting t_2 tend to infinity yields

$$-\left\{\bar{z}_+(t_1)^2 - z_+(t_1)^2\right\} \geq \tfrac{1}{2} a \int_{t_1}^{\infty} (\bar{z}_+^2 - z_+^2)\, dt.$$

Hence $\bar{z}_+$ and z_+ are equal on $t \geq t_1$.

Because solutions of (7.15) are locally unique as long as z remains nonzero, we can continue z_+ for decreasing t in a unique manner as long as $z_+(t) > 0$.

This proves the theorem as far as z_+ is concerned. The result for z_- is derived in a similar way. •

The uniqueness shown above implies that when $\lambda = -1/L$, the only orbit in the phase plane for which z tends to L as $\xi \to \infty$ is the line $z = L$. Obviously, this orbit can never match up with a lower solution z_-. In a similar way, a solution can not have $\lambda = 1$, either. This proves the strictness of the inequalities of Lemma 7.3.

Define the functions $T_+(\lambda)$ and $T_-(\lambda)$ as follows:

$$T_\pm(\lambda) = \inf\{t \in \mathbb{R} : z_\pm(t) > 0\}.$$

A priori these functions need not be finite, and there is no reason why $T_+(\lambda)$ should be equal to $T_-(\lambda)$ for any λ. The next Lemma leads the way to the conclusion that there exists exactly one value of λ such that $T_+(\lambda) = T_-(\lambda)$.

Lemma 7.9 —

1. *For all* $-1/L < \lambda < 1$, $T_+(\lambda)$ *and* $T_-(\lambda)$ *are finite;*

2. T_+ *is a strictly increasing function of* λ, *and* T_- *a strictly decreasing one;*

3. *We have the following upper bounds:*

$$T_+(\lambda) \le \hat{T}_+(\lambda) \quad \textit{and} \quad T_-(\lambda) \le \hat{T}_-(\lambda),$$

where $\hat{T}_+$ *and* $\hat{T}_-$ *are defined by*

$$e^{-2\hat{T}_+(\lambda)} = \begin{cases} \dfrac{2}{\lambda^2}\{\lambda L - \log(1+\lambda L)\} & \textit{for } \lambda \neq 0 \\ L^2 & \textit{for } \lambda = 0 \end{cases}$$

$$e^{-2\hat{T}_-(\lambda)} = \begin{cases} -\dfrac{2}{\lambda^2}\{\lambda + \log(1-\lambda)\} & \textit{for } \lambda \neq 0 \\ 1 & \textit{for } \lambda = 0; \end{cases}$$

4. $\begin{cases} T_+(\lambda) \to -\infty \textit{ as } \lambda \downarrow -\dfrac{1}{L}; \\ T_-(\lambda) \to -\infty \textit{ as } \lambda \uparrow 1; \end{cases}$

5. $T_+(\lambda)$ *and* $T_-(\lambda)$ *are continuous in* λ.

Proof. We shall only prove the assertions for T_+, as the extension to T_- is straightforward. Assume the converse of part 1 of the lemma: $z_+(t)$ exists and is positive for all $t \in \mathbb{R}$. Since $z'_+(t) > 0$ for all t, this implies $z'_+(t) \downarrow 0$ as $t \to -\infty$. This contradicts equation (7.15).

For part two, suppose that $T_+(\lambda_1) \ge T_+(\lambda_2)$ while $\lambda_1 < \lambda_2$. From Lemma 7.5 we conclude that for t_0 large enough, $z_+(t_0, \lambda_1) > z_+(t_0, \lambda_2)$. Between $t = T_+(\lambda_1)$ and $t = t_0$, the solutions $z_+(t, \lambda_1)$ and $z_+(t, \lambda_2)$ must intersect in such a way that

$$\frac{d}{dt} z_+(t, \lambda_1) \ge \frac{d}{dt} z_+(t, \lambda_2).$$

Again we find a contradiction with equation (7.15).

Part three is proved by considering the solution ζ of the problem

$$\begin{cases} \zeta'(t) = e^{-2t}\left(\lambda + \dfrac{1}{\zeta(t)}\right) & \text{for } t \in \mathbb{R} \\ \zeta(\infty) = L. \end{cases}$$

The function ζ can be calculated explicitly:

$$e^{-2t} = \frac{2}{\lambda^2}\left\{\lambda(L - \zeta(t)) + \log\frac{1 + \lambda\zeta(t)}{1 + \lambda L}\right\}. \qquad (7.25)$$

From (7.25) we calculate that ζ tends to L as $L - (1 + \lambda L)/(2L)e^{-2t}$, which implies by Lemma 7.5 that $z_+(t) > \zeta(t)$ for large t. Then, if t_0 is the largest value of t for which the graphs of z_+ and ζ intersect, we have $z'_+(t_0) \geq \zeta'(t_0)$. This is contradicted by equation (7.15).

