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PROVING TOTAL DUAL INTEGRALITY WITH CROSS-FREE FAMILIES – A GENERAL FRAMEWORK

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We present a theorem stating that certain classes of linear programming problems have integer optimal (primal and dual) solutions. The theorem includes as special cases earlier results of Johnson, Edmonds and Giles, Frank, Hoffman and Schwartz, Gröflin and Hoffman, and Lawler and Martel. The proof method consists of deriving total dual integrality for the corresponding system of linear inequalities from the total unimodularity of certain 'cross-free' subsystems. The scheme presented here differs from the one proposed earlier by Grishuhin in that Grishuhin requires the total unimodularity of cross-free subsystems in the axioms, whereas here this follows from easier verifiable axioms.

Key words: Total Dual Integrality, Crossing Family, Cross-Free Family, Total Unimodularity, Edmonds–Giles, Kernel System, Lucchesi–Younger, Submodular, Lattice Polyhedra, Polymatroidal Network Flows.

1. Introduction

A system $Ax \leq b$ of linear inequalities is called *totally dual integral* if the right hand side of the linear programming duality equation

$$\max\{cx \mid Ax \leq b\} = \min\{yb \mid y \geq 0, yA = c\} \tag{1}$$

has an integer optimal solution y for each integer vector c for which the minimum exists. (By giving expressions like $Ax \leq b$ and (1) we implicitly assume compatibility of the sizes of the matrix A and the vectors b and c.) Hoffman [19] and Edmonds and Giles [3] showed the interesting fact that if $Ax \leq b$ is totally dual integral and b is integral, then also the left hand side of (1) has an integer optimal solution.

In recent years several authors introduced frameworks to derive the total dual integrality of certain systems of linear inequalities: see Johnson [21], Edmonds and Giles [3], Frank [5], Hoffman and Schwartz [20], Gröflin and Hoffman [16], Lawler and Martel [22] (see [26] for a survey). The basic idea of these schemes is that total dual integrality follows from the total unimodularity of certain 'cross-free' subsystems. The frameworks contain as special cases several min-max relations from combinatorial optimization, like Ford and Fulkerson's max-flow min-cut theorem, König's matching theorem, Edmonds' matroid intersection theorem, Fulkerson's branching theorem, Lucchesi and Younger's directed cut theorem, Dilworth's chain decomposition theorem, Nash–Williams' orientation theorem.

We here give a theorem which contains each of the schemes mentioned above as special cases. It differs from the one described by Grishuhin [15] in that Grishuhin requires the total unimodularity of certain subsystems a priori in the axioms, whereas in our approach this follows from some easier verifiable axioms (see Section 4 below). Our proof is based on the more or less standard methods introduced in the papers mentioned above.

As usual, two subsets S and T of a set V are called *crossing* if $S \not\subseteq T \not\subseteq S$, $S \cap T \neq \emptyset$, $S \cup T \neq V$. A collection of subsets of a set V is called *cross-free* if no two sets in the collection are crossing.

2. Total dual integrality from cross-free families

Let V be a finite set, let $\mathscr{C}$ be a collection of subsets of V, let n be a natural number, let $b, c \in (\mathbb{R} \cup \{\pm\infty\})^n$, let $f\colon \mathscr{C} \to \mathbb{R}$ and let $h\colon \mathscr{C} \to \{0, \pm 1\}^n$ be such that:

for each $j = 1, \ldots, n$, and for all S, T, U in $\mathscr{C}$:
(a) if $S \subseteq T$ and $h(S)_j \neq 0 \neq h(T)_j$, then $h(S)_j = h(T)_j$;
(b) if $S \cap T = \emptyset$ or $S \cup T = V$, and $h(S)_j \neq 0 \neq h(T)_j$, then $h(S)_j = -h(T)_j$;
(c) if $S \subseteq T \subseteq U$, or $S \subseteq T \subseteq V \backslash U$, or $S \subseteq V \backslash T \subseteq U$, or $V \backslash S \subseteq T \subseteq U$, and $h(S)_j \neq 0 \neq h(U)_j$, then $h(T)_j \neq 0$; (2)

for any two crossing sets S and T in $\mathscr{C}$ there exist S' and T' in $\mathscr{C}$ such that $S' \subset S$ and

$$(h(S) + h(T) - h(S') - h(T'))x \leqslant f(S) + f(T) - f(S') - f(T') \qquad (3)$$

for each vector x with $b \leqslant x \leqslant c$.

(Here we consider the vectors $h(S)$, $h(T)$, and so on, as row vectors, and the vectors b, c and x as column vectors, thus allowing notation like used in (3) above.)

Now consider the system of linear inequalities:

$$\begin{aligned} &\text{(i)} \quad b \leqslant x \leqslant c, \\ &\text{(ii)} \quad h(S)x \leqslant f(S) \quad (S \in \mathscr{C}). \end{aligned} \qquad (4)$$

Theorem. *The system* (4) *is totally dual integral.*

Corollary 1. *If f, b and c are integral, then each face of the polyhedron determined by* (4) *contains integer points.*

Note that if the polyhedron determined by (4) has vertices, then the conclusion of Corollary 1 is equivalent to all vertices of the polyhedron being integral.

