



Ordering sequences by permutation transducers

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Abstract

To extend a natural concept of equivalence of sequences to two-sided infinite sequences, the notion of permutation transducer is introduced. Requiring the underlying automaton to be deterministic in two directions, it provides the means to rewrite bi-infinite sequences. The first steps in studying the ensuing hierarchy of equivalence classes of bi-infinite sequences are taken, by describing the classes of ultimately periodic two-sided infinite sequences. It is important to make a distinction between unpointed and pointed sequences, that is, whether or not sequences are considered equivalent up to shifts. While one-sided ultimately periodic sequences form a single equivalence class under ordinary transductions, which is shown to split into two under permutation transductions, in the two-sided case there are three unpointed and seven pointed equivalence classes under permutation transduction.

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1. Introduction

Finite state transducers provide a natural way to compare (one-sided) infinite sequences, as was observed in [3] (and previously, by Rayna [5], for Mealy machines). The main goal of this paper is to extend this comparison to two-sided infinite sequences. A finite state transducer proceeds forward in a deterministic way. In order to apply it to two-sided sequences it is natural to

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add a requirement of co-determinism in order also to be able to proceed backward deterministically. This means that not only for every state and symbol there is exactly one outgoing arrow (as is required by determinism), but also exactly one incoming arrow. This requirement is exactly the same as the restriction for a deterministic finite automaton (DFA, [1,6]) to be a permutation automaton, see for example [7]. That is why we call the resulting kind of state transducer a *permutation transducer*: a permutation transducer is a permutation automaton with output, just like a finite state transducer is a finite automaton with output.

Equivalence under transducers organizes infinite sequences into a hierarchy with interesting properties, as ongoing research is revealing, see for example [2,4]. Under the stricter notion of equivalence under permutation transducers a finer hierarchy of one-sided sequences arises, as well as a novel hierarchy of bi-infinite sequences (in fact, two distinct versions of them, see below). We take the first steps here to explore the lower regions of these new hierarchies. Explicitly, we completely describe the set of ultimately periodic infinite sequences and how it splits into equivalence classes.

Although our main goal is to study the effect of permutation automata on two-sided sequences, first in Section 2 we investigate their effect on one-sided sequences. In Section 3 we investigate the effect of permutation automata on pointed two-sided sequences, reusing results from Section 2 as much as possible. Next, in Section 4 we do the same for unpointed two-sided sequences, being pointed two-sided sequences modulo shifting. Finally, we give some further results relating to the new hierarchy and some open problems in Section 5.

2. One-sided sequences

A (one-sided infinite) sequence σ over an alphabet Σ is a mapping $\sigma : \mathbb{N} \rightarrow \Sigma$, where $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. It is often more convenient to denote $\sigma(i)$ by σ_i , just as indexing is used on finite sequences in Σ^* . For $u = u_0u_1 \dots u_{n-1} \in \Sigma^*$ and $\sigma : \mathbb{N} \rightarrow \Sigma$ we then define concatenation $u\sigma : \mathbb{N} \rightarrow \Sigma$ by $(u\sigma)_i = u_i$ for $i < n$ and $(u\sigma)_i = \sigma_{i-n}$ for $i \geq n$. Also, u^ω is defined by $u_i^\omega = u_{i \bmod n}$ for $u \in \Sigma^*$; the empty sequence of length 0 is denoted by ϵ . A sequence of the form u^ω for $u \neq \epsilon$ is called (purely) *periodic*; a sequence of the form uv^ω for $v \neq \epsilon$ is called *ultimately periodic*.

A representation uv^ω of an ultimately periodic sequence is called *canonical* if either $u = \epsilon$ or the last element of v is distinct from the last element of u : u is the smallest prefix of the sequence such that the remainder of the sequence is periodic. Moreover we take the size of v to be minimal. It is straightforward to show that every ultimately periodic sequence has a unique canonical representation.

In the following we may think of $\Sigma = \{0, 1\}$, but all our claims hold for any finite Σ containing two distinct elements 0, 1.

A finite state transducer (FST, or transducer for short) $T = (Q, q_0, \delta, \lambda)$ over Σ is defined to consist of a finite set Q of states, an initial state $q_0 \in Q$, a transition function $\delta : Q \times \Sigma \rightarrow Q$ and an output function $\lambda : Q \times \Sigma \rightarrow \Sigma^*$.

The standard way to draw a transducer $T = (Q, q_0, \delta, \lambda)$ is by a directed graph in which Q is the set of nodes, and for every $q \in Q, a \in \Sigma$ an arrow from q to $\delta(q, a)$ is drawn, labeled by $a|\lambda(q, a)$. The initial state $q_0 \in Q$ is marked by an incoming arrow.

