Stellingen behorende bij het proefschrift

Percolation and Coalescing Particle Systems

Alexei Ermakov

1. Beschouw onafhankelijke punt-percolatie op het vlakke zeshoekige (honingraat-achtige) rooster. Laat $\rho(x, y)$ de lengte zijn van het kortste pad tussen de roosterpunten x en y. Laat O, x_1, x_2 drie roosterpunten zijn zodanig dat $\rho(O, x_1) = 1$, $\rho(O, x_2) = 2$ en zij $C(O)$ de bij O behorende open cluster. Dan geldt:

   $$P_p(\{|C(O)| = \infty\} \cap \{x_1 \text{ is open}\}) \geq P_p(\{|C(O)| = \infty\} \cap \{x_2 \text{ is open}\})$$

   voor iedere $p \in [0, 1]$. [1]

2. Neem vier gehele getallen x_1, x_2, x_3, x_4 zodanig dat $x_1 \leq x_2 \leq x_3 \leq x_4$. Beschouw de graaf G met als knopen alle punten in $([x_1, x_4] \times [0, 1]) \cap \mathbb{Z}^2$ en met kanten tussen elke twee knopen op afstand één. Kleur elke kant óf rood óf blauw met kans 1/2 onafhankelijk van de anderen. Neem een verzameling $C \subseteq ([x_2, x_3] \times [0, 1]) \cap \mathbb{Z}^2$, en noem haar leden de ***wisselpunten***. Neem twee punten, $A \in \{(x_2, 0), (x_2, 1)\}$ en $B \in \{(x_3, 0), (x_3, 1)\}$. Noem A en B ***verbonden middels een rood-rood*** (resp. ***rood-blauw***) ***pad*** als er een pad in G bestaat dat begint in A en eindigt in B, zodanig dat de eerste kant rood en laatste kant rood (resp. blauw) is, en elke twee opeenvolgende kanten van gelijke kleur zijn, tenzij er een wisselpunt tussen de kanten zit. Dan geldt

   $$\begin{aligned} & P(A \text{ is verbonden met } B \text{ middels een rood-rood pad}) \\ \geq\ & P(A \text{ is verbonden met } B \text{ middels een rood-blauw pad}). \end{aligned}$$

[1] Nadat het eerste bewijs van de stelling door de auteur was gevonden, suggereerde J. van den Berg een ander, heel kort bewijs daarvan. Geen van de twee bewijzen kan echter uitgebreid worden voor langere afstanden of voor $\mathbb{Z}^d, d \geq 2$.

3. In the Skorokhod's Representation Theorem (see [1], p.361):

   > [Consider] B_t, a Brownian motion starting from $B_0 = 0$.
   >
   > ...
   >
   > *Skorokhod's Representation Theorem.* If $EX = 0$ and $EX^2 < \infty$, then there is a stopping time T for Brownian motion so that $B_T \stackrel{d}{=} X$.

   the condition "$EX = 0$" is unnecessary and can be removed.

[1] DURRETT, R. (1991). *Probability: theory and examples*, Wadsworth.

4. It is very likely that all the known forms of life on our planet have come into existence at one single place. The probability that it happened at n different places independently is exponentially bounded in n. Indeed, every known form of life has the same optical dissymmetry. For instance, DNA molecules always have the shape of a *right* helix. The emergence of each of the two mirror-symmetric forms is equally probable. Hence we have

$$P\left(\begin{array}{l|l} \text{All known forms of life} & \text{the life forms we can ob-} \\ \text{on Earth have the same} & \text{serve came into existence} \\ \text{optical dissymmetry} & \text{at } n \text{ places independently} \end{array}\right) = \frac{1}{2^{n-1}}, \quad n \geq 1.$$

5. The more time one has the less usefully one spends it.

6. The less you know of other people the easier it is to critisize them.

7. A research institute should be accessible 24 hours a day.

Theorems, as a supplement to the thesis

Percolation and Coalescing Particle Systems

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1. Consider independent site percolation on the planar hexagonal (honeycomb) lattice. Let $\rho(x,y)$ be the length of the shortest path between the sites x and y. Let O, x_1, x_2 be three sites of the lattice, such that $\rho(O,x_1) = 1$, $\rho(O,x_2) = 2$ and denote by $C(O)$ the open cluster containing O. Then the inequality

   $$P_p(\{|C(O)| = \infty\} \cap \{x_1 \text{ is open}\}) \geq P_p(\{|C(O)| = \infty\} \cap \{x_2 \text{ is open}\})$$

   holds for any $p \in [0,1]$. [1]

2. Fix four integers x_1, x_2, x_3, x_4 such that $x_1 \leq x_2 \leq x_3 \leq x_4$. Consider the graph G with vertex set $([x_1, x_4] \times [0,1]) \cap \mathbb{Z}^2$ and edges between each two vertices on distance one from each other. Colour each edge either red or blue i.i.d. with probability 1/2. Fix a set $C \subseteq ([x_2, x_3] \times [0,1]) \cap \mathbb{Z}^2$, and call its elements *change points.* Fix two vertices, $A \in \{(x_2,0),(x_2,1)\}$ and $B \in \{(x_3,0),(x_3,1)\}$. Call A and B *connected by a red-red* (resp. *red-blue*) *path* if there exists a path in G which starts at A and ends at B such that the first edge is red and the last edge is red (resp. blue), and each two successive edges have the same colour unless they have a *change point* inbetween. Then

   $$\begin{aligned} &P(A \text{ is connected to } B \text{ by a red-red path}) \\ \geq\ &P(A \text{ is connected to } B \text{ by a red-blue path}). \end{aligned}$$

[1] After the first proof of the theorem was found by the author, J. van den Berg suggested another, very easy proof of it. Neither of the proofs, however, can be extended to longer distances or to $\mathbb{Z}^d, d \geq 2$.

3. In de Skorokhod Representatie-Stelling (zie [1], p.361):

   > [Beschouw] B_t, een Brownse beweging beginnend in $B_0 = 0$.
   >
   > ...
   >
   > *Skorokhod Representatie-Stelling.* Als $EX = 0$ en $EX^2 < \infty$, dan is er een stop-tijd T voor de Brownse beweging zodanig dat $B_T \overset{d}{=} X$.

   is de voorwaarde "$EX = 0$" overbodig en kan dus weggelaten worden.

   [1] DURRETT, R. (1991). *Probability: theory and examples*, Wadsworth.

4. Alle bekende vormen van leven op onze planeet zijn naar alle waarschijnlijkheid op één enkele plaats ontstaan. De kans dat het op n verschillende plaatsen onafhankelijk gebeurde is exponentieel begrensd in n. Immers, alle bekende vormen van leven hebben een identieke optische dissymmetrie. Bijvoorbeeld, de moleculen van het DNA hebben altijd de vorm van een *rechtse* spiraal. Beide spiegel-symmetrische vormen zouden met dezelfde kans (1/2) kunnen ontstaan. Dus wij hebben

$$P\left(\begin{array}{l|l} \text{Alle bekende vormen van le-} & \text{de bekende vormen van} \\ \text{ven op aarde hebben een iden-} & \text{leven zijn onafhankelijk} \\ \text{tieke optische dissymmetrie} & \text{ontstaan op } n \text{ plaatsen} \end{array}\right) = \frac{1}{2^{n-1}}, \; n \geq 1.$$

5. Hoe meer tijd men heeft, hoe minder doelmatig men haar besteedt.

6. Hoe minder goed je anderen kent, hoe makkelijker het is hen te bekritiseren.

7. Een onderzoeksinstituut zou 24 uur per dag toegankelijk moeten zijn.

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AND
COALESCING PARTICLE SYSTEMS

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Dr. J. van den Berg heeft als begeleider in belangrijke mate aan het totstandkomen van het proefschrift bijgedragen.

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Chapter 1

Introduction

In this thesis we explore qualitative and quantitative properties of certain particle systems. We mainly deal with systems of coalescing particles, discrete and continuous in space and time, and with site percolation. The main part of this thesis consists of four articles which may be read independently of each other. The purpose of this introductory chapter is threefold. At first, we give the reader the notion of the considered models and some important background. Then we discuss briefly the main results of the thesis. And finally, we relate the following chapters to each other, and present some ideas and results which motivated the choice of the topics of our research, but are not included in the four articles.

The models we study in this thesis either come from or are related to statistical physics. Statistical physics deals with large random media, or large systems with random fine structure. A typical example may be a piece of an inhomogeneous material. Such systems have two 'levels': microscopic and macroscopic. On the microscopic level we see 'the building blocks' of the system. They may be atoms or molecules, magnetic domains, organic cells, neurons, individuals in a population, trees in a forest, nodes in a telephone or computer network, messages in these networks etc. The number of these building blocks in the models either is infinite or tends to infinity. These building blocks can be identical to each other, or all of them can be elements of the same simple state space. Their interaction is usually local, quite simple, and assumed to be known. The microscopic configuration must have some sort of randomness. If a system is dynamical, i.e. it evolves in time, then either the initial state, or the dynamics, or both, may include randomness. In dynamical particle systems, particle positions can be either fixed or changing in time. The first way of representation is mainly used for models of solid state, while the second one can be appropriate in liquid or gas phase models. We shall use both approaches, sometimes even for one and the same model, in order to better apprehend its properties.

On the macroscopic level we look at global properties of the system. It may be statistical properties like density of a certain type of particles, conductivity, free energy, pressure or magnetisation. It may also be the existence of a global structure, like an unbounded connected component, and properties of such a structure.

These macroscopic properties are usually non-random. The simplest case in which this can be seen is the Kolmogorov 0-1 law (Kolmogorov(1933), see [Fel68b], p.122):

If an event A is determined by a countable system of mutually independent random variables, and A is independent of any finite subset of them, then either $P(A) = 0$ or $P(A) = 1$.

What are the macroscopic properties of a system and how do they depend on the microscopic properties and the system parameters – that is the main question statistical physics tries to answer.

1.1 Percolation

Percolation theory studies connectivity properties of large random graphs. The first model we consider is the classical *site percolation* model on the d-dimensional integer lattice. This lattice is the graph with vertex set $\mathbb{Z}^d$ and edges between each pair of *nearest neighbours*, i.e. the vertices on Eucledian distance one from each other. Each vertex can be either *open* or *closed.* It is open with probability $p \in [0, 1]$, independently of the others.

A *path* is a finite or infinite sequence of vertices such that every two consecutive vertices are adjacent in the graph (i.e. they are nearest neighbours). A path is called *open* if all its vertices are open. Let C be the *open cluster* of the origin, i.e the maximal connected component, containing the origin, of the subgraph of the integer lattice induced by the set of open vertices. The vertex set of C contains those vertices from which there is an open path to the origin. If the origin is closed, then C is the null-graph. The macroscopic property we are looking at in this model is whether or not C can be infinite. We say that *percolation occurs* if $P(|C| = \infty) > 0$.

In a similar model called *bond percolation* the edges instead of vertices are open or closed. It was observed by Fisher and Essam [FE61] that site percolation models are more general, because bond percolation can always be reformulated as site percolation on an appropriate lattice.

The percolation model was first introduced by Broadbent and Hammersley [BH57]. They proposed it as a model of a porous medium, namely the activated carbon which is used in gas mask filters. The open vertices represent pores (empty spaces), and the closed vertices represent solid spaces in the medium. The occurrence of percolation means that there is a global system of connected pores, and a gas can filter through the medium.

Percolation is one of the simplest statistical physics models, and it has been intensively studied over the last 40 years, and especially after a breakthrough due to Kesten [Kes80]. For recent surveys please see [Gri89] and [Gri96]. The percolation model has served as a playground for developing new approaches and techniques, which later have found many other applications within probability, statistical physics, combinatorics and ergodic theory. Despite all the thorough research efforts, some key questions about the model remain still unanswered. One of these questions is whether or not the percolation function which is defined below is continuous in dimension three and higher. This continuity is proven in $d = 2$ and $d \geq 19$ (see [Har60], [HS90], [HS94]).

The percolation model and its derivations have numerous direct applications in natural sciences, and they also serve as an instrument for studying other statistical physics models (see e.g. Bunimovich and Troubetzkoy [BT92], van den Berg and Maes [BM94], Grimmett, Menshikov and Volkov [GMV96]). Another example is discussed in Chapter 3 of this thesis.

The percolation model has one parameter, the probability p with which the vertices are open. The *percolation function* $P_p(|C| = \infty)$ is clearly increasing in p, and hence there exists the *critical probability* $p_c = \inf\{p : P_p(|C| = \infty) > 0\}$, such that percolation occurs whenever $p > p_c$.

It has been known since Hammersley [Ham57], [Ham59] and Fisher [Fis61] that $0 < p_c < 1$ for $d \geq 2$, but often an exact value or sharp bounds are desired. For bond percolation on the square lattice (and also for site percolation on the 2-dimensional triangular lattice) the critical probability is known to be exactly $1/2$ [Kes80]. This is due to the "self-duality" of the first lattice and the "self-matching" property (introduced in [FE61], [Fis61]) of the second one. For almost all other cases, including our case of *site percolation on the square lattice* ($\mathbb{Z}^2$), the exact value of the critical probability is not known. Several upper and lower bounds have been found. The best known so far are

$$p_c > 0.5416 \quad \text{(Menshikov and Pelikh [MP89])},$$
$$p_c < 0.6819 \quad \text{(Zuev [Zue87])}.$$

In Chapter 2 of the thesis we streamline the Menshikov and Pelikh reasoning and improve their lower bound to 0.556 (see Theorem 1.1), which is notably closer to the Monte-Carlo estimated value of 0.593. Along with the original geometric idea of [MP89] we extensively use coupling and stochastic dominance of multivariate probability distributions to obtain our bound for p_c.

1.2 Systems of coalescing particles

Percolation is an example of a static 'particle system'. Let us now consider systems which evolve with time. Time in these systems may be either discrete or continuous. In the first case such systems are called *probabilistic cellular automata* (see [Wol86]), in the second case – *(continuous time) interacting particle systems* (see [Lig85]).

First we consider the cellular automata. A probabilistic cellular automaton is a discrete-time Markov chain with state space $S^{\mathbb{Z}^d}$. Here d is dimension, and S is the single-site configuration space. Typical choices for S are $\{0,1\}$, $\mathbb{Z}^+$ or $\mathbb{R}^+$. Let us denote by $\xi_{x,t}$ the state of the site $x \in \mathbb{Z}^d$ at time $t \in \mathbb{Z}^+$. We consider models with random initial state and/or random dynamics. At each time step, all the site states are updated simultaneously, according to an *updating rule*. This rule must be

- *local,* i.e. the new site state $\xi_{x,t+1}$ depends only on the neighbour states $\xi_{y,t}$, for y within a fixed distance from x,
- *space-time homogeneous,* i.e. invariant under space and time shifts, and
- *independent* for different sites. i.e. the conditional distributions of the updated states$\{\xi_{x,t+1}\}_{x\in\mathbb{Z}^d}$ given $\xi_{\cdot,t}$ are independent.

A special class of (probabilistic) cellular automata are *coalescing particle systems.* Let S be either $\mathbb{Z}^+$ or $\mathbb{R}^+$, and let positive $\xi_{x,t}$ represent the mass of a particle at the site x at time t. If $\xi_{x,t} = 0$, we say that the site x is empty. At each time t each occupied vertex x chooses a destination $a_t(x)$ of the movement of the particle from x. $a_t(x)$ must satisfy the same requirements as the updating rule above: it must be local, space-time homogeneous, independent for different sites and 'Markovian', i.e. given the present, independent of the past. The *displacement* $||a_t(x) - x||_1$ must be uniformly bounded (in all our examples it is bounded by one). At each time step, all the particles jump simultaneously to their destinations, thus making

$$\xi_{x,t+1} = \sum_{y:\, a_t(y)=x} \xi_{y,t}. \tag{1.1}$$

If two or more particles move to the same site x at time t, then they *coalesce*, i.e. merge into one single particle with mass equal to the sum of the incoming masses. Hence we call such a system discrete-time *coalescing weighted particle system.* If we map S into $S' = \{0,1\}$ by

$$\xi'_{x,t} = \mathbf{1}_{\{\xi_{x,t}>0\}}, \tag{1.2}$$

we obtain a discrete-time *coalescing weightless particle system.* Several examples of such systems will be considered in the following sections. Note that the process $\xi'_{\cdot,t}$, $t \geq 0$ is in general not Markovian.

Coalescing particle systems are natural models for a wide variety of aggregation processes. Examples may include coagulation of milk or other suspensions, polymerisation, nest construction by social insects, flows of capital and monopolisation, processes of collecting information or picking up packages for shipment, formation of stars or evolution of a system of black holes etc. Also, as it happens with percolation, the coalescing particle systems may be useful for studying other statistical physics objects.

1.3 Clustering process of Coffman, Courtois et al.

In Chapter 3 of the thesis we consider a simple particular example of a coalescing weighted particle system, namely the *distributed clustering process* introduced by Coffman, Courtois, Gilbert and Piret [CCGP91]). In this model the particle masses (or 'resources') $\xi_{x,t}$ are elements of $\mathbb{R}^+$, and at each time every vertex gives its resource to the 'richest' (heaviest) vertex among itself and its nearest neighbours:

$$a_t(x) \in \operatorname*{Argmax}_{y:||y-x||_1\leq 1} \xi_{y,t} := \{y : ||y-x||_1 \leq 1,\, \xi_{y,t} = \max_{z:||z-x||_1\leq 1} \xi_{z,t}\}.$$

If the set in the r.h.s. has more than one element, $a_t(x)$ is chosen from them randomly, with equal probabilities. So the vertices with large resources tend to attract resources from their neighbours.

The initial distribution of resources is i.i.d. If it has no atoms, then the evolution is deterministic a.s., and all the randomness is contained in the initial state of the system.

The behaviour of this simple model resembles a number of different clustering processes occurring in nature, for instance the construction of nest or honeycomb by social insects, as explained in [CCGP91].

As time grows, the local configuration of resources stabilise at least at some places. A natural question to ask is:

Question A. *Does the resource of each vertex change only finitely many times a.s.?*

The authors of the article [CCGP91] deal mainly with the one-dimensional case, in which the answer is obviously positive. They also make some remarks about higher dimensions, in which they implicitly assume that the answer is still positive in these cases, and that 'the resources do not get lost', i.e. the following question has also positive answer:

Question B. *Is the expectation of the final resource value at any site equal to the expectation of the initial value?*

However, neither assertion is rigorously proven so far. The questions (A) and (B) are central for our further discussion. They were first pointed out by van den Berg and Meester [BM91], who proved some weaker stability results. In particular, they proved that, for the Coffman-Courtois clustering process,

- $\lim_{t\to\infty} \xi_{x,t}$ exists and is finite for any x a.s.;

- If $d = 2$, and the initial resource distribution has no atoms, then for any $x \in \mathbb{Z}^2$ eventually either $\xi_{x,t} = 0$ or $a_t(x)$ remains constant.

The proofs of these results are based on ergodicity and symmetry arguments.

These results do not rule out the existence of *limiting flows*, or directed nearest-neighbour chains of vertices, in which every vertex eventually stays occupied and passes its resource (or, particle) to the next one in the chain. Note that at any fixed time such chains are a.s. finite. However, the length of such a chain can grow with time. Any attempts to prove that limiting flows do not exist a.s. have failed so far (see [BM91]). If such flows do exist, then Question A has negative answer. Moreover, these flows can 'carry some resources away to infinity', so that Question B may also have negative answer. Example 2 in Section 1.4 below shows that (B) can fail if we replace the i.i.d. initial condition by a translation invariant one.

In Chapter 3 we study the same clustering process by using a percolation approach. We consider the corresponding coalescing weightless particle system ξ' (1.2) at a small fixed time t. The 'percolation of ones' of $\{\xi'_{x,t}\}_{x\in\mathbb{Z}^d}$ is the percolation model in which a vertex x is declared open if $\xi'_{x,t} = 1$ and closed otherwise. This percolation model is *finite dependent*: if two sets of vertices are more than $4t$ edges away from each other, than the resource values on these sets are mutually independent.

It is trivial to see that all closed vertices are 'frozen': if $\xi'_{x,t} = 0$ then $\xi'_{x,s} = 0 \; \forall s \geq t$. Therefore, after time t no transition of resources can occur between the clusters of ones of $\xi'_{\cdot,t}$. If $\xi'_{\cdot,t}$ does not percolate, then all these clusters are finite, and therefore the answers on questions (A) and (B) are positive.

So we need to study the (finite dependent) percolation of ones in the coalescing weightless particle system $\xi'_{\cdot,t}$, and to show that for some $t > 0$ (for instance for $t = 1$) percolation does not occur. One obvious approach is to compare $\xi'_{\cdot,t}$ with the independent percolation model described in Section 1.1, and to try to show that $\xi'_{\cdot,t}$ is in a certain sense *dominated* by the independent percolation with parameter $p < p_c$. This approach leads to a requirement of a lower bound for p_c. However, since the construction of such a comparison appeared infeasible, we have pursued another approach, which makes use of *crossing probabilities* for percolation of ones of $\xi'_{\cdot,1}$.

From now on we assume that the dimension d equals 2.

Definition 1.1 *For a given (site) percolation model on $\mathbb{Z}^2$ and for two positive integers n and m, $n \leq m$, the* crossing probability $p_{n,m}$ *is the probability that within the rectangle $([0,n] \times [0,m]) \cap \mathbb{Z}^2$ there is an open path which connects its left and right sides.*

In Chapter 3 we prove

Theorem 1.2 *Consider k-dependent ($k \geq 0$) site percolation on $\mathbb{Z}^2$. If there exists an $n \geq 1$ with $p_{n,2n+k+1} < 1/13$, then the system does not percolate, and the size of the open cluster is exponentially bounded:*

$$\exists \lambda > 0 : \; P(|C| > n) < e^{-\lambda n} \quad \forall n \in \mathbb{N}.$$

Results of this kind are well-known in the literature; see e.g. [Kes81] and [CC86]. However, our theorem 1.2 has the sharpest bound on the crossing probability and a short and clear proof which can straightforwardly be generalised to higher dimensions.

From Theorem 1.2 it follows directly that if, for the (4-dependent) percolation of weightless particles $\xi'_{\cdot,1}$ of Coffman–Courtois clustering process at time one, the crossing probability $p_{n,2n+5}$ is less that $1/13$ for some n, then the process stabilises locally (the questions (A) and (B) above have positive answer), and the resource at the origin $\xi_{0,t}$ stops changing after a random time with exponentially bounded tail.

It remains to verify that $\xi'_{\cdot,1}$ satisfies the conditions of Theorem 1.2. This is a matter of a *finite* computation, which is, however, still too long for our computers. Instead, we applied Monte-Carlo simulation, which gave overwhelming evidence that the condition of Theorem 1.2 is indeed satisfied for $n = 5$.

1.4 Non-dissipation of masses in coalescing particle systems.

The lack of a rigorous proof in the last (computational) step of the above reasoning prompted our further studies in this direction. We conjectured that an analog of Question B has positive answer in a much more general setup.

Conjecture 1.3 *(Non-dissipation of masses). Let $\{\xi_{x,t}\}_{x\in\mathbb{Z}^d, t\in\mathbb{Z}^+}$ be an arbitrary discrete-time coalescing weighted particle system, as defined in Section 1.2. Suppose the initial resources $\{\xi_{x,0}\}_{x\in\mathbb{Z}^d}$ are i.i.d. and uniformly bounded, and the dynamics (1.1) is invariant under all the symmetries of the lattice.*

If the mass at any $x \in \mathbb{Z}^d$ a.s. tends to a limit:

$$\xi_{x,\infty} = \lim_{t\to\infty} \xi_{x,t} \quad a.s., \tag{1.3}$$

then

$$E\xi_{x,\infty} = E\xi_{x,0}. \tag{1.4}$$

This conjecture can be easily proven in the one-dimensional case (even when the initial distribution is stationary and ergodic but not i.i.d.):

Proof of Conjecture 1.3 for $d = 1$.

Suppose a system $\{\xi_{x,t}\}_{x\in\mathbb{Z}^d, t\in\mathbb{Z}^+}$ satisfies the conditions of Conjecture 1.3. Note that, at any finite time t and for any x,

$$E\xi_{x,t} = E\xi_{x,0} = E\xi_{0,0}.$$

If $\xi_{\cdot,\cdot}$ is constant a.s. then the statement (1.4) is trivially true. Otherwise, if $\xi_{\cdot,\cdot}$ is not a.s. constant then the configuration at time one $\xi_{\cdot,1}$ is not a.s. constant, and

$$\exists\varepsilon : \forall t \geq 1 \quad P(\xi_{0,t} > E\xi_{0,0} + \varepsilon) =: \alpha(t) > 0.$$

However, by Fatou's lemma we have

$$E\xi_{x,\infty} \leq E\xi_{x,0},$$

which, together with (1.3), implies that

$$\exists T \in \mathbb{Z}^+ : \; P(\forall t \geq T : \xi_{0,t} \leq E\xi_{0,0} + \varepsilon) =: \tilde{\alpha} > 0.$$

Fix ε and $T \geq 1$ which satisfy the above conditions. Call a vertex $x \in \mathbb{Z}$ *blue* if $\xi_{x,T} > E\xi_{0,0}+\varepsilon$, and *red* if $\forall t \geq T : \xi_{x,t} \leq E\xi_{0,0}+\varepsilon$. By the individual ergodic theorem, blue and red vertices appear on $\mathbb{Z}$ with positive frequencies $\alpha(T)$ and $\tilde{\alpha}$. After time T no particle can pass through two red vertices with at least one blue vertex in the interval between them. The motion of every particle becomes confined to a finite interval. (1.4) follows easily by applying the individual ergodic theorem again. □

A similar conjecture can be formulated for the continuous time systems as well. Conjecture 1.3 answers Question (B) affirmatively. It has some other interesting implications which will be stated later. Now we present a few examples which show that the conjecture is not trivial in $d \geq 2$, and how it can fail when we relax some of its conditions.