Part four follows from the observation that

$$\lim_{\lambda\downarrow -1/L} \frac{2}{\lambda^2}\{\lambda L - \log(1 + \lambda L)\} = \infty.$$

To prove the continuity of T_+ with respect to λ, suppose that for some $-1/L < \lambda_0 < 1$, $\ell_\ell = \lim_{\lambda\uparrow\lambda_0} T_+(\lambda)$ and $\ell_r = \lim_{\lambda\downarrow\lambda_0} T_+(\lambda)$ do not coincide. In proving part 2 of this Lemma we not only showed that T_+ decreases, but also that the function $z_+(t)$ increases when λ increases. We can therefore define the limit functions $w_\ell(t) = \lim_{\lambda\uparrow\lambda_0} z_+(\lambda, t)$ and $w_r(t) = \lim_{\lambda\downarrow\lambda_0} z_+(\lambda, t)$. By considering a weak formulation of (7.15) and passing to the limit in λ, we find that w_ℓ and w_r both satisfy equation (7.15) for $\lambda = \lambda_0$. Since they both lie between $z \equiv L$ and $z = \phi_+(\lambda_0, t)$, and therefore they both tend to L as $t \to \infty$, this is in contradiction with the uniqueness of z_+. •

To conclude the proof of Theorem 7.7, let us draw T_+ and T_- in one diagram (Figure 7.5). Lemma 7.9 guarantees that there is exactly one value of λ such that $T_+(\lambda) = T_-(\lambda)$. For this value of λ, z_+ and z_- match up continuously at $t = T_\pm(\lambda)$. Using Lemma 7.6 we conclude that there exists exactly one solution (f, α, β) of Problem (7.10). •

7.4 Qualitative properties

In the previous section we have proved that for every value of n between $\frac{2}{N}$ and 1, there exists exactly one solution (f, α, β) of Problem (7.10). In this section we study the behaviour of β, or equivalently, $\lambda = 2\beta$, as we vary n. We will write $\lambda^*(n)$ for the value of λ given by Theorem 7.7.

First we prove continuity of λ^* with respect to n.

Lemma 7.10 — *λ^* is a continuous function of n.*

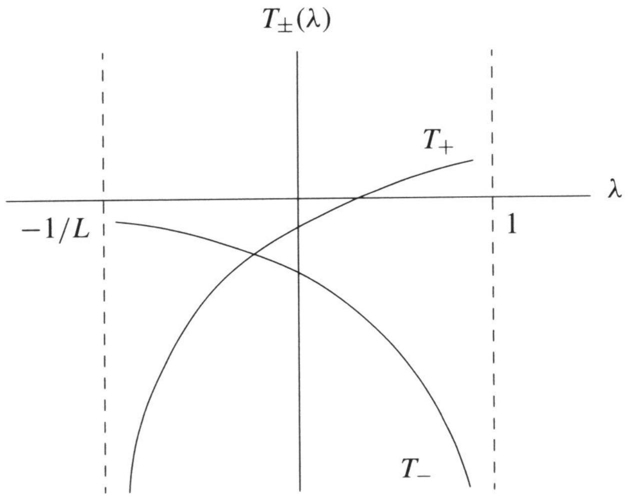

Figure 7.5: The functions T_+ and T_-

Proof. In Lemma 7.9 it was proved that for fixed n, $T_+(\lambda)$ and $T_-(\lambda)$ are continuous functions of λ. One can extend this result in a straightforward way to state that the functions T_+ and T_- are continuous in the variable pair (λ, n) for all (λ, n) in the appropriate range.

Now suppose that λ^* is discontinuous in $\tilde{n}$. Then we can choose a sequence $\{n_i\}$ converging to $\tilde{n}$ such that $\lambda_i = \lambda^*(n_i) \to \lambda_1 \neq \lambda^*(\tilde{n})$. Therefore, by definition, $T_+(\lambda_i, n_i) = T_-(\lambda_i, n_i)$, and

$$0 = \lim_{i\to\infty} \{T_+(\lambda_i, n_i) - T_-(\lambda_i, n_i)\} = T_+(\lambda_1, \tilde{n}) - T_-(\lambda_1, \tilde{n}),$$

which implies that there exists a solution (f_1, λ_1) other than the one given by Theorem 7.7. This is contradicted by the uniqueness. •

It has been known for some time (see [Pel81] or [Kin93a]) that when n equals

$$n_0 \stackrel{\text{def}}{=} \frac{4}{N+2},$$

the solution of Problem (7.10) can be calculated explicitly:

$$f(\eta) = \left(1 + \frac{\eta^2}{4N}\right)^{-\frac{1}{2}N-1}.$$

By substituting f into equation (7.5) one finds that $\lambda^*(n_0) = 0$. The values of λ^* are ordered with respect to $n = n_0$:

Lemma 7.11 — *If $n < n_0$ then $\lambda^*(n) < 0$, and if $n > n_0$ then $\lambda^*(n) > 0$.*

Proof. Suppose that $n < n_0$; when $n > n_0$ the argument is similar. We shall show that $T_+(0) > T_-(0)$. This implies by the monotonicity of T_+ and T_- that $\lambda^* < 0$ (see Figure 7.5).

Let z_+ and z_- be the upper and lower solution of equation (7.15) in which we have set $\lambda = 0$. Then $z_-(t) \to -1$ as $t \to \infty$. Set $\hat{z}_- = -z_-$. Then $\hat{z}_-(t) \to 1$ as $t \to \infty$, and since $L < 1$ because $n < n_0$, it follows that $\hat{z}_-(t) > z_+(t)$ when t is sufficiently large. Plainly, it is enough to show that $\hat{z}_-(t) > z_+(t)$ for all $T_+(0) < t < \infty$.