Corollary 2. *If f, b, c and w are integral, then the dual linear programming problems*

$$\max\{wx \mid b \leqslant x \leqslant c, h(S)x \leqslant f(S) (S \in \mathscr{C})\}$$
$$= \min\left\{ y_1 c - y_2 b + \sum_{S \in \mathscr{C}} z(S) f(S) \mid y_1, y_2 \in \mathbb{R}^n_+, z \in \mathbb{R}^{\mathscr{C}}_+, \right.$$
$$\left. y_1 - y_2 + \sum_{S \in \mathscr{C}} z(S) h(S) = w \right\} \tag{5}$$

have integer optimal solutions x, y_1, y_2, z.

Remark. To facilitate the interpretation of conditions (2) and (3), we give som examples.

If $D = (V, A)$ is a directed graph (with vertex set V and arrow set A), let a collection of subsets of V, and let h: $\mathscr{C} \to \{0, \pm 1\}^A$ be given by

$$h(S)_a = +1 \quad \text{if arrow } a \text{ leaves } S,$$
$$h(S)_a = -1 \quad \text{if arrow } a \text{ enters } S,$$
$$h(S)_a = 0 \quad \text{otherwise,}$$

for $S \in \mathscr{C}$, $a \in A$. Then h satisfies condition (2). Alternatively, let h: $\mathscr{C} \to \{0, \text{:}$ given by

$$h(S)_a = -1 \quad \text{if arrow } a \text{ enters } S,$$
$$h(S)_a = 0 \quad \text{otherwise,}$$

for $S \in \mathscr{C}$, $a \in A$. Then h satisfies condition (2), provided that there a T, U in $\mathscr{C}$ and an arrow a in A such such that $S \subseteq V \backslash T \subseteq U$ and a and U.

There are two prime examples of condition (3). First, suppos are crossing sets in $\mathscr{C}$, then $S \cap T$ and $S \cup T$ also belong to $\mathscr{C}$, an

$$f(S) + f(T) \geqslant f(S \cap T) + f(S \cup T), \qquad h(S) + h(T) = h(S \cap T) +$$

(f is *submodular* and h is *modular*). Then condition (3) is satisfied and $T' = S \cup T$. Second, suppose that $b \geqslant 0$, and that if S and T a in $\mathscr{C}$, then $S \cap T$ and $S \cup T$ also belong to $\mathscr{C}$, and

$$f(S) + f(T) \geqslant f(S \cap T) + f(S \cup T), \qquad h(S) + h(T) \leqslant h(S \cap T) +$$

(f is submodular and h is *supermodular*). Then again condition (3) is $S' = S \cap T$ and $T' = S \cup T$.

In (6) h is modular, and in (7) h is supermodular.

As usual, for infinite components of b and c the corresponding inequalities in (4) vanish, and the corresponding dual variables in (5) will be zero in any optimum solution.

Proof of the Theorem. Let w be an integer vector for which the optima in (5) exist. We have to show that the right hand side of (5) has an integer optimal solution y_1, y_2, z. Let M be such that $\sum_{S\in\mathscr{C}} z(s) \leq M$ for at least one optimal solution y_1, y_2, z of the minimum in (5). Order the sets in $\mathscr{C}$ as $S_1, \ldots, S_t$ such that if $S_i \subseteq S_j$ then $i \leq j$. Now let y_1, y_2, z attain the minimum in (5) such that $\sum_{S\in\mathscr{C}} z(S) \leq M$ and such that

$$(z(S_1), \ldots, z(S_t)) \tag{10}$$

is lexicographically maximal (this maximum exists by simple compactness arguments). Define

$$\mathscr{C}' := \{S \in \mathscr{C} \mid z(S) > 0\}. \tag{11}$$

We show that $\mathscr{C}'$ is cross-free, i.e., does not contain crossing sets.

Indeed, suppose to the contrary there are crossing sets S, T in $\mathscr{C}'$, where we assume that S has a smaller index than T in the ordering of $\mathscr{C}$. Let $\varepsilon := \min\{z(S), z(T)\} > 0$, and let S', T' as in (3). Define $z' : \mathscr{C} \to \mathbb{R}$ by

$$\begin{aligned}
&z'(S) := z(S) - \varepsilon, \qquad z'(T) := z(T) - \varepsilon,\\
&z'(S') := z(S') + \varepsilon, \qquad z'(T') := z(T') + \varepsilon,\\
&z'(U) := z(U) \quad \text{if } U \in \mathscr{C}\backslash\{S, T, S', T'\}.
\end{aligned} \tag{12}$$

Now notice that by linear programming duality the inequality (3) implies:

$$\begin{aligned}
&\min\{y_1''c - y_2''b \mid y_1'', y_2'' \in \mathbb{R}_+^n,\ y_1'' - y_2'' = h(S) + h(T) - h(S') - h(T')\}\\
&\quad = \max\{(h(S) + h(T) - h(S') - h(T'))x \mid b \leq x \leq c\}\\
&\quad \leq f(S) + f(T) - f(S') - f(T').
\end{aligned} \tag{13}$$