Both δ and λ are extended to $Q \times \Sigma^*$ by defining, recursively

$$\begin{aligned} \delta(q, \epsilon) &= q, & \delta(q, au) &= \delta(\delta(q, a), u), \\ \lambda(q, \epsilon) &= \epsilon, & \lambda(q, au) &= \lambda(q, a)\lambda(\delta(q, a), u) \end{aligned}$$

for $a \in \Sigma$ and $u \in \Sigma^*$. It is straightforward from this definition that if u is prefix of v then $\lambda(q_0, u)$ is a prefix of $\lambda(q_0, v)$. This ensures well-definedness of the following definition:

$$T(\sigma) = \lim_{n \rightarrow \infty} \lambda(q_0, u_0 u_1 \cdots u_{n-1}).$$

Here $T(\sigma)$ is either finite or infinite. If q_1, q_2, \dots is defined by $q_{i+1} = \delta(q_i, \sigma_i)$ for $i \geq 0$, then $T(\sigma)$ is the limit concatenation of $\lambda(q_0, \sigma_0)\lambda(q_1, \sigma_1)\lambda(q_2, \sigma_2) \cdots$.

A key notion of [3,2] is the pre-order \geq defined on sequences by

$$\sigma \geq \tau \iff \exists \text{ finite state transducer } T : \tau = T(\sigma).$$

Equivalence classes of $\geq \cup \leq$ are called *degrees*, and \geq implies an order on degrees. The main goal of [3,2] is to investigate the structure of this order, in particular the investigation of *atoms*: elements that are strictly greater than the bottom element, but not strictly greater than any other sequence strictly greater than the bottom.

In order to extend the action of transducers to two-sided infinite sequences, it is natural to consider finite state transducers for which the transition function δ not only deterministically defines forward processing of any sequence, but also backward processing. For automata without output this extra requirement has been investigated in *permutation automata* [7], therefore the corresponding variant of transducers will be called *permutation transducer*. More precisely, we have the following definition.

Definition 1. A permutation transducer over Σ is a finite state transducer $T = (Q, q_0, \delta, \lambda)$ with the additional requirement that for every $a \in \Sigma$ the function $q \mapsto \delta(q, a)$ is a bijection from Q to Q .

For $\sigma, \tau : \mathbb{N} \rightarrow \Sigma$ we define \geq_p by

$$\sigma \geq_p \tau \iff \exists \text{ permutation transducer } T : \tau = T(\sigma).$$

A *partial permutation transducer* $T = (Q, q_0, \delta, \lambda)$ consists of a finite set Q and initial state $q_0 \in Q$, together with a partial function $\delta : Q \times \Sigma \rightarrow Q$ such that for every $q \in Q, a \in \Sigma$ there is at most one $q' \in Q$ such that $\delta(q', a) = q$, and $\lambda : Q \times \Sigma \rightarrow \Sigma^*$ is a partial function that is defined on the same pairs that δ is defined for.

Thus, in a permutation transducer for every symbol $a \in \Sigma$ there is exactly one incoming and exactly one outgoing a -arrow for every state $q \in Q$, but in a partial permutation transducer ‘exactly one’ is weakened to ‘at most one’.

Lemma 2. Every partial permutation transducer can be extended to a permutation transducer; that is, if $(Q, q_0, \delta, \lambda)$ is a partial permutation transducer then a permutation transducer $(Q, q_0, \delta', \lambda')$ exists such $\delta(q, a) = \delta'(q, a)$ and $\lambda(q, a) = \lambda'(q, a)$ whenever $\delta(q, a)$ is defined.

Proof. Initialize $\delta' = \delta$ and $\lambda' = \lambda$. If $q \in Q, a \in \Sigma$ exists for which $\delta'(q, a)$ is undefined, then there is a $q' \in Q$ not in the image of $x \mapsto \delta(x, a)$; define $\delta(q, a) = q'$ and $\lambda(q, a) = \epsilon$ (or any arbitrary string). Repeat this until (after finitely many steps) $\delta'(q, a)$ is defined for all $q \in Q, a \in \Sigma$. Then $(Q, q_0, \delta', \lambda')$ is a permutation transducer satisfying the requirements. \square

The identity transducer $I = (\{q_0\}, q_0, \delta, \lambda)$ defined by $\delta(q_0, a) = q_0$ and $\lambda(q_0, a) = a$ for all $a \in \Sigma$ is a permutation transducer satisfying $I(\sigma) = \sigma$ for all σ , proving that \geq_p is reflexive.

For $0 \in \Sigma$ let $0^\omega : \mathbb{N} \rightarrow \Sigma$ be defined by $0_i^\omega = 0$ for all $i \in \mathbb{N}$. Clearly $T(\sigma) = 0^\omega$ for the permutation transducer $T = (\{q_0\}, q_0, \delta, \lambda)$ defined by $\delta(q_0, a) = q_0$ and $\lambda(q_0, a) = 0$ for $a \in \Sigma$, so $\sigma \geq_p 0^\omega$ for all $\sigma : \mathbb{N} \rightarrow \Sigma$.

Transitivity of \geq_p follows from the following lemma.