Example 1. Let $d = 2$. In this example the initial resources $\xi_{\cdot,0}$ are stationary, but not i.i.d. First we note that one can choose a spanning tree for the square lattice in a (space-)stationary way (see [Pem91]). This spanning tree defines for each vertex a unique nearest-neighbour path to infinity ('the root'). By "magnifying" this tree by factor 2 we get another tree, which covers all sites with both coordinates even and "half of" the sites whose coordinates have different parity. By shifting such a tree by a vector uniformly chosen from $\{(0,0),(0,1),(1,0),(1,1)\}$, we obtain a stationary tree on $\mathbb{Z}^2$ which covers each vertex with probability 1/2, such that if two neighbour vertices are covered by the tree, then one of them lies on the path from the other one to the root.

Having chosen such a tree, we define $\xi_{x,0} = \mathbf{1}_{\{x \text{ is covered by the tree}\}}$. Let the dynamics be the following:

- if at time t an occupied site x has *exactly one* nearest neighbour y which is occupied (i.e. $\xi_{y,t} > 0$), then x sends its mass to that neighbour: $a_t(x) = y$.
- otherwise the particle at x does not move ($a_t(x) = x$).

This dynamics ensures that the mass flows from 'leaves' to 'the root', so that the tree 'shrinks', and eventually any finite region becomes empty. So we have $E\xi_{x,0} = 1/2$, but $\xi_{x,\infty} = 0$ a.s.

Example 2. The above example can be modified in such a way that its dynamics becomes identical to the dynamics of Coffman, Courtois, Gilbert and Piret's clustering process introduced in Section 1.3. For the initial state, we take the stationary tree constructed in Example 1 and define

$$\xi_{x,0} = \begin{cases} 1 + \sum\limits_{\substack{y \in \mathbb{Z}^2 : \text{ the path from } y \text{ to} \\ \text{the tree root goes through } x}} \xi_{y,0}, & \text{if } x \text{ is covered by the tree,} \\ 0, \quad \text{otherwise.} & \end{cases}$$

In this case $\xi_{x,0}$ is not bounded. Again, as in Example 1, the resources drain towards the root, and $\xi_{x,\infty} = 0$ a.s.

Example 3. In this example the dynamics is not invariant under rotation of the lattice $\mathbb{Z}^2$ through the angle $\pi/2$ or π. The particles are allowed to move only to the right or upwards. This process has 'one step' memory: every vertex remembers the direction, in which the last particle has left it. The initial masses are all ones: $\xi_{x,0} \equiv 1$. At time 0, each vertex x chooses the direction of the first step of its particle $a_0(x)$ to be either $x + (0,1)$ or $x + (1,0)$, with probability 1/2 each. After time 0, each vertex sends the particles passing through it in the same direction: $a_t(x) = a_0(x)$, $t \geq 0$. We show that every site eventually becomes vacant:

Fix an $x \in \mathbb{Z}^2$. Let ω_t be the set of vertices whose particles are in x at time t. Clearly, $|\omega_t| = \xi_{x,t}$. It is not difficult to see (draw a picture), that ω_t is an inclined 'linear interval' on $\mathbb{Z}^2$, and that, as t grows, the size $|\omega_t|$ of this interval makes a symmetric simple random walk which starts at one and is absorbed at zero. The increments of this random walk are i.i.d. and take values $-1, 0$ and 1 with probabilities $\frac{1}{4}, \frac{1}{2}$ and $\frac{1}{4}$ resp. This random walk is recurrent, from which our claim immediately follows.

Example 4. The previous example can be modified so that the process becomes Markovian. We can either let $a_t(x)$ point to the one of two neighbours $x + (0,1)$ and $x + (1,0)$ which has the biggest mass at time t, or select $a_t(x)$ at random from the (non-empty) subset of $\{x + (0,1), x + (1,0)\}$ occupied by particles at time t. The qualitative behaviour of these systems is expected to be the same as in Example 3: all the mass flows away to infinity. But we have not rigorously proved it.

Now we shall state an important consequence of Conjecture 1.3 and look at its implications for some particular coalescing particle systems.

Definition 1.4 *We call a discrete (or, continuous) coalescing weighted particle system* non-rigid, *if* $\forall x \in \mathbb{Z}^d$, $\exists \varepsilon = \varepsilon(x)$, *s.t.* $P(\exists t > 0 : \xi_{0,t} > 0) > \varepsilon$ *for any possible starting configuration* $\xi_{\cdot,0}$ *with* $\xi_{x,0} > 0$.

Conjecture 1.5 (Consequence of Conjecture 1.3) *If a discrete coalescing weighted particle system* $\{\xi_{x,t}\}_{x\in\mathbb{Z}^d,t\in\mathbb{Z}^+}$ *with positive occupation probability (particle density) at time zero* ($P(\xi_{0,0}>0)>0$) *is non-rigid and satisfies the assumptions of Conjecture 1.3 (but not necessary the if-clause (1.3)), then*

i) the origin is visited infinitely often a.s. ($\sum_t \xi'_{0,t}=\infty$ *a.s.*)*;*

ii) $\sum_t P(\xi_{0,t}>0)=\infty$.

Proof (that this is indeed a consequence of Conjecture 1.3).
(i) We consider two cases: (a): Suppose that P(the origin is visited infinitely often) $=0$. Then $\xi_{0,\infty}=0$ a.s. This contradicts Conjecture 1.3, which states that $E\xi_{0,\infty}=E\xi_{0,0}>0$.
(b) If P(the origin is visited infinitely often) >0 then a.s. there is a site $x\in\mathbb{Z}^d$ which is visited infinitely often. The non-rigidity condition ensures that in this case the origin is also visited i.o.
(ii) follows from (i) by the first Borel-Cantelli lemma. □

1.5 Coalescing and annihilating random walks on $\mathbb{Z}^d$

Now let us consider some particular cases of coalescing particle systems, which satisfy the conditions of Conjecture 1.5. The simplest example of such kind is the system of *coalescing (independent) simple random walks* on $\mathbb{Z}^d$. It can more naturally be formulated in continuous time, as follows. At time zero there is a particle of mass one at each site: $\xi_{x,0}=1, \quad \forall x\in\mathbb{Z}^d$. Each particle performs a continuous time simple random walk, i.e. it jumps with equal probability to any neighbour site after waiting for a mean one exponentially distributed time. Each particle moves independently of the others until it collides with another particle. There are two simple collision rules: *coalescence* and *annihilation.* In the coalescing model the colliding particles merge into one particle with the mass equal to the sum of the two colliding masses. In the annihilating model the two colliding particles disappear. Usually one disregards particle masses and considers weightless particle system ξ', as in (1.2). Note that in these special cases ξ' is a (continuous time) Markov process.

The coalescing and annihilating random walks are among the most widely studied interacting particle systems. The coalescing random walks model was first introduced in [HL76] and [Har76] as an auxiliary process in the study of other particle systems. The authors have established and exploited a *duality relation* between the system of coalescing random walks and the *voter model* (see [Gri79], [Lig85]). The system of coalescing random walks soon received independent attention ([Gri78], [Arr81]).

The annihilating random walks model was first introduced by Erdős and Ney [EN74]. They conjectured that in the one-dimensional, discrete time simple annihilating random walk model, which starts with a particle at each site except for the origin, the origin is a.s. visited by at least one particle. This conjecture was proved by Adelman [Ade76]. More general results were obtained in [Gri78] and [Arr83]. They showed, in particular, that in both coalescing and annihilating random walk model on $\mathbb{Z}^d$, $d \geq 1$, the origin is visited infinitely often, provided the initial condition is such that there exists a constant $R > 0$ s.t. at time zero there is a particle within distance R of any site $x \in \mathbb{Z}^d$. This result shows that our Conjecture 1.5 holds for the coalescing random walks model.

Conjecture 1.5 also called our attention to the *particle density* $\rho_t = P(\xi_{0,t} > 0)$ for coalescing particle systems. It states that the particle density is *non-summable*. Bramson and Griffeath [BG80a] have computed the asymptotic particle density for the coalescing random walks model for $d \geq 2$ (the one-dimensional case is easy and it was already known):

$$\rho_t^{CRW} \sim \begin{cases} (\pi t)^{-1/2}, & d = 1, \\ (\pi t/\log t)^{-1}, & d = 2, \\ (\gamma_d t)^{-1}, & d \geq 3, \end{cases} \tag{1.5}$$

where γ_d is the probability that a d-dimensional simple random walk never returns to the origin.

We are interested in the following

Question C. *Can the particle density ρ_t for a locally-dependent coalescing particle system (introduced in Section 1.2) be asymptotically essentially smaller than that of the system of independent coalescing random walks (1.5)?*

By being *asymptotically essentially smaller* we mean that the ratio of the two densities tends to zero. This question remains still open.

Note that, by introducing local dependence, ρ_t can be easily made bigger. It can even have a positive limit, as it does in the case of Coffman–Courtois et al. clustering process [CCGP91].

1.6 Coalescing ideal gas

Question C in the previous section inspired us to consider the following one-dimensional coalescing system, which we call *coalescing ideal gas* (CIG), and which was first mentioned by Fisch [Fis92]. The initial state has a particle at every even site of $\mathbb{Z}$.[1]

Each particle starts moving linearly with speed equal to $+1$ or -1 with probability $1/2$, independently of the other particles. When two particles collide, they coalesce into one particle, which chooses the speed of further movement to be $+1$ or -1 w. p. $1/2$, independently of anything else.

Let $\hat{\eta}_x(t)$ be the position at time t of the particle which starts at time 0 from the site x. It is easy to see that the increments of the movement of a particle at different times are positively correlated:

$$\begin{aligned}&\operatorname{Cov}\left(\hat{\eta}_x(t+1)-\hat{\eta}_x(t),\,\hat{\eta}_x(t+s+1)-\hat{\eta}_x(t+s)\right)=\\ &\qquad P(\text{the particle starting at } x \text{ has no collisions between } t+1 \text{ and } t+s),\end{aligned}$$

for $t \in \mathbb{Z}^+$, $s \in \mathbb{Z}^+ \setminus \{0\}$. Therefore a CIG particle 'in the average moves further' than a simple random walk. Indeed, from the results of Chapter 4 it follows that $\operatorname{Var}\hat{\eta}_x(t) \sim \frac{1}{2}t^2$, whereas for a simple random walk the variance of the particle position at time t is equal to t. Hence one can naively expect that the CIG particles will collide with each other much quicker, and the *CIG particle density*,

$$\rho_t^{CIG} = \frac{1}{2}P(\text{site 0 or site 1 is occupied at time } t),\ t \in \mathbb{Z}^+,$$

will decrease faster than that of the coalescing random walks model. However Theorem 4.1 in Chapter 4, which presents exact values for ρ_t^{CIG} and other probabilities describing the system, shows that, for $t \in \mathbb{Z}^+$,

$$\rho_t^{CIG} \sim (\pi t)^{-1/2}.$$

So the asymptotics of the particle density of our system are the same as those of coalescing random walks model in one dimension (1.5). This supports the feeling that the answer to Question C is probably negative.

Among other results of Chapter 4 is Lemma 4.3, which states that the time intervals between consecutive collisions of a tagged CIG particle are i.i.d., which at first sight looks quite counterintuitive.

1.7 Coalescing Brownian motions

The system of *coalescing Brownian motions* on the real line (CBM) was introduced and extensively studied by R.Arratia in his Ph.D. thesis [Arr79].

[1]This initial condition is used in Chapter 4. It ensures that all the collisions occure at integer times. In Chapter 5 we turn to another initial condition of *all* integer sites being occupied.

This system can intuitively be defined as a system of continuous particle trajectories $c_x(t) = c_x(t,\omega), x \in \mathbb{R}, t \in \mathbb{R}^+$, such that

- $c_x(0) = x,\ \forall x \in \mathbb{R}$,
- $c_x(t) = c_y(t) \Rightarrow c_x(s) = c_y(s),\ \forall s > t$ (coalescence),
- for every finite $A \subset \mathbb{R}$, the family of paths $(c_x(\cdot),\ x \in A)$ must be a family of coalescing Brownian motions.

A finite family of coalescing Brownian motions can be introduced as a system of Brownian motions which are independent except for coalescing interference upon collisions.

Arratia [Arr79] has shown the existence of CBM, the Markov property of the time evolution of the set of particles and the self-similarity under space-time scaling

$$\left\{\frac{1}{\sqrt{s}}c_{x\sqrt{s}}(ts)\right\}_{x\in\mathbf{R},t\in\mathbf{R}^+} \stackrel{d}{=} \{c_x(t)\}_{x\in\mathbf{R},t\in\mathbf{R}^+}, \quad \text{for any } s \in \mathbb{R}^+. \tag{1.6}$$

He has also shown that the system of coalescing simple random walks on $\mathbb{Z}^1$ converges under the scaling as above to the system of coalescing Brownian motions in a certain weak sense.

The system of CBM is essentially one-dimensional. In higher dimensions, the probability that two Brownian motions will ever collide is zero. In these cases one has to redefine the interaction mechanism. For instance, Smoluchowski [Smo16] introduced a model of coagulation in colloids based on coalescing Brownian motions in $\mathbb{R}^3$. In this model each Brownian particle has a fixed diameter R. When two particles collide, i.e. come within distance R from each other, one of them (chosen uniformly from the two) disappears and the other one continues its Brownian motion. The set of particles in this model is discrete, and the particle density is limited from above at any time.

In the one-dimensional system of CBM this is not the case. At time zero we can have a particle at every point of the real line. However, at any positive time the set of aggregated particles is a.s. *discrete*, or locally-finite:

$$|\{c_x(t), x \in \mathbb{R}\} \cap [-l,l]| \text{ is finite a.s. for any } t > 0, l > 0 \quad ([\text{Arr79}], \text{p. } 74).$$

The construction of CBM can be derived from the construction of a countable system of coalescing Brownian motions starting from rationals or dyadic rationals. When the latter is done, an arbitrary trajectory $c_x(\cdot), x \in \mathbb{R}$ can be defined by

$$c_x(t) = \sup\{c_y(t),\, y \in \mathbb{Q},\, y \le x\}.$$

This is due to the fact that $\mathbb{R}^1$ is linearly ordered, and the dynamics of CBM preserves the ordering of the particle positions.

1.8 Coalescing flight processes

In Chapter 5 we define the system of *coalescing flight processes* (CFP) $\{\eta_x(t)\}_{x\in\mathbb{R},t\in\mathbb{R}^+}$ (see (5.16) and Figure 5.2 on page 89. An outline of our construction of CFP is given in the second half of this section), and show that this system is the hydrodynamic limit of the coalescing ideal gas (Theorem 5.13), much like CBM is the hydrodynamic limit of the system of coalescing random walks.

Along with certain similarities, there are some striking differences between CFP and CBM. The CFP system is self-similar in distribution under the scaling

$$\left\{\frac{\eta_{sx}(st)}{s}\right\}_{x\in\mathbb{R},t\in\mathbb{R}^+} \stackrel{d}{=} \{\eta_x(t)\}_{x\in\mathbb{R},t\in\mathbb{R}^+}, \quad \text{for any } s \in \mathbb{R}^+.$$

This scaling is *non-diffusive*, that is to say that, as time scales by s, space also scales by s, and not by $\sqrt{s}$, as in the case of CBM (1.6).

The set of CFP particles at time 1, $\{\eta_x(1), x \in \mathbb{R}\}$, is *uncountable* and its Hausdorff dimension is 1/2 (Proposition 5.3). But this set is not self-similar in distribution: the event "$\{\eta_x(1), x \in \mathbb{R}\} \cap [-l, l] = \varnothing$" has positive probability for $l < 1$ and probability zero for $l \ge 1$.

On the other hand, the set of occupation times of the origin by a CFP particle, $\{t \in \mathbb{R}^+ : \exists x \in \mathbb{R} : \eta_x(t) = 0\}$, is a random fractal set (i.e. it is self-similar in distribution).

Looking at the CFP system at a positive time, we see finite-width flows, where each flow contains uncountably many particles, which all move at unit speed in the same direction. Call the $\frac{1}{2}$-Hausdorff measure of a flow its *mass*. When two opposite flows "collide", all particles of the "lighter" flow coalesce into one, while some particles of the "heavier" flow remain untouched and fly further separately. Their mass is equal to the difference of the masses of the two flows before collision.

At a positive time, only countably many particles have emerged from a coalescence, while the rest came from time zero without a collision. This is an intuitive reason for the fact that the hydrodynamic limits of the particle sets of coalescing and annihilating ideal gas coincide (Proposition 5.4).

The **construction of CFP** in Chapter 5 is done in the framework of a general construction of annihilating and coalescing ballistic particle systems, which applies both to annihilating and coalescing ideal gas (Section 5.3) and to CFP (Sections 5.4, 5.6–5.7). The idea of this construction first appeared in [BF95], where it was used for studying the scaling limit of the set of particles of annihilating ideal gas (some more details in [BF97]).

Fix a time $t \geq 0$. Call a càdlàg function $\Phi_t : \mathbb{R} \to \mathbb{R}$ the *profile function* of an annihilating or coalescing system of particles, if for any finite interval $[a, b] \subset \mathbb{R}$, the total mass of all particles which are between a and b at time t and go to the right minus the total mass of the particles from the same interval which go to the left is equal to $\Phi_t(b) - \Phi_t(a)$. So Φ_t is a generalised *counting measure* of the set of particles whose masses can be positive or negative.

Consider a semigroup of functional operators $S_t : D(\mathbb{R}) \to D(\mathbb{R})$, $t \geq 0$, defined by

$$(S_t F)(x) = \inf\{F(x + y) : y \in [-t, t)\}, \ x \in \mathbb{R}.$$

Now, as one can check, the time evolution of the profile function Φ_t can be expressed by the action of S_t on Φ_0:

$$\Phi_t = S_t \Phi_0.$$

If the initial masses of the particles are ± 1, the construction produces the annihilating dynamics, if the masses have a continuous distribution, we get a coalescing dynamics, and if the masses are "infinitesimal" and the initial profile function Φ_0 is a Brownian motion indexed by $\mathbb{R}$, then we get the dynamics of the CFP (Figure 5.2 on page 89), which is the main topic of Chapter 5.

The above representation allows us to derive various properties of CFP, including the distributions of a particle trajectory and of the collision time of two particles, and to prove that CFP is the hydrodynamic limit of CIG, both in terms of particle sets and particle trajectories.

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Chapter 2

Lower bound for the critical probability of site percolation on $\mathbb{Z}^2$

A New Lower Bound for the Critical Probability of Site Percolation on the Square Lattice [1]

By Jacob van den Berg and Alexei Ermakov

Abstract

The critical probability for site percolation on the square lattice is not known exactly. Several authors have given rigorous upper and lower bounds. Some recent lower bounds are (each displayed here with the first three digits) 0.503 [Tót85], 0.522 [Zue88] and, the best lower bound so far, 0.541 [MP89]. By a modification of the method of Menshikov and Pelikh we get a significant improvement, namely 0.556.

Apart from a few classical results on percolation and coupling, which are explicitly stated in the Introduction, this paper is self-contained.

Key words: percolation, critical probability, coupling, stochastic dominance.

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2.1 Introduction

Site percolation on the square lattice is one of the most studied percolation processes. In contrast to bond percolation on this lattice (for which the critical probability is 1/2 [Kes80], the critical probability for site percolation is not known exactly. Monte Carlo simulations suggest that it is about 0.593. Several authors have obtained rigorous upper and lower bounds. In this paper we concentrate on lower bounds, and we start with some history: From the general arguments of [Ham57] it followed that the above mentioned critical probability is larger than 1/3. The main result of [Har60] (combined with a comparison result of [Ham61] yields that it is at least 1/2. About twenty years later, it was rigorously proved that 1/2 is also a *strict* lower bound [Hig82]. After this, improvements were made more frequently: 0.503478 [Tót85], 0.522105 [Zue88] and, finally, 0.5416 [MP89].

Improving bounds for critical percolation probabilities is not only interesting in itself, but also useful for other fields (see, for instance, [vdBM94]).

In the present paper we use the main idea of Menshikov and Pelikh, but modify their method and make more extensive use of stochastic dominance and coupling arguments. This leads to a significant improvement, stated in Theorem 1.1 below. Moreover, while many details in the paper of Menshikov and Pelikh have been omitted, we give a complete account.

Theorem 2.1 *The critical probability for site percolation on the square lattice is larger than* 0.556.

In the remainder of this section we give a short introduction to the key notions in this paper: percolation, coupling and stochastic dominance. We will state two classical results (Theorems 2.2 and 2.3 below) which will be used later. Apart from these, our paper is self-contained. In Section 2 we discuss the main ideas of the Menshikov-Pelikh method. The proof of Theorem 2.1 involves a "global" comparison argument (Section 3) and is completed in Section 4 by "local" comparison arguments.

- **Percolation.** Let G be a vertex transitive, countably infinite graph, for instance the d-dimensional hypercubic lattice. Suppose each vertex is, independently of all others, open (usually denoted by 1) with probability p and closed (0) with probability $1-p$. An open cluster is a maximal connected subgraph of which all vertices are open. Let v be any vertex of G and define the percolation probability

$$\theta(p) = P(v \text{ belongs to an infinite open cluster}). \qquad (2.1)$$

Next we define the critical probability

$$P_c = inf\{p : \theta(p) > 0\}. \tag{2.2}$$

To emphasize the dependence of θ and P_c on the graph G, we will sometimes write θ_G and $P_c(G)$.

The model above is called *site percolation.* If not the vertices but the edges are randomly open or closed we speak of *bond* percolation. For more information on percolation in general see [Gri89] or [Kes87]. Some papers which deal specifically with bounds on critical probabilities are (besides those mentioned above) [BBS94] and [LW88].

- **Site percolation on the square lattice.** The square lattice, which we denote by S in this paper, is the graph whose vertices can be viewed as points with integer coordinates in the plane, and where two vertices v and w share an edge (are adjacent; are neighbours) iff their Euclidean distance $\|v - w\|$ equals 1 (see fig. 1a). The so-called *matching lattice* of S, denoted by S^*, is obtained from S by "adding the diagonals" in each unit square (see fig.1b). More precisely, S^* has the same vertices as S, but in S^* two vertices v and w share an edge iff $\|v-w\| = 1$ or $\sqrt{2}$. So in S each vertex has four neighbours and in S^* eight neighbours.

  Fisher [Fis61], using arguments of [Har60] proved the following result, which is of essential importance in the Menshikov-Pelikh method.

  Theorem 2.2

$$P_c(S) + P_c(S^*) \geq 1. \tag{2.3}$$

  Remarks

  (i) Since the early eighties it is known that the reversed inequality also holds, but that inequality plays no role in this paper.

  (ii) Since S^* contains S, it is clear that $P_c(S^*) \leq P_c(S)$. So Theorem 2.2 immediately implies $P_c(S) \geq 1/2$, which is one of the lower bounds listed in the beginning of this section.

- **Stochastic dominance and coupling.** Let $\Omega = \{0,1\}^K$ with K a finite or countable set. Elements of Ω are typically denoted by $\omega \equiv (\omega_i, i \in K)$. We say that $\omega \leq \omega'$ if $\omega_i \leq \omega'_i$ for all $i \in K$. A set $A \subset \Omega$ is *increasing* if $\omega \in A$ and $\omega \leq \omega'$ implies $\omega' \in A$. Let μ and μ' be two probability measures on Ω (equipped with the natural sigma-algebra generated by finite cylinder events). We say that μ is *stochastically dominated* by μ' (which we denote by $\mu \overset{st}{\leq} \mu'$) if $\mu(A) \leq \mu'(A)$ for every increasing event A. If the distribution μ depends on a real-valued parameter p, and $p \leq p'$ implies $\mu_p \overset{st}{\leq} \mu_{p'}$ then we say that μ is stochastically increasing in p. A *coupling* of μ and μ' is a probability measure P on $\Omega \times \Omega$ whose two marginals on Ω are μ and μ' (i.e., which satisfies $P(A \times \Omega) = \mu(A)$ and $P(\Omega \times A) = \mu'(A)$ for all events $A \subset \Omega$). The following result, which gives a connection between coupling and stochastic dominance, is well-known (but not easy to prove) and goes back to more general results by [Str65] (see e.g. [Lig85]).

Theorem 2.3 *$\mu \overset{st}{\leq} \mu'$ if and only if there exists a coupling P of μ and μ' with the property $P(\{(\omega, \omega') \in \Omega \times \Omega : \omega \leq \omega'\}) = 1$.*

Note that the if-part of this Theorem is obvious.

Since subsets of K correspond to elements of $\{0,1\}^K$, the above definitions give naturally rise to analogous definitions for random subsets.

2.2 The Menshikov-Pelikh method

A key idea in the Menshikov-Pelikh paper is that by deleting every other vertical edge of S^* we obtain the lattice in fig.1c, which can also be viewed (as we will do in the remainder of this paper) as the lattice in fig.1d, which we denote by L. This lattice L consists of two layers of S with vertical connections, and one diagonal connection in every vertical face. More precisely, the set of vertices of L is $\mathbb{Z}^2 \times \{0,1\}$ and two different vertices $v = (x_1, x_2, x_3)$ and $w = (y_1, y_2, y_3)$ share an edge in L if their Euclidean distance is 1, or ($y_1 = x_1$ and $y_2 - x_2 = x_3 - y_3$) or ($y_2 = x_2$ and $y_1 - x_1 = x_3 - y_3$).

From Theorem 2.2 we have

Lemma 2.4 *Suppose p is such that either $\theta_L(p) > 0$ or $\theta_L(p) = \theta_S(1-p) = 0$. Then $P_c(S) \geq 1 - p$.*

Proof. If the assumption of Lemma 2.4 holds, but the conclusion does not, then $\theta_S(1-p) > 0$ and hence $\theta_L(p) > 0$. Hence $P_c(L) \leq p$. Since we also have $P_c(S) < 1-p$, we get $P_c(L) + P_c(S) < 1$ and hence (since L, as a graph, is contained in S^*), $P_c(S) + P_c(S^*) < 1$, which contradicts Theorem 2.2. □

If $p = 1/2$, then the assumption of Lemma 2.4 is clearly true (since L contains S) but this only gives the lower bound $1/2$. However, the special structure of L makes it possible to compare percolation on L and S in a suitable way, giving rise to values of p which are considerably smaller than $1/2$ but still have the property that percolation on L with parameter p is "easier" than percolation on S with the (larger) parameter $1 - p$. By Lemma 2.4 we then get a lower bound for $P_c(S)$ considerably larger than $1/2$. This is, roughly speaking, what is done in the Menshikov-Pelikh paper, and will also be done in our paper (but in a different way; see Remark (iii) at the end of Section 4).