To prove that this is indeed the case, suppose to the contrary that

$$\tau = \inf\{t > T_+(0) : \hat{z}_- > z_+ \text{ on } (t, \infty)\} > T_+(0).$$

Then

$$\hat{z}_-(\tau) = z_+(\tau) \quad \text{and} \quad \hat{z}'_-(\tau) \geq z'_+(\tau). \tag{7.26}$$

Hence, from (7.15) we deduce that at $t = \tau$,

$$\begin{aligned} \hat{z}'_- &= a\hat{z}_- - a(1-L) - (aL - e^{-2\tau})\frac{1}{\hat{z}_-} \\ &< a\hat{z}_- + a(1-L) - (aL - e^{-2\tau})\frac{1}{\hat{z}_-} \\ &= az_+ + a(1-L) - (aL - e^{-2\tau})\frac{1}{z_+} = z'_+, \end{aligned}$$

which contradicts (7.26). •

The ordering given by Lemma 7.11 has an important consequence for the behaviour of the solution u. When $n > n_0$, $\lambda^*(n) > 0$, which is equivalent with $\beta(n) > 0$, and therefore the solution u given by (7.3) contracts as t approaches T. When $n < n_0$, the profile of u spreads out when $t \to T$. When n equals n_0, η is in fact equal to $|x|$, and the solution u is given by

$$u(x,t) = (T-t)^{(N+2)/4}\left(1 + \frac{|x|^2}{4N}\right)^{-(N+2)/2}.$$

The remainder of this section is devoted to the calculation of the two limits

$$\lim_{n\downarrow\frac{2}{N}} \lambda^*(n) \quad \text{and} \quad \lim_{n\uparrow 1} \lambda^*(n).$$

To simplify the notation, we shall drop the superscript '$*$' from λ^*, and write $T = T(\lambda(n)) = T(n)$ for the common vanishing point of z_+ and z_-.

First we consider the limit process $n \downarrow \frac{2}{N}$. Recall that

$$a = \frac{2}{n} - 2 > 0 \quad \text{and} \quad L = \frac{nN-2}{2(1-n)} > 0.$$

Hence $n \downarrow \frac{2}{N}$ implies that

$$a \uparrow \underline{a} \overset{\text{def}}{=} N - 2 \quad \text{and} \quad L \downarrow 0.$$

Therefore the upper half of the phase plane 'collapses': the line $z = L$ descends to zero. In addition, the isocline $\phi_+(t)$ vanishes for the value of t given by

$$aL = e^{-2t}, \tag{7.27}$$

and this vanishing point clearly 'runs off' to plus infinity when $n \downarrow \frac{2}{N}$. These observations suggest the following scaling of z_+:

$$e^{-2t} = aLe^{-2\sigma} \iff \sigma = t + \tfrac{1}{2}\log(aL) \quad \text{and} \quad w(\sigma) = \frac{1}{L}z_+(t). \tag{7.28}$$

For every n, the function w tends to 1 as σ tends to infinity, and satisfies the following equation (in which primes denote differentiation with respect to σ):

$$\frac{L}{a}ww' = (Lw+1)(w-1) + e^{-2\sigma}(1-\gamma w), \tag{7.29}$$

where we have written γ for $-\lambda L$. Note that by Lemmas 7.3 and 7.11, $0 < \gamma < 1$. The coefficient of the derivative w' in (7.29) tends to zero as $n \downarrow \frac{2}{N}$. We therefore introduce a second scaling of the independent variable. Define Σ as the vanishing point of w (the analogue of T in the variable σ):

$$\Sigma = T + \tfrac{1}{2}\log(aL),$$

and set

$$\sigma = \Sigma + aL\tau \quad \text{and} \quad x(\tau) = w(\sigma).$$

We find that x satisfies the following equation (where the prime now denotes differentiation with respect to τ):

$$\frac{1}{a^2}xx' = (Lx+1)(x-1)e^{-2\Sigma-2aL\tau}(1-\gamma x) \quad \text{for } \tau > 0. \tag{7.30}$$

Before we can continue with this equation, we have to consider the lower part of the phase plane. We shall see later that $\gamma \to 1$, and therefore $\lambda = -\gamma/L \to -\infty$. We can rid equation (7.15) of this parameter blow-up by introducing a scaling of z_- which is different from the one we use for z_+:

$$-\lambda e^{-2t} = e^{-2s} \iff s = t - \tfrac{1}{2}\log(-\lambda) \quad \text{and} \quad y(s) = z_-(t),$$

which results in the equation

$$yy' = a(y-L)(y+1) - e^{-2s}\left(y + \frac{1}{\lambda}\right). \tag{7.31}$$

We also define

$$S = T - \tfrac{1}{2}\log(-\lambda).$$

Note that S and Σ are linked in the following way:

$$S = \Sigma - \tfrac{1}{2}\log(a\gamma). \tag{7.32}$$

Now we are in a position to formulate our result. To facilitate the notation, the functions x and y are defined equal to zero outside of their domain of definition.