Lemma 3. *Let $T_i = (Q_i, q_{i0}, \delta_i, \lambda_i)$ be permutation transducers for $i = 1, 2$. Then the wreath product of T_1, T_2 , defined by*

$$T_1 \cdot T_2 = (Q_1 \times Q_2, (q_{10}, q_{20}), \delta, \lambda)$$

with

$$\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, \lambda_1(q_1, a))),$$

where $\lambda((q_1, q_2), a) = \lambda_2(q_2, \lambda_1(q_1, a))$, is a permutation transducer satisfying

$$T_2(T_1(\sigma)) = (T_1 \cdot T_2)(\sigma)$$

for every $\sigma : \mathbb{N} \rightarrow \Sigma$.

Proof. By definition $T_1 \cdot T_2$ is a transducer. That $T_2(T_1(\sigma)) = (T_1 \cdot T_2)(\sigma)$ was already observed in [3]. So it remains to prove that $(q_1, q_2) \mapsto \delta((q_1, q_2), a)$ is bijective for every $a \in \Sigma$. Since it is a map from the finite set $Q_1 \times Q_2$ to itself, it suffices to prove injectivity.

Suppose $\delta((q_1, q_2), a) = \delta((q'_1, q'_2), a)$. Since

$$\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, \lambda_1(q_1, a)))$$

and $\delta((q'_1, q'_2), a) = (\delta_1(q'_1, a), \delta_2(q'_2, \lambda_1(q'_1, a)))$ and $q \mapsto \delta_1(q, a)$ is injective we conclude $q'_1 = q_1$. The remaining proof obligation $q'_2 = q_2$ follows from the observation that $q \mapsto \delta_2(q, \lambda_1(q_1, a))$ is a composition of bijective functions, hence injective. \square

For $\sigma, \tau : \mathbb{N} \rightarrow \Sigma$ we define \sim_p by

$$\sigma \sim_p \tau \iff \sigma \geq_p \tau \wedge \tau \geq_p \sigma.$$

Since \geq_p is reflexive and transitive, \sim_p is an equivalence relation, and \geq_p implies an order on the equivalence classes that we will also denote by \geq_p . We write $[\sigma]_p$ for the equivalence class of σ .

In the order based on general finite state transducers, the bottom element $[0^\omega]$ consists of all ultimately periodic sequences. In our setting based on permutation transducers, the bottom element consists only of the proper subset of all (purely) periodic sequences, as is stated in the following lemma.

Lemma 4. *Let $\sigma, \tau : \mathbb{N} \rightarrow \Sigma$. Then:*

$$\sigma \sim_p 0^\omega \iff \sigma \text{ is periodic};$$

moreover,

$$\sigma \text{ is periodic and } \sigma \geq_p \tau \implies \tau \text{ is periodic}.$$

Proof. As observed above, we have $\sigma \geq_p 0^\omega$ for all $\sigma : \mathbb{N} \rightarrow \Sigma$. So for the first claim it remains to prove that $0^\omega \geq_p \sigma$ if and only if σ is periodic.

If σ is periodic then $\sigma = u^\omega$ for some non-empty string u , and $T(0^\omega) = \sigma$ for the permutation transducer $T = (\{q_0\}, q_0, \delta, \lambda)$ defined by $\delta(q_0, a) = q_0$ and $\lambda(q_0, a) = u$ for $a \in \Sigma$, proving $0^\omega \geq_p \sigma$.

Conversely, if $T(0^\omega) = \sigma$ for some permutation transducer $T = (Q, q_0, \delta, \lambda)$, then the permutation $q \mapsto \delta(q, 0)$ has some order k , and hence $\delta(q_0, 0^k) = q_0$. Let $u = \lambda(q_0, 0^k)$; then $\sigma = T(0^\omega) = u^\omega$ is periodic.

For the second claim, let σ be periodic and $\sigma \geq_p \tau$. By the first claim we know that $\sigma \sim_p 0^\omega$, so $0^\omega \geq_p \sigma \geq_p \tau$. Since $\tau \geq_p 0^\omega$ for all τ , we conclude that $\tau \sim_p 0^\omega$, so $\tau \in [0^\omega]_p$ is periodic. \square

In an order \geq on a set having a minimum \perp , an element x is called an *atom* if $\{y \mid x \geq y\} = \{\perp, x\}$.

It is a natural problem to find atoms in our modified setting, that is, with respect to ‘permutation equivalence’ \sim_p . An immediate consequence of the following lemma is that $[10^\omega]_p$ is such an atom.

Lemma 5. *For $\sigma : \mathbb{N} \rightarrow \Sigma$ we have $10^\omega \geq_p \sigma$ if and only if σ is ultimately periodic.*

Proof. If σ is ultimately periodic then $\sigma = uv^\omega$ for some $u, v \in \Sigma^*$, $u \neq \epsilon$. Then $T(10^\omega) = \sigma$ for the permutation transducer $T = (\{q_0\}, q_0, \delta, \lambda)$ defined by $\delta(q_0, 0) = \delta(q_0, 1) = q_0$ and $\lambda(q_0, 1) = u$ and $\lambda(q_0, 0) = v$, proving that $10^\omega \geq_p \sigma$.