In Section 3 we describe a "global comparison" between percolation on S and L involving certain growth processes. In Section 4 we study the local properties of these growth processes and complete the proof of Theorem 2.1.

2.3 The global comparison

Consider site percolation on L with parameter p. Menshikov and Pelikh observed that, using the interpretation of L as two layers of S (with extra connections between the two layers) this can also be described as "percolation on S with four defect types": Assign to each vertex v of S the random variable $\varepsilon(v) \in \{0,1\}^2$ which describes the states of its "lower corresponding vertex" $(v, 0)$ and "upper corresponding vertex" $(v, 1)$ in L. For instance, if $(v, 0)$ is open (1) and $(v, 1)$ closed (0), then we set $\varepsilon(v) = \binom{0}{1}$. So the $\varepsilon(v), v \in \mathbb{Z}^2$, are i.i.d. random variables taking values $\binom{0}{1}, \binom{1}{0}, \binom{0}{0}$ and $\binom{1}{1}$ with probability $p(1-p), p(1-p), (1-p)^2$ and p^2 respectively. Further, we say that two adjacent vertices $v = (x_1, x_2)$ and $w = (y_1, y_2)$ of S are ε-adjacent if there exist corresponding vertices (x_1, x_2, x_3) and (y_1, y_2, y_3) in L of v and w respectively, which are open and adjacent in L. More precisely, if for two adjacent vertices v and w we denote the one with smallest sum of coordinates by min and the other by max, then v and w are ε-adjacent if $(\varepsilon(min), \varepsilon(max)) = (\binom{1}{*}, \binom{1}{*})$ or $(\binom{*}{1}, \binom{*}{1})$ or $(\binom{1}{*}, \binom{*}{1})$. Here $*$ stands for "0 or 1". We say that two vertices v and w in S are ε-connected if there exists a path from v to w such that each pair of consecutive vertices in this path is ε-adjacent. This gives, in an obvious way, rise to a definition of ε-connected clusters. From the definition of the ε-variables it is clear that the occurrence of an infinite open cluster on L implies the occurrence of an infinite ε-connected cluster on S and vice versa.

Now consider the following growth process on S. First define a total order on S with minimal element O. Let the process $\varepsilon_v, v \in \mathbb{Z}^2$, be as above. The first step in the growth process consists of "inspecting" the ε-value of O. If it is $\binom{1}{*}$ or $\binom{*}{1}$, then we assign $\xi(O) = 1$, otherwise $\xi(O) = 0$. More generally, at the n'th step we select the vertex v which is lowest in order, has been inspected before and received ξ-value 1, and has at least one neighbour which has not yet received a ξ-value (such a neighbour will be called a child of v and v its parent). If no such vertex v exists, we say that the process has died out, and each vertex which has not yet received a ξ-value, gets ξ-value 0. Otherwise assign to each child of v the ξ-value 1 or 0, depending on whether or not it is ε-adjacent to v. It is clear that each vertex with ξ-value 1 is ε-connected to O (although the reverse is not generally true). Therefore, if the above growth process does not die out, the ε-cluster of O is infinite, and hence, in the corresponding percolation process on L, $(0,0,0)$ or $(0,0,1)$ belongs to an infinite open cluster, so that $\theta_L(p) > 0$. Now suppose that p is such that, at each step, the joint conditional distribution of the ξ-values assigned to the children at that step, given all the ξ-values of the vertices which have been treated at previous steps, stochastically dominates the product distribution with parameter $1-p$. Then, using Theorem 2.3 for each step (with K the set of children at that step) and standard arguments, we can construct a coupling $\mathcal{P}$ of the set of vertices which eventually have ξ-value 1, and the open cluster containing O in the percolation process on S with parameter $1-p$, in such a way that with $\mathcal{P}$-probability 1 the former set contains the latter. In particular, we then have that $\theta_S(1-p) > 0$ implies $\theta_L(p) > 0$, so that, by Lemma 2.4, $1-p$ is a lower bound for $P_c(S)$.

In view of the above, it is natural to investigate the conditional "offspring" distributions in the growth process. This will be done in the next section.

2.4 The local comparison and the proof of Theorem 2.1

In this section we study the (conditional) offspring distribution in a step in the growth process described in the previous section. The amount of conditioning information is, in some sense, unbounded: the larger the number of previous steps, the more information we have. By "conditioning out" all information except certain "local information", we get a finite problem as will be pointed out below (see also Remark (iv) at the end of this paper).

First we give some more notation. From now on we let, for $k \in \mathbb{N}$ and $p_1, \ldots, p_k \in [0,1]$, $\pi_{(p_1,\ldots,p_k)}$ denote the product distribution on $\{0,1\}^k$ with parameters $p_1, \ldots, p_k$. If $p_1 = p_2 = \ldots = p_k$, and the value of k is clear from the context, we will write just π_{p_1}.

In the previous section we defined the random variables $\varepsilon_v, v \in \mathbb{Z}^2$, and the notion of ε-adjacency. Define, for each edge $e = \{v, w\}$ of S,

$$\eta(e) = I(v \text{ and } w \text{ are } \varepsilon - \text{adjacent}), \tag{2.4}$$

where $I(\cdot)$ denotes the indicator function. Since $\eta(\{v,w\})$ is a function of $\varepsilon(v)$ and $\varepsilon(w)$ and the ε's are independent random variables, we have the following version of a general well-known property:

Lemma 2.5 *Let $W \subset V$ be finite sets of vertices of S, and X a set of edges of S with both endpoints in V. Let ∂W be the set of all $w \in W$ for which there exists a $v \in V \setminus W$ with $\{v,w\} \in X$. Further, let $X(W)$ be the edges in the set X of which both endpoints are in W. Finally, let Y be a set of edges with one endpoint in V^c and the other in W. Then, given $(\varepsilon(w), w \in \partial W)$, $(\eta(z), z \in Y \cup X(W))$ is independent of $(\eta(x), x \in X \setminus X(W))$. Hence, if Ω denotes $\{\binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1}\}^{\partial W}$, then for any $\alpha \in \{0,1\}^Y$ and $\beta \in \{0,1\}^X$,*

$$\begin{aligned}
& P\left((\eta(y) = \alpha(y), y \in Y) \mid (\eta(x) = \beta(x), x \in X)\right) \\
= \; & \sum_{\gamma \in \Omega} P\left((\varepsilon(w) = \gamma(w), w \in \partial W) \mid (\eta(x) = \beta(x), x \in X)\right) \\
& \times P((\eta(y) = \alpha(y), y \in Y) \mid (\varepsilon(w) = \gamma(w), w \in \partial W), \\
& \qquad\qquad (\eta(x) = \beta(x), x \in X(W))).
\end{aligned} \tag{2.5}$$

This Lemma, which is intuitively obvious and easy to prove, will be used frequently in this paper (mostly without explicitly referring to it).

Now suppose we are at a certain step in the growth process described in Section 3. Let b_2 denote the vertex selected at this step (we reserve the notation b_1 and b_3 for possible "brothers" of b_2). Suppose b_2 has three children a_1, a_2 and a_3. In view of the arguments at the end of Section 3, we are interested in whether the conditional distribution of $(\xi(a_1), \xi(a_2), \xi(a_3))$, given all the ξ-values assigned in previous steps, stochastically dominates π_{1-p}. From the information on which we condition, we can reconstruct the past of the growth process in the sense that, for each step in the past, we know the selected vertex at that step and the assignment of ξ-values to its children. In other words, it tells us for each vertex v which already has a ξ-value, who its parent w is and what the value of $\eta(\{v,w\})$ is. In particular, we know from this information who the parent c of b_2 is, who the parent d of c is, and who the "brothers" of b_2 (i.e. other children of c) are (if there are any). To be more explicit, suppose we are in the situation of fig.2a, where b_2 has two brothers, named b_1 and b_3. Note that if $\eta(\{b_1, c\}) = 0$, then $\xi(b_1) = 0$, so b_1 has not been the selected vertex at any step, and hence the only neighbour v of b_1 for which we know $\eta(\{b_1, v\})$ is c. The same observation holds for b_3. Now apply Lemma refl2-4.1 (with V the set of all vertices which have already obtained a ξ-value, $W = \{b_1, b_2, b_3, c, d\}$, X the set of edges $\{v, v'\}$ with $v, v' \in V$ and v the parent of v' (or vice versa), and Y the set consisting of the edges $\{a_1, b_2\}$, $\{a_2, b_2\}$ and $\{a_3, b_2\}$. Then it is easy to see that the conditional distribution we are looking at is a mixture of conditional distributions of $(\eta(\{a_1, b_2\}), \eta(\{a_2, b_2\}), \eta(\{a_3, b_2\})$ given $\eta(\{b_2, c\}) = 1$, $\eta(\{c, d\}) = 1$, and the values of $\eta(\{b_1, c\})$, $\eta(\{b_3, c\})$ and $\varepsilon(d)$, and the ε-values of those brothers of b which are ε-adjacent to c. If each such 'local conditional' distribution stochastically dominates π_{1-p}, then clearly the mixture itself also dominates π_{1-p}. Therefore we will study these local conditional distributions. Of course the configuration in fig.2a is only one of the configurations of a_1, a_2, a_3, b_1, b_2, b_3, c and d which can occur. However, it is not difficult to see that there is essentially only one other configuration, namely that in fig.2b. This follows from the following arguments: first of all, by the symmetry properties of L, it does not matter which of the four neighbours of b_2 we choose for c. Then there are three choices for d: one where b_2, c and d are on one line, and two where the line through b_2 and c and that through c and d are perpendicular. Of the last two cases, one is, from a graph-theoretical viewpoint, equivalent to the case where b_2, c and d are on one line. Further, our assumption that b_2 has three children is sufficiently general: if it has fewer children, say only a_2 and a_3, then we can add an "imaginary copy" of a_1 (in fact this has been done in fig.2b, where a_1 is adjacent to d and hence must be a child of d or of a vertex whose ξ-value is "at least as old" as that of d). This does not disturb our arguments, because, if the joint conditional distribution, which we study, of the three children dominates π_{1-p}, then so do its marginals, in particular the joint conditional distribution of the two 'real' children. For similar reasons we have assumed that c has, besides b, two other children. Since, if it has fewer, then we add "imaginary" ones. The conditional distribution which we are studying is, in the real situation, a convex combination of the relevant conditional distributions in the cases with 'imaginary' b_i's. If π_{1-p} is stochastically dominated by each of these, then it is also stochastically dominated by the mixture.

Summarizing the above arguments, we get the Proposition below, where we abbreviate $\eta(\{a_i, b_2\})$ by $\eta(a_i)$ and $\eta(\{b_i, c\})$ by $\eta(b_i)$, $i = 1, \ldots, 3$, and $\eta(\{c, d\})$ by $\eta(c)$.

Proposition 2.6 *Let $a_1, a_2, a_3, b_1, b_2, b_3, c$ and d be as in fig.2a or 2b. Let $\varepsilon(v)$, with v in the above set, be independent random variables taking values $\binom{0}{0}$, $\binom{0}{1}$, $\binom{1}{0}$ and $\binom{1}{1}$ with probability $(1-p)^2$, $p(1-p)$, $p(1-p)$ and p^2 respectively. Let $\eta(a_i) = I(a_i \text{ is } \varepsilon - \text{adjacent to } b_2)$, $i = 1, 2, 3$, $\eta(b_i) = I(b_i \text{ is } \varepsilon - \text{adjacent to } c)$, $i = 1, 2, 3$, and $\eta(c) = I(c \text{ is } \varepsilon - \text{adjacent to } d)$, with ε-adjacency as defined in Sect. 3. Let $\mathcal{M}$ be the set of all events of the form $\{\eta(b_2) = 1, \eta(c) = 1\} \cap E_d \cap E_{b_1} \cap E_{b_3}$. Here E_d is one of the events $\{\varepsilon(d) = \binom{1}{0}\}$, $\{\varepsilon(d) = \binom{0}{1}\}$, or $\{\varepsilon(d) = \binom{1}{1}\}$, and each $E_{b_i}, i = 1, 3$ is one of the events $\{\eta(b_i) = 0\}$, $\{\eta(b_i) = 1, \varepsilon(b_i) = \binom{1}{0}\}$, $\{\eta(b_i) = 1, \varepsilon(b_i) = \binom{0}{1}\}$, or $\{\eta(b_i) = 1, \varepsilon(b_i) = \binom{1}{1}\}$.*
If p is such that in both cases (fig. 2a and 2b), for each $E \in \mathcal{M}$, the conditional distribution $P\left((\eta(a_1), \eta(a_2), \eta(a_3)) \in \cdot \mid E\right)$ stochastically dominates π_{1-p}, then $P_c(S) \geq 1 - p$.

By this proposition we have reduced our problem to a finite one, but there is still much work to be done. To prove Theorem 2.1, we want to show that the condition of Proposition 2.6 holds for $p = 1 - 0.556 = 0.444$. In principle we could, for each $E \in \mathcal{M}$, check if the stochastic dominance property in the condition of the Proposition indeed holds. For each choice of E this would lead (as we will see later) to checking 13 inequalities, for each of the two cases I (fig.2a) and II (fig.2b). Since there are 48 possibilities for E, one would have to check $2 \times 13 \times 48 = 1248$ inequalities. Although this is possible by computer, we proceed in a different way, which leads to checking "only" 13 inequalities. We will construct a suitable distribution $\hat{\mu}$ on $\{0,1\}^3$ which, for each $E \in \mathcal{M}$, is stochastically dominated by $P((\eta(a_1), \eta(a_2), \eta(a_3)) \in \cdot \mid E)$. The only thing we then have to do is to check if, for $p = 0.444$, $\hat{\mu}$ stochastically dominates π_{1-p}. (One may think that checking the 1248 inequalities may lead to a larger lower bound of $P_c(S)$, but that appears not to be the case. See Remark ii) at the end of this section).

We start by giving some extra notation. First, q will denote $1-p$. Let, for $\alpha \in \{\binom{0}{1}, \binom{1}{0}, \binom{1}{1}\}$, ν_α be the conditional distribution of $(\eta(a_1), \eta(a_2), \eta(a_3))$ given $\varepsilon(b_2) = \alpha$. It is easy to see that in both case I and II,

$$\begin{aligned} \nu_{\binom{1}{1}} &= \pi_{1-q^2}, \\ \nu_{\binom{1}{0}} &= \pi_{(1-q^2, 1-q^2, p)}, \\ \nu_{\binom{0}{1}} &= \pi_{(p, p, 1-q^2)}. \end{aligned} \tag{2.6}$$

Moreover, let for $\alpha = \binom{1}{0}, \binom{0}{1}$ and $\binom{1}{1}$, μ_α denote the conditional probability of $(\eta(a_1), \eta(a_2), \eta(a_3))$ given $\varepsilon(c) = \alpha$ and $\eta(b_2) = 1$. By elementary manipulations (sum over the possibilities for $\varepsilon(b_2)$ and use that the conditional distribution of $((\eta(a_1), \eta(a_2), \eta(a_3))$ given $\varepsilon(c)$, $\eta(b_2)$ and $\varepsilon(b_2)$ depends only on $\varepsilon(b_2)$) we can express the μ_α's as mixtures of the ν_α's:

$$\begin{array}{rcccccl} \mu_{\binom{1}{*}} & = & \frac{p}{1+q}\nu_{\binom{1}{1}} & + & \frac{q}{1+q}\nu_{\binom{1}{0}} & + & \frac{q}{1+q}\nu_{\binom{0}{1}}; \\ \mu_{\binom{0}{1}} & = & p\nu_{\binom{1}{1}} & & + & & q\nu_{\binom{0}{1}}, \end{array} \tag{2.7}$$

where $\mu_{\binom{1}{*}}$ stands for $\mu_{\binom{1}{1}}$ and $\mu_{\binom{1}{0}}$ which appear to be equal.
Now define

$$\tilde{\mu} := \left(1 - q\frac{2+q}{1+q}\right)\nu_{\binom{1}{1}} + \frac{q}{1+q}\,\nu_{\binom{1}{0}} + q\,\nu_{\binom{0}{1}}. \tag{2.8}$$

From (2.7) and (2.8), using that $\nu_{\binom{0}{1}}$ and $\nu_{\binom{1}{0}}$ are both stochastically smaller than $\nu_{\binom{1}{1}}$ and that $q \geq q/(1+q)$, we have

Lemma 2.7

$$\tilde{\mu} \overset{st}{\leq} \mu_{\binom{1}{*}} \text{ and } \tilde{\mu} \overset{st}{\leq} \mu_{\binom{0}{1}}.$$

Our next step in the construction of a distribution $\hat{\mu}$ as mentioned in the discussion after Proposition 2.6, is the following Lemma.

Lemma 2.8 *Let $r \in [0,1]$. If $min_{E\in\mathcal{M}} P\left(\varepsilon(c) = \binom{1}{*} \mid E\right) \geq r$, then, for all $E \in \mathcal{M}$,*

$$P\left((\eta(a_1, \eta(a_2), \eta(a_3)) \in \cdot \mid E\right) \overset{st}{\geq} r\mu_{\binom{1}{*}} + (1-r)\tilde{\mu}.$$

Proof. Suppose the assumption of Lemma 2.8 holds. Let $A \subset \{0,1\}^3$ be an increasing event. We have

$$\begin{aligned} & P\left((\eta(a_1), \eta(a_2), \eta(a_3)) \in A \mid E\right) \\ = & \sum_{\alpha\in\{\binom{0}{1},\binom{1}{*}\}} P\left(\varepsilon(c) = \alpha \mid E\right) P\left((\eta(a_1), \eta(a_2), \eta(a_3)) \in A \mid E, \varepsilon(c) = \alpha\right) \\ = & \sum_{\alpha\in\{\binom{0}{1},\binom{1}{*}\}} P\left(\varepsilon(c) = \alpha \mid E\right) \mu_\alpha(A) \\ \geq & P\left(\varepsilon(c) = \tbinom{1}{*} \mid E\right) \mu_{\binom{1}{*}}(A) + P\left(\varepsilon(c) = \tbinom{0}{1} \mid E\right) \tilde{\mu}(A) \\ \geq & r\mu_{\binom{1}{*}}(A) + (1-r)\tilde{\mu}(A). \end{aligned} \tag{2.9}$$

The first equality is trivial, the second follows from the definition of μ_α and the fact that, given $\varepsilon(c)$ and $\eta(b_2)$, $(\eta(a_1), \eta(a_2), \eta(a_3))$ is independent of $\varepsilon(d)$, $\eta(c)$, $\eta(b_1)$, $\eta(b_3)$, $\varepsilon(b_1)$ and $\varepsilon(b_3)$. The first inequality follows from the second statement in Lemma 2.7. The second inequality follows from the first statement in Lemma 2.7 and the assumption of Lemma 2.8. □

Proposition refp2-4.2 and Lemma 2.8 motivate the search for an appropriate r. We find

Lemma 2.9 *Let, for $p \in (0,1)$, $r_o(p) = (1-q^2)/(2-q^2)$. We have*

$$min_{E \in \mathcal{M}} P\left((\varepsilon(c) = \tbinom{1}{*} \mid E\right) = r_0(p).$$

Proof. We give separate proofs for case I (fig.2a) and II (fig.2b). We start with case I. First we observe that if $\varepsilon(b_1) \equiv \binom{1}{0}$ on E, then E implies $\varepsilon(c) = \binom{1}{*}$. Also note that if $\varepsilon(b_1) \equiv \binom{*}{1}$ on E, then the information concerning b_1 in E is redundant (for the event $\{\varepsilon(c) = \binom{1}{*}\}$). Hence, by the previous observation, $P\left((\varepsilon(c) = \binom{1}{*} \mid E\right)$ is then a convex combination of 1 and $P\left((\varepsilon(c) = \binom{1}{*} \mid E'\right)$, where $E' \in \mathcal{M}$ satisfies $\eta(b_1) \equiv 0$. Therefore we will assume $\eta(b_1) \equiv 0$ on E. Also note that, if $\varepsilon(d) \equiv \binom{0}{1}$, then E implies $\varepsilon(c) = \binom{*}{1}$, and the the information concerning b_3 is redundant. It is clear that we can then replace E by $E_1 := \{\eta(b_1) = 0, \eta(b_2) = 1, \varepsilon(c) = \binom{*}{1}\}$ without changing the conditional probability that $\varepsilon(c) = \binom{1}{*}$. This is also the case when $\varepsilon(b_3) \equiv \binom{0}{1}$ on E. Further, if $\varepsilon(b_3) \equiv \binom{1}{*}$, then the information concerning b_3 is redundant. By these observations it follows that we only have to compute $P(\varepsilon(c) = \binom{1}{*} \mid E_i), i = 1, 2$, with E_1 as above and $E_2 = \{\varepsilon(d) = \binom{1}{*}, \eta(b_1) = 0, \eta(b_2) = 1, \eta(b_3) = 0, \eta(c) = 1\}$. It is elementary to verify that these are equal to $r_0(p)$ and $\frac{p(1-q^2)+(1-q^2)}{p(2-q^2)+(1-q^2)}$ (which is clearly larger than $r_0(p)$) respectively. This completes the proof for case I.

As to case II, first note that if $\varepsilon(d) \equiv \binom{1}{0}$ on E, then E implies $\varepsilon(c) = \binom{1}{*}$, so we will assume that $\varepsilon(d) \equiv \binom{*}{1}$. Further, if $\varepsilon(b_1) \equiv \binom{1}{*}$, then the information concerning b_1 is redundant, and if $\varepsilon(b_1) \equiv \binom{0}{1}$, then E implies $\varepsilon(c) = \binom{*}{1}$, and we can replace E by the event $F_1 := \{\varepsilon(c) = \binom{*}{1}, \eta(b_2) = 1\}$. If $\eta(b_1) \equiv 0$ on E, then $\varepsilon(b_1)$ is either $\binom{0}{0}$, in which case the information on b_1 is redundant, or $\binom{0}{1}$, in which case $\varepsilon(c)$ must be $\binom{*}{0}$ and hence $\binom{1}{0}$. By symmetry, similar observations hold for b_3. By these arguments it suffices to compute $P(\varepsilon(c) = \binom{1}{*} \mid F_1)$, which is equal to $(1-q^2)/(2-q^2-p)$, which clearly is larger than $r_0(p)$. □

Proposition 2.10 Proposition 4.6

Let $p \in (0,1)$ and $r_0(p) = (1-q^2)/(2-q^2)$ as in Lemma 2.9. Let $\hat{\mu}$ be the following mixture of three product distributions on $\{0,1\}^3$

$$\begin{aligned}\hat{\mu} &= \left(r_0(p)\frac{p}{1+q} + (1-r_0(p))(1-q\frac{2+q}{1+q})\right)\pi_{1-q^2} \\ &+ \frac{q}{1+q}\pi_{(1-q^2,1-q^2,p)} \\ &+ \left(r_0(p)\frac{q}{1+q} + (1-r_0(p))q\right)\pi_{(p,p,1-q^2)}.\end{aligned} \tag{2.10}$$

If $\hat{\mu} \overset{st}{\geq} \pi_{1-p}$, then $P_c(S) \geq 1-p$.

Proof. Use (2.6), (2.7) and (2.8) to see that

$$\hat{\mu} = r_0(p)\mu_{\binom{1}{*}} + (1-r_0(p))\tilde{\mu}. \tag{2.11}$$

The proposition is now a straightforward consequence of Proposition 2.6, Lemma 2.8 and Lemma 2.9. □

Proof of Theorem 2.1

Theorem 2.1 will now be proved by checking that, if $p = 0.444$, for each increasing event $A \subset \{0,1\}^3$,

$$\hat{\mu}(A) > \pi_{1-p}(A). \tag{2.12}$$

In fact, since $\hat{\mu}$ is symmetric in a_1 and a_2, we do not have to check all increasing events (only up to $a_1 - a_2$ symmetry). It appears that we have to check (2.12) for thirteen events A. These events and their probabilities under $\hat{\mu}$ and π_{1-p} (for $p = 0.444$) are given in the following table (with the notation $A_i = \{(x_1,x_2,x_3) \in \{0,1\}^3 : x_i = 1\}$). Although we used the computer to make these calculations, the inequalities can also be checked by hand in a "reasonable" time (note that, for each A, the r.h.s of (2.12) is a very simple polynomial and the l.h.s. a relatively simple rational expression in p; also note that the probabilities for case 6-13 in the table can be very simply expressed in terms of those for case 1-5). Therefore, in our opinion, our proof of Theorem 2.1 should be considered as a "classical" proof and not a computer-assisted proof.