Lemma 7.12 — *Let $n \downarrow \frac{2}{N}$. Then*

1. $\gamma \to 1$;

2. $S \to \underline{S} = -\frac{1}{2}\log N$, *which is equivalent to* $\Sigma \to \underline{\Sigma} = \frac{1}{2}\log\dfrac{N-2}{N}$;

3. *x tends to the solution of the problem*

$$\begin{cases} \underline{x}\,\underline{x}' = \dfrac{2}{N-2}(1-\underline{x}) & \text{for } \tau > 0 \\ \underline{x}(\tau) = 0 & \text{for } \tau \le 0; \end{cases}$$

4. y tends to the limit function $\underline{y}$ given by

$$\underline{y}(s) = \begin{cases} -1 + \frac{1}{N}e^{-2s} & \textit{for } s > \underline{S} \\ 0 & \textit{for } s \le \underline{S}; \end{cases} \tag{7.33}$$

Here the convergence of x and y is uniform on compact subsets of the real line.

Proof. In the same way as the existence and uniqueness of solutions was shown by matching the upper half of the phase plane with the lower half, we prove this lemma by studying, separately, first the functions y and then the functions x, and then combining the results.

Step 1: The lower half of the phase plane. Throughout step one *we shall assume that λ is bounded away from zero as $n \downarrow \frac{2}{N}$*. In step two we shall prove, independently of these results, that $\gamma \to 1$ and therefore $\lambda = -\gamma/L \to -\infty$, thereby justifying this assumption.

First we prove that S can not tend to plus infinity as $n \downarrow \frac{2}{N}$. This follows from Lemma 7.9, in the following way:

$$e^{-2S} = -\lambda e^{-2T} \ge -\lambda e^{-2\hat{T}_-(\lambda)} = 2 + \frac{2}{\lambda}\log(1-\lambda),$$

and since we assume that λ stays bounded away from zero, this last expression is positive and bounded away from zero. This implies that S is bounded from above.

Now choose a sequence $\{n_i\}$, converging to $\frac{2}{N}$, such that $S \to \underline{S} \in [-\infty, \infty)$ and $\lambda \to \underline{\lambda} \in [-\infty, 0]$ along that sequence. Equation (7.31) implies that the sequence of functions y^2 is equicontinuous. By the Arzelà-Ascoli theorem we can extract a subsequence such that y^2 converges uniformly on compact subsets of $\mathbb{R}$ along that subsequence. The same holds for the sequence of functions y, because the function $t \mapsto \sqrt{t}$ is uniformly continuous, and the limit function $\underline{y}$ is continuous on $\mathbb{R}$. We integrate equation (7.31) from $s_1 > \underline{S}$ to $s_2 > s_1$:

$$\tfrac{1}{2}y^2(s_2) - \tfrac{1}{2}y^2(s_1) = a\int_{s_1}^{s_2} \left\{(y-L)(y+1) - e^{-2\tilde{s}}\left(y + \frac{1}{\lambda}\right)\right\} d\tilde{s},$$

and by passing to the limit we deduce that the limit function $\underline{y}$ satisfies

$$\begin{cases} \underline{y}\underline{y}' = a\underline{y}(\underline{y}+1) - e^{-2s}\left(\underline{y} + \frac{1}{\underline{\lambda}}\right) & \text{for } s > \underline{S} \\ \underline{y}(s) = 0 & \text{for } s \le \underline{S} \text{ (if } \underline{S} > -\infty). \end{cases} \tag{7.34}$$

If, for the moment, we assume that $\lambda \to \infty$, then we can integrate the equation for $\underline{y}$ to obtain

$$\underline{y}(s) = \begin{cases} -1 + \dfrac{1}{N} e^{-2s} & \text{for } s > \underline{S} \\ 0 & \text{for } s \leq \underline{S}. \end{cases} \tag{7.35}$$

The continuity of $\underline{y}$ implies that $\underline{S}$ is equal to either $-\infty$ or $-\frac{1}{2}\log N$. Since all y are positive, the former is ruled out. We conclude that $S \to -\frac{1}{2}\log N$.

Step 2: The upper half of the phase plane. For all n, the solution z_+ lies above the isocline ϕ_+, and therefore (7.27) implies that $T \leq -\frac{1}{2}\log(aL)$, or $\Sigma \leq 0$. Choose a sequence $\{n_i\}$, converging to $\frac{2}{N}$, such that $\Sigma \to \underline{\Sigma} \in [-\infty, 0]$ along that sequence. Then integrate (7.30) from $\tau_1 > 0$ to $\tau_2 > \tau_1$:

$$\frac{1}{2a^2}\left(x^2(\tau_2) - x^2(\tau_1)\right) =$$
$$= -\int_{\tau_1}^{\tau_2} (Lx+1)(1-x)\,d\tau - e^{-2\Sigma}\int_{\tau_1}^{\tau_2} e^{-2aL\tau}(\gamma x - 1)\,d\tau \tag{7.36}$$

First we use (7.36) to prove that Σ does not tend to minus infinity. Suppose that it does. Then $e^{-2\Sigma}$ becomes very large, while the first two terms in (7.36) remain bounded. Since x and γ both are less than or equal to one, this implies that $\gamma \to 1$ as $n \downarrow \frac{2}{N}$. But then $\lambda = -\gamma/L$ tends to minus infinity, and we have previously calculated that in that case S tends to $-\frac{1}{2}\log N$. Using (7.32), we find that

$$\Sigma \to -\tfrac{1}{2}\log N + \tfrac{1}{2}\log \underline{a} > -\infty,$$

which contradicts the assumption. Therefore Σ is bounded from below (and also from above, since $\Sigma \leq 0$).