For the converse, assume that $T(10^\omega) = \sigma$ for some permutation transducer $T = (Q, q_0, \delta, \lambda)$. Let $q_1 = \delta(q_0, 1)$ and $u = \lambda(q_0, 1)$. Let k be the order of the permutation $q \mapsto \delta(q, 0)$, and write $v = \lambda(q_1, 0^k)$. Then $\sigma = uv^\omega$ is ultimately periodic. \square

Theorem 6. *The class of ultimately periodic one-sided sequences splits into exactly two equivalence classes of \sim_p : $[0^\omega]_p$ consisting of the periodic sequences and $[10^\omega]_p$ consisting of the remaining ultimately periodic sequences.*

Proof. After Lemmas 4 and 5 it remains to show that $\sigma \geq_p 10^\omega$ for every ultimately periodic sequence σ that is not periodic. Write $\sigma = uv^\omega$. Since v may be replaced by v^k for any $k > 0$, we may assume $|u| \leq |v|$. Next we may replace u by the prefix of uv^ω of length $|v|$ and v by a cyclic shift, yielding the same uv^ω , but with both $u, v \in \Sigma^n$ for some $n > 0$. Since σ is not periodic, there exists $k < n$ such that $u_k \neq v_k$.

Let T be the permutation transducer with n states q_0, \dots, q_{n-1} , $\delta(q_i, a) = q_{i+1 \bmod n}$ for $i = 0, 1, \dots, n - 1$, and $\lambda(q_k, u_k) = 1$, $\lambda(q_k, a) = 0$ for $a \neq u_k$, and $\lambda(q_j, a) = \epsilon$ for $j \neq k$, $a \in \Sigma$. Then $T(\sigma) = T(uv^\omega) = 10^\omega$, so $\sigma \geq_p 10^\omega$. \square

3. Pointed two-sided sequences

A *pointed two-sided sequence* is a map $\sigma : \mathbb{Z} \rightarrow \Sigma$. Similar to the convention in the one-sided case, we write σ_i for $\sigma(i)$, so

$$\sigma = \cdots \sigma_{-2} \sigma_{-1} . \sigma_0 \sigma_1 \sigma_2 \cdots$$

where we write a dot ‘.’ left from σ_0 to mark the 0-position, as there is no obvious starting point anymore.

For a non-empty string u we write ${}^\omega u$ for infinitely many copies of u extending infinitely to the left. Hence ${}^\omega 0.1^\omega$, for example, denotes $\sigma : \mathbb{Z} \rightarrow \Sigma$ defined by $\sigma_x = 0$ for all $x < 0$ in \mathbb{Z} and $\sigma_x = 1$ for $x \geq 0$.

For a permutation transducer the function $q \mapsto \delta(q, a)$ is bijective for every $a \in \Sigma$, so for a permutation transducer $T = (Q, q_0, \delta, \lambda)$ and for $\sigma : \mathbb{Z} \rightarrow \Sigma$ we uniquely define q_i for all $i \in \mathbb{Z}$

satisfying $q_{i+1} = \delta(q_i, \sigma_i)$ for all $i \in \mathbb{Z}$. Then we define

$$T(\sigma) = \cdots \lambda(q_{-2}, \sigma_{-2})\lambda(q_{-1}, \sigma_{-1})\lambda(q_0, \sigma_0)\lambda(q_1, \sigma_1)\lambda(q_2, \sigma_2) \cdots$$

Note that $T(\sigma)$ may be finite, bounded on the left and infinite on the right or the other way around, or infinite in both directions. Only in the last case $T(\sigma)$ is a map $\mathbb{Z} \rightarrow \Sigma$.

For $\sigma, \tau : \mathbb{Z} \rightarrow \Sigma$ we define

$$\sigma \geq_p \tau \iff \text{there exists a permutation transducer } T : T(\sigma) = \tau.$$

Then

$$\sigma \sim_p \tau \iff \sigma \geq_p \tau \wedge \tau \geq_p \sigma,$$

and

$$[\sigma]_p = \{\tau \mid \tau \sim_p \sigma\}.$$

Reflexivity of \geq_p follows from choosing the identity transducer as we did for one-sided sequences. Transitivity of \geq_p follows from the following lemma that is a straightforward extension of Lemma 3, from which we also reuse the definition of the wreath product.

Lemma 7. *Let T_i be permutation transducers for $i = 1, 2$. Then $T_2(T_1(\sigma)) = (T_1 \cdot T_2)(\sigma)$ for every $\sigma : \mathbb{Z} \rightarrow \Sigma$.*

The reverse σ^R of $\sigma : \mathbb{Z} \rightarrow \Sigma$ is defined by $\sigma_i^R = \sigma_{-1-i}$ for all $i \in \mathbb{Z}$: this way reversing means reflection in the dot. For a set A of pointed two-sided sequences we define $A^R = \{\sigma^R \mid \sigma \in A\}$.