Case	A	$\hat{\mu}(A)$ $(p = 0.444)$	$\pi_{0.556}(A)$
1	A_1	0.573647	0.556000
2	A_3	0.602653	0.556000
3	$A_1 \cap A_2$	0.344268	0.309136
4	$A_1 \cap A3$	0.335370	0.309136
5	$A_1 \cap A_2 \cap A_3$	0.195740	0.171880
6	$A_1 \cup A_2$	0.803026	0.802864
7	$A_1 \cup A_3$	0.840930	0.802864
8	$A_1 \cup (A_2 \cap A_3)$	0.713277	0.693256
9	$(A_1 \cap A_2) \cup A_3$	0.751181	0.693256
10	$(A_1 \cap A_2) \cup (A_1 \cap A_3)$	0.483898	0.446392
11	$(A_1 \cap A_3) \cup (A_2 \cap A_3)$	0.475001	0.446392
12	$A_1 \cup A_2 \cup A_3$	0.930678	0.912472
13	$(A_1 \cap A_2) \cup (A_2 \cap A_3) \cup (A_1 \cap A_3)$	0.623529	0.583649

In each of the thirteen cases, the probability under the distribution $\hat{\mu}$ (with $p = 0.444$) is, according to the table, indeed larger than under $\pi_{0.556}$ (note that the difference is minimal for case 6; see also Remark (i) below). Hence Theorem 2.1 follows. □

Remarks

i) The distributions ν_α (see (2.6)) are clearly stochastically increasing in p. This, together with the obvious facts that $\nu_{\binom{1}{1}}$ dominates both $\nu_{\binom{1}{0}}$ and $\nu_{\binom{0}{1}}$, and that p and $p/(1+q)$ are increasing in p, implies that also $\mu_{\binom{1}{*}}$ and $\mu_{\binom{0}{1}}$ are stochastically increasing in p. In the same way we see from (2.8) that $\tilde{\mu}$ is stochastically increasing in p. Finally, since $\mu_{\binom{1}{*}}$ stochastically dominates $\tilde{\mu}$ and both are stochastically increasing in p, and $r_0(p)$ is also increasing in p, we have from (2.11) that $\hat{\mu}$ is stochastically increasing in p. Hence, for each increasing event A, $\hat{\mu}(A)$ is an increasing function of p. Clearly, $\pi_{1-p}(A)$ is decreasing in p. It is also easy to check that each of these increasing (decreasing) functions is 0 (1) for $p = 0$ and 1 (0) for $p = 1$. So each equation $\hat{\mu}(A) = \pi_{1-p}(A)$ has a unique solution in [0,1]. Our lower bound 0.556 is taken (a little bit smaller than) 1 minus the largest of the thirteen solutions (namely, the solution for the event $A_1 \cup A_2$).

ii) One may think that we may have taken $\tilde{\mu}$ in (2.8) "too stochastically small" and that checking the 1248 inequalities mentioned after Proposition 2.6 may give a better (i.e. larger) lower bound for $P_c(S)$. This is not true because of the following: As we said in Remark (i), the event $A_1 \cup A_2$ is the worst one, in the sense that, if we decrease p, then this event is the first for which (2.12) will fail. Also, from (6), (7) and (8) it follows immediately that the probability of this event under $\tilde{\mu}$ is the same as under $\mu_{\binom{0}{1}}$. Now let E_0 be an event for which the minimum in Lemma 2.9 is reached. Then it is easy to see that, with $A = A_1 \cup A_2$, $E = E_0$, and $r = r_0(p)$, we have equality in (2.9), and from (2.11) that the last expression equals $\hat{\mu}(A)$. Hence, if p decreases, then as soon as (2.12) fails for some A, then one of the 1248 inequalities mentioned above also fails.
iii) As we said in the Introduction, many details have been omitted in the Menshikov-Pelikh paper. Therefore it is not easy to compare our paper with theirs precisely. However, it seems that the improvement we have obtained is mainly due to the fact that we study the *joint* conditional distribution of all children of a parent, while they study the conditional distribution of one child (and therefore have to include information on possible brothers in the event they condition on). The growth process in their paper is slightly different from ours. For instance, in their setup a child can have more than one parent, which is seemingly an advantage but appears, so far, to make things only unnecessary complicated. Generally speaking, the comparison arguments in their paper are formulated in the framework of Markov chain theory rather than coupling theory.
iv) In the beginning of this section we reduced the problem to a finite one, by "conditioning out all information outside a_1, a_2, a_3, b_1, b_2, b_3, c and d". This choice looks somewhat arbitrary and one may wander whether including more information leads to a better lower bound for $P_c(S)$. By remark (ii) above and because of the form of the events for which the minimum in Lemma 2.9 is reached, it follows that, to improve the bound in this way, one has to include not only the parent of d and the brothers of c, but also (at least) the children of b_3. Based on calculations we made, we found the improvement which can be obtained in that way not sufficient to justify the extra amount of work.

Acknowledgment: We thank M.V. Menshikov for giving us an informal explanation of the main ideas in his paper with K.D. Pelikh.

2.5 Figures

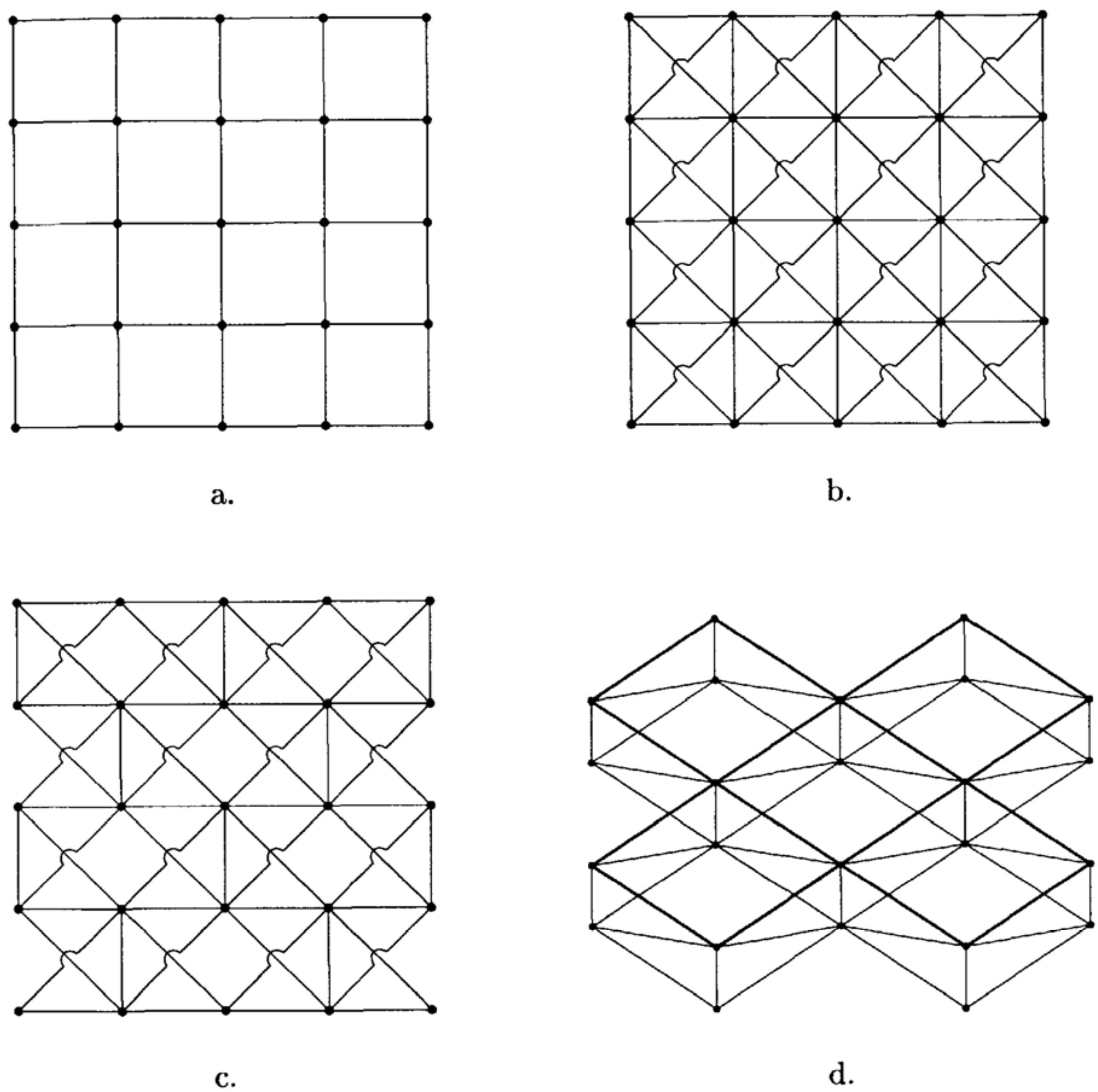
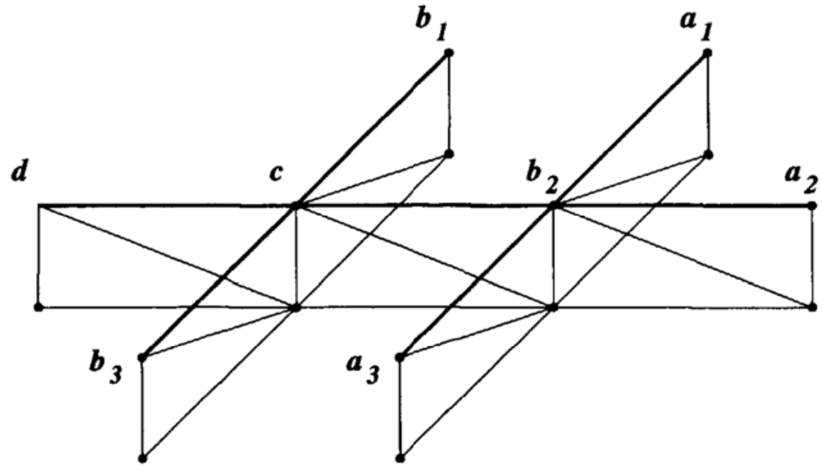
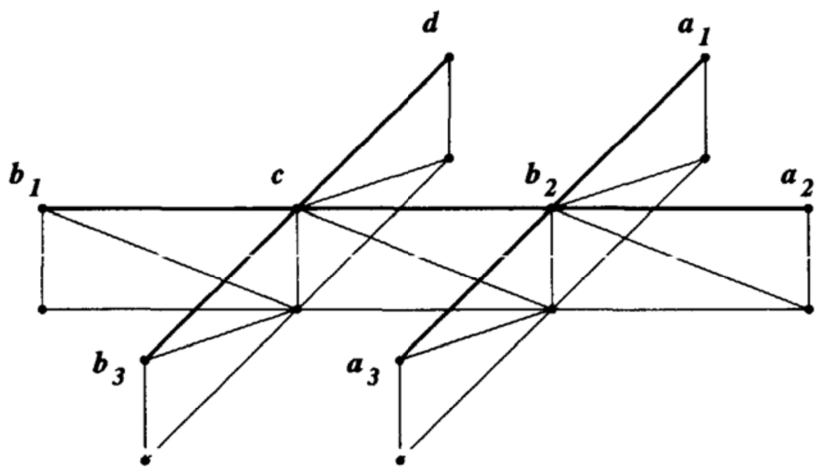


Figure 1.

a. – Lattice S
b. – Lattice S^*
c. and d. – Two spatial representations of the lattice L



a). Conf. I



b). Conf. II

Figure 2.

The local configurations I and II.

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Chapter 3

Distributed clustering process of Coffman, Courtois, Gilbert and Piret

On a distributed clustering process of Coffman, Courtois, Gilbert and Piret [1]

By Jacob van den Berg and Alexei Ermakov

Abstract

Coffman et al [CCGP91] have introduced a flow process in graphs, where each vertex gets an initial random resource, and where at each time vertices with large resources tend to attract resources from neighbours. The initial resources are assumed to be i.i.d., with a continuous distribution.

We are mainly interested in the following question: does, with probability 1, the resource of each vertex change only finitely many times?

Coffman et al mainly concentrate on the case where the graph corresponds with the integer points on the line, in which case it is easily seen that the answer to the above question is positive. For more-dimensional lattices they make general remarks which suggest that the answer to the above question is still positive. However, no proof seems to be known.

We restrict to the case of the square lattice, and, by a percolation approach, we reduce the question above to the question whether a certain quantity, which can be obtained from a *finite* computation, is sufficiently small. This computation is, however, still too long to be executed in an acceptable time. Therefore we applied

[1] Submitted for publication

Monte Carlo simulation for this finite problem, which gave overwhelming evidence that, for the square lattice, the answer to the main question is positive.

Keywords: *flow of resources, clustering, percolation*

3.1 Introduction

Coffman et al [CCGP91] have introduced the following model of "distributed clustering". Let G be a (finite or countable), locally finite, connected graph.

We denote by $\rho(v, w)$ the graphical distance between the vertices v and w (i.e., the minimal number of edges in a path from v to w). Further, we define $B_n(v) := \{w : \rho(v, w) \leq n\}$, and $\partial B_n(v) := \{w : \rho(v, w) = n\}$.

Now assign to each vertex v an initial "resource" $r_0(v)$. We assume that these initial resources are non-negative i.i.d. random variables with a continuous distribution. Now we define the resources at time t, denoted by $r_t(v), t = 1, 2, \ldots$, inductively as follows: Let, for each v with $r_t(v) \neq 0$, $a_t(v)$ be the vertex $w \in B_1(v)$ for which $r_t(w)$ is maximal (note that $a_t(v)$ may be v itself). Now define, for each v, $r_{t+1}(v) = \sum_{w:a_t(w)=v} r_t(w)$. In other words, at each time each vertex (simultaneously with all others) gives its resource to the richest vertex in its neighbourhood. We are mainly interested in the following questions:

Question 3.1

a) Does, with probability 1, each vertex eventually reach a final resource value?

b) If the answer to question a) is positive, is the expectation of this final value equal to the expectation of the initial value? (In other words: "Is it impossible for resources to get lost"?).

Remarks

1. Clearly, once a vertex has resource value 0, its value remains 0 forever. Further, it is not difficult to see that, if G is finite, then at each time that at least one vertex changes its value, the value of *some* vertex changes from positive to 0. Hence, on a graph with n vertices, all resource values remain unchanged after time n.

2. Coffman et al mainly concentrate on the case where G is the graph of which the vertices are the integer points on the line, and where each pair of consecutive integers has an edge. For that case the evolution of the system

is, in some sense, very regular. In particular, the answer to Question 3.1 (a and b) is positive, and several quantities of interest can be explicitly computed.
3. As Coffman et al pointed out (in the last section of their paper), the evolution in the more-dimensional cases is typically much more complicated, and there is no hope for exact calculations as done for the line. Without explicitly saying so, they suggest that the answers to Question 3.1 (a and b) above are still positive for these cases. However, it seems that no proof is known.
4. Van den Berg and Meester [BM91] prove weaker stability properties than those in Question 3.1(a and b). In particular, for the square lattice they proved that (a.s.) for each vertex v, eventually either $r_t(v) = 0$, or $a_t(v)$ remains constant. However, Question 3.1 remained open.

In the present paper we concentrate, like Van den Berg and Meester (see Remark 4 above) on the square lattice. However, we use a percolation-like approach (Van den Berg and Meester used ergodicity and symmetry arguments). In Section 3.2 we reduce, by an adaptation of quite standard percolation arguments, Question 3.1 to a *finite* problem. Although this problem is finite, it appears to be too large to be solved rigorously in a realistic time. Therefore we applied Monte Carlo simulation to it, which strongly convinced us that, for the square lattice, the answer to Question 3.1 (a and b) is positive and, as a side result, that, for the evolution on an $n \times n$ torus instead of the full lattice, the expected time until *all* vertices have reached their final value is at most of order $\log n$.

3.2 Results

In the remainder of this paper G is the square lattice, i.e., the graph of which the vertices are the elements of $\mathbb{Z}^2$, and where two vertices $v = (v_1, v_2)$ and $w = (w_1, w_2)$ share an edge iff $|v_1 - w_1| + |v_2 - w_2| = 1$.

Let $r_0(v), v \in \mathbb{Z}^2$, and $r_1(v), v \in \mathbb{Z}^2$ be the initial resources and the resources at time 1 respectively, as described in Section 3.1. In particular, the $r_0(v)$'s are non-negative i.i.d. random variables with a continuous distribution. Further, let the random variables X_v be defined by $X_v = 1$ if $r_1(v) > 0$, and 0 otherwise, $v \in \mathbb{Z}^2$. Note that the X-values depend only on the *order statistics* of the r_0-values. Also note that the X-process is *4-dependent*: if $V, W \subset \mathbb{Z}^2$ and $\min\{\|v - w\| : v \in V, w \in W\} > 4$, then the collection $(X_i, i \in V)$ is independent of the collection $(X_j, j \in W)$.

We are particularly interested in the percolative properties of this X-process. (For a quite general treatment of percolation, see Grimmett [Gri89]). When a vertex v has $X(v) = 1$, we say that v is *open*. An *open path* is a path in the graph of which every vertex is open. The set $\{v : X_v = 1\}$ can be partitioned in maximal connected components. We say that the X-process *percolates* if one or more of these components are infinite. Note that the future evolution of the resources on a *finite* component C no longer depends on the values outside C (because all vertices on the outer boundary of C have resource 0 at time 1 and hence, as mentioned in Section 3.1, Remark 1, will have resource 0 forever; this boundary with 0 resource values isolates C from the outside). Hence (see again Section 3.1, Remark 1) after time $|C| + 1$ the resource values of C remain unchanged. So if, with probability 1, X does not percolate, then the answer to Question 3.1a is positive. Moreover, we then also have that no resource "escapes to infinity" and hence (by standard arguments) that the answer to Question 3.1b is also positive.

To state the theorem below, we need some extra terminology and notation. Let $k \neq l$ be positive integers. By a *crossing* of the *rectangle* $([0, k] \times [0, l]) \cap \mathbb{Z}^2$ we mean an open path which starts on one of the long sides of the rectangle, ends on the opposite long side, and which lies entirely in the rectangle.

By $p_{n,m}$ we denote the probability that there exists a crossing of a given $n \times m$ rectangle, say the rectangle $([0, n] \times [0, m]) \cap \mathbb{Z}^2$.

We say that a function $f : \mathbb{N} \to \mathbb{R}^+$ decays exponentially if $\exists \lambda > 0$ such that $f(n) < e^{-\lambda n}$, $n \in \mathbb{N}$.

Theorem 3.2

If there exist $N > 0$, $K > 4$ with $p_{N,2N+K} < 1/13$, then a) - d) below hold.

a) The probability of an open path from O to ∂B_n decays exponentially in n. In particular, the X-process does not percolate and the answer to Question 3.1 (a and b) (in Section 3.1) is positive.

b) $P(|C| > n)$ decays exponentially in n, where C is the open component of the origin.

c) $P(\tau > n)$ decays exponentially in n, where τ is the smallest time after which the resource value of O remains unchanged.

d) Consider the distributed clustering process on an $n \times n$ torus (instead of the full square lattice), and let T be the smallest time after which all *resource values remain unchanged. Then $E(T) = O(\log n)$.*

Remarks

1. It is a well-known result in the percolation literature that if crossing probabilities are smaller than a certain κ, then no percolation occurs and the cluster radius and cluster size distributions have an exponential decay. However, in that literature one is mainly interested in the *existence* of such κ rather than its value. This explains why the values which arise from the computations in the early publications were extremely small (see the proof of Theorem 3.2b). Later, in the work of Chayes and Chayes [CC86] a scaling argument was used to show that 1/16 suffices. Below we refine the technique used in [Kes81] and obtain the threshold of 1/13. In the dimensions higher than two, our method produces thresholds on the crossing probabilities, which are by a power of magnitude larger than those obtained by the argument of Chayes and Chayes.

2. Since $p_{n,2n+k}$ is non-decreasing in k, it makes sense to check the conditions of Theorem 3.2 only for $K = 5$, the smallest value beyond the radius of dependence of the X-process.

Proof of Theorem 3.2

First we need some more notation. Let N and K be fixed and satisfy the condition of the theorem. Let $\alpha(i) = \lfloor \frac{i}{2} \rfloor (K+N) + \mathbf{1}_{\{i \text{ is odd}\}} K$, $i \in \mathbb{Z}$. Let, for integer n, m, $S_{n,m}$ be the rectangle $([\alpha(n), \alpha(n+1)] \times [\alpha(m), \alpha(m+1)]) \cap \mathbb{Z}^2$. In the simplest case of $K = N$, $\alpha(i) = Ni$, and all $S_{n,m}$ are $N \times N$ squares. In the general case, if both indices n and m are even, or both are odd, or one is even and the other is odd, $S_{n,m}$ is a $K \times K$, $N \times N$ or $N \times K$ rectangle respectively. The union of all $S_{n,m}$ covers the integer lattice, and each two of them are either disjoint or share a common side.

Let $H_{n,m}$ be the horizontal rectangle given by $H_{n,m} = S_{n-1,m} \cup S_{n,m} \cup S_{n+1,m}$. We call n, m the *coordinates* of $H_{n,m}$. Further let $V_{n,m}$ be the vertical analog of $H_{n,m}$, i.e. $V_{n,m} = S_{n,m-1} \cup S_{n,m} \cup S_{n,m+1}$. Finally, for n and m even, let $\partial_{n,m}$ be the set of twenty $(2N+K) \times N$ rectangles which "surrounds" $S_{n,m}$ at "scaled distance" 5 (see Fig. 3.1). More precisely,

$$\partial_{n,m} = \{H_{n+i,m+j}, V_{n+j,m+i} \,:\, |i| + |j| = 5,\ i \text{ is even}\}.$$

Proof of Theorem 3.2a

Suppose there is an open path Π from O to ∂B_n. If n is sufficiently large, compared to N and K, Π goes outside $\cup_{-6 \le i,j \le 6} S_{i,j}$. Hence there must

be a crossing of some rectangle belonging to $\partial_{0,0}$. Let R_1 be the rectangle which corresponds to the *last* of such crossings appearing on the path Π (it is not difficult to check that this is well-defined), and let (x_1, y_1) be the coordinates of R_1. Remark: It is important to note that after this last crossing Π will never visit the set $\cup_{|i|,|j|\leq 3} S_{i,j}$ again. Let $\tilde{x}_1$ and $\tilde{y}_1$ be two even numbers such that $|\tilde{x}_1| + |\tilde{y}_1| = 4$ and $|x_1 - \tilde{x}_1| + |y_1 - \tilde{y}_1| = 1$. So $(\tilde{x}_1, \tilde{y}_1)$ are the coordinates of the $K \times K$ square $S_{\tilde{x}_1,\tilde{y}_1}$ neighbouring R_1 at the side of the origin. Now, for similar reasons as above, if n is sufficiently large there must be a crossing of a rectangle in $\partial_{\tilde{x}_1,\tilde{y}_1}$. Let x_2, y_2 be the coordinates of the rectangle R_2 corresponding to the last such crossing. By the remark above, each vertex of this crossing is at distance $\geq K$ from R_1. Carrying on like this we construct a sequence of rectangles $R_1, R_2, \ldots R_k$ with $k = \lfloor n/(2N + 2K) \rfloor - 1$, and a sequence $(\tilde{x}_i, \tilde{y}_i), i = 0, 1, \ldots, k$ such that:
i) $\tilde{x}_0 = \tilde{y}_0 = 0$; $R_i \in \partial_{\tilde{x}_{i-1},\tilde{y}_{i-1}}$, $1 \leq i \leq k$; $\tilde{x}_i$ and $\tilde{y}_i$ are even and such that $|\tilde{x}_i - \tilde{x}_{i-1}| + |\tilde{y}_i - \tilde{y}_{i-1}| = 4$ and $|x_i - \tilde{x}_i| + |y_i - \tilde{y}_i| = 1$, where (x_i, y_i) are the coordinates of R_i, $1 \leq i \leq k$.

ii) There is a crossing of R_1; For each i, $2 \leq i \leq k$, there is a crossing of R_i which has distance $\geq K$ from each $R_j, j < i$.

Note that, since K is larger than 4 (and the X-process is 4-dependent), the events in (ii) are independent. Each of these events has probability $\leq p_{N,2N+K}$. Further, $\partial_{\tilde{x}_i,\tilde{y}_i}$ has 20 elements (rectangles) for any i, but if $i \geq 1$ then, as the reader can check, at least 7 of them are covered or surrounded by $\partial_{\tilde{x}_{i-1},\tilde{y}_{i-1}}$. So there are at most 13 "fresh" rectangles, and at least one of them is crossed by the path Π. Hence, there are at most $20 \cdot 13^{k-1}$ sequences which satisfy (i). Hence, the probability that there exists an open path from O to ∂B_n is at most $20 \cdot 13^{k-1} p(N)^k$, with k as defined above. From this the result follows immediately.

Proof of Theorem 3.2b

As mentioned before, in the literature on percolation there are theorems analogous to 3.2b but with the condition "$\exists N : p_{N,2N+K} < 1/13$" replaced by "$\exists N : p_{N,3N} < \kappa$", where κ is a fixed small number (for instance, the κ in Theorem 1 of Kesten [Kes81] is (in the case of the square lattice) smaller than 10^{-100}). The proof of these theorems can be straightforwardly adapted (possibly to the cost of an even smaller κ) to finite-dependent models. However, the smallness of κ is no longer important to us, because $p_{m,3m}$ is clearly at most the product of the number of vertices on a long side of a given $3m \times m$

rectangle (i.e., $3m+1$) and the probability of an open path from O to ∂B_m, which by (1a) goes to 0 as m goes to ∞.

Proof of Theorem 3.2c

This follows immediately from 1b and the earlier observation that $\tau \leq |C| + 1$.

Proof of Theorem 3.2d

Note that, for l sufficiently small (compared to n), the distribution of the configuration at time 1 on a set of vertices of radius l "does not feel" whether we work on an $n \times n$ torus or on the full lattice. In particular, the probability that the component of O is smaller than l is then the same for the torus as for the infinite lattice. Let, for each vertex v on the torus, T_v be the smallest time after which its resource value remains unchanged. By the observation in Remark 1 in Sect. 1, $T \leq$ the size of the $n \times n$ torus, which is n^2. Therefore we trivially have, for each $a > 0$,

$$ET = \sum_{l=1}^{n^2} P(T \geq l) \leq a \log n + n^2 P(T \geq a \log n). \tag{3.1}$$

Further, $n^2 P(T \geq a \log n) = n^2 P(\exists$ a vertex v on the torus with $T_v \geq a \log n)$ $\leq n^4 P(T_O \geq a \log n)$. However, by the observation in the beginning of this proof, for each a, for all sufficiently (depending on a) large n, $P(T_O \geq a \log n) = P(\tau \geq a \log n)$, with τ as in 1c. From this and 1c, we obtain that, for a sufficiently large, the second term in the r.h.s. of (3.1) goes to 0 as $n \to \infty$.
This completes the proof of Theorem 3.2. □

Conjecture

$p_{5,15} < 1/13$.