It follows that γ can not tend to zero. For if it did, then using (7.32) and the boundedness of Σ, S would tend to plus infinity, a contradiction. This implies that $\lambda = -\gamma/L$ indeed tends to infinity.

To prove the convergence of x, we pass to the limit in equation (7.36). Since γ is bounded between zero and one, we can extract a subsequence such that $\gamma \to \underline{\gamma} \in (0, 1]$. We find the following differential equation for the limit function $\underline{x}$:

$$\begin{cases} \frac{1}{a^2}\underline{x}\,\underline{x}' = -(1-\underline{x}) - e^{-2\underline{\Sigma}}(\underline{\gamma}\,\underline{x} - 1) & \text{for } \tau > 0 \\ \underline{x}(\tau) = 0 & \text{for } \tau \leq 0. \end{cases} \tag{7.37}$$

It follows from (7.37) that

$$\underline{x} \to \frac{e^{-2\underline{\Sigma}} - 1}{\underline{\gamma} e^{-2\underline{\Sigma}} - 1} \quad \text{as} \quad \tau \to \infty. \tag{7.38}$$

Note that

$$\underline{\Sigma} \to -\tfrac{1}{2}\log N + \tfrac{1}{2}\log(\underline{a\gamma}) \leq \tfrac{1}{2}\log\frac{N-2}{N} < 0,$$

and therefore $e^{-2\underline{\Sigma}} > 1$. The limit value (7.38) can only be less or equal to one (as is necessary, since all x are less or equal to one) if $\underline{\gamma} = 1$.

A final remark to conclude the proof. We have liberally taken subsequences to arrive at this result. Because of the exact characterisation of the limit functions $\underline{x}$ and $\underline{y}$ and of the limit value 1 of γ, however, the assertions automatically apply to any sequence. •

In Figure 7.6 the convergence of γ towards 1 is plotted from numerical calculations.

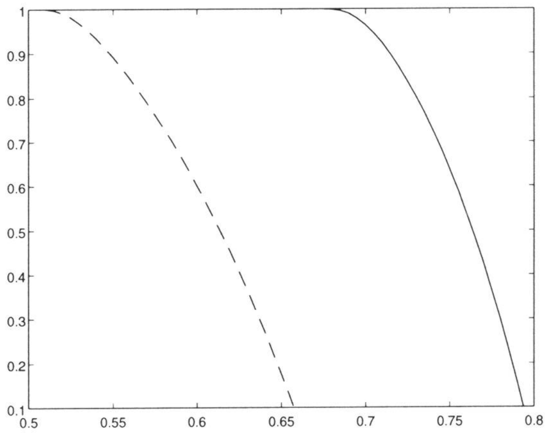

Figure 7.6: Plot of γ against n, for two values of N. The continuous line is for $N = 3$, the dashed line for $N = 4$.

If we translate the results of Lemma (7.12) back in terms of z_+ and z_-, we find the following statements:

Theorem 7.13 — *Let $n \downarrow \frac{2}{N}$, and write $\varepsilon(n) = n - \frac{2}{N}$. Then*

1. $\varepsilon(n)\lambda(n) \to -2\dfrac{N-2}{N^2}$;

2. $T(n) + \frac{1}{2}\log\varepsilon(n) \to \frac{1}{2}\log\Big(2\frac{N-2}{N^3}\Big)$;

3. $\frac{1}{\varepsilon(n)} z_+\Big(T(n) + \varepsilon(n)\frac{N}{n}\tau\Big) \to \frac{N^2}{2(N-2)}\underline{x}(\tau)$ *for all* $\tau \in \mathbb{R}$;

4. $z_-\big(s + \frac{1}{2}\log(-\lambda(n))\big) \to \underline{y}(s)$ *for all* $s \in \mathbb{R}$.

The convergence is uniform on compact subsets of $\mathbb{R}$ *in the variables* τ *and* s.

Let us now direct our attention towards the other limit, $n \uparrow 1$. As n approaches 1, the parameter $a = \frac{2}{n} - 2$ tends to zero and $L = (nN-2)/2(1-n) \to \infty$. Note that $aL \to N-2$. In the previous limit, λ converged to its lower bound $(-\infty)$; here we therefore expect λ to tend to its upper bound, one. We shall show that this is indeed the case.

Since $n \geq n_0$, the values of λ are confined to the interval $[0, 1]$, and we can choose a sequence $\{n_i\}$, converging to one, such that $\lambda \to \bar{\lambda} \in [0, 1]$ along that sequence. When L tends to infinity, $\hat{T}_+(\lambda)$—as defined in Lemma 7.9—tends to minus infinity uniformly in λ, thereby forcing $T_+(\lambda) = T_-(\lambda)$ to minus infinity, too. We shall write $T = T(n) = T_\pm(\lambda(n))$.