Let y_1'', y_2'' attain this minimum, and define $y_1' := y_1 + \varepsilon y_1''$ and $y_2' := y_2 + \varepsilon y_2''$. Then y_1', y_2', z' again form a feasible solution for the minimum in (5), since

$$\begin{aligned}
&y_1' - y_2' + \sum_{U\in\mathscr{C}} z'(U)h(U)\\
&\quad = y_1 - y_2 + \varepsilon(y_1'' - y_2'' + \sum_{U\in\mathscr{C}} z(U)h(U) - \varepsilon(h(S) + h(T) - h(S') - h(T'))\\
&\quad = y_1 - y_2 + \sum_{U\in\mathscr{C}} z(U)h(U) = w,
\end{aligned} \tag{14}$$

and in fact form an optimum solution since

$$y_1'c - y_2'b + \sum_{U \in \mathscr{C}} z'(U)f(U)$$

$$= y_1c - y_2b + \varepsilon(y_1''c - y_2''b) + \sum_{U \in \mathscr{C}} z(U)f(U) - \varepsilon(f(S) + f(T)$$

$$-f(S') - f(T')) \leq y_1c - y_2b + \sum_{U \in \mathscr{C}} z(U)f(U). \tag{15}$$

But replacing z by z' makes (10) lexicographically larger, contradicting our assumption.

So $\mathscr{C}'$ is cross-free, which implies, as we now show, that the matrix made up by the vectors $h(S)$ for S in $\mathscr{C}'$ is totally unimodular. Indeed, following Edmonds and Giles [3], there exists a 'tree-representation' for $\mathscr{C}'$, i.e., a directed tree T with vertex set W and arc set A, and a function $\phi : V \to W$, such that $\mathscr{C}' = \{V_a \mid a \in A\}$, where, for $a = (w', w'') \in A$,

$$V_a := \{v \in V \mid \phi(v) \text{ is in the component of } (W, A\backslash\{a\}) \text{ containing } w''\}. \tag{16}$$

So V_a consists of all v in V such that arc a of T points towards $\phi(v)$. Now it follows easily from condition (2) that for each $j = 1, \ldots, n$, the arcs a of the tree T for which $h(V_a)_j = \pm 1$ form a path in T, such that the sign of $h(V_a)_j$ corresponds to arc a being 'forward' or 'backward'. To see this, observe that condition (2)(c) implies that the arcs a with $h(V_a)_j = \pm 1$ form a subtree of T. Condition (2)(a) implies that if two such arcs a' and a'' are oriented in series, then $h(V_{a'})_j = h(V_{a''})_j$. Condition (2)($b$) implies that if they are not in series (i.e., they point towards each other, or they are oriented away from each other), the signs are opposite. In particular this gives that no three arcs meeting in one vertex all have $h(V_a)_j \neq 0$.

Hence the matrix of constraints corresponding to $\mathscr{C}'$ is totally unimodular (cf. Tutte [28]). Since the minimum in (5) does not change if we put the extra condition that $z(S) = 0$ if $S \notin \mathscr{C}'$ (as z above attains the minimum), the total unimodularity of the remaining constraint matrix gives that the minimum has an integer optimal solution. □

Remark. Rafael Hassin suggested the following. The theorem holds true if we replace condition (3) by the weaker condition

> the sets in $\mathscr{C}$ can be ordered as $S_1, \ldots, S_t$ such that for any two crossing sets S and T in $\mathscr{C}$ there exist S' and T' in $\mathscr{C}$ such that S' has smaller index than S and
>
> $$(h(S) + h(T) - h(S') - h(T'))x \leq f(S) + f(T) - f(S') - f(T') \tag{3'}$$
>
> for each vector x with $b \leq x \leq c$.

The proof above still applies, except for deleting the sentence 'Order the sets in $\mathscr{C}$ as $S_1, \ldots, S_t$ such that if $S_i \subseteq S_j$ then $i \leq j$'.

3. Applications

We now mention a number of applications of the theorem above, in order of increasing generality (cf. [26]).

I. *Max-flow min-cut theorem.* Ford and Fulkerson [4] proved the following Max-flow min-cut theorem. Let $D=(V, A)$ be a directed graph, let $r, s \in V$, and let $c: A \to \mathbb{R}$ be a 'capacity' function. Then the maximum value of a flow from r to s subject to the capacity c, is equal to the minimum capacity of a cut separating r from s. This follows from Corollary 2 by taking

$$\mathscr{C} := \{\{v\} \mid v \in V\backslash\{r, s\}\} \cup \{V\backslash\{v\} \mid v \in V\backslash\{r, s\}\},$$

$f(S) := 0$ for $S \in \mathscr{C}$, h as in (6), $b \equiv 0$, and $w \in \mathbb{R}^A$ with $w_a = +1$ if arc a leaves r, $w_a = -1$ if a enters r, and $w_a = 0$ otherwise.