The reverse T^R of a permutation transducer $T = (Q, q_0, \delta, \lambda)$ is defined by $T^R = (Q, q_0, \delta^R, \lambda^R)$ where $\delta^R(q, a)$ is the unique $q' \in Q$ for which $\delta(q', a) = q$, and $\lambda^R(q', a)$ is the reverse $(\lambda(q, a))^R$ of the string $\lambda(q, a)$. The following lemma is straightforward.

Lemma 8. *For every $\sigma, \tau : \mathbb{Z} \rightarrow \Sigma$ and every permutation transducer T we have*

- $(\sigma^R)^R = \sigma$,
- $(T^R)^R = T$,
- $T^R(\sigma^R) = (T(\sigma))^R$,
- $\sigma \geq_p \tau \iff \sigma^R \geq_p \tau^R$, and
- $[\sigma]_p^R = [\sigma^R]_p$.

Every pointed two-sided sequence can uniquely be written as

$$\alpha^R \cdot \beta = \cdots \alpha_2 \alpha_1 \alpha_0 \cdot \beta_0 \beta_1 \beta_2 \cdots$$

for $\alpha, \beta : \mathbb{N} \rightarrow \Sigma$. The following lemma relates \geq_p on one-sided sequences to \geq_p on two-sided sequences.

Lemma 9. *If $\alpha^R \cdot \beta \geq_p \alpha'^R \cdot \beta'$ for $\alpha, \alpha', \beta, \beta' : \mathbb{N} \rightarrow \Sigma$, then $\alpha \geq_p \alpha'$ and $\beta \geq_p \beta'$.*

Proof. Let $\alpha^R \cdot \beta \geq_p \alpha'^R \cdot \beta'$. Then there is a permutation transducer T such that $T(\alpha^R \cdot \beta) = \alpha'^R \cdot \beta'$. Ignoring everything left from the dot yields $T(\beta) = \beta'$, proving $\beta \geq_p \beta'$. Applying Lemma 8 gives $T^R(\beta^R \cdot \alpha) = (T(\alpha^R \cdot \beta))^R = \beta'^R \cdot \alpha'$. Thus $T^R(\alpha) = \alpha'$, proving $\alpha \geq_p \alpha'$. \square

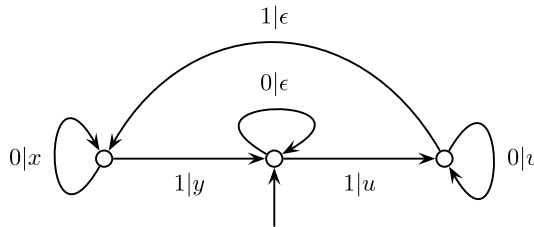
The converse of **Lemma 9** does not hold: for $\alpha = \beta = \beta' = 0^\omega$ and $\alpha' = 1^\omega$ we have $\alpha \geq_p \alpha'$, $\beta \geq_p \beta'$ but $\alpha^R \cdot \beta \not\geq_p \alpha'^R \cdot \beta'$, as we will see in **Theorem 12**.

Pointed two-sided sequences of the shape ${}^\omega xy.uv^\omega$ for strings x, y, u, v , with x and v non-empty, are called *ultimately periodic*. Note that x and v are not required to be related in any way! Every ultimately periodic pointed two-sided sequence has a unique *canonical representation* ${}^\omega xy.uv^\omega$: for the right part uv^ω either $u = \epsilon$ or the last element of u is distinct from the last element of v , and v is of minimal size, and similarly for the left part ${}^\omega xy$. If $u = \epsilon$, then u is omitted, and similarly for y .

Lemma 10. *A pointed two-sided sequence $\sigma : \mathbb{Z} \rightarrow \Sigma$ is ultimately periodic if and only if*

$${}^\omega 01.10^\omega \geq_p \sigma.$$

Proof. Let $\sigma = {}^\omega xy.uv^\omega$ be ultimately periodic. Then $\sigma = T({}^\omega 01.10^\omega)$ for the permutation transducer T defined as follows:



This proves ${}^\omega 01.10^\omega \geq_p \sigma$.

For the converse, let $\sigma = T({}^\omega 01.10^\omega)$ for some permutation transducer T . Then $T(0^\omega) = v^\omega$ for some non-empty v , and $T({}^\omega 0) = {}^\omega x$ for some non-empty x , so σ will be ultimately periodic. \square

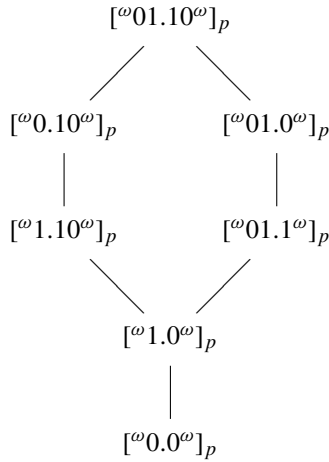
Lemma 11. *Every ultimately periodic pointed two-sided sequence can be written as ${}^\omega xy.uv^\omega$ with x, y, u, v all having the same length.*