With this remark in mind we introduce the following variable transformations:

$$\alpha = e^{2T}, \quad t = T + \alpha\sigma, \quad \text{and} \quad y(\sigma) = z_-(t).$$

This leads to

$$yy' = \alpha a(y - L)(y+1) + e^{-2\alpha\sigma}(\lambda y + 1) \quad \text{for} \quad \sigma > 0, \tag{7.39}$$

while $y(0) = 0$. Define $y(\sigma) = 0$ for all $\sigma < 0$, too. Equation (7.39) implies that the sequence of functions y^2 is equicontinuous as $n \uparrow 1$. The Arzelà-Ascoli theorem then implies that we can extract a subsequence such that the functions y^2, and therefore also the functions y, converge uniformly on compact sets. The limit function $\bar{y}$ is continuous and by passing to the limit in the equivalent integral equation we find that $\bar{y}$ satisfies

$$\begin{cases} \bar{y}' = \bar{\lambda} + \dfrac{1}{\bar{y}} & \text{for } \sigma > 0 \\ \bar{y}(\sigma) = 0 & \text{for } \sigma \leq 0. \end{cases} \tag{7.40}$$

From (7.40) it follows that the limit function $\bar{y}$ tends to $-\bar{\lambda}^{-1}$ as $\sigma \to \infty$, and by the fact that $y \geq -1$ for all σ and n, we conclude that $\bar{\lambda}$ must be equal to one.

The behaviour of z_+ can be retrieved with the following scaling:

$$ae^{-2t} = e^{-2\tau} \iff \tau = t - \tfrac{1}{2}\log a \quad \text{and} \quad x(\tau) = \frac{1}{L} z_-(t),$$

leading to

$$xx' = a(x-1)\left(x + \frac{1}{L}\right) + \frac{1}{aL} e^{-2\tau}\left(\lambda x + \frac{1}{L}\right), \tag{7.41}$$

for $\tau > T - \frac{1}{2}\log a$. Again we set $x(\tau)$ equal to zero for $\tau \leq T - \frac{1}{2}\log a$. The Arzelà-Ascoli theorem yields the convergence of a subsequence of the functions x, uniform on compact subsets of $\mathbb{R}$, to a continuous limit function $\bar{x}$. Define

$$\bar{\tau} = \limsup_{n \uparrow 1} \left(T - \tfrac{1}{2}\log a\right).$$

It follows from the upper bound $\hat{T}_+$ defined in Lemma 7.9 that $\bar{\tau} < \infty$. On $\{\tau > \bar{\tau}\}$ we can pass to the limit in equation (7.41), finding

$$\bar{x}' = \frac{1}{N-2} e^{-2\tau},$$

which results in

$$\bar{x}(\tau) = 1 - \frac{1}{2(N-2)} e^{-2\tau} \quad \text{for all} \quad \tau > \bar{\tau}.$$

The continuity of the limit function $\underline{x}$ now implies that $\bar{\tau} = -\frac{1}{2}\log 2(N-2)$. It follows from the explicitness of this value that $T - \frac{1}{2}\log a$ converges to $-\frac{1}{2}\log 2(N-2)$ along every sequence $n \uparrow 1$.

Let us summarise our results in the

Lemma 7.14 — *Let $n \uparrow 1$. Then*

1. $\lambda \to 1$;
2. $T - \frac{1}{2}\log a \to -\frac{1}{2}\log 2(N-2)$;
3. *x tends to the limit function*

$$\begin{cases} \bar{x}(\tau) = 1 + \dfrac{1}{2(N-2)} e^{-2\tau} & \textit{for } \tau > -\frac{1}{2}\log 2(N-2) \\ \bar{x}(\tau) = 0 & \textit{for } \tau \leq -\frac{1}{2}\log 2(N-2) \end{cases}$$

4. *y tends to the function* $\bar{y}$ *given by*

$$\begin{cases} \bar{y}(\sigma) - \log(1 + \bar{y}(\sigma)) = \sigma & \textit{for } \sigma > 0 \\ \bar{y}(\sigma) = 0 & \textit{for } \sigma \leq 0 \end{cases}$$

Here convergence is uniform on compact subsets of the real line.

Figure 7.7 shows the convergence of λ to 1 as n tends to 1.

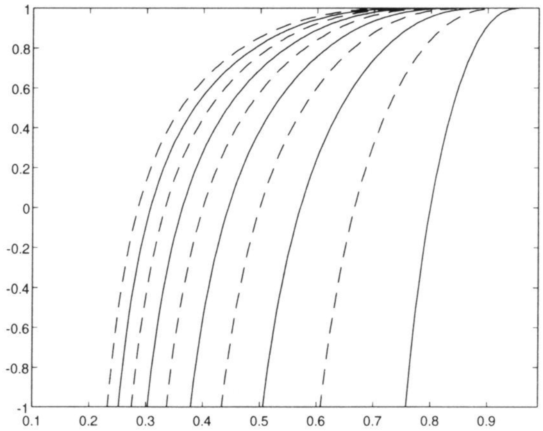

Figure 7.7: Graph of λ as a function of n. The curves are for $N = 3, 4, \ldots, 12$, where the dimension increases from right to left.

We conclude with the translation of these assertions into the original variables. We again define $z_+(t) = z_-(t) = 0$ for all $t < T(n)$.