Remark: Clearly (by Theorem 3.2), under this conjecture the conclusions in Theorem 3.2 a-d hold (and hence the answers to Question 3.1 (a and b) are positive).

Although we have not computed a rigorous convenient upper bound for $p_{5,15}$, the word "conjecture" is somewhat too weak here. One should note that the exact computation of $p_{5,15}$ is a *finite* task: As we remarked before, the X-process depends only on the *order statistics* of the initial resource process. More precisely, the $X_i, i \in H_{0,0}$, depends only on the order statistics of the $r_0(j), \rho(j, H_{0,0}) \leq 2$. So the exact computation of $p_{5,15}$ can be done by enumerating all permutations of the set $\{1, 2, \ldots, |W|\}$ (where $W = \{j : \rho(j, H_{0,0}) \leq 2\}$, and counting the number of permutations which give rise to a crossing of $H_{0,0}$ in the induced X-configuration. However, since $|W| > 100$, not even the fastest computer is able to do this in a realistic time. Maybe by clever inclusion-exclusion arguments one could obtain a rigorous *upper bound* $< 1/13$ for $p_{5,15}$ (or some other $p_{n,2n+5}$). However, we have not done this, but instead used Monte Carlo simulation to estimate $p_{n,2n+5}$ for various values of n (see table below and compare the crossing probabilities with $1/13 = 0.07692...$), which very strongly indicates that the conjecture is true.

n	*Monte Carlo estimate of* $p_{n,2n+5}$	standard deviation	# simulations
4	0.13048	0.00005	5×10^7
5	0.06798	0.00004	5×10^7
6	0.03467	0.00003	5×10^7

Final remark

Even if there were faster computers which could compute (or sufficiently bound) $p_{5,15}$ rigorously, the proof would still not be entirely satisfactory, because for other lattices one would have to make all calculations again. It would be nice to have a more elegant and general proof of absence of percolation for the X-process. For instance by showing (in some sense) domination by a subcritical "ordinary" percolation process. However, in spite of serious attempts, we have not succeeded to do this.

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Figure 1

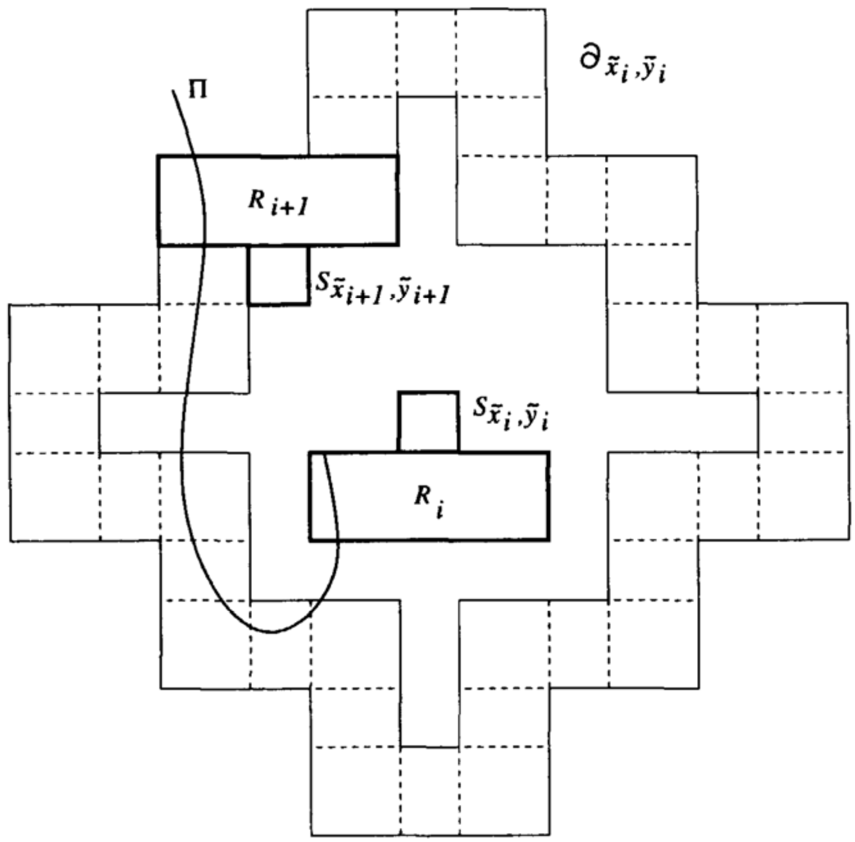

Figure 3.1: A part of a path Π, which crosses the rectangles R_i and R_{i+1}. The second rectangle is an element of $\partial_{\tilde{x}_i,\tilde{y}_i}$.

Chapter 4

Coalescing Ideal Gas

Exact Probabilities and Asymptotics for the One-dimensional Coalescing Ideal Gas [1]

By Alexei Ermakov

Abstract

We consider a modification of the well-known system of coalescing random walks in one dimension, both in discrete and continuous time. In our models each particle moves with unit speed, and it can change its direction of movement only at times of collisions with other particles. At these times (and at time 0) the direction is chosen randomly, with equal probability to the left or to the right, independently of anything else. In this article, we exhibit the exact distributions of particle density and of other relevant quantities at finite time t, and their asymptotics as $t \to \infty$. In particular, it appears that the density of particles at time t is equal to the probability of the event that a simple random walk starting at site one first hits the origin after time t. It is noteworthy that a relation of the same kind is known to hold for the standard system of coalescing random walks in one dimension, though the proof is quite different in that case.

Key words: Interacting particle systems; coalescing random walks; clustering; asymptotic density.

4.1 Introduction and statement of results

Before introducing the coalescing ideal gas model, let us first consider the system of *coalescing random walks* (CRW), which is one of the simplest inter-

[1] To appear in *Stochastic Processes and their Applications*

acting particle systems. Its state space is $\mathbb{Z}^* = \{\text{all subsets of } \mathbb{Z}^1\}$, and time is continuous. At time 0 there is a particle in every site. Each particle performs an independent, continuous time simple random walk with jump rate 1, until it runs into another particle. When two particles meet, they coalesce into one particle, which resumes the same random walk. The behaviour of this system is well understood (see, for instance [Har76], [Gri79], [BG80b], [Arr81]). Let S_t be the position of a continuous time simple random walk at time t: it starts at 0 and makes jumps to each of the 2 neighbour sites at rate 1/2 (so the total jump rate is 1). Let $\tau(x), x \in \mathbb{Z}$ be the first hitting time of site x:

$$\tau(x) = \inf\{t : S_t = x\}.$$

Let $\rho_t(\textit{CRW})$ be the "particle density", or the probability that the origin in the system of coalescing random walks described above is occupied at time t. The value of $\rho_t(\textit{CRW})$ can be computed using a duality relation [Har76], [BG80a]: it is equal to the probability that in the classical one-dimensional *voter model* (see e.g. [Lig85]), the opinion of the individual at the origin survives at time t. Let n_t be the size at time t of the interval of $\mathbb{Z}$ made up of those sites which have inherited the opinion of $(0,0)$. n_t can be viewed as a rate-2 simple random walk on $\mathbb{Z}^+$ (or birth and death process) starting at 1, with absorption at 0.

Hence

$$\begin{aligned} \rho_t(\textit{CRW}) &= P(n_t > 0) = P(\tau(-1) > 2t) \\ &= e^{-2t}(I_0(2t) + I_1(2t)) \sim \frac{1}{\sqrt{\pi}} t^{-1/2}, \quad t \to \infty. \end{aligned} \tag{4.1}$$

The third equality can be obtained by using the Laplace transform (or more directly, see [Fel68b], p.60). $I_\nu(t)$ is the modified Bessel function of the first kind:

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)} \tag{4.2}$$

The asymptotics for the particle density of CRW in higher dimensions was obtained in [BG80a]; duality relations are also exploited, but in this case the relation between CRW and simple random walk (the second equality in (4.1)) is distorted by a random factor and is hardly usable at all.

In [BG80b] it was noted that the asymptotics of $\rho_t(\textit{CRW})$ (4.1) remains the same if we start the CRW from an initial configuration distributed according to an arbitrary mixing measure $\mu \neq \delta_\varnothing$ on $\mathbb{Z}^*$ where $\delta_\varnothing$ denotes the measure which assigns measure 1 to the empty set.

It is thus natural to ask, to which extent the asymptotics of $\rho_t(\textsc{crw})$ can be influenced by introducing local dependence in the dynamics of the underlying coalescing random walk. One simple example of such process is the following. We start at time 0 from $2\mathbb{Z}$, i.e. from the configuration in which all even sites are occupied by particles, and assign to each particle a direction of movement: either $+1$ or -1, with equal probabilities, independently. The particles begin moving in the assigned directions with unit speed, and each particle keeps its direction until it collides with another particle. At the collision, the two particles coalesce into one particle, which chooses its direction with equal probability in each way, independently of anything else, and moves further in this direction. We shall call this interacting particle system *coalescing ideal gas* (CIG), and denote the set of x-coordinates of the particles at time $i \in \mathbb{Z}^+$ by $\hat{\xi}_i$. (Since all the collisions in this process occur at integer times, we shall restrict to integer times). We shall also consider the continuous space and time coalescing ideal gas ξ_t^λ, $\lambda \in (0,\infty)$, $t \in [0,\infty)$ which has the same dynamics as described above, but its initial distribution is a Poisson point process of density λ on $\mathbb{R}^1$. Here and below we mark discrete processes and variables with a hat, to distinguish them from the continuous ones.

This system can be considered as a highly simplified model for aggregation processes at low pressure, such as an early stage of polymerisation in gas phase or aggregation of mist particles. The chemical bonds in the first and the surface tension in the second case make the particles coalesce when they meet, and the absence of air lets them move forward without deviating.

If we change the collision rule from coalescence to annihilation, we obtain the *annihilating ideal gas*, or the *deterministic surface growth* model. It has random initial state and deterministic dynamics. This model and its scaling limit are rather good understood (see [BF95] and references therein). B.Fisch [Fis92] has found the asymptotic rate of the particle density, and conjectured that for the coalescing case it is the same. This conjecture is now proved in Theorem 4.1 in this paper.

First we examine the discrete system $\hat{\xi}_t$. All the particles have even coordinates at even times, and odd coordinates at odd times.
Introduce

$$\begin{aligned}
\hat{\theta}_i &= P\ \text{(the particle which started from the origin at time 0 has its first collision with another particle at time } i\text{)},\ i > 0,\\
\hat{\varphi}_i &= P\ \text{(a collision takes place at time } i \text{ at } x = (i \bmod 2)\text{)},\ i > 0,\\
\hat{\rho}_i &= P\ \text{(there is a particle at time } i \text{ at } x = (i \bmod 2)\text{)}\\
&= P\,(\hat{\xi}_i \cap \{0,1\} \neq \emptyset),\ i \geq 0.
\end{aligned}$$

Here (i mod 2) denotes the remainder of dividing i by 2. For convenience, we define $\hat{\theta}_0 = 0$, $\hat{\varphi}_0 = 1$.

We shall also need an (independent of $\hat{\xi}_t$) discrete time *simple random walk* $\hat{S}_n$ on $\mathbb{Z}$ starting at 0.

$$X_i = \text{i.i.d.},\ P(X_i = -1) = P(X_i = 1) = \tfrac{1}{2},\ i \in \mathbb{Z}^+,$$
$$\hat{S}_0 = 0,\ \hat{S}_n = \textstyle\sum_{i=0}^{n-1} X_i$$

Let $\hat{\tau}_k(x)$ be the *k-th return time* of $\hat{S}_n$ to the site $x \in \mathbb{Z}$:

$$\begin{aligned} \hat{\tau}_1(x) &= \min\{i > 0 : \hat{S}_i = x\} \\ \hat{\tau}_k(x) &= \min\{i > \hat{\tau}_{k-1}(x) : \hat{S}_i = x\}, \quad k > 1. \end{aligned}$$

In the discrete CIG, the increments of a trajectory of a particle are distributed like X_i, but they are positively correlated: the covariance of two increments is equal to the probability that there is no collision inbetween. One could expect therefore that the particles in CIG will aggregate at a higher rate, and the density of the particles will be asymptotically smaller than that of CRW, but this is not the case. In Section 4.2 we shall prove

Theorem 4.1 *For the discrete coalescing ideal gas the following relations hold:*

$$\text{for } n \geq 1,\ \hat{\theta}_n = P(\hat{\tau}_1(0) = 2n) = \frac{1}{2n-1}\binom{2n}{n}2^{-2n} \underset{n\to\infty}{\sim} \frac{1}{2\sqrt{\pi}}n^{-3/2} \tag{4.3}$$

$$\hat{\varphi}_n = 2\hat{\theta}_{n+1} = \frac{1}{n+1}\binom{2n}{n}2^{-2n} \underset{n\to\infty}{\sim} \frac{1}{\sqrt{\pi}}n^{-3/2} \tag{4.4}$$

$$\begin{aligned} \hat{\rho}_n &= 2P(\hat{\tau}_1(0) > 2(n+1)) = 2P(\hat{\tau}_1(-1) > 2(n+1)) \\ &= \binom{2n+1}{n+1}2^{-2n} \underset{n\to\infty}{\sim} \frac{2}{\sqrt{\pi}}n^{-1/2}. \end{aligned} \tag{4.5}$$

Note the similarity between (4.5) and (4.1), which we shall discuss later.

The key step in the proof of the Theorem 4.1 will be establishing a coupling relation which yields the first equality in (4.3).

We now proceed with the continuous version of CIG, ξ_t^λ, which was introduced above. Let $\mathring{\xi}_t^\lambda$ be the Palm version of this process, which is obtained by adding a particle at the origin to the initial configuration ($\mathring{\xi}_0^\lambda = \xi_0^\lambda \cup \{0\}$) and then using the same dynamics. Let P_λ and $\mathring{P}_\lambda$ be the probability measures associated with these processes.

We define the following densities which characterise these processes:

$$\begin{aligned}
\theta_t^\lambda \, dt &= \overset{\circ}{P}_\lambda \text{ (the particle which started from the origin at time 0 has its} \\
&\qquad \text{first collision with another particle at time } s \in [t, t+dt]) \\
\varphi_t^\lambda \, dx \, dt &= P_\lambda \text{ (there is a collision in the space-time area } [0, dx] \times [t, t+dt]) \\
\rho_t^\lambda \, dx &= P_\lambda \text{ (there is a particle at time } t \text{ in the interval } [0, dx]) \\
&= P\left(\xi_t^\lambda \cap [0, dx] \neq \emptyset\right).
\end{aligned}$$

It is clear that θ_t^λ, φ_t^λ and ρ_t^λ are finite, positive, continuous functions, and that the process ξ_t^λ is space-stationary, so that these definitions are consistent.

Now we formulate the continuous analog of Theorem 4.1:

Theorem 4.2 *For the processes $\overset{\circ}{\xi}{}_t^\lambda$, ξ_t^λ (continuous CIG) the following relations hold:*

$$\theta_t^\lambda = \lambda e^{-\lambda t} \left[I_0(-\lambda t) + I_1(-\lambda t) \right] \underset{t \to \infty}{\sim} \frac{1}{2\sqrt{2\pi\lambda}} t^{-3/2}, \tag{4.6}$$

$$\varphi_t^\lambda = \lambda t^{-1} e^{-\lambda t} I_1(\lambda t) \underset{t \to \infty}{\sim} \sqrt{\frac{\lambda}{2\pi}} \, t^{-3/2}, \tag{4.7}$$

$$\rho_t^\lambda = \lambda P\left(\tau(-1) \geq \lambda t\right) = \lambda e^{-\lambda t} \left[I_0(\lambda t) + I_1(\lambda t) \right] \underset{t \to \infty}{\sim} \sqrt{\frac{2\lambda}{\pi}} \, t^{-1/2} \tag{4.8}$$

where $I_\nu(z)$ is defined by (4.2).

Note that although at $t \to 0$, both θ_t^λ and ρ_t^λ are clearly linear in λ:

$$\lim_{t \downarrow 0} \theta_t^\lambda = \rho_0^\lambda = \lambda,$$

this linearity is not preserved when t is big.

It is not surprising that both continuous and discrete CIG have essentially the same asymptotics. More remarkable is the fact that the asymptotics of the particle densities of these two versions of CIG, $\hat{\rho}_n$ and ρ_t^λ, are, up to a constant, equal to those of the particle density $\rho_t(\textsc{crw})$ (4.1), despite the higher mean square displacements of the particles in CIG. Furthermore, all the three densities ($\hat{\rho}_n$, ρ_t^λ and $\rho_t(\textsc{crw})$) can be expressed through return times of a simple random walk (see (4.1), (4.5) and (4.8)). However, while (4.1) follows from a natural coupling between the processes, and another less direct coupling exists in (4.3), we do not see a direct probabilistic relation between CIG and the simple random walk, which yields the first equality in (4.8).

Along with the asymptotic similarity of the densities of CIG and CRW, there is also a fundamental difference between them: unlike the latter system ([BG80b], thm.1), the asymptotics of our system are sensitive to the initial condition, i.e. to the initial density of particles.

4.2 Further properties and proofs for the discrete case

Lemma 4.3 *Consider the trajectory of a CIG particle which starts at site x_0 at time $t_0 = 0$. Let (x_n, t_n), $n \geq 1$ be the coordinates of the n-th collision of the particle.*

Then

1. $(x_{n+1} - x_n)_{n\in\mathbb{Z}^+}$ *(and hence also $(t_{n+1} - t_n)_{n\in\mathbb{Z}^+}$) is an i.i.d. sequence,*
2. $$P\left(x_{n+1} - x_n = k\right) = \frac{1}{2}P\left(t_{n+1} - t_n = |k|\right) = \frac{1}{2}\hat{\theta}_{|k|}, \tag{4.9}$$
3. $$\hat{\varphi}_n = 2\hat{\theta}_{n+1}, \quad n \geq 0, \tag{4.10}$$
4. $$\hat{\rho}_n = \sum_{i>n} \hat{\varphi}_i, \quad n \geq 0. \tag{4.11}$$

Proof. The system of CIG is based on an underlying collection of i.i.d. random variables

$$\begin{array}{l} \{\alpha_{x,t} \,:\, (x,t) \in \mathbb{Z} \times \mathbb{Z}^+,\ x + t \text{ is even}\}, \quad \text{such that} \\ P\left(\alpha_{x,t} = 1\right) = P\left(\alpha_{x,t} = -1\right) = \frac{1}{2} \end{array} \tag{4.12}$$

If there is a collision at (x, t) then after it the particle takes the direction $\alpha_{x,t}$.

Fix an integer $n > 0$ and a sequence $\{a_i\}_{i=1}^{n+1}$. Let $b_i = |a_i|$, $a = \sum_{i=1}^n a_i$, $b = \sum_{i=1}^n b_i$. By translation invariance we can assume that $x_0 = -(a + b)$. We have to show that the events

$$\begin{aligned} A &= \{x_i - x_{i-1} = a_i,\ i = 1, \ldots, n\} \quad \text{and} \\ B &= \{x_{n+1} - x_n = a_{n+1}\} \end{aligned}$$

are independent of each other. Note that A implies $(x_n, t_n) = (x_0 + a, b) = (-b, b)$. Assume without loss of generality that $\alpha_{-b,b} = +1$. Let

$$t^* = \min\{t \in \mathbb{Z}^+ : \text{ there is a collision at } (t+2, t) \text{ and } \alpha_{t+2,t} = -1\}.$$

Under our assumptions, if $(x_n, t_n) = (-b, b)$ then we have

$$x_{n+1} - x_n = t^* + 1. \tag{4.13}$$

The event A is independent of $\{\alpha_{x,t} : x > 0\}$, since any particle which passes through the region $\{(x,t) : x > 0\}$ can not visit the point $(-b, b)$. On the other hand, t^* is independent of $\{\alpha_{x,t} : x \leq 0\}$. Hence t^* is independent of A, and so is B, because of (4.13). This proves assertion 1 of the lemma.

Now we drop all the assumptions made above and instead assume that $\alpha_{0,0} = +1$. The definition of t^* remains the same. One can see that the first collision of the particle which starts at the origin occurs at time $t^* + 1$. Hence

$$\hat{\theta}_{i+1} = P(t^* = i).$$

This observation together with (4.13) proves the last equation in (4.9).

From the definition of t^* it is clear that

$$\{t^* = i\} = \{\text{there is a collision at } (i+2, i)\} \cap \{\alpha_{i+2,i} = -1\}.$$

Taking the probabilities we obtain $\hat{\theta}_{i+1} = \frac{1}{2}\hat{\varphi}_i$, which proves (4.10).

Equation (4.11) follows from the individual ergodic theorem and the fact that each collision replaces two particles by one, so that the particle density decreases each time by $\hat{\varphi}_n$. □

Proof of Theorem 4.1. We shall prove (4.3) by constructing a bijection between the two configuration spaces.

Let T be the time of the first collision of the CIG particle which starts from the origin. Assume for convenience that $\alpha_{0,0} = +1$. Then all the $T+1$ particles which start from $0, 2, \dots, 2T$ coalesce by time T into a single particle with the coordinates (T, T). These particles undergo exactly T collisions up to time T, since each collision reduces the number of particles by one. Let

$$R_T = \{(2i, 0),\ i = 0, \dots, T\} \cup \begin{array}{l}\{\text{the points of collisions, up to time } T,\\ \text{of the particles which visit } (T,T)\}.\end{array}$$

We have just seen that $\operatorname{card}(R_T) = 2T + 1$.

Let us now order the elements of R_T:

$$R_T = \{(\tilde{x}_0, \tilde{t}_0),\ (\tilde{x}_1, \tilde{t}_1), \dots, (\tilde{x}_{2T}, \tilde{t}_{2T})\} \tag{4.14}$$

in such way that

$$\{i < j\} \text{ iff } \{\tilde{x}_i + \tilde{t}_i < \tilde{x}_j + \tilde{t}_j \text{ or } (\tilde{x}_i + \tilde{t}_i = \tilde{x}_j + \tilde{t}_j \text{ and } \tilde{t}_i < \tilde{t}_j)\}.$$

Such an ordering exists and is unique.

Consider the simple random walk

$$\tilde{S}_n = \sum_{i=0}^{n-1} \alpha_{\tilde{x}_i,\tilde{t}_i}, \quad \tilde{S}_0 = 0.$$

(We have assumed that $\alpha_{0,0} = 1$, and hence $\tilde{S}_1 = 1$. The other case can be treated similarly).

The mapping between R_T and $(\tilde{S}_i)_{i=0}^{2T}$ is bijective, since, from knowing $(\tilde{x}_j, \tilde{t}_j)_{j=0}^{i}$ and $(\alpha_{\tilde{x}_j,\tilde{t}_j})_{j=0}^{i}$, $i < 2T$ one can determine $(\tilde{x}_{i+1}, \tilde{t}_{i+1})$. Note that $2T$ is the first return time of the random walk $\tilde{S}_i$ to the origin. Indeed, each step up ($\alpha_{\tilde{x}_i,\tilde{t}_i} = +1$) means that the next particle in the ordering (4.14), $(\tilde{x}_{i+1}, \tilde{t}_{i+1})$ is a new particle with $\tilde{t}_{i+1} = 0$, so we can say that such a step adds one particle to the system. On the other side, a step down ($\alpha_{\tilde{x}_i,\tilde{t}_i} = -1$) means that a collision takes place at $(\tilde{x}_{i+1}, \tilde{t}_{i+1}) : \tilde{t}_{i+1} > 0$ and the number of particles decreases by one. Hence the number of CIG particles after i steps of construction is equal to $\tilde{S}_i + 1$. At step $2T$ the number of particles reduces to one, and the random walk returns to zero for the first time. Therefore

$$\hat{\theta}_n = P(T = n) = P(\hat{\tau}_1(0) = 2n),$$

which yields (4.3).

This, with (4.10), gives (4.4). The first equation in (4.5) follows from (4.3), (4.10), (4.11). The second equation follows from the reflection principle ([Fel68a], p.77). □

4.3 Proofs for the continuous case

In order to prove Theorem 4.2 we shall establish the continuous analogs of some relations in Lemma 4.3 and add one more relation (4.15), which is necessary to resolve the system.

Lemma 4.4 *For the processes $\mathring{\xi}_t^\lambda$, ξ_t^λ the following relations hold:*

$$\theta_t^\lambda = \lambda e^{-2\lambda t} + \int_0^t \varphi_s^\lambda e^{-2\lambda(t-s)} ds, \tag{4.15}$$

$$\varphi_t^\lambda = \frac{\lambda}{2}\theta_t^\lambda + \frac{1}{2}\int_0^t \varphi_s^\lambda \theta_{t-s}^\lambda ds, \tag{4.16}$$

$$\rho_t^\lambda = \int_t^\infty \varphi_s^\lambda ds. \tag{4.17}$$

We shall first show how Theorem 4.2 follows from Lemma 4.4.

Proof of Theorem 4.2. By applying the Laplace transform to both sides of the equations (4.15)–(4.17) we get:

$$\begin{cases} L_\theta(p) &=& \dfrac{\lambda + L_\varphi(p)}{2\lambda + p}, \\ L_\varphi(p) &=& \dfrac{\lambda}{2}\, L_\theta(p) + \dfrac{1}{2}\, L_\varphi(p)\, L_\theta(p), \\ L_\rho(p) &=& \dfrac{\lambda - L_\varphi(p)}{p}. \end{cases}$$

The only solution of this system of equations which satisfies the natural conditions
$\lim_{p\to\infty} L_X(p) = 0$ is

$$\begin{array}{rcl} L_\theta(p) &=& 1 - p^{1/2}(p+2\lambda)^{-1/2}, \\ L_\varphi(p) &=& \lambda + p - p^{1/2}(p+2\lambda)^{1/2}, \\ L_\rho(p) &=& p^{-1/2}(p+2\lambda)^{1/2} - 1. \end{array}$$

Now we can use the fact that the Laplace transform of a sum of two modified Bessel functions I_0 and I_1 is given by

$$L_{I_0(\lambda t)+I_1(\lambda t)}(p) \;=\; \frac{1}{\lambda}\,((p+\lambda)^{1/2}\,(p-\lambda)^{-1/2} - 1),$$

to invert the Laplace transforms of θ_t^λ and ρ_t^λ and obtain (4.6) and (4.8). The density of collisions φ_t^λ can be computed by taking the derivative of the particle density ρ_t^λ. The asymptotics of these probability densities are then obtained by application of a Tauberian theorem ([Fel68b], thm.4, p.423). The first equation in (4.8) can be checked by comparing the expression for ρ_t^λ with (4.1). This completes the proof of Theorem 4.2. □

Proof of Lemma 4.4. First we note that the first two statements of Lemma 4.3 have straightforward continuous analogs.