Proof. First the repeating parts x and v are made of the same length, for example by replacing x by $|v|$ copies of x and v by $|x|$ copies of v . Next the length of x and v is increased so that it exceeds the lengths of u and y , by replacing both by a sufficient number of copies of themselves. Finally, u and y^R are replaced by the prefix of uv , respectively $(xy)^R$ of length $|v| = |x|$, while v and x are replaced by an appropriate cyclic shift. \square

Theorem 12. *The class of ultimately periodic pointed two-sided sequences under \sim splits into the following seven equivalence classes under \sim_p , assuming ${}^\omega xy.uv^\omega$ (with y and/or u possibly omitted) to be in canonical representation:*

- $[{}^\omega 0.0^\omega]_p = \{{}^\omega v.v^\omega\}$.
- $[{}^\omega 1.0^\omega]_p = \{{}^\omega x.v^\omega \mid x \neq v\}$.
- $[{}^\omega 1.10^\omega]_p = \{{}^\omega x.uv^\omega \mid u \neq \epsilon \wedge u \text{ is a prefix of } x^\omega\}$.
- $[{}^\omega 01.1^\omega]_p = \{{}^\omega xy.v^\omega \mid y \neq \epsilon \wedge y \text{ is a postfix of } {}^\omega v\}$.
- $[{}^\omega 0.10^\omega]_p = \{{}^\omega x.uv^\omega \mid u \neq \epsilon \wedge u \text{ is not a prefix of } x^\omega\}$.
- $[{}^\omega 01.0^\omega]_p = \{{}^\omega xy.v^\omega \mid y \neq \epsilon \wedge y \text{ is not a postfix of } {}^\omega v\}$.
- $[{}^\omega 01.10^\omega]_p = \{{}^\omega xy.uv^\omega \mid y \neq \epsilon \neq u\}$.

The Hasse diagram of the order \geq_p on these classes is as follows.

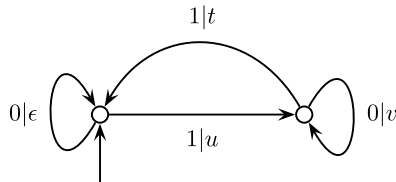


Proof. The seven given sets form a partition of the sequences of the shape $^\omega xy.uv^\omega$ in canonical form, being the set of all ultimately periodic sequences. The theorem will follow from:

- (A) For each of the seven claims of the shape $[\sigma]_p = S$ we prove that all elements of S are equivalent to σ : for every $\tau \in S$ two permutation transducers T, U exist such that $T(\sigma) = \tau$ and $U(\tau) = \sigma$.
- (B) For every $[\sigma]_p$ directly above $[\tau]_p$ in the Hasse diagram, we give a permutation transducer T such that $T(\sigma) = \tau$ and for every $[\sigma]_p$ not above $[\tau]_p$ in the Hasse diagram, we prove that $\sigma \not\geq_p \tau$.

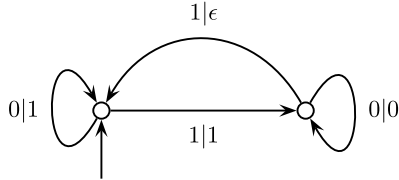
The sequence $^\omega 0.0^\omega$ maps to $^\omega v.v^\omega = ^\omega xy.uv^\omega$ for $x = v, y = u = \epsilon$ by any permutation transducer with one state q_0 satisfying $\delta(q_0, 0) = q_0$ and $\lambda(q_0, 0) = v$. Conversely $^\omega v.v^\omega$ maps to $^\omega 0.0^\omega$ by any permutation transducer with one state q_0 in which $\delta(q_0, a) = q_0$ and $\lambda(q_0, a) = 0$ for $a \in \Sigma$. This proves (A) for the first claim. The sequence $^\omega 1.0^\omega$ maps to $^\omega x.v^\omega$ by the permutation transducer with one state q_0 satisfying $\delta(q_0, 0) = q_0, \lambda(q_0, 0) = v$ and $\lambda(q_0, 1) = x$. Conversely, let $m = |x|$ and $n = |v|$. Since $x \neq v$ originate from the canonical representation, we have $x^n \neq v^m$. So there is an $i, 0 \leq i < mn$ such that $(x^n)_i \neq (v^m)_i$. Now $^\omega x.v^\omega$ maps to $^\omega 1.0^\omega$ by the permutation transducer with mn states $q_0, \dots, q_{mn-1}, \delta(q_k, a) = q_{(k+1) \bmod mn}$ for all k , and $\lambda(q_i, x_i) = 1, \lambda(q_i, a) = 0$, and $\lambda(q_j, a) = \epsilon$ for $j \neq i, a \in \Sigma$. This proves (A) for the second claim.