II. *Fulkerson's branching theorem.* Fulkerson [11] proved the following. Let $D=(V, A)$ be a directed graph, let $r \in V$, and let $l: A \to \mathbb{Z}_+$ be a 'length' function. Then the minimum length of an r-branching is equal to the maximum number t of r-cuts $C_1, \ldots, C_t$ (repetition allowed) such that no arrow a of D is in more than $l(a)$ of these cuts. (Here an *r-branching* is a spanning directed tree rooted at r. An *r-cut* is a set of arcs entering some nonempty subset of $V\backslash\{r\}$.)

This follows from Corollary 2 by taking $\mathscr{C} := \{S \mid \emptyset \neq S \subseteq V\backslash\{r\}\}$, $f(S) := -1$ for $S \in \mathscr{C}$, h as in (7), $b \equiv 0$, $c \equiv \infty$ and $w = -l$.

III. *A bi-branching theorem.* In [24] the following was proved. Let $D=(V, A)$ be a directed graph, let V be partitioned into classes V_1 and V_2, and let $l: A \to \mathbb{Z}_+$ be a 'length' function. Then the minimum length of a bi-branching is equal to the maximum number t of sets $C_1, \ldots, C_t$ (repetition allowed), where each C_i intersects all bi-branchings, such that no arrow a is in more than $l(a)$ of the C_i. (Here a *bi-branching* is a set A' of arrows of D such that each point in V_1 is the end point of a directed path contained in A' starting in V_2, and each point in V_2 is the starting point of a directed path contained in A' ending in V_1.)

This follows from Corollary 2 by taking $\mathscr{C} := \{S \mid \emptyset \neq S \subseteq V_1$ or $V_1 \subseteq S \neq V\}$, $f(S) := -1$ for $S \in \mathscr{C}$, h as in (7), $b \equiv 0$, $c \equiv \infty$, and $w = -l$. This application contains application II as special case.

IV. *Lucchesi–Younger theorem.* Lucchesi and Younger [23] showed the following. Let $D=(V, A)$ be a directed graph. Then the maximum number of pairwise disjoint directed cuts is equal to the minimum size of a set intersecting all directed cuts. (A *directed cut* is a set of arrows entering some nonempty proper subset of V, provided that no arrow of D leaves this subset.)

This follows from Corollary 2 by taking $\mathscr{C} := \{S \subseteq V \mid \emptyset \neq S \neq V$, no arrow of D leaves $S\}$, $f(S) := -1$ for $S \in \mathscr{C}$, h as in (7), $b \equiv 0$, $c \equiv \infty$, $w \equiv -1$. By varying w we obtain a weighted version.

V. *A strong connector theorem.* In [24] the following was proved. Let $D=(V, A)$ and $D'=(V, A')$ be directed graphs such that for all (v_1, v_2) in A there are v_3, v_4

in V and directed paths in D' from v_1 to v_3, from v_4 to v_3, and from v_4 to v_2. Let $l: A \to \mathbb{Z}_+$ be a 'length' function. Then the minimum length of a strong connector for D' is equal to the maximum number t of sets $C_1, \ldots, C_t$ (repetition allowed), each intersecting all strong connectors, such that no arrow a of D is in more than $l(a)$ of the C_i. (Here a *strong connector* (*for* D') is a set A'' of arrows of D such that the directed graph $(V, A' \cup A'')$ is strongly connected.)

This follows from Corollary 2 by taking $\mathscr{C} := \{S \subseteq V \mid \emptyset \neq S \neq V$, no arrow of D' enters $S\}$, $f(S) := -1$ for $S \in \mathscr{C}$, h as in (7) (with respect to the graph D), $b \equiv 0$, $c \equiv \infty$ and $w = -l$. This application contains II, III, IV.

VI. *Kernel systems.* Frank [5] showed the following. Let $D = (V, A)$ be a directed graph, let $\mathscr{C}$ be a collection of subsets of V and let $g: \mathscr{C} \to \mathbb{Z}$ be such that if S and T are in $\mathscr{C}$, and $S \cap T \neq \emptyset$, then $S \cap T$ and $S \cup T$ also belong to $\mathscr{C}$, and $g(S) + g(T) \leqslant g(S \cap T) + g(S \cup T)$. Let h be as in (7). Then the system: $-h(S)x \geqslant g(S) (S \in \mathscr{C})$, $x \geqslant 0$, is totally dual integral.

This follows from the Theorem by taking $f = -g$, $b \equiv 0$, $c \equiv \infty$. Applications II and III are special cases.

VII. *Matroid intersection.* Edmonds [2] showed the following 'matroid intersection theorem'. Let $M_1 = (X, \mathscr{I}_1)$ and $M_2 = (X, \mathscr{I}_2)$ be matroids, with rank functions r_1 and r_2, respectively, such that $r_1(X) = r_2(X)$. Then the maximum size of a common independent set in M_1 and M_2 is equal to $\min_{X' \subseteq X}(r_1(X') + r_2(X \backslash X'))$.

This follows from Corollary 2 by taking V to be the disjoint union of two copies X_1 and of X_2 of X, $\mathscr{C} := \{S \subseteq V \mid S \subseteq X_1$ or $X_1 \subseteq S\}$, $f(S) := r_1(S)$ if $S \subseteq X_1$, $f(S) := r_2(X_2 \backslash S)$ if $S \supseteq X_1$, $h(S) \in \{0, 1\}^X$ to be the incidence vector of S if $S \subseteq X_1$, and the incidence vector of $X_2 \backslash S$ if $S \supseteq X_1$, $b \equiv 0$, $c \equiv 1$, $w \equiv 1$. Similarly, weighted and 'polymatroidal' versions follow.