For given x, t, $\triangle x > 0$ and $\triangle t > 0$ let us denote by $M^\pm[x,t,\triangle x,\triangle t]$ the interior of the space-time region surrounded by the parallelogram with vertices $\{(x,t),\ (x+\triangle x,t),\ (x+\triangle x \pm \triangle t, t+\triangle t),\ (x\pm\triangle t, t+\triangle t)\}$, together with its lower and left boundaries $[(x,t),\ (x+\triangle x,t)) \cup [(x,t),\ (x\pm\triangle t, t+\triangle t))$. The area of $M^\pm[x,t,\triangle x,\triangle t]$ is $\triangle x\,\triangle t$, and hence

$$P_\lambda\,(\text{there is a collision in } M^\pm[x,t,\triangle x,\triangle t]) = \varphi_t^\lambda\,\triangle x\,\triangle t + \triangle x\, o(\triangle t).$$

In the case $t = 0$ the lemma is trivial.

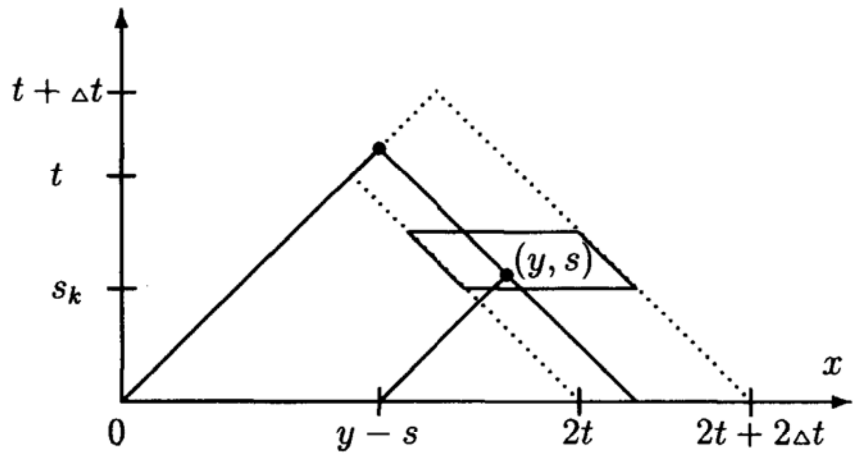

Figure 4.1: Computation of θ_t^λ.

Let us now fix some $t > 0$, and take $\triangle t > 0$, which shall later be shrunk to zero.

In order to prove (4.15) we take a look at the Palm version of the process, $\mathring{\xi}^\lambda_\cdot$, and assume, without loss of generality, that the particle starting at 0 went initially to the right.

Consider the event

$$C = \{\text{the number of particles at time 0 in the interval } [2t,\, 2t{+}2\triangle t) \text{ is at most } 1\}.$$

Note that $P(C) = 1 - O((\triangle t)^2)$.

Fix $N \geq 1$, and break the interval $[0, t)$ into N subintervals of the length $\triangle s = t/N$ by the points $s_k = kt/N$, $0 \leq k \leq N$. Under the condition C we have (see Fig. 4.1)

$$\begin{aligned}
&\{\text{the particle starting at 0 has its first collision in } [t,\, t + \triangle t)\} \\
&\quad = \Big(\{\text{the first particle to the right of 0 is at the time 0 in } [2t,\, 2t + 2\triangle t)\} \\
&\qquad \cap \{\text{it starts going to the left}\} \Big) \\
&\qquad \cup \Bigg(\Big(\bigcup_{k=0}^{N-1} \Big(\{\text{there is a collision at some point } (y, s) \\
&\qquad\qquad \text{in } M^-[2t - s_k, s_k, 2\triangle t, \triangle s]\,\} \\
&\qquad\qquad \cap \{\text{after this collision the particle goes to the left}\} \\
&\qquad\qquad \cap \{\text{at time 0 there are no particles between 0 and } y - s\}\Big)\Big)\Bigg).
\end{aligned}$$

Now, we let first $\triangle s$ and then $\triangle t$ tend to 0. In the limit the complement of the condition C becomes negligible, and we obtain the integral equation (4.15).

To prove (4.16) we turn to the process $\xi^{\lambda}_{\cdot}$. For given t we introduce the same partition of $[0, t)$ as above. Let $\triangle x$ be of order $\triangle t$. Now we shall condition on the event

$$D = \{\text{the number of particles at time 0 in the interval } [0, \triangle x) \text{ is at most } 1\}.$$

Given D, we have

$$\begin{aligned}
&\{\text{there is a collision in } M^+[t, t, \triangle x, \triangle t]\} \\
&\quad = \ \Big(\{\text{there is a particle in } [0, \triangle x) \text{ at time } 0\} \\
&\qquad\quad \cap \{\text{it starts going to the right}\} \\
&\qquad\quad \cap \{\text{its first collision after time 0 takes place} \\
&\qquad\qquad \text{within the time interval } [t,\ t + \triangle t)\} \Big) \\
&\qquad \cup \left(\bigcup_{k=0}^{N-1} \Big(\{\text{there is a collision at some time } s \text{ in } M^+[s_k, s_k, \triangle x, \triangle s]\} \right. \\
&\qquad\qquad\qquad \cap \{\text{after this collision the particle goes to the right}\} \\
&\qquad\qquad\qquad \cap \{\text{the interval between this and the next collision} \\
&\qquad\qquad\qquad\quad \left. \text{is in } [t - s,\ t - s + \triangle t)\} \Big) \right).
\end{aligned}$$

Now we obtain the integral equation (4.16) by the same limiting procedure as above. □

Final remarks

- Our technique can not be applied to higher dimensions. It seems unlikely that exact results can be obtained there for finite time, but perhaps asymptotics can be computed by a different technique. We can guess that in higher dimensions the particle density has asymptotic order of t^{-1}, with a possible logarithmic correction coefficient, as in the CRW case ([BG80a]).

- More realistic models should have more than just two possible speeds with which a particle can move.

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Chapter 5

Annihilating and coalescing ballistic particle systems and their hydrodynamic limits

On some annihilating and coalescing systems [1]

By Alexei Ermakov, Bálint Tóth and Wendelin Werner

Abstract

In the present paper we continue the investigation of the so-called Coalescing Ideal Gas in one dimension, initiated by the first author in [7]. The model consists of point-like particles moving with velocities ± 1 which coalesce and chose a fresh velocity with the same distribution, when colliding. In the previous paper [7] various identities in law were derived for the infinitely extended system. In the present paper we consider the scaling limit of the model in its various guises. The main result is the derivation of the scaling limit (invariance principle) for the joint law of an arbitrary finite number of individual particle trajectories in this system. We also obtain the scaling limit of the density profile of the system, which strongly resembles earlier results of Belitsky-Ferrari [2].

Key Words: *Interacting particle systems; coalescing and annihilating ideal gas; ballistic coalescence and annihilation; random walks; Brownian motion; hydrodynamic limit; invariance principles.*

[1]This article is submitted for publication

5.1 Introduction

In the present paper we study the asymptotic behaviour of the time evolution of one-dimensional systems of coalescing/annihilating ballistic particles. The two basic models discussed are the following:
(1) Coalescing Ideal Gas: at $t = 0$ at every point of integer coordinate there is a particle. Particles have independent identically distributed *velocities* v_i, with distribution $P(v_i = \pm 1) = 1/2$. The time evolution is the following: particles move rectilinearly and uniformly till first collision, when two particles collide they coalesce into one single particle, this single particle chooses freshly a new velocity with the same distribution and continues flying with this new velocity till the next collision, when the same procedure is repeated. The first author studied this *process* (of infinitely many particles) in [7]. The scaling limit of the process (in its various guises) was not treated there.
(2) Annihilating Ideal Gas: the initial conditions are the same, but the time evolution differs. Now, when two particles collide they annihilate. This process was studied by Belitsky and Ferrari in [2], where they prove a scaling limit for the time evolution of the density profile of the system.

The two models can be formulated in a unified way: The particles, beside their i.i.d. ± 1 velocities, also have i.i.d. *masses* $M_i > 0$, $i \in \mathbb{Z}$, which are independent from the velocities. Call $m_i = v_i \cdot M_i$ the *charge* of particle i. Now, define the dynamics in the following natural way: when two particles collide they form one new particle with charge equal to the sum of the two incoming charges. I.e.:

$$
\begin{aligned}
m^{(out)} &= m_1^{(in)} + m_2^{(in)} \\
v^{(out)} &= \mathrm{sgn}(m^{(out)}) \\
M^{(out)} &= |m^{(out)}| = |M_1^{(in)} \quad M_2^{(in)}|
\end{aligned}
$$

If ever $m^{(out)} = 0$, then the two particles annihilate. Now, if the initial masses of particles are identically $M_i = 1$ then clearly we get the model of annihilating ideal gas studied by Belitsky and Ferrari. On the other hand, it is easy to see that if initially the masses have i.i.d. exponential distributions of parameter 1 then this model will mimic the coalescing ideal gas. Note, that the dynamics now is strictly deterministic. The randomness of the dynamics formulated in the first paragraph is now encoded in the random initial masses.

Using this observation we can apply the basic construction of Belitsky and Ferrari to show that the charge-density profile (or the *profile function* i.e. the function $x \mapsto$ the total charge between 0 and x at time t) of the coalescing ideal gas obeys *the same scaling limit* (when $t \to \infty$) as that of the annihilating model of Belitsky and Ferrari. This is just a simple remark to the paper [2].

We then study in more detail the limiting process of profile functions. We prove, inter alia, that in the scaling limit, the set occupied by particles has Hausdorff dimension 1/2 and the profile function is exactly the distribution function of the 'flat' 1/2-Hausdorff measure on this set. These statements are straightforward translations of well-known facts about sample path properties of one-dimensional Brownian motion.

We also study the scaling limit of individual trajectories in the coalescing ideal gas. Note that this question makes sense only in the case of the *coalescing* system: in the annihilating gas individual trajectories die out at the first collision. We prove that the properly rescaled trajectory of a tagged particle in the coalescing ideal gas converges in distribution to the 'Brownian flight process' $\eta(\cdot)$ defined as follows:

$$\eta(t) = \int_0^t \operatorname{sgn}(W_s)\, ds$$

where W denotes a one-dimensional Brownian motion started from 0. We also prove joint invariance principles for any finite number of tagged particles in the system of coalescing ideal gas.

The paper is structured as follows: In Section 2 we reformulate the basic construction of Belitsky and Ferrari. In Section 3 we give precise mathematical meaning to what has been said in this introduction, i.e. we formulate the models of annihilating/coalescing ideal gas, in a joint formalism. In Section 4 we study in detail the limiting object, what we call 'Brownian continuous system'. In Section 5 we formulate the invariance principle for the rescaled profile functions. In the last three sections we study the particle paths properties of coalescing systems. These sections make a *genuinely new* contribution, while Sections 2 to 5 may be considered just as remarks to the paper of Belitsky and Ferrari. In Section 6 we give a general definition of particle paths for a coalescing system and study their properties. In Section 7 we deal with particle paths of the Brownian continuous system, or 'coalescing flight processes'. Finally, in Section 8 we prove the invariance principle for the individual trajectories in coalescing ideal gas.

Notation

$D(\mathbb{R})$ denotes the set of càdlàg real-valued functions on $\mathbb{R}$. Throughout this paper, we will use only the topology on $D(\mathbb{R})$ induced by uniform convergence on compact intervals. A continuity statement on the space $D(\mathbb{R})$ without any further detail will mean continuity with respect to this topology.

$C(\mathbb{R})$ denotes the set of continuous real-valued functions on $\mathbb{R}$ (also endowed with the topology induced by uniform convergence on compact intervals).

When A and B are two sets, $T(A,B) := B^A$ will denote the set of mappings of A into B. We will for instance use the set $T(\mathbb{R}^+, D(\mathbb{R}))$.

If $\alpha > 0$, then h_α denotes the standard α-Hausdorff measure in $\mathbb{R}$ (see e.g. [12] for a precise definition).

5.2 The deterministic semi-group

Consider a càdlàg function $\Phi \in D(\mathbb{R})$. For all $t \geq 0$, we define the function $S_t(\Phi)$ as follows:

$$\begin{aligned} S_0(\Phi) &= \Phi \\ S_t(\Phi)(x) &= \inf\{\Phi(x+y) \ : \ y \in [-t,t)\} \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

Note that S_t maps $C(\mathbb{R})$ into itself and $D(\mathbb{R})$ into itself.

In the following proposition we list some straightforward properties of S_t.

Proposition 5.1 *For any $\Phi \in D(\mathbb{R})$:*

i) For all $t \geq 0$ and $s \geq 0$, $S_t(S_s(\Phi)) = S_{t+s}(\Phi)$. In other words, $(S_t)_{t\geq 0}$ is a semi-group of transformations.

ii) For all $t > 0$, $S_t(\Phi)$ has locally bounded variation. Moreover, there exist two strictly increasing sequences $(x_n)_{n\in\mathbb{Z}}$ and $(y_n)_{n\in\mathbb{Z}}$ such that $\lim_{n\to-\infty} x_n = -\lim_{n\to+\infty} x_n = -\infty$ and for all $n \in \mathbb{Z}$, $x_n < y_n < x_{n+1}$ and $S_t(\Phi)$ is non-decreasing in $[x_n, y_n]$ and non-increasing in $[y_n, x_{n+1}]$.

iii) For all $a > 0$, $S_t(a\Phi) = aS_t(\Phi)$.

iv) For all $a > 0$, define $\lambda_a : D(\mathbb{R}) \to D(\mathbb{R})$ by $(\lambda_a\Phi)(x) = \Phi(ax)$. Then

$$S_{at} \circ \lambda_a = \lambda_a \circ S_t.$$

We now state some 'contractivity' properties of $S.$, which follow immediately from the fact that for $x \in \mathbb{R}$, and for any Φ, Ψ in $D(\mathbb{R})$,

$$|S_t\Phi(x) - S_t\Psi(x)| \leq \sup_{y\in[x-t,x+t]} |\Phi(y) - \Psi(y)|.$$

Proposition 5.2 *i) Let $T(\mathbb{R}^+, D(\mathbb{R}))$ be the set of trajectories of profile functions endowed with the 'uniform uniform' topology induced by uniform convergence on compact subsets of $\mathbb{R}^+ \times \mathbb{R}$.*

The mapping $\mathcal{S} : D(\mathbb{R}) \to T(\mathbb{R}^+, D(\mathbb{R}))$ defined by

$$\mathcal{S}(\Phi)(t) = S_t(\Phi)$$

is continuous. In particular, for any fixed $t \geq 0$, $S_t : D(\mathbb{R}) \to D(\mathbb{R})$ is continuous.

ii) For any fixed *$\Phi \in C(\mathbb{R})$ the mapping $\mathbb{R}^+ \ni t \mapsto S_t\Phi \in C(\mathbb{R})$ is continuous. (the same is also true for $\Phi \in D(\mathbb{R})$ endowed with Skorokhod topology, which we otherwise do not consider in this paper).*

5.3 Discrete Examples

5.3.1 Annihilating particles

Consider now the following deterministic setting: Define two disjoint locally finite subsets of $\mathbb{R}$: A_0^+ and A_0^-. Assume that at time 0, at each point of A_0^+ (respectively A_0^-) a particle starts with unit speed to the right (respectively to the left), and when two particles meet, they annihilate. This type of models has been studied by Fisch [8], Belitsky-Ferrari [2] and it is closely related to the so-called three-colour cellular automaton. As pointed out in [2] it is very easy to express the positions of living particles at time $t > 0$ using S_t.

Define the measure

$$\mu_0 = \sum_{x \in A_0^+} \delta_x - \sum_{x \in A_0^-} \delta_x$$

and the right-continuous function of locally bounded variation $\Phi_0 : \mathbb{R} \to \mathbb{R}$, such that $\Phi_0(0) = 0$ and that the derivative Φ_0' (in the sense of distributions) of Φ_0 is μ_0.

Φ_0 is a step-function with jumps of magnitude ± 1. It is straightforward to check that for any $t > 0$, the function $\Phi_t := S_t(\Phi_0)$ is also a step-function, with jumps of magnitude ± 1. In fact, it is very easy to see that if we define

$$\begin{aligned}
\mu_t &= (\Phi_t)' \\
A_t &= \{x \in \mathbb{R} \ : \ \lim_{y \to x-} \Phi_t(y) \neq \lim_{y \to x+} \Phi_t(y)\} \\
A_t^- &= \{x \in \mathbb{R} \ : \ \lim_{y \to x-} \Phi_t(y) = \lim_{y \to x+} \Phi_t(y) + 1\}
\end{aligned}$$

$$A_t^+ \quad = \quad \{x \in \mathbb{R} \ : \ \lim_{y \to x-} \Phi_t(y) = \lim_{y \to x+} \Phi_t(y) - 1\}$$

then A_t (respectively A_t^+, A_t^-) correspond to the set of particles living at time t (respectively living at time t that move to the right, living at time t that move to the left).

5.3.2 Coalescing particles

Assume now that we modify the previous model in the following way: Each particle is assigned a (positive or negative) charge and moves to the right or to the left at unit speed according to the sign of their charges: It moves to the right if its charge is positive and it moves to the left if the charge is negative. When two particles of charges m^+ and m^- collide, then they stick together and become a single particle of charge $m^+ + m^-$ that moves on with unit speed to the left or to the right depending on the sign of $m^+ + m^-$. Again, once the initial data (the locally finite set of particles A_0 with their respective charges m_x) is given, this system evolves deterministically. We define this time

$$\mu_0 = \sum_{x \in A_0} m_x \delta_x$$

and the function Φ_0 as above. In this case again, the system at time t is described by $\Phi_t := S_t(\Phi_0)$. More precisely, it is easy to check that the set of particles (of non-zero charge) living at time t is the set

$$A_t = \{x \in \mathbb{R} \ : \ \lim_{y \to x-} \Phi_t(y) \neq \lim_{y \to x+} \Phi_t(y)\}$$

and that the charge of the particle located at $x \in A_t$ at time t is

$$m_{x,t} = \lim_{\varepsilon \downarrow 0} \{\Phi_t(x + \varepsilon) - \Phi_t(x - \varepsilon)\} .$$

An interesting subcase here is the case where the absolute value of the charges of the initial particles are independent identically distributed variables, with an exponential law of parameter 1. Note that when two particles meet, their charges have *different* signs; hence,

- The 'outcoming' particle moves to the left (resp. to the right) with probability 1/2.

- The absolute value of the charge of the outcoming particle is again an exponentially distributed random variable of parameter 1, which is independent of the charges of all other particles living at the same time (this is simply because the absolute value of the difference of two independent identically distributed exponential variables of parameter 1 is again an exponentially distributed variable of parameter 1) and is independent of its velocity.

In other words, the law of $(A_t, t \geq 0)$ is exactly that corresponding to the positions of coalescing particles moving at unit speed that choose randomly (with probability 1/2-1/2) whether they go to the right or to the left when they collide (and coalesce). This system, with initial state $A_0 = \mathbb{Z}$, is also mentioned in Fisch [8] and has been studied in Ermakov [7], where it was called *coalescing ideal gas.* Let us stress that the system of coalescing particles which choose randomly their direction when they coalesce is not deterministic, but it is equivalent to the deterministic system of particles of i.i.d. exponential randomly signed mass. In the latter deterministic case, all the collision rules are contained in the information provided by the initial data, i.e. the charges of particles living at time 0.

We shall see how this interpretation of the system in terms of S_t provides an economic way of deriving limit results.

5.4 The Brownian continuous system

We now briefly study the continuous counterpart of the systems that we just described. This continuous system has been introduced in [2]. In the next sections, we shall see that it corresponds to the scaling limit of these discrete systems.

Suppose now that $(B_x, x \in \mathbb{R})$ is a two-sided linear Brownian motion with $B_0 = 0$ (i.e. $(B_x, x \geq 0)$ and $(B_{-x}, x \geq 0)$ are two independent Brownian motions started from 0). Define then, for all $t \geq 0$,

$$\Phi_t = S_t(B).$$

As mentioned in Section 2, Φ_t is of bounded variation for all $t > 0$. In particular, one can define the signed measure $\mu_t = (\Phi_t)'$ (in the sense of distributions) defined on intervals as

$$\mu_t((a,b]) = \Phi_t(b) - \Phi_t(a).$$

This measure μ_t can be loosely speaking interpreted as a regularisation of the white noise.

Let us stress again that the only random part comes from the initial data $\Phi_0 = B$, and that the evolution of Φ_t given Φ_0 is then deterministic.

We now state some results that give some insight into the process $(\Phi_t, t \geq 0)$. As μ_t is a signed measure (when $t > 0$), it is the difference of two non-negative measures μ_t^+ and μ_t^- so that $\mu_t = \mu_t^+ - \mu_t^-$.

Proposition 5.3 *i) For all $t > 0$, the supports Λ_t, Λ_t^+ and Λ_t^- of μ_t, μ_t^- and μ_t^+ are sets of Hausdorff dimension $1/2$, and of locally finite $1/2$-Hausdorff measure.*

ii) The measure μ_t^+ (resp. μ_t^-) is exactly the $1/2$-Hausdorff measure supported by the set Λ_t^+ (resp. Λ_t^-). In other words, if the interval $I = (a, b)$ does not intersect Λ_t^- (i.e. that Φ_t is non-decreasing on I), then $\Phi_t(b) - \Phi_t(a)$ is precisely the $1/2$-Hausdorff measure of $\Lambda_t^+ \cap I$.

The sets Λ_t^+ and Λ_t^- should be interpreted as the sets of particles moving to the right and to the left at time t. As opposed to the previous discrete cases, these sets are uncountable, and their 'mass' is measured by the 1/2-Hausdorff measure.

Proof of Proposition 5.3. Let us first recall some well-known facts concerning the level-sets of Brownian motion. Define the one-dimensional Brownian motion $(W_x, x \geq 0)$ started from 0 and denote its local time at level 0 and time x by ℓ_x. Define also for all $x \geq 0$,

$$W_x^* = \sup_{y \in [0,x]} W_y.$$

The law of $((W_x^* - W_x, W_x^*), x \geq 0)$ is identical to that of $((|W_x|, \ell_x), x \geq 0)$ (see e.g. [11]). On the other hand, the local time at 0 of Brownian motion can be exactly defined as the 1/2-Hausdorff measure of the set of zeros of this Brownian motion (see [11], section 2.5); more precisely, for all $x \geq 0$,

$$\ell_x = h_{1/2}\left(\{y \in [0, x] \ : \ W_y = 0\}\right).$$

Combining these two facts shows immediately that

$$\left(h_{1/2}\left(\left\{y \in [0, x] \ : \ W_y = W_y^*\right\}\right), x \geq 0\right) = (W_x^*, x \geq 0). \tag{5.1}$$

Let us now come back to the actual proof of Proposition 5.3: We now say that x is a point of right-increase (resp. left-increase) of Φ_t if and only if there exists $\varepsilon > 0$ such that such that for all $y \in (x, x + \varepsilon)$ (resp. $y \in (x - \varepsilon, x)$) $\Phi_t(y) > \Phi_t(x)$ (resp. $\Phi_t(y) < \Phi_t(x)$). Note that points of right-increase and left-increase play here a different role due to the non-symmetry of the definition of S_t (we used inf and not sup).

Clearly, the definition of Φ_t implies that $x \in \mathbb{R}$ is a point of right-increase of Φ_t if and only if, for all $y \in (x-t, x+t]$, $B_y > B_{x-t}$.

Let us now define the set

$$H_t = \{x \in \mathbb{R} \ : \ x \text{ is a point of right-increase of } \Phi_t\}.$$

We are now going to show that H_t and Λ_t^+ differ by at most countably many points. Clearly, $H_t \subset \Lambda_t^+$. Suppose for a moment that $x \in \Lambda_t^+ \setminus H_t$. As $x \notin H_t$, there exists $y \in (x-t, x+t]$ such that $B_y \le B_{x-t}$. As $x \in \Lambda_t^+ \setminus H_t$, it is a point of left-increase of Φ_t, and this implies readily that $B_{x-t} = \Phi_t(x)$. Hence one of the following two events is necessarily true:

- $B_{x-t} = B_{x+t}$ (in other words $y = x + t$).
- y is a local minimum of B (this happens if $y \neq x + t$).

Note now that for any rational number $q \in \mathbb{Q}$, there can exist only one $x \in (q-t, q+t)$ such that $B_{x-t} = B_{x+t}$ and for all $z \in (x-t, x+t)$, $B_z \ge B_{x-t}$. Hence, almost surely, for all $t > 0$,

$$\{x \in H_t \setminus \Lambda_t^+ \ : \ B_{x-t} = B_{x+t}\} \text{ is at most countable.} \tag{5.2}$$

Let us now consider the case where y is a local minimum of B. Note that any two local minima of B do occur at different heights (as B only countably many local minima). Hence,

$$x - t = \sup\{z < y \ : \ B_z = B_y\}.$$

In other words, there exists a surjection of the set of local minima of B onto the set

$$\{x \in H_t \setminus \Lambda_t^+ \ : \ \exists y \in (x-t, x+t), \ B_{x-t} = B_y = \Phi_t(x)\}$$

which is therefore also at most countable.