If $u \neq \epsilon$ is a prefix of x^ω , then it is also a prefix of x^n for some $n > 0$, and we can write $x^n = ut$. Then the following permutation transducer maps $^\omega 1.10^\omega$ to $^\omega (ut).uv^\omega = ^\omega x.uv^\omega$.



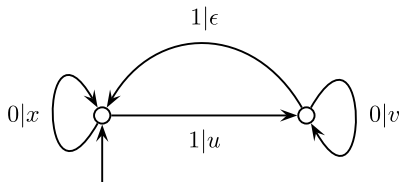
Conversely, let $\sigma = ^\omega x.uv^\omega$. Then by Lemma 11 we can redefine x, y, u, v such that $\sigma = ^\omega xy.uv^\omega$ for $x, y, u, v \in \Sigma^n$, for some $n > 0$. Since the left part of σ is periodic and the right part is not, we have $x = y$ and $\sigma = ^\omega x.uv^\omega$ for $u \neq v$. Choose $i < n$ such that $u_i \neq v_i$.

Now ${}^\omega x.uv^\omega$ maps to either ${}^\omega 1.10^\omega$ or ${}^\omega 0.10^\omega$ by the permutation transducer with n states q_0, \dots, q_{n-1} , $\delta(q_k, a) = q_{(k+1) \bmod n}$ for all k , and $\lambda(q_i, u_i) = 1$, $\lambda(q_i, a) = 0$ for $a \neq u_i$, and $\lambda(q_j, a) = \epsilon$ for $j \neq i$, $a \in \Sigma$. If it is ${}^\omega 1.10^\omega$ we are done, otherwise ${}^\omega 0.10^\omega$ is further mapped to ${}^\omega 1.10^\omega$ by the following permutation transducer.

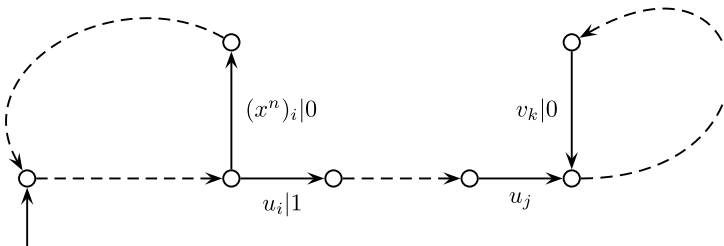


This proves (A) for the third claim; the fourth holds by symmetry (Lemma 8).

The following permutation transducer maps ${}^\omega 0.10^\omega$ to ${}^\omega x.uv^\omega$.



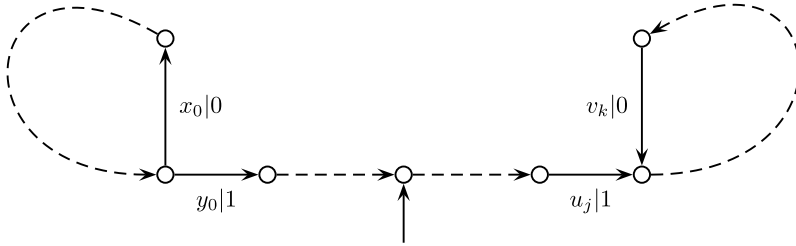
Conversely, if $\sigma = {}^\omega x.uv^\omega$ for $u \neq \epsilon$ not a prefix of x^ω , we have to find a permutation transducer mapping σ to ${}^\omega 0.10^\omega$. Let $u = u_0u_1 \dots u_j$. Since ${}^\omega x.uv^\omega$ is in canonical representation for the last element v_k of v we have $v_k \neq u_j$. Since u is not a prefix of x^ω , there is $n > 0$ such that $|x^n| \geq |u|$, and $(x^n)_i \neq u_i$ for some $i \leq j$; let i be minimal with this property. Now consider the following partial permutation transducer:



The horizontal part of the left cycle is labeled by $(x^n)_0 = u_0, (x^n)_1 = u_1, \dots, (x^n)_{i-1} = u_{i-1}$; this part may be empty. From the indicated $(x^n)_i$ -arrow to the initial state, the remaining part of this cycle is labeled by the consecutive elements of the remainder of x^n . The right cycle is labeled by the consecutive elements v_1, v_2, \dots of v , ending in the indicated v_k -arrow. The connection between the two cycles is labeled by u_i, \dots, u_j ; in case $i = j$, this consists of a single arrow. The three arrows with output are the only ones with non-empty output: all other arrows have output ϵ . Note that this satisfies the requirements of a partial permutation transducer since there is only one state with two outgoing arrows, respectively labeled by the distinct values u_i and $(x^n)_i$, and only one state with two incoming arrows, respectively labeled by the distinct values u_j and v_k . Next apply Lemma 2 to extend this partial permutation transducer to a permutation transducer. Applying this permutation transducer to $\sigma = {}^\omega x.uv^\omega$ produces ${}^\omega 0.10^\omega$. This proves (A) for the fifth claim; the sixth holds by symmetry (Lemma 8).