VIII. *Generalized polymatroids* (Frank [6]). Let $\mathscr{B}_1$, $\mathscr{P}_1$, $\mathscr{B}_2$, $\mathscr{P}_2$ be collections of subsets of a finite set X, and let $b_1: \mathscr{B}_1 \to \mathbb{Z}$, $p_1: \mathscr{P}_1 \to \mathbb{Z}$, $b_2: \mathscr{B}_2 \to \mathbb{Z}$, $p_2: \mathscr{P}_2 \to \mathbb{Z}$, such that, for $i = 1, 2$

$$\begin{aligned}
&\text{(i) if } S, T \in \mathscr{B}_i \text{ and } S \cap T \neq \emptyset, \text{ then } S \cap T,\ S \cup T \in \mathscr{B}_i \text{ and} \\
&\quad b_i(S \cap T) + b_i(S \cup T) \leqslant b_i(S) + b_i(T); \\
&\text{(ii) if } S, T \in \mathscr{P}_i \text{ and } S \cap T \neq \emptyset, \text{ then } S \cap T,\ S \cup T \in \mathscr{P}_i \text{ and} \\
&\quad p_i(S \cap T) + p_i(S \cup T) \geqslant p_i(S) + p_i(T); \\
&\text{(iii) if } S \in \mathscr{B}_i,\ T \in \mathscr{P}_i, \text{ and } S \backslash T \neq \emptyset,\ T \backslash S \neq \emptyset, \text{ then } S \backslash T \in \mathscr{B}_i, \\
&\quad T \backslash S \in \mathscr{P}_i \text{ and } b_i(S \backslash T) - p_i(T \backslash S) \leqslant b_i(S) - p_i(T).
\end{aligned} \tag{17}$$

Frank [6] showed the total dual integrality of the following system of linear inequalities:

$$\begin{aligned}
&\sum_{s \in S} x(s) \leqslant b_i(S) \quad (i = 1, 2;\ S \in \mathscr{B}_i), \\
&\sum_{s \in S} x(s) \geqslant p_i(S) \quad (i = 1, 2;\ S \in \mathscr{P}_i).
\end{aligned} \tag{18}$$

This follows from the theorem by letting V be the union of two disjoint copies X_1 and X_2 of V, and

$$\mathscr{C}=\{S\subseteq V \mid S\subseteq X_1 \text{ and } S\in\mathscr{B}_1, \text{ or } S\subseteq X_2 \text{ and } X_1\backslash S\in\mathscr{P}_1,\\ \text{or } S\supseteq X_1 \text{ and } X_2\backslash S\in\mathscr{B}_2, \text{ or } S\subseteq X_2 \text{ and } S\in\mathscr{P}_2\}.$$

Let $f:\mathscr{C}\to\mathbb{R}$ and $h:\mathscr{C}\to\{0,\pm 1\}^X$ be given by:

$$\begin{array}{lll} f(S)=b_1(S), & h(S)=\chi(S) & \text{if } S\subseteq X_1,\ S\in\mathscr{B}_1,\\ f(S)=-p_1(X_1\backslash S), & h(S)=-\chi(X_1\backslash S) & \text{if } S\supseteq X_2,\ X_1\backslash S\in\mathscr{P}_1,\\ f(S)=b_2(X_2\backslash S), & h(S)=\chi(X_2\backslash S) & \text{if } S\supseteq X_1,\ X_2\backslash S\in\mathscr{B}_2,\\ f(S)=-p_2(S), & h(S)=-\chi(S) & \text{if } S\subseteq X_2,\ S\in\mathscr{P}_2, \end{array} \tag{19}$$

where $\chi(S)$ denotes the incidence vector of a set S. Again, conditions (2) and (8) are satisfied, and the theorem above is equivalent to Frank's theorem. This application includes application VII.

IX. *A theorem from [24].* In [24] the following was proved. Let $\mathscr{C}$ be a collection of subsets of a set V, and let $f:\mathscr{C}\to\mathbb{Z}$ be such that if S, T are crossing sets in $\mathscr{C}$, then $S\cap T$, $S\cup T$ also belong to $\mathscr{C}$, and $f(S\cap T)+f(S\cup T)\geqslant f(S)+f(T)$. Let furthermore a directed graph $D=(V,A)$ be given such that if S, T, U are in $\mathscr{C}$ such that $S\subseteq V\backslash T\subseteq U$, then no arrow of D enters both S and U. Let $l:A\to\mathbb{Z}_+$ be a 'length' function. Then the minimum length of a set $A'\subseteq A$ such that each S in $\mathscr{C}$ is entered by at least $f(S)$ arrows in A', is equal to the maximum value of $\sum_{i=1}^{k} f(V_i)$, where $S_1,\ldots,S_k$ are sets in $\mathscr{C}$ such that each arrow a of D enters at most $l(a)$ of the S_i.