Finally, putting the pieces together (using (5.2)), we get that almost surely, for all $t > 0$,

$$H_t \setminus \Lambda_t^+ \text{ is at most countable.} \tag{5.3}$$

Suppose now for a moment that $x \in H_t$. In particular, this implies (by continuity of B and because for all $y \in (x-t, x+t]$, $B_y > B_{x-t}$) that there exists a rational $q > x + t$, such that

$$x - t = \sup\{y \le q \ : \ B_y \le B_{x-t}\}.$$

Similarly, for all $z \in (x-t, x)$ that is a point of right-increase of Φ_t, the previous observation yields readily that

$$z - t = \sup\{y \leq x - t \ : \ B_y = B_{z-t}\}.$$

But as for all $y \in (x-t, q]$, $B_y > B_{x-t} > B_{z-t}$, we get

$$z - t = \sup\{y \leq q \ : \ B_y = B_{z-t}\}.$$

Hence, $H_t \cap (x-t, x)$ corresponds exactly to hitting times of its maximum of the reversed process started at q. More precisely,

$$H_t \cap (x-t, x) = \{y + t \in (x-t, x) \ : \ W^q_{q-y} = \sup_{[0,q-y]} W^q(\cdot)\}$$

where $W^q(\cdot) = B(q) - B(q - \cdot)$. Hence, combining this with (5.1) implies that for all $a \in (x-t, x)$,

$$h_{1/2}(H_t \cap (a, x)) = B_{x-t} - \inf_{y \in [a-t, x-t]} B_y = \Phi_t(x) - \min_{[a,x]} \Phi_t(\cdot).$$

This implies (using Proposition 1-(ii) and (5.3)) all results dealing with Λ_t^+ stated in the Proposition. Those concerning Λ_t^- are derived via a symmetry argument. □

5.5 Invariance principle for the profile function

Given a discrete (annihilating or coalescing) particle system started from the integer lattice points, as described in Section 5.3, denote by $\hat{\Phi}_0(x)$ its profile function, i.e. the total charge in the interval between the origin and the point of coordinate x, at time 0:

$$\hat{\Phi}_0(x) := \begin{cases} \sum_{i \in \mathbf{Z} \cap (0,x]} m_i & \text{if} \quad x > 0 \\ -\sum_{i \in \mathbf{Z} \cap (x,0]} m_i & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0, \end{cases} \tag{5.4}$$

where the m_i are i.i.d. charges with $E(m_i) = 0$ and $E\left(m_i^2\right) = 1$. We now introduce the time-evolved profile

$$\hat{\Phi}_t = S_t(\hat{\Phi}_0),$$

that is the profile function at time t.

The formalism introduced in the previous sections helps us to formulate and prove immediately functional limit theorems (invariance principles) for the rescaled profile function $\hat{\Phi}^{(N)}$ defined as follows:

$$\hat{\Phi}_t^{(N)}(x) := N^{-1/2}\hat{\Phi}_{Nt}(Nx).$$

Using Proposition 5.1, (iii) and (iv), it is straightforward to check that

$$\hat{\Phi}_t^{(N)} = S_t\hat{\Phi}_0^{(N)}.$$

Hence we directly conclude the following functional limit theorem for the profile function evolving in time:

Proposition 5.4 *The sequence* $(\hat{\Phi}_\cdot^{(N)}(\cdot)) = \mathcal{S}(\hat{\Phi}^{(N)})$ *converges weakly in* $T(\mathbb{R}^+, D(\mathbb{R}))$ *(endowed with the 'uniform uniform topology') to* $\mathcal{S}(B)$ *when* $N \to \infty$*, where* B *is a two-sided Brownian motion with* $B_0 = 0$.

Indeed: This proposition follows directly from the weak convergence $\hat{\Phi}_0^{(N)} \Rightarrow B$ in $D(\mathbb{R})$ and the continuity of $\mathcal{S} : D(\mathbb{R}) \to T(\mathbb{R}^+, D(\mathbb{R}))$.

5.6 Brownian flight process. Particle paths in coalescing systems

From now on we shall concentrate on the systems of coalescing particles. More precisely, we are going to study the scaling limit of trajectories of *individual* particles in the coalescing ideal gas. For this purpose we introduce

Definition 5.5 *The (Brownian) flight process is*

$$\eta(t) = \int_0^t sgn\,(W_s)\,ds, \quad t \in \mathbb{R}^+. \tag{5.5}$$

Here, as earlier, $W.$ *is a one-dimensional Brownian motion started from 0.*

This process consists of countably many linear segments with slopes ± 1 ("flights"). The length of each segment is distributed as the length of an excursion of Brownian motion.

Proposition 5.6 *i) The flight process is self-similar in distribution*

$$\frac{\eta(\alpha\cdot)}{\alpha} \stackrel{d}{=} \eta(\cdot), \quad \forall\alpha \in \mathbb{R}^+ \setminus \{0\};$$

ii) The density of $\eta(t)$ on $(-t,t)$ is

$$\frac{dy}{\pi\sqrt{t^2-y^2}.}$$

Proof. i) follows from the self-similarity of Brownian motion, and ii) is a direct consequence of the Arcsine law of $\int_0^1 1_{\{W_s>0\}}ds$ (see e.g. [14], p.255). □

The Brownian flight process is important for us, because, as we shall see below in (5.7), it is the scaling limit of a trajectory of a coalescing ideal gas particle.

Consider the coalescing ideal gas system (CIG), as described in Section 5.3.2, with the initial particle set $A_0 = \mathbb{Z}$, their masses M_x distributed exponentially with parameter 1, and velocities v_x equal to $+1$ or -1 with probability 1/2. Let $\hat{\Phi}_0(x)$ be the corresponding profile function, as defined in (5.4). Let us denote by $\{\hat{\eta}_x(t)\}_{t\in\mathbb{R}^+}$ the path of the particle which starts at $x \in \mathbb{Z}$. The motion of the particle follows the motion of the corresponding discontinuity of $\hat{\Phi}_t = S_t\hat{\Phi}_0$.

Theorem 1 and Lemma 3 in [7] state that, for the CIG with initial particle set $\mathbb{Z}$, a particle trajectory starting from $x \in \mathbb{Z}$ can be expressed by

$$\{\hat{\eta}_x(n)\}_{n\in\mathbb{Z}^+} \stackrel{d}{=} \left\{x + \frac{1}{4}\sum_{k=0}^{4n-1}\operatorname{sgn}(S_k + S_{k+1})\right\}_{n\in\mathbb{Z}^+}, \tag{5.6}$$

where $(S_n, n \geq 0)$ is a simple symmetric random walk started from $S_0 = 0$.

By Donsker's theorem, this implies the scaling limit result

$$\frac{\hat{\eta}_{nx}(n\cdot)}{n} \underset{n\to\infty}{\Longrightarrow} x + \eta(\cdot) \quad \text{in } C(\mathbb{R}^+) \quad \text{for any } x \in \mathbb{Z}. \tag{5.7}$$

Now, we would like to generalise (5.7) to deal with *joint distributions of finitely many trajectories*, and in particular we want to compute the scaling limit of the collision time of two particles. But the simple representation (5.6) works only for one particle path. So we have to construct multiple particle paths on the same probability space. This can be done by using the profile function machinery of Section 5.3.2 as follows.

Let $\Phi_0(\cdot) \in D(\mathbb{R})$ be an arbitrary initial underlying profile function. Further on we shall leave out the index 0 and write it simply as $\Phi(\cdot)$. We assume the following natural condition:

$$\liminf_{x\to+\infty}\Phi(x) = \liminf_{x\to-\infty}\Phi(x) = -\infty.$$

As we shall see, this assumption in fact means that any two particles will eventually collide. One could (with very slight modifications) adapt the following construction to a general Φ.

We also use the notation

$$\Phi(x\pm) = \lim_{\varepsilon\downarrow 0} \Phi(x \pm \varepsilon),$$

and similarly for any other function.

We say that a particle starts from $x \in \mathbb{R}$ if $\Phi(\cdot)$ is not constant in the neighbourhood of x. Let us denote by $\{\xi(\Phi, x, t)\}_{t\in\mathbb{R}^+}$ the trajectory of the *tagged particle* which starts at such a point x. It is defined as follows. Let $h \in \mathbb{R}$ be an auxiliary variable.

$$\begin{aligned}
l_x(h) &= \sup\{y < x : \Phi(y) < h\},\\
r_x(h) &= \inf\{y > x : \Phi(y) < h\},\\
\theta_x(h) &= \frac{1}{2}(r_x(h) - l_x(h)),\\
\chi_x(h) &= \frac{1}{2}(r_x(h) + l_x(h)),\\
h^* = h^*(\Phi, x, t) &= \sup\{h \in \mathbb{R} : \theta_x(h) > t\},\\
\xi_x(t) = \xi(\Phi, x, t) &= \\
= \begin{cases} \chi_x(h^*) & \text{if } \theta_x(h^*) = \theta_x(h^*+) \\ \chi_x(h^*+) + \frac{\chi_x(h^*) - \chi_x(h^*+)}{\theta_x(h^*) - \theta_x(h^*+)}(t - \theta_x(h^*+)) & \text{if } \theta_x(h^*) \neq \theta_x(h^*+). \end{cases} & \qquad (5.8)
\end{aligned}$$

This construction is illustrated by Fig. 5.1 and 5.2 in the end of the article.

Note that ξ is invariant under scaling of Φ: For all $\lambda > 0$, $x \in \mathbb{R}$ and $t \geq 0$,

$$\xi(\lambda\Phi, x, t) = \xi(\Phi, x, t). \qquad (5.9)$$

It is straightforward to see that h^* is the minimum of the underlying profile function Φ_t in the vicinity of the tagged particle position at time t: $h^* = \min\{\Phi_t(\xi(\Phi, x, t)),\ \Phi_t(\xi(\Phi, x, t)-)\}$. $\theta_x(h^*)$ and $\chi_x(h^*)$ are the time and the spatial location of the first collision of the tagged particle at or after time t. $l_x(h^*)$ and $r_x(h^*)$ are the starting points of the two particles which take part in this collision and which did not change their direction of movement before it.

Note that $h \mapsto l_x(h)$ is non-decreasing and that $h \mapsto r_x(h)$ is non-increasing. Therefore $\theta_x(\cdot)$ is non-increasing, and for any $\Phi \in D(\mathbb{R})$ and $x \in \mathbb{R}$

$$|\theta_x(h) - \theta_x(h')| \geq |\chi_x(h) - \chi_x(h')| \quad \forall h,\ h' \in \mathbb{R}. \qquad (5.10)$$

From this by a simple reasoning it is clear that all trajectories $\xi(\Phi, x, \cdot)$ are Lipschitz-continuous of order 1:

$$|\xi_x(t) - \xi_x(t')| \leq |t - t'| \quad \forall \Phi \in D(\mathbb{R}),\ x \in \mathbb{R},\ t \in \mathbb{R}^+. \tag{5.11}$$

Note however that some particles can move with speed slower than 1: For instance if x is a local minimum of Φ, then $|\eta_x'(0)| < 1$.

Suppose that $\Phi(\cdot)$ is not constant in the neighbourhoods of x and $y \in \mathbb{R}$, $x < y$. Let us define the *coalescence time* $T_{x,y}(\Phi)$ by the formulae:

$$\begin{aligned} g_{x,y} &= \inf\{\Phi(z),\ z \in (x,y)\}, \\ T_{x,y} &= T_{x,y}(\Phi) \ = \ \theta_x(g_{x,y}). \end{aligned} \tag{5.12}$$

The name "coalescence time" is explained by

Proposition 5.7 *Let x and y be as in the above definition. Then the particles which start from x and y* coalesce *at time $T_{x,y}(\Phi)$, i.e.*

$$\begin{aligned} \xi(\Phi,x,t) \neq \xi(\Phi,y,t) \quad &\textit{if} \quad t < T_{x,y}(\Phi), \\ \xi(\Phi,x,t) = \xi(\Phi,y,t) \quad &\textit{if} \quad t \geq T_{x,y}(\Phi). \end{aligned}$$

First we shall prove a technical lemma.

Lemma 5.8 *For all $h \in \mathbb{R}$, for all $x \in \mathbb{R}$,*

$$\xi_z(\theta_x(h)) = \chi_x(h) \quad \forall z \in [l_x(h), r_x(h)]. \tag{5.13}$$

Moreover, if $l_x(h+) < r_x(h+)$ then

$$\xi_z(\theta_x(h+)) = \chi_x(h+) \quad \forall z \in (l_x(h+), r_x(h+)). \tag{5.14}$$

Proof of Lemma 5.8. By the definition, $l_\cdot(h), r_\cdot(h), \theta_\cdot(h)$ and $\chi_\cdot(h)$ are constant in the interval $[l_x(h), r_x(h)]$. (5.13) follows directly. (5.14) follows from (5.13) and the left-continuity of $l_x(\cdot), r_x(\cdot), \theta_x(\cdot)$ and $\chi_x(\cdot)$ □

Proof of Proposition 5.7.
(i) First we consider the case $t = T_{x,y}$. Since $x, y \in [l_x(g_{x,y}),\ r_x(g_{x,y})]$, by (5.13) we have $\xi_x(T_{x,y}) = \xi_y(T_{x,y})$.
(ii) Now we consider the case $t > T_{x,y}$. It follows directly from the definitions, that

$$\begin{aligned} h^*(\Phi,x,t) < h^*(\Phi,x,T_{x,y}) &\leq g_{x,y}, \\ \theta_x(h) = \theta_y(h) \quad &\forall h \leq g_{x,y}, \\ \chi_x(h) = \chi_y(h) \quad &\forall h \leq g_{x,y}. \end{aligned}$$

By (5.8) it follows that $\xi_x(t) = \xi_y(t)$.
(iii) And finally we consider the case $t < T_{x,y}$. Let us assume first that $h^*(\Phi, x, 0) > g_{x,y}$ and $h^*(\Phi, y, 0) > g_{x,y}$; (Recall that $h^*(\Phi, z, 0) = \min\{\Phi(z), \Phi(z-)\}$). By (5.14) we have $\xi_x(\theta_x(g_{x,y}+)) = \chi_x(g_{x,y}+)$. By the Lipschitz-continuity (5.11), this means that

$$\xi_x(t) \leq \chi_x(g_{x,y}+) + (\theta_x(g_{x,y}+) - t) = r_x(g_{x,y}+) - t, \quad \forall t \in [0, \theta_x(g_{x,y}+)].$$

On the other hand, the assumption $h^*(\Phi, y, 0) > g_{x,y}$ implies that $y > r_x(g_{x,y}+)$, and hence

$$\xi_y(t) > r_x(g_{x,y}+) - t, \quad \forall t \in \mathbb{R}^+.$$

This proves that $\xi_x(t) < \xi_y(t),\ t \in [0, \theta_x(g_{x,y}+)]$. An analogous argument shows that the same inequality holds for $t \in [0, \theta_y(g_{x,y}+)]$. In the remaining interval $[\max\{\theta_x(g_{x,y}+), \theta_y(g_{x,y}+)\},\ T_{x,y}]$ both particles move along non-parallel rectilinear paths, which start at two different points and collide at time $T_{x,y}$, according to (i). Therefore, the two paths do not intersect before $T_{x,y}$.

In the case $h^*(\Phi, x, 0) = g_{x,y}$: $\theta_x(g_{x,y}+) = 0$, $\chi_x(g_{x,y}+) = x$, and the particle moves rectilinearly from time 0 to $T_{x,y}$, which makes the proof just simpler. The same holds if $h^*(\Phi, y, 0) = g_{x,y}$. □

The definition (5.8) works correctly for the discrete CIG (see Fig. 5.1 in the end of the article):

Proposition 5.9 *The CIG particle trajectories $\hat{\eta}_x(t)$ defined in the beginning of this section can be expressed in terms of the underlying profile function $\hat{\Phi} = \hat{\Phi}_0$ in the following way:*

$$\hat{\eta}_x(t) = \xi(\hat{\Phi}, x, t),\ x \in \mathbb{Z},\ t \in \mathbb{R}^+. \tag{5.15}$$

The proof is straightforward, since in Section 5.3.2 it is shown that the profile function $\hat{\Phi}$ indeed corresponds to the CIG.

5.7 Coalescing flight processes

The particle path construction of the previous section (5.8) can also be applied to the Brownian continuous system: In this case we define the particle trajectories by

$$\eta_x(t) = \xi(B, x, t),\ x \in \mathbb{R},\ t \in \mathbb{R}^+, \tag{5.16}$$

where $\mathbb{R} \ni x \mapsto B_x$ is the Brownian profile function, i.e. a two-sided Brownian motion, just as in section 5.4. We call the family $\{\eta_x(\cdot)\}_{x\in\mathbf{R}}$ a *system of coalescing flight processes* (in short: CFP). This name is justified by Propositions 5.7 and 5.10.

Proposition 5.10 *i) The CFP particle trajectories are equidistributed with the Flight Process: For all fixed $x \in \mathbb{R}$,*

$$\eta_x(\cdot) \stackrel{d}{=} x + \eta(\cdot). \tag{5.17}$$

ii) CFP is self-similar in distribution, as space and time scale by the same factor: For all $\alpha > 0$,

$$\left\{\frac{\eta_{\alpha x}(\alpha t)}{\alpha}\right\}_{x\in\mathbf{R},t\in\mathbf{R}^+} \stackrel{d}{=} \{\eta_x(t)\}_{x\in\mathbf{R},t\in\mathbf{R}^+}. \tag{5.18}$$

Proof. (i) Without loss of generality we assume $x = 0$. Define the sets:

$$\begin{aligned} M_R &= \{\alpha \geq 0 : (\exists\varepsilon > 0) \text{ such that } B_\alpha = \inf\{B_z : z \in [0, \alpha + \varepsilon]\}\} \\ M_L &= \{\alpha \leq 0 : (\exists\varepsilon > 0) \text{ such that } B_\alpha = \inf\{B_z : z \in [\alpha - \varepsilon, 0]\}\} \end{aligned}$$

The sets M_R and M_L are countable, so we can list their elements as

$$M_R = \{\alpha_i : i = 1, 2, 3, \dots\} \qquad M_L = \{\alpha_i : i = -1, -2, -3, \dots\}$$

For convenience we shall denote $\alpha_0 = 0$. Denote also:

$$\begin{aligned} h_i &= B_{\alpha_i}, \qquad i \in \mathbb{Z}, \\ \beta_i &= \begin{cases} \inf\{z > \alpha_i : B_z = h_i\}, & \text{if } i \geq 0 \\ \sup\{z < \alpha_i : B_z = h_i\}, & \text{if } i < 0, \end{cases} \\ \gamma_i &= \begin{cases} \sup\{z \leq 0 : B_z = h_i\}, & \text{if } i \geq 0 \\ \inf\{z \geq 0 : B_z = h_i\}, & \text{if } i < 0, \end{cases} \\ \rho_i &= |\alpha_i - \gamma_i|, \qquad i \in \mathbb{Z}, \\ \sigma_i &= |\beta_i - \gamma_i|, \qquad i \in \mathbb{Z}. \end{aligned}$$

Note that by the above definitions we have

$$\begin{aligned} \gamma_i < 0 < \alpha_i < \beta_i \qquad & \text{for } i > 0, \\ \alpha_0 = \beta_0 = \gamma_0 = 0, & \\ \beta_i < \alpha_i < 0 < \gamma_i \qquad & \text{for } i < 0. \end{aligned}$$

The following lemma states well known pathwise properties of Brownian motion:

Lemma 5.11 *For almost all Brownian profile functions $B.$,*
(i) $\{h_i : i \in \mathbb{Z}\}$ *is dense in* $(-\infty, 0]$,
(ii) for any $i, j \in \mathbb{Z}$ *with* $i \neq j$ *one of the following two alternatives holds:*

- *either* $h_i < h_j$ *and in this case* $\sigma_j < \rho_i$,
- *or* $h_j < h_i$ *and in this case* $\sigma_i < \rho_j$.

Indeed, *(i)* follows from the fact that a.s. the heights of the local minima of Brownian motion form a dense set in $\mathbb{R}$; *(ii)* is a consequence of the fact that a.s. there are no two local extrema of the same hight.

From this simple lemma it immediately follows that the closed intervals $[\rho_i, \sigma_i]$, $i \in \mathbb{Z}$ are pairwise disjoint and their union is dense in $\mathbb{R}^+$. Given this fact we can define the function $\mathbb{R}^+ \ni s \mapsto X_s \in \mathbb{R}$ as follows:

$$X_s = \begin{cases} B_{s+\gamma_i} - h_i & \text{if } s \in [\rho_i, \sigma_i] \quad \text{for some } i \geq 0 \\ -B_{-s+\gamma_i} + h_i & \text{if } s \in [\rho_i, \sigma_i] \quad \text{for some } i < 0 \\ 0 & \text{if } s \notin \cup_{i \in \mathbb{Z}} [\rho_i, \sigma_i]. \end{cases}$$

In plain words this definition means the following: we take the two independent (one-sided) Brownian paths B_s, $s \geq 0$ and $B'_s = B_{-s}$, $s \geq 0$ and we define the processes

$$R_t = B_t - \min_{0 \leq s \leq t} B_s, \qquad R'_t = B'_t - \min_{0 \leq s \leq t} B'_s.$$

It is well known that $R.$ and $R'_.$ defined this way will be two independent Brownian motions reflecting from 0. $X.$ is obtained by "gluing together $R.$ and $-R'_.$ according to their local time at 0", i.e. the excursions of $R.$ and $-R'_.$ away from 0 are glued together according to the a.s. well defined order of their occurrence. $X.$ obtained this way is an other Brownian motion. Finally, it is straightforward to see that in case of a Brownian profile function (which is a.s. continuous, has no points of increase or decrease and has no two local extrema of the same height) the definition (5.15) of $\eta_0(t)$ is equivalent to

$$\eta_0(t) = \int_0^t \operatorname{sgn}(X_s) ds.$$

(ii) Self-similarity follows from the self-similarity of Brownian motion and of ξ (5.9), and from (5.16):

$$\begin{aligned} \left\{ \frac{\eta_{\alpha x}(\alpha t)}{\alpha} \right\}_{x \in \mathbb{R}, t \in \mathbb{R}^+} &= \left\{ \frac{\xi(B, \alpha x, \alpha t)}{\alpha} \right\}_{x \in \mathbb{R}, t \in \mathbb{R}^+} \\ &\stackrel{d}{=} \left\{ \frac{\xi(x \mapsto \alpha^{-1/2} B_{\alpha x}, \alpha x, \alpha t)}{\alpha} \right\}_{x \in \mathbb{R}, t \in \mathbb{R}^+} \\ &= \{\xi(\alpha^{-1/2} B, x, t)\}_{x \in \mathbb{R}, t \in \mathbb{R}^+} \\ &= \{\xi(B, x, t)\}_{x \in \mathbb{R}, t \in \mathbb{R}^+} = \{\eta_x(t)\}_{x \in \mathbb{R}, t \in \mathbb{R}^+}. \quad \square \end{aligned}$$

Now we can compute the coalescence time $T_{x,y} = T_{x,y}(B)$ of two tagged CFP particles (5.12):

Proposition 5.12 *Let Z denote a standard stable random variable of index 1/2 (i.e. that has the same law as the hitting time of 1 by a standard Brownian motion started from 0) and let W denote a Brownian motion with $W_0 = 0$, that is independent of Z.*

For the CFP system, for any $x, y \in \mathbb{R} : x < y$,

$$\frac{T_{x,y}}{y-x} \stackrel{d}{=} T_{0,1} \stackrel{d}{=} \frac{1}{2}(1 + (W_1 - 2I_1)^2 Z), \tag{5.19}$$

where $I_1 = \inf_{s\in[0,1]} W_s$. In other words, $2T_{0,1} - 1$ is distributed with the density

$$\frac{2t^{1/2}}{\pi(t+1)^2} \text{ on } \mathbb{R}^+. \tag{5.20}$$

Proof. The first relation in (5.19) is clear from the self-similarity of CFP (5.18). From the definition of $T_{x,y}$ (5.12) it is clear that

$$T_{0,1} = \frac{1}{2}(\sup\{s \le 0 : B_s \le I_1\} + \inf\{s \ge 1 : B_s \le I_1\}).$$

Conditionally on $\{B_s, s \in [0,1]\}$, $\sup\{s \le 0 : B_s \le I_1\}$ and $\inf\{s \ge 1 : B_s \le I_1\} - 1$ are independent and their laws are respectively identical to that of $I_1^2 Z$ (i.e. the hitting time of I_1 by $x \mapsto B_{-x}$) and $(B_1 - I_1)^2 Z$ (i.e. the hitting time of $I_1 - B_1$ by $x \mapsto B_{1+x} - B_1$). Hence, as $I_1 \le 0$ and $I_1 \le B_1$, we get (using the fact that Z is stable of order 1/2) that

$$T_{0,1} \stackrel{d}{=} \frac{1}{2}(1 + (B_1 - 2I_1)^2 Z).$$

Let us compute the density of $(B_1 - 2I_1)^2 Z$. By integrating the joint density of (B_1, I_1),
$\sqrt{\frac{2}{\pi}}\,(a-2b)\,e^{-(a-2b)^2/2}$ on $\{(a,b) : a \ge b,\ b \le 0\}$ (see [14], p.105) we obtain

$$\begin{aligned}
\frac{P(B_1 - 2I_1 \in [\alpha, \alpha + d\alpha])}{d\alpha} &= \frac{1}{d\alpha}\int_{-\alpha}^{0} P(I_1 \in [b, b+db], \\
&\qquad B_1 \in [2I_1 + \alpha,\ 2I_1 + \alpha + d\alpha])\, db \\
&= \int_{-\alpha}^{0} \sqrt{\frac{2}{\pi}}\,(2b + \alpha - 2b)e^{-(2b+\alpha-2b)^2/2} db \\
&= \sqrt{\frac{2}{\pi}}\,\alpha e^{-\alpha^2/2} \int_{-\alpha}^{0} db \;=\; \sqrt{\frac{2}{\pi}}\,\alpha^2 e^{-\alpha^2/2}.
\end{aligned}$$

It is well-known that

$$\frac{P(Z \in [t, t+dt])}{dt} = \frac{1}{\sqrt{2\pi t^3}} e^{-1/2t}$$

(see [6], p.353). Therefore, one easily gets (5.20). □

5.8 Invariance principle for particle paths

The following theorem shows that, under the natural scaling (5.18), any finite set of CIG trajectories converges weakly to the corresponding CFP trajectories:

Theorem 5.13 *(Invariance principle for particle paths).*
For any finite set $X = \{x_1, \dots, x_k\} \subset \mathbb{R}$,

$$\left\{ \frac{\hat{\eta}_{\lfloor nx \rfloor}(nt)}{n} \right\}_{x \in X,\, t \in \mathbb{R}^+} \underset{n\to\infty}{\Longrightarrow} \{\eta_x(t)\}_{x \in X,\, t \in \mathbb{R}^+}$$

in the topology of uniform convergence on compacts on $(C(\mathbb{R}^+))^k$.