Theorem 7.15 — *Let* $n \uparrow 1$. *Then*

1. $\lambda(n) \to 1$;
2. $T(n) - \frac{1}{2}\log(1-n) \to -\frac{1}{2}\log(N-2)$;
3. $(1-n)z_+\big(\tau + \frac{1}{2}\log(1-n)\big) \to (N-2)\bar{x}\big(\tau - \frac{1}{2}\log 2\big)$ *for all* $\tau \in \mathbb{R}$;
4. $z_-(T(n) + \alpha\sigma) \to \bar{y}(\sigma)$ *for all* $\sigma \in \mathbb{R}$.

The convergence is uniform on compact subsets of $\mathbb{R}$ *in the variables* τ *and* σ.

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Samenvatting

In dit proefschrift beschouwen we vier wiskundige problemen. Deze vinden hun oorsprong in modellen van vloeistofstroming in poreuze media en hebben niet-lineaire diffusie als gemeenschappelijk kenmerk.

Hoofdstuk 1 heeft als doel een overzicht te geven van de resultaten tegen de achtergrond van bestaande theorie.

Hoofdstukken 2–5 hebben betrekking op een model voor de verspreiding van reactieve stoffen in de bodem als gevolg van grondwaterstroming.

In Hoofdstuk 2 beschouwen we het Cauchyprobleem geassocieerd met dit model. We bewijzen dat dit probleem een unieke oplossing heeft.

In Hoofdstuk 3 gebruiken we dit resultaat om het lange-termijngedrag van oplossingen te bestuderen. We laten zien dat dit gedrag wordt gegeven door een lopende golf.

In Hoofdstuk 4 bestuderen we een probleem dat ontstaat bij het injecteren van water in de grond. Hierbij ontstaat een radiaal stromingsprofiel. We laten zien dat hier het lange-termijngedrag van de oplossing wordt gegeven door een gelijkvormigheidsoplossing. Dit is een speciale oplossing van het probleem die volgens een vast profiel met de tijd uitdijt.

In Hoofdstuk 5 leiden we voorwaarden af die garanderen dat de opgeloste stoffen zich met eindige snelheid verspreiden. Hierdoor ontstaan grensvlakken tussen gebieden met en gebieden zonder opgeloste stoffen.

In Hoofdstuk 6 bestuderen we de grensvlakken van een aanverwant probleem. We leiden een criterium af dat aangeeft of de grensvlakken in eindige tijd verdwijnen.

Hoofdstuk 7 behandelt gelijkvormigheidsoplossingen van een probleem met 'snelle' diffusie. Deze zijn oplossingen van een niet-linear eigenwaardeprobleem. Met behulp van fasevlaktechnieken bewijzen we dat dit probleem een eenduidige oplossing heeft, en onderzoeken we twee limietgevallen.

Hoofdstukken 2–7 zijn—behoudens enkele wijzigingen—transscripties van artikelen:

Hoofdstukken 2 en 3: *Convergence to Travelling Waves in a Reaction-Diffusion System Arising in Contaminant Transport*, ter publicatie aangeboden, met D. Hilhorst;

Hoofdstuk 4: *Asymptotic Behaviour of Solutions of a Nonlinear Transport Equation*, Journal für die reine und angewandte Mathematik **479** (1996), pp. 77–98, met C. J. van Duijn;

Hoofdstuk 5: *Spatial Localization for a General Reaction-Diffusion System*, te verschijnen in Annales de la Faculté des Sciences de l'Université de Toulouse, met G. Galiano;

Hoofdstuk 6: *Disappearing Interfaces in Nonlinear Diffusion*, te verschijnen in Advances in Mathematical Sciences and Applications, met M. Guedda en D. Hilhorst;
A Supersolution for the Porous Media Equation with Nonuniform Density, Applied Mathematics Letters **7** (1994), pp. 29–32;

Hoofdstuk 7: *Self-similar Solutions of a Fast Diffusion Equation That Do Not Conserve Mass*, Differential and Integral Equations, **8** (1995), pp. 2045–2064, met Hongfei Zhang.

Curriculum Vitae

De auteur van dit proefschrift werd op 27 januari 1969 geboren in Brighton in Groot-Brittannië. Hij behaalde het VWO-diploma aan het Stedelijk Gymnasium te Leiden in 1987 en vervolgde zijn opleiding met de studie Wiskunde aan de Rijksuniversiteit Leiden. Als onderdeel hiervan heeft hij het Diplôme d'Études Approfondies d'Analyse Numérique behaald aan de Université Pierre et Marie Curie (Paris VI) te Parijs. De studie Wiskunde in Leiden werd in 1992 met lof afgesloten.

Van 1 december 1992 tot 1 december 1996 was hij werkzaam als onderzoeker in opleiding, eerst aan de Technische Universiteit Delft, en vanaf oktober 1995 aan het Centrum voor Wiskunde en Informatica in Amsterdam. Het onderzoeksproject was getiteld 'Nonlinear Convection and Diffusion of Contaminants in Porous Media' en werd afgesloten met dit proefschrift. Dit project werd uitgevoerd onder begeleiding van prof. dr. ir. C. J. van Duijn.