This result follows easily from Corollary 2 by taking h as given in (7) and $b\equiv-\infty$, $c\equiv 0$, $w=l$. This application contains II, III, IV, V, VI, VII.

X. *Polymatroidal network flows* (Hassin [18], Lawler and Martel [22]). Let $D=(X,A)$ be a directed graph, let $r,s\in X$, and let, for each v in X, f_v^+ and f_v^- be monotone submodular set functions on $\delta^+(v)$ and $\delta^-(v)$, respectively. That is, $f_v^+:\mathscr{P}(\delta^+(v))\to\mathbb{R}_+$ and $f_v^-:\mathscr{P}(\delta^-(v))\to\mathbb{R}_+$, for $v\in X$, such that:

$$\begin{array}{ll} f_v^+(A'\cap A'')+f_v^+(A'\cup A'')\leqslant f_v^+(A')+f_v^+(A'') & \text{if } A',A''\subseteq\delta^+(V),\\ f_v^+(A')\leqslant f_v^+(A'') & \text{if } A'\subseteq A''\subseteq\delta^+(v); \end{array} \tag{20}$$

and similarly for f_v^- and $\delta^-(v)$.

Now a flow x from r to s is called *independent* if for all v in X one has:

$$\begin{array}{ll} \sum_{a\in A'} x(a)\leqslant f_v^+(A') & \text{if } A'\subseteq\delta^+(v),\\ \sum_{a\in A'} x(a)\leqslant f_v^-(A') & \text{if } A'\subseteq\delta^-(v). \end{array} \tag{21}$$

Lawler and Martel [22] showed that the maximum value of an independent flow is equal to the minimum value of

$$\sum_{v\in X} (f_v^+(A_v^+)+f_v^-(A_v^-)), \tag{22}$$

where, for each v in X, $A_v^+\subseteq \delta^+(v)$ and $A_v^-\subseteq \delta^-(v)$, such that $\bigcup_{v\in X}(A_v^+\cup A_v^-)$ contains a cut separating r from s. Moreover, if f_v^+ and f_v^- are integral, there exists an integer maximum flow.

This follows from Corollary 2 by letting $V=\{a^+ \mid a\in A\}\cup\{a^- \mid a\in A\}$, where a^+ and a^- denote new abstract symbols, for each a in A. Let $\mathscr{C}$ consist of the following sets:

$$\begin{array}{lll} \text{(i)} & \{a^+ \mid a\in A'\} & \text{for } v\in X,\ A'\subseteq\delta^+(v), \\ \text{(ii)} & V\backslash\{a^- \mid a\in A'\} & \text{for } v\in X,\ A'\subseteq\delta^-(v), \\ \text{(iii)} & \{a^+ \mid a\in\delta^+(v)\}\cup\{a^- \mid a\in\delta^-(v)\} & \text{for } v\in X,\ v\neq r, s. \end{array} \tag{23}$$

Define a function $f:\mathscr{C}\to\mathbb{R}_+$ as follows.

$$\begin{array}{lll} \text{(i)} & f(S)=f_v^+(A') & \text{if } S \text{ has form (23)(i)}, \\ \text{(ii)} & f(S)=f_v^-(A') & \text{if } S \text{ has form (23)(ii)}, \\ \text{(iii)} & f(S)=0 & \text{if } S \text{ has form (23)(iii)}. \end{array} \tag{24}$$

Define a function $h:\mathscr{C}\to\{0,\pm1\}^A$ as follows.

$$\begin{array}{ll} \text{(i)} & h(S)_a=1, \text{ if } a\in A',\ h(S)_a=0, \text{ if } a\notin A', \text{ if } S \text{ has} \\ & \text{form (23)(i) or (23)(ii)}, \\ \text{(ii)} & h(S)_a=1, \text{ if } a\in\delta^-(v),\ h(S)_a=-1, \text{ if } a\in\delta^+(v), \\ & h(S)_a=0 \text{ otherwise, if } S \text{ has form (23)(iii)}. \end{array} \tag{25}$$

Then (2) and (8) are satisfied. Let $w:A\to\mathbb{Z}$ be defined by: $w(a)=1$ if $a\in\delta^+(r)$, $w(a)=-1$ if $a\in\delta^-(r)$, and $w(a)=0$ otherwise. For $c\equiv 0$ and $d\equiv\infty$, the theorem and Corollary 2 imply the result of Lawler and Martel mentioned above. This application contains I and VII.

XI. *The Edmonds–Giles theorem.* Edmonds and Giles [3] showed the following. Let $D=(V,A)$ be a directed graph, let $\mathscr{C}$ be a collection of subsets of V, and let $f:\mathscr{C}\to\mathbb{Z}$ be such that for each two crossing sets S, T in $\mathscr{C}$, the sets $S\cap T$ and $S\cup T$ also belong to $\mathscr{C}$, and $f(S)+f(T)\geqslant f(S\cap T)+f(S\cup T)$. Let h be as in (6). Then the system: $h(S)x\leqslant f(S)$ $(S\in\mathscr{C})$, $b\leqslant x\leqslant c$ is totally dual integral, for all $b, c\in\mathbb{R}^A$.