First we state the B-a.s. continuity of a CFP particle path $D(\mathbb{R}) \ni B \mapsto \xi(B, x, \cdot)$ in $T(\mathbb{R}, C(\mathbb{R}^+))$:

Lemma 5.14 *For almost all Brownian profiles $B.$, for all $T > 0$, for all $x \in \mathbb{R}$ and all $\varepsilon > 0$ there exist $\delta = \delta(B, x, \varepsilon, T) > 0$, $L = L(B, x, \varepsilon, T) \in (-\infty, x)$ and $R = R(B, x, \varepsilon, T) \in (x, \infty)$ such that for any profile function $\tilde{B}. \in D(\mathbb{R})$ if* $\sup\{|\tilde{B}_z - B_z| : L \le z \le R\} < \delta$ *then* $\sup\{|\xi(\tilde{B}, x, t) - \xi(B, x, t)| : 0 \le t \le T\} < \varepsilon$.

Proof of Lemma 5.14. In order to simplify notation we shall denote $\xi_x(t) = \xi(B, x, t)$ and $\tilde{\xi}_x(t) = \xi(\tilde{B}, x, t)$. Note first that due to Lipschitz continuity of $t \mapsto \xi(\tilde{B}, x, t)$ (5.11) for any initial position x and profile function $\tilde{B}$, it suffices to prove $|\xi(\tilde{B}, x, t) - \xi(B, x, t)| < \varepsilon$ for any $t \in \mathbb{R}^+$ fixed. We shall prove it for $t = 1$. Let the Brownian profile $B.$ and the initial position $x \in \mathbb{R}$ be fixed. We shall exploit the following almost sure properties of the Brownian profile function: (1) B is a.s. continuous; (2) B a.s. does not have two or more local extrema at the same hight; (3) almost surely $t = 1$ is not a collision time of the trajectory $\xi_x(t)$, i.e. $\theta^- < 1 < \theta^+$, where $\theta^\pm$ are defined below.

Denote

$$\begin{aligned}
h^* &= h^*(B,x,1) = \inf\{h:\theta_x(h)<1\} = \sup\{h:\theta_x(h)>1\},\\
l^\pm &= l_x(h^*\mp) = \lim_{\varepsilon\downarrow 0} l_x(h^*\mp\varepsilon),\\
r^\pm &= r_x(h^*\mp) = \lim_{\varepsilon\downarrow 0} r_x(h^*\mp\varepsilon),\\
\theta^\pm &= \frac{1}{2}(r^\pm - l^\pm),\\
\chi^\pm &= \frac{1}{2}(r^\pm + l^\pm).
\end{aligned}$$

Note that due to the continuity of $B_{\cdot}$, we have

$$h^* = B_{l^\pm} = B_{r^\pm}.$$

From the definition of the particle trajectories (5.8) it follows that

$$\xi_x(\theta^\pm) = \chi^\pm, \qquad \xi_x(1) = \frac{\theta^+ - 1}{\theta^+ - \theta^-}\chi^- + \frac{1-\theta^-}{\theta^+ - \theta^-}\chi^+.$$

From the fact that a.s. there are no two local minima of the same hight, it follows that almost surely one of the following two alternatives holds: **either** $l^+ = l^- < x < r^- < r^+$, in which case $l^\pm$ and r^+ are not local extrema and r^- is a local minimum of B; **or** $l^+ < l^- < x < r^- = r^+$, in which case $r^\pm$ and l^+ are not local extrema and l^- is a local minimum of B. Assume the first alternative and denote $l = l^\pm$. The proof for the second alternative is analogous. Under this assumption we easily find

$$\xi_x(1) = \chi^- + (1-\theta^-) = \chi^+ - (\theta^+ - 1).$$

Further on we denote

$$\begin{aligned}
\mu &= \inf\{B_z : l+\varepsilon < z < r^- - \varepsilon\},\\
\nu &= \min\{\inf\{B_z : l-\varepsilon \le z \le l\}, \inf\{B_z : r^+ \le z \le r^+ + \varepsilon\}\}.
\end{aligned}$$

Since there are no two local minima of $B_{\cdot}$ having the same hight and l and r^+ are not local extrema of B, we have

$$\nu < h^* < \mu.$$

We denote

$$\delta = \min\{\mu - h^*, h^* - \nu\} > 0$$

and prove that for any $\tilde{B} \in D(\mathbb{R})$

$$\sup\{|\tilde{B}_z - B_z| : z \in (l-\varepsilon, r^+ + \varepsilon]\} < \frac{\delta}{3} \quad \Longrightarrow \quad |\tilde{\xi}_x(1) - \xi_x(1)| < 2\varepsilon, \quad (5.21)$$

from where the assertion of the lemma follows.

Indeed, it is straightforward to check that with this choice of δ the assumption in (5.21) directly implies the following inequalities:

$$\begin{aligned} &\min\{\tilde{B}_l, \tilde{B}_{r^-}\} < h^* + \frac{\delta}{2} < \inf\{\tilde{B}_z : z \in [l+\varepsilon, < r^- - \varepsilon]\}; \\ &\min\{\inf\{\tilde{B}_z : z \in [l-\varepsilon, l]\}, \inf\{\tilde{B}_z : z \in [r^+, r^+ + \varepsilon]\}\} \\ &\qquad\qquad\qquad\qquad < h^* - \frac{\delta}{2} < \inf\{\tilde{B}_z : z \in [l, r^+]\}. \end{aligned}$$

From the first set of inequalities it follows that

$$\begin{aligned} \tilde{l}^- &:= \tilde{l}_x(h^* + \frac{\delta}{2}) \in [l, l+\varepsilon], \\ \tilde{r}^- &:= \tilde{r}_x(h^* + \frac{\delta}{2}) \in [r^- - \varepsilon, r^-]. \end{aligned}$$

Similarly, from the second set of inequalities we get:

$$\begin{aligned} \tilde{l}^+ &:= \tilde{l}_x(h^* - \frac{\delta}{2}) \in [l-\varepsilon, l], \\ \tilde{r}^+ &:= \tilde{r}_x(h^* - \frac{\delta}{2}) \in [r^+, r^+ + \varepsilon]. \end{aligned}$$

Denoting

$$\begin{aligned} \tilde{\theta}^\pm &:= \frac{1}{2}(\tilde{r}^\pm - \tilde{l}^\pm), \\ \tilde{\chi}^\pm &:= \frac{1}{2}(\tilde{r}^\pm + \tilde{l}^\pm), \end{aligned}$$

by the definition of the particle trajectories (5.8) we have

$$\tilde{\xi}_x(\tilde{\theta}^-) = \tilde{\chi}^- \quad \text{and} \quad \tilde{\xi}_x(\tilde{\theta}^+) = \tilde{\chi}^+.$$

Clearly, $\tilde{\theta}^- \leq \theta^-$ and $\tilde{\theta}^+ \geq \theta^+$, so we have $\tilde{\theta}^- < 1 < \tilde{\theta}^+$. From Lipschitz-continuity of the trajectory $\tilde{\xi}_x(t)$ it follows that

$$\tilde{\chi}^+ - (\tilde{\theta}^+ - 1) \leq \tilde{\xi}_x(1) \leq \tilde{\chi}^- + (1 - \tilde{\theta}^-).$$

Putting all the ingredients together we find

$$(\tilde{\theta}^+ - \theta^+) + (\tilde{\chi}^+ - \chi^+) \leq \tilde{\xi}_x(1) - \xi_x(1) \leq (\theta^- - \tilde{\theta}^-) + (\tilde{\chi}^- - \chi^-).$$

From the obvious bounds $|\theta^\pm - \tilde{\theta}^\pm| \leq \varepsilon$, $|\chi^\pm - \tilde{\chi}^\pm| \leq \varepsilon$ finally we get

$$|\tilde{\xi}_x(1) - \xi_x(1)| \leq 2\varepsilon.$$

□

Proof of Theorem 5.13. We shall first show that the finite-dimensional distributions of $\hat{\eta}$ converge under the scaling to those of η:

$$\left\{\frac{\hat{\eta}_{\lfloor nx \rfloor}(nt)}{n}\right\}_{x\in X,\, t\in T} \underset{n\to\infty}{\Longrightarrow} \{\eta_x(t)\}_{x\in X,\, t\in T}\,, \tag{5.22}$$

for any finite sets $X = \{x_1, \ldots, x_k\} \subset \mathbb{R}$ and $T = \{t_1, \ldots, t_l\} \subset \mathbb{R}^+$.

Let us extend the definition of $\xi_x(t)$ (5.8), and hence that of $\hat{\eta}_x(t)$ (5.15), to all $x \in \mathbb{R}$ by

$$\xi_x(t) = \xi(\Phi, x, t) = \xi(\Phi, \sup\{y < x : \Phi(y) \neq \Phi(x)\}, t),\ \Phi \in D(\mathbb{R}),\ x \in \mathbb{R},\ t \in \mathbb{R}^+.$$

For CIG it implies that

$$\hat{\eta}_x(t) = \hat{\eta}_{\lfloor x \rfloor}(t),\ x \in \mathbb{R},\ t \in \mathbb{R}^+.$$

Note that

$$\frac{\hat{\eta}_{\lfloor nx \rfloor}(nt)}{n} = \frac{\xi(\hat{\Phi}_0, \lfloor nx \rfloor, nt)}{n} = \frac{\xi(\hat{\Phi}_0, nx, nt)}{n} = \xi(\hat{\Phi}_0^{(n)}, x, t).$$

Now the scaling limit (5.22) follows from Donsker's theorem ([3], p.151) i.e.

$$\hat{\Phi}_0^{(n)}(\cdot) \underset{n\to\infty}{\Longrightarrow} B.$$

and from the B-a.s. continuity of the functional $\xi(\cdot, x, t) : D(\mathbb{R}) \to \mathbb{R}$ (Lemma 5.14), by [6], Theorem 6.7 on page 365.

Now, let us note that, for any fixed $x \in \mathbb{R}$, $k > 0$, the sequence of probability measures

$$\left\{P\left(\left\{\frac{\hat{\eta}_{\lfloor nx \rfloor}(nt)}{n}\right\}_{t\in[0,k]}\right)\right\}_{n\geq 1}$$

is tight, since the trajectories are Lipschitz-continuous of order 1 (5.11) and uniformly bounded in n. By Corollary 7 in [15], this tightness and (5.22) are enough to prove Theorem 5.13. □

Let us denote, as in the previous section, the coalescence time of CFP trajectories by $T_{x,y} = T_{x,y}(B)$, and that of CIG trajectories by $\hat{T}_{x,y} = T_{x,y}(\hat{\Phi})$. As one can expect from Theorem 5.13, an invariance principle holds for the coalescence times also:

Theorem 5.15 *For any $x, y \in \mathbb{R}$:*

$$\frac{\hat{T}_{\lfloor nx \rfloor, \lfloor ny \rfloor}}{n} \underset{n\to\infty}{\Longrightarrow} T_{x,y}$$

This theorem can be proven along similar lines as Theorem 5.13. We safely leave it to the reader.

Recall that the explicit law of $T_{x,y}$ was derived in Proposition 5.12.

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Figures

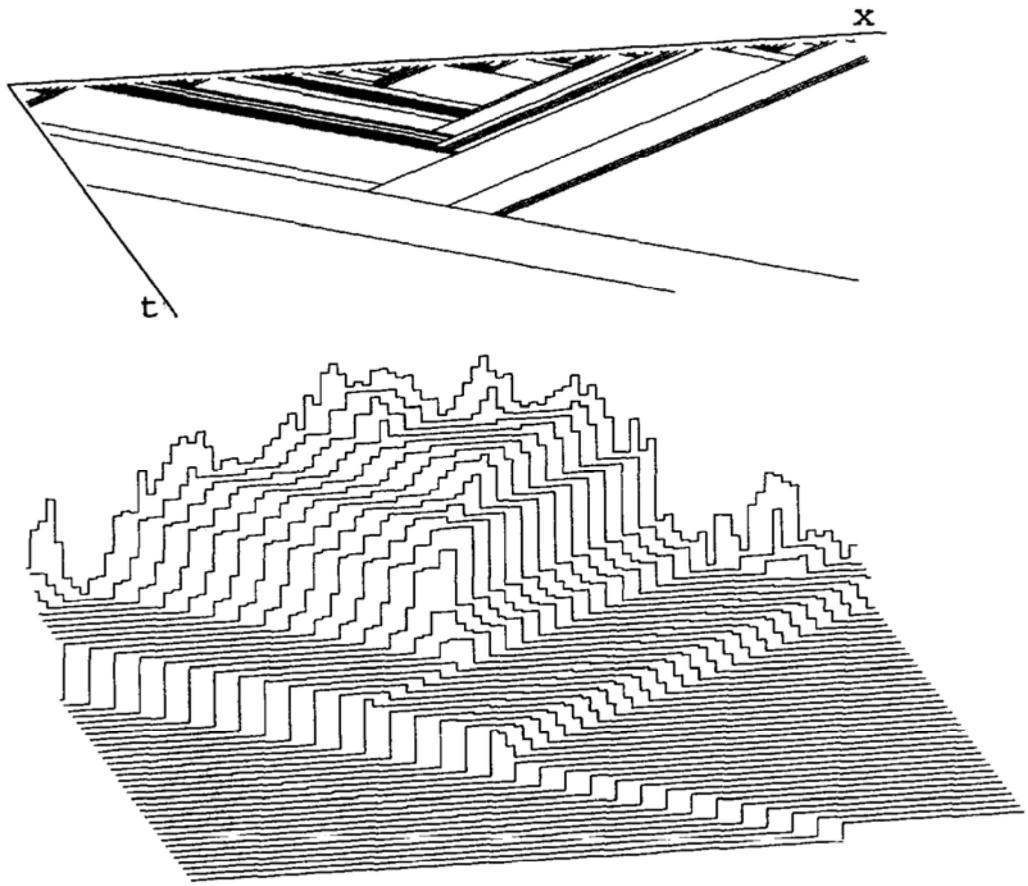

Figure 5.1: Coalescing ideal gas trajectories $\hat{\eta}_x(t) = \xi(\hat{\Phi}_0, x, t)$ and the underlying profile functions $\hat{\Phi}_t(x) = (S_t\hat{\Phi}_0)(x)$.

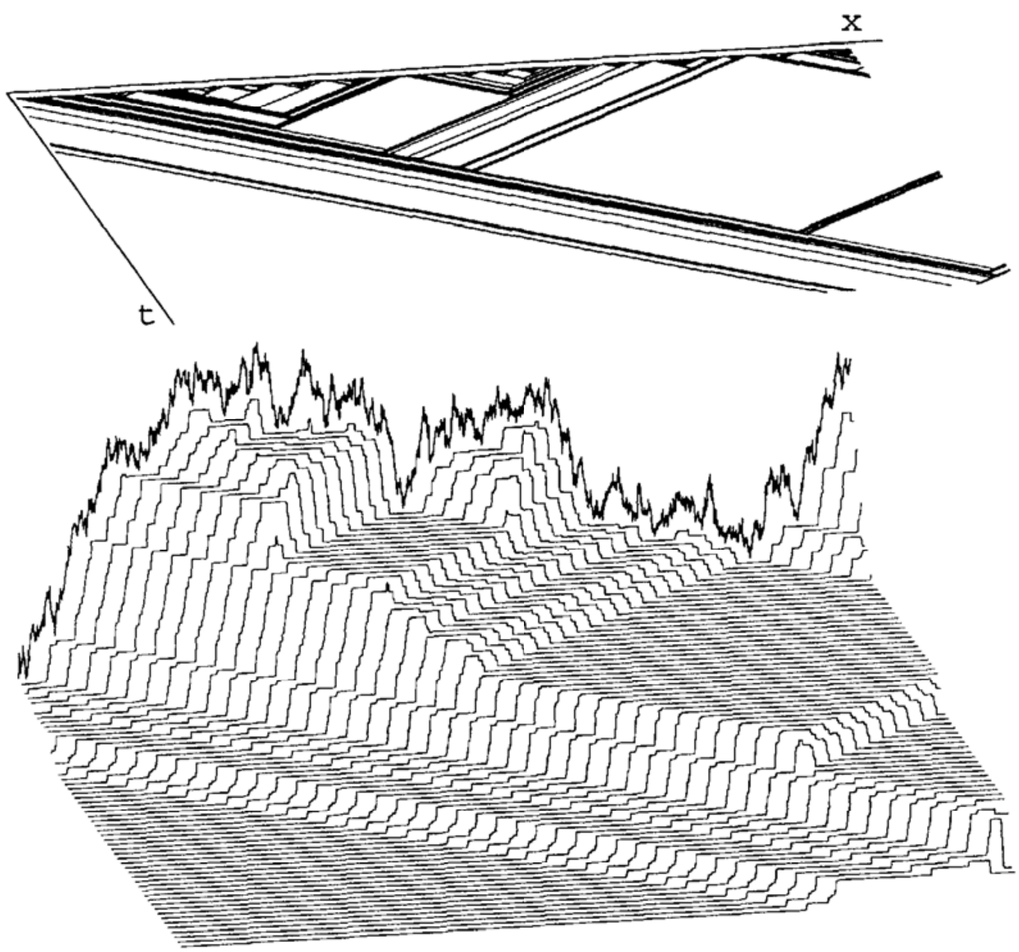

Figure 5.2: Coalescing flight processes $\eta_x(t) = \xi(B, x, t)$ and the underlying profile functions $\Phi_t(x) = (S_t B)(x)$.

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Summary

Percolation and Coalescing Particle Systems

In this thesis we study the site percolation model, which describes connectivity properties of infinitely large random graphs, and coalescing particle systems, which model various aggregation and clustering processes in nature.

In Chapter 1 we introduce to the reader the considered models in the context of current and previous research in the area. We discuss briefly the main results of the following chapters and state several intriguing open questions, which we were not able to answer in full generality, but which inspired much of the research in the thesis.

In Chapter 2 (joint work with J. van den Berg) we obtain a new lower bound for the critical probability of two-dimensional site percolation, by refining a method of Menshikov and Pelikh [MP89].

In Chapter 3 (joint work with J. van den Berg) we use a percolation approach to study a simple clustering process (coalescing particle system) introduced by Coffman, Courtois, Gilbert and Piret [CCGP91]. We reduce the problem of proving the local stabilisation of the process to a finite computational problem. As an intermediate result, we improve the classical necessary condition, involving crossing event probabilities, for percolation in k-dependent random fields [CC86].

In Chapter 4 we study discrete and continuous variants of the one-dimensional coalescing ideal gas model first mentioned by Fisch [Fis92]. We prove an isomorphism between the discrete model and a simple random walk of one particle, and compute several key quantities characterising the model, including the particle density at any time. In particular, we prove the conjecture of Fisch [Fis92] about the asymptotic behaviour of the particle density in this model.

In Chapter 5 (joint work with B. Tóth and W. Werner) we generalise a representation, due to Belitsky and Ferrari [BF95], of the annihilating ideal gas (Ballistic Annihilation) in terms of an underlying simple random walk trajectory (profile function), so that it can deal with coalescing ideal gas. This provides a unified view of annihilating and coalescing systems of ballistic particles. We study the hydrodynamic limit of the systems, and establish the weak limits of scaled particle trajectories and collision times.

Samenvatting

Percolatie en Systemen van Samensmeltende Deeltjes

Dit proefschrift is gewijd aan het punt-percolatie model, dat de samenhangseigenschappen van oneindig grote stochastische grafen beschrijft, en aan systemen van samensmeltende deeltjes, die verscheidene aggregatie- en clusterprocessen in de natuur modellen.

Hoofdstuk 1 geeft een inleiding tot de beschouwde modellen in de context van het huidige en vroegere onderzoek op het gebied. We geven een overzicht van de belangrijkste resultaten van de volgende hoofdstukken en stellen enkele boeiende vragen, die we niet in volledige algemeenheid konden beantwoorden, maar die een groot deel van het onderzoek in dit proefschrift inspireerden.

In Hoofdstuk 2 (gezamenlijk werk met J. van den Berg) wordt een nieuwe ondergrens voor de kritische waarschijnlijkheid van twee-dimensionale punt-percolatie verkregen, door het verfijnen van een methode van Menshikov en Pelikh [MP89].

In Hoofdstuk 3 (gezamenlijk werk met J. van den Berg) gebruiken we een percolatie-benadering voor het bestuderen van een eenvoudig clusterproces (systeem van samensmeltende deeltjes), dat door Coffman, Courtois, Gilbert en Piret [CCGP91] was geintroduceerd. We brengen het probleem van het bewijzen van de locale stabilisatie van het proces terug tot een eindig rekenprobleem. Als een tussenresultaat, verbeteren we de klassieke noodzakelijke voorwaarde, uitgedrukt in kansen van oversteekgebeurtenissen, voor percolatie in k-afhankelijke stochastische velden [CC86].

In Hoofdstuk 4 bestuderen wij de discrete en continue varianten van het één-dimensionale samensmeltende ideale gas, dat voor het eerst in [Fis92] werd genoemd. We bewijzen een isomorfisme van het discrete model met een simpele stochastische wandeling van een enkel deeltje, en berekenen enkele belangrijke grootheden die dit model karakteriseren, waaronder de dichtheid van de deeltjes op ieder tijdstip. In het bijzonder bewijzen we het vermoeden van Fisch [Fis92] over het asymptotische gedrag van de deeltjesdichtheid in dit model.

In Hoofdstuk 5 (gezamenlijk werk met B. Tóth en W. Werner) generaliseren we een door Belitsky en Ferrari [BF95] gevonden uitdrukking voor het annihilerende ideale gas (Ballistische Annihilatie) in termen van de baan van een simpele stochastische wandeling (profiel-functie). Op deze wijze verkrijgen we een universele beschrijving van de systemen van annihilerende en samensmeltende ballistische deeltjes. We bestuderen de hydrodynamisch limiet van deze systemen en vinden de zwakke limieten van de (geschaalde) deeltjesbanen en botsingstijden.

Резюме

Перколяция и системы соединяющихся частиц

Эта диссертация посвящена модели перколяции, или просачивания, описывающей свойства связности бесконечных случайных графов, и системам соединящихся (слипающихся) частиц, моделирующим различные процессы объединения и конгломерации в природе.

В первой главе читателю предлагается введение в рассматриваемые модели в контексте текущих и предыдущих научных исследований в этой области. Мы вкратце обсуждаем основные результаты последующих глав, и ставим несколько интересных открытых вопросов, на которые мы не нашли исчерпывающих ответов, но которые побудили нас к исследованиям, представленным в этой диссертации.

Вторая глава (совместная работа с Я. ван ден Бергом) содержит новую нижнюю оценку порога перколяции для двумерной задачи узлов, полученную путем усовершенствования метода, предложенного М. В. Меньшиковым и К. Д. Пелихом [MP89].

В третьей главе (совместная работа с Я. ван ден Бергом) мы применяем перколяционный подход в исследовании простого процесса конгломерации, предложенного Е. Г. Коффманом с соавторами [CCGP91]. Мы сводим проблему доказательства локальной стабилизации этого процесса к конечной вычислительной проблеме. В качестве промежуточного результата, мы улучшаем классическое необходимое условие перколяции для k-зависимых случайных полей, содержащее вероятности пересечения прямоугольников [CC86].

В главе 4 мы изучаем дискретный и непрерывный варианты одномерной модели соединяющегося идеального газа, впервые упоминавшейся Р. Фишем [Fis92]. Мы доказываем изоморфизм между вышеназванной дискретной моделью и простым случайным блужданием одной частицы, и вычисляем несколько ключевых величин, описывающих эту модель, в том числе плотность частиц в произвольный момент времени. В числе прочего, мы доказываем гипотезу Р. Фиша [Fis92] об асимптотическом поведении плотности частиц в этой модели.

В главе 5 (совместная работа с Б. Тотом и В. Вернером), мы обобщаем предложенную В. Белицким и А. Феррари [BF95] конструкцию аннигилирующего идеального газа (баллистической аннигиляции) на основе траектории простого случайного блуждания одной частицы (функции-профиля), так что она распространяется и на соединяющийся идеальный газ. Таким образом, мы предлагаем единый взгляд на системы аннигилирующих и соединяющихся баллистических частиц. Мы исследуем гидродинамический предел этих систем, и находим слабые пределы масштабированных траекторий частиц и времен столкновения.

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Curriculum Vitae

De schrijver van dit proefschrift werd geboren op 3 december 1967 te Moskou. In 1985 maakte hij de middelbare mathematische school af. Daarna begon hij de studie toegepaste wiskunde aan het Moskou Instituut voor Fysica en Technologie, waar hij in 1991 het diploma van ingenieur-mathematicus cum laude behaalde. In het studiejaar 1992/1993 heeft hij het Master Class programma "kansrekening en statistiek" aan de Universiteit Utrecht met succes voltooid. Van 1993 tot 1997 was hij als onderzoeker in opleiding werkzaam, voor het eerste halfjaar aan de Technische Universiteit Delft en daarna aan het Centrum voor Wiskunde en Informatica te Amsterdam.