STELLINGEN

behorende bij het proefschrift
'PROBLEMS IN DEGENERATE DIFFUSION'

MARK A. PELETIER
9 JANUARI 1997

1. In his *Principia Mathematica* Isaac Newton derives the form of the solid of revolution that experiences the least resistance in moving through a 'rare medium' with a constant velocity in the direction of the axis of revolution. We can restate his formulation in modern notation, and we abandon the restriction to axially symmetrical functions:

$$\underset{\substack{u:\Omega\to[0,M]\\ u \text{ concave}}}{\text{Min}} \int_\Omega \frac{dx}{1+|\nabla u|^2}.$$

Here $M > 0$ is a parameter and $\Omega \subset \mathbb{R}^2$ denotes the basis of the solid, which is perpendicular to the velocity.

If $u : \Omega \to [0, M]$ is the solution of this minimisation problem, then u is piecewise linear.

2. Consider the problem

$$\begin{aligned}
&u_t = u_{xx} + \alpha r(t) u_x && x > 0,\ t > 0, && (1)\\
&r(t) = \frac{d}{dt}\int_0^\infty u(x,t)\,dx && t > 0, && (2)\\
&u(0,t) = 1,\ u(\infty, t) = 0 && t > 0, &&\\
&u(x,0) = u_0(x) && x > 0. &&
\end{aligned}$$

Here $\alpha \in \mathbb{R}$ is a parameter.

- If $\alpha > -1$, then for all $0 \le u_0 \le 1$ this problem has a unique solution which is defined and positive for all $t > 0$;
- If $\alpha = -1$, then (1-2) only has constant solutions;
- If $\alpha < -1$, then every solution with finite mass $\int_0^\infty u(x,t)\,dx$ will become identically equal to zero in finite time.

3. The system of equations

$$\begin{aligned}
&u_t - u_{xx} = -k(\varphi(u) - v) && -1 < x < 1,\ t > 0,\\
&v_t = k(\varphi(u) - v) && -1 < x < 1,\ t > 0,\\
&u(-1,t) = -1 \quad \text{en} \quad u(1,t) = 1 && t > 0,
\end{aligned}$$

with $\varphi(s) = -2s + 3s^3$ has a unique solution for every $k > 0$ and initial datum $(u_0, v_0) \in L^\infty(-1, 1)$.

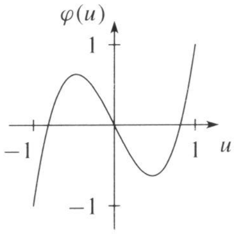

Figure 1: The function $\varphi(u)$.

When $k \to \infty$, the functions u_k converge to a solution of the problem

$$\begin{aligned} &\beta(u)_t - u_{xx} = 0 && -1 < x < 1,\ t > 0, \\ &u(-1,t) = -1 \quad \text{en} \quad u(1,t) = 1 && t > 0, \end{aligned}$$

where $\beta(u)$ is obtained from the function $u \mapsto u+\varphi(u)$ as depicted in Figure 2:

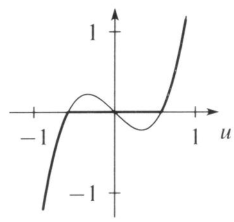

Figure 2: The functions $u \mapsto u + \varphi(u)$ and $u \mapsto \beta(u)$ (thick line).

4. The results obtained in Chapter 6 of this thesis have an elliptical analogue. Let $N \geq 1$ en let $\rho \in L^\infty(\mathbb{R}^N)$, $\rho > 0$, be radially symmetrical if $N \geq 2$. Let u be the solution of the problem

$$\begin{aligned} &-\Delta u + \lambda\rho(x)u^p = f && x \in \mathbb{R}^N, \\ &u(x) \to 0 && |x| \to \infty. \end{aligned}$$

The exponent p satisfies $0 < p < 1$, and $\lambda > 0$ is a parameter. We define I by

$$N = 1: \ I = \int_{|x|>1} \rho(x)\,|x|dx, \qquad N = 2: \ I = \int_1^\infty \rho(r)\,r \log r dr,$$

$$N \geq 3: \ I = \int_1^\infty \rho(r)\,r^{N-1} dr,$$

where $r = |x|$.

Suppose that the support of f is contained in the unit ball in $\mathbb{R}^N$. Then the following holds.

1. If $I = \infty$, then u has compact support for every $\lambda > 0$;
2. If $I < \infty$, then there exists $\lambda_c > 0$ such that u has compact support if and only if $\lambda > \lambda_c$.

5. The solution of the problem

$$u_t + u_x = (u^m)_{xx} \qquad 0 < x < 1,\ t > 0$$

with $m > 1$ for an initial datum $u_0 \geq 0$ and a boundary condition $u(0, t) = u(1, t) = 0$ vanishes in finite time. If T is the extinction time, then the behaviour of the solution u just before $t = T$ is given by a self-similar solution of the form

$$u(x, t) = (T - t)^{\frac{1}{m-1}} f\left(\frac{x}{T - t}\right).$$

6. The most effective way to disseminate the content of a report seems to be by marking it as 'confidential'.

7. A rapidly flashing bicycle light is more effective and at the same time less distracting than a slowly flashing light.

8. The current restriction of telephone numbers to digits is both technically and conceptually out of date.