This follows directly from the Theorem, and contains I, IV, VII, X and XIII as special cases.

XII. *Lattice polyhedra* (Hoffman and Schwartz [20], cf. Gröflin and Hoffman [16]). Let $(V,\leqslant)$ be a partially ordered set, let $f':V\to\mathbb{Z}$ and $h':V\to\{0,\pm1\}^n$ be such that:

$$\begin{aligned}&\text{(i) if } a\leq b\leq c \text{ then } |h'(a)_j-h'(b)_j|\leq 1 \text{ and}\\&\quad |h'(a)_j-h'(b)_j+h'(c)_j|\leq 1 \text{ for } j=1,\ldots,n;\\&\text{(ii) for all } a, b \text{ in } V \text{ with } a\not\leq b\not\leq a \text{ there exist } a\wedge b \text{ and } a\vee b \text{ in}\\&\quad V \text{ such that } a\wedge b<a \text{ and}\\&\quad f'(a\wedge b)+f'(a\vee b)\leq f'(a)+f'(b),\\&\quad h'(a\wedge b)+h'(a\vee b)=h'(a)+h'(b).\end{aligned} \tag{26}$$

It is shown in [20] and [16] that the system of linear inequalities:

$$b\leq x\leq c, \qquad h'(a)x\leq f'(a) \quad (a\in V), \tag{27}$$

is totally dual integral, for all $b, c\in(\mathbb{R}\cup\{\pm\infty\})^n$. Moreover, if $b\geq 0$ we may relax the 'modularity' condition for h' in (26)(ii) to a 'supermodularity' condition (i.e., $=$ is replaced by $\geq$).

This follows from the theorem above by taking $\mathscr{C}$ to be the collection of prime ideals, i.e., sets of the form $V_{v'}:=\{v\in V\mid v\leq v'\}$, for $v'\in V$. By defining $f(V_v)=f'(v)$ and $h(V_v)=h'(v)$ for v in V, conditions (2) and (3) are satisfied. This case includes the applications I and VII.

XIII. *Distributive lattice polyhedra* (Gröflin and Hoffman [16]). Let $\mathscr{L}$ be a collection of subsets of a set V closed under taking intersections and unions such that $\emptyset$ and V belong to $\mathscr{L}$ (so $\mathscr{L}$ forms a distributive lattice). Let $h:\mathscr{L}\to\{0,\pm1\}^n$ be such that:

$$\begin{aligned}&\text{(i) } h(\emptyset)=h(V)=0;\\&\text{(ii) if } S\subseteq T\subseteq U \text{ are in } \mathscr{L}, \text{ then } |h(S)_j-h(T)_j|\leq 1 \text{ and}\\&\quad |h(S)_j-h(T)_j+h(U)_j|\leq 1 \text{ for all } j=1,\ldots\ n;\\&\text{(iii) if } S, T\in\mathscr{L} \text{ then } h(S)+h(T)=h(S\cap T)+h(S\cup T).\end{aligned} \tag{28}$$

Let $\mathscr{C}\subseteq\mathscr{L}$ and $f:\mathscr{C}\to\mathbb{Z}$ be such that:

$$\begin{aligned}&\text{if } S \text{ and } T \text{ are crossing sets in } \mathscr{C}, \text{ then } S\cap T \text{ and } S\cup T \text{ also}\\&\text{belong to } \mathscr{C} \text{ and } f(S\cap T)+f(S\cup T)\leq f(S)+f(T).\end{aligned} \tag{29}$$

Then the following system of linear inequalities is totally dual integral, for each $b, c\in(\mathbb{Z}\cup\{\pm\infty\})^n$:

$$b\leq x\leq c, \qquad h(S)x\leq f(S) \quad (S\in\mathscr{C}). \tag{30}$$

Moreover, if $b\geq 0$, and $h:\mathscr{L}\to\{0,+1\}^n$, we may replace the 'modularity' condition (28)(iii) by a 'supermodularity' condition (i.e., $=$ is replaced by $\leq$).

To derive these results from the theorem, observe that h restricted to $\mathscr{C}$ satisfies (2).

Indeed, first let $S, T\in\mathscr{C}$ and $j=1,\ldots,n$, with both $h(S)_j$ and $h(T)_j$ nonzero. If $S\subseteq T$, then by (28)(ii) $h(S)_j=h(T)_j$. If $S\cap T=\emptyset$ then by (28)(iii) $h(S)+h(T)=h(\emptyset)+h(S\cup T)=h(S\cup T)$, which is a $\{0,\pm1\}$-vector, and hence $h(S)_j=-h(T)_j$. This is derived similarly if $S\cup T=V$.

Second, let $S, T, U \in \mathscr{C}$ and $j = 1, \ldots, n$, such that $S \subseteq T \subseteq U$, or $S \subseteq T \subseteq V \backslash U$, or $S \subseteq V \backslash T \subseteq U$, or $V \backslash S \subseteq T \subseteq U$, and suppose both $h(S)_j$ and $h(U)_j$ are nonzero. We have to show that $h(T)_j$ also is nonzero. If $S \subseteq T \subseteq U$, then by (28)(ii) $h(T)_j$ is nonzero. If $S \subseteq T \subseteq V \backslash U$, then $h(T)_j = h(T \cup U)_j + h(\emptyset)_j - h(U)_j = h(T \cup U)_j + h(S)_j$. If $h(T)_j = 0$ then $h(T \cup U)_j = -h(S)_j$, contradicting the fact that $S \subseteq T \cup U$ and $h(S)_j \neq 0$. If $S \subseteq V \backslash T \subseteq U$, then $h(T)_j = h(T \cap U)_j + h(V)_j - h(U)_j = h(T \cap U)_j - h(S)_j$. If $h(T)_j = 0$ then $h(T \cap U)_j = h(S)_j$, contradicting the fact that $(T \cap U) \cap S = \emptyset$ and $h(S)_j \neq 0$. If $V \backslash S \subseteq T \subseteq U$, then $h(T)_j = h(S \cap T)_j + h(V)_j - h(S)_j = h(S \cap T)_j + h(U)_j$. If $h(T)_j = 0$ then $h(S \cap T)_j = -h(U)_j$, contradicting the fact that $S \cap T \subseteq U$ and $h(U)_j \neq 0$.

One similarly shows (2) if h is nonnegative and supermodular.

Since (3) (in fact, (8) or (9)) also is satisfied, we may apply the theorem. This application contains applications I, IV, VI, VII, X, XI.

4. Some final remarks

There are some obvious generalizations of the above scheme. It is straightforward to see that condition (3) may be replaced by:

for any two crossing sets S and T in $\mathscr{C}$ there exist $S_1, \ldots, S_k$ in $\mathscr{C}$ such that $S_1 \subset S$ and

$$\begin{aligned} & f(S) + f(T) - f(S_1) - \cdots - f(S_k) \\ & \geqslant (h(S) + h(T) - h(S_1) - \cdots - h(S_k))x \end{aligned} \tag{31}$$

for each vector x with $b \leqslant x \leqslant c$.

We may allow $\mathscr{C}$ to be a 'family', in which a set may occur more than once, with possibly different values under the functions f.

Among the problems still left is how to abstract $\mathscr{C}$ to just an order-theoretic structure. How necessary is it that $\mathscr{C}$ be a collection of sets?

Moreover, can the scheme be extended so that it also yields the total dual integrality of the matching polyhedron (Cunningham and Marsh [1]), which can be shown along similar lines (cf. [27, 25]), and even of the matching forest inequalities (Giles [12, 13, 14])?

Among others, in Frank [7, 8], Fujishige [8, 10], Grötschel, Lovász and Schrijver [17], Hassin [18], Lawler and Martel [22], Zimmermann [29] (polynomial) algorithms are described for solving some of the special cases mentioned in Section 3. Can these methods be extended to the general problem (5)? One of the problems may be to find a short way to describe the problem, i.e., to describe the collection $\mathscr{C}$ and the functions f and h. Typically, $\mathscr{C}$ is given not by listing all of its members (which may be exponential in $|V|$), but by a generating structure of size polynomially bounded by $|V|$ (which is the case in most of the applications given above).

The following totally dual integral system is a special case of (4), but is not contained in any of the schemes I–XIII:

$$\begin{aligned} x_1 + x_2 \qquad\quad &\leqslant 1, \\ x_1 - x_2 \qquad\quad &\leqslant 2, \\ x_1 \qquad + x_3 \quad &\leqslant 2, \\ - x_3 - x_4 &\leqslant -3, \\ x_2 \qquad\quad &\geqslant 0, \\ x_3 \quad &\geqslant 0. \end{aligned} \tag{32}$$

To see that (32) is a special case of (4), take

$V := \{s, t, u, v, w\}$, $\quad \mathscr{C} := \{\{s, t, w\}, \{t, u, w\}, \{t, w\}, \{s, u, v, w\}\}$,

$h(\{s, t, w\}) = (1, 1, 0, 0)$, $\quad h(\{t, u, w\}) = (1, -1, 0, 0)$, $\quad h(\{t, w\}) = (1, 0, 1, 0)$,

$h(\{s, u, v, w\}) = (0, 0, -1, -1)$, $\quad b = (-\infty, 0, 0, -\infty)^{\mathrm{T}}$, $\quad c = (\infty, \infty, \infty, \infty)^{\mathrm{T}}$.

To see that (32) is not contained in any of the schemes I–XIII, first observe that (32) cannot be contained in any scheme where not both $+1$ and -1 are allowed as coefficients in one inequality. Then, by the results of [26] it suffices to check that (32) is not contained in XI or XII, which we leave to the reader.

Also Grishuhin's framework [15], which is partly more general (as it contains any system with a totally unimodular constraint matrix), is partly more restrictive, as it does not contain (32). If we replace axiom (2) above by the weaker axiom:

if $\mathscr{C}' \subseteq \mathscr{C}$ is cross-free, then the matrix with rows $h(S)$ $(S \in \mathscr{C}')$ is totally unimodular, (33)

the Theorem above is still true (as follows directly from its proof), which gives a common generalization of Grishuhin's scheme and the one presented above.

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