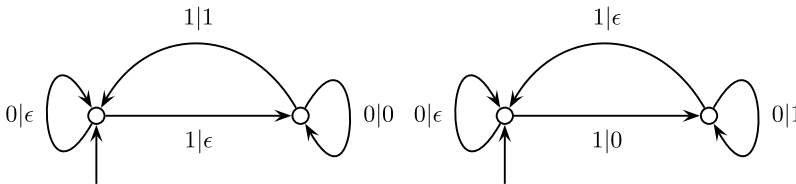
For the remaining claim in (A), a permutation transducer mapping ${}^\omega 01.10^\omega$ to ${}^\omega xy.uv^\omega$ was given in the proof of Lemma 10.

Conversely, for $y \neq \epsilon \neq u$ we have to give a permutation transducer mapping ${}^\omega x y . u v {}^\omega$ to ${}^\omega 01 . 10 {}^\omega$. Due to the canonical representation we know that the last element v_k of v may be assumed to be distinct from the last element u_j of u , and the first element x_0 of x to be distinct from the first element y_0 of y . Now consider the following partial permutation transducer:



The left cycle is labeled by the consecutive elements of x , starting in the indicated x_0 -arrow. The right cycle is labeled by the consecutive elements of v , ending in the indicated v_k -arrow. The connection between the left cycle and the initial state is labeled by the consecutive elements from y , starting in the indicated y_0 -arrow. The connection between the initial state and the right cycle is labeled by the consecutive elements from u , ending in the indicated u_j -arrow. The four arrows shown with output are the only ones with non-empty output: all other arrows have output ϵ . Since $x_0 \neq y_0$ and $u_j \neq v_k$, this is a partial permutation transducer that can be extended to a permutation transducer by Lemma 2. Applying this permutation transducer to $\sigma = {}^\omega x y . u v {}^\omega$ produces ${}^\omega 01 . 10 {}^\omega$. This concludes the proof of (A) for the final claim.

For part (B) we start by giving permutation transducers transforming σ to τ for every $[\sigma]_p$ in the Hasse diagram directly above $[\tau]_p$. For $\sigma = {}^\omega 01 . 10 {}^\omega$ and $\tau = {}^\omega 0 . 10 {}^\omega$ or $\tau = {}^\omega 01 . 0 {}^\omega$ this was already done in the proof of Lemma 10. For $\sigma = {}^\omega 0 . 10 {}^\omega$ and $\tau = {}^\omega 1 . 10 {}^\omega$ this was already done as part of the proof of the fifth claim in (A); by taking its reverse it is done for $\sigma = {}^\omega 01 . 0 {}^\omega$ and $\tau = {}^\omega 01 . 1 {}^\omega$. For $\sigma = {}^\omega 1 . 10 {}^\omega$ or $\sigma = {}^\omega 01 . 1 {}^\omega$ and $\tau = {}^\omega 1 . 0 {}^\omega$ it is done by the following permutation transducers, respectively.



Finally, for $\sigma = {}^\omega 1 . 0 {}^\omega$ and $\tau = {}^\omega 0 . 0 {}^\omega$ it is done by any permutation transducer producing 0 in every output.

It remains to prove that $\sigma \not\leq_p \tau$ for every $[\sigma]_p$ not above $[\tau]_p$ in the Hasse diagram. For most cases the key argument is that a periodic one-sided sequence is not \geq_p a non-periodic one by Lemma 4; by applying Lemma 9 this argument can be applied both on the left part and the right part of σ and τ . For $\tau = {}^\omega 01 . 10 {}^\omega$ the claim follows since for all other $[\sigma]_p$ in the Hasse diagram, either the left part or the right part is periodic. For $\tau = {}^\omega 1 . 10 {}^\omega$ all four $[\sigma]_p$ that are not above $[\tau]_p$ have a periodic right part, so the same argument applies. By symmetry, the same holds for $\tau = {}^\omega 01 . 1 {}^\omega$. The only remaining claims that we have to prove are

1. ${}^\omega 0 . 0 {}^\omega \not\leq_p {}^\omega 1 . 0 {}^\omega$,
2. ${}^\omega 1 . 10 {}^\omega \not\leq_p {}^\omega 0 . 10 {}^\omega$,
3. ${}^\omega 01 . 1 {}^\omega \not\leq_p {}^\omega 01 . 0 {}^\omega$.

For Claim 1 assume that a permutation transducer $T = (Q, q_0, \delta, \lambda)$ exists such that $T(\omega 0.0^\omega) = \omega 1.0^\omega$. As before, choose $k > 0$ such that $\delta(q_0, 0^k) = q_0$. Let $u = \lambda(q_0, 0^k)$. Then $T(\omega 0.0^\omega) = \omega u.u^\omega$, contradicting $T(\omega 0.0^\omega) = \omega 1.0^\omega$.

For Claim 2 assume that a permutation transducer $T = (Q, q_0, \delta, \lambda)$ exists such that $T(\omega 1.10^\omega) = \omega 0.10^\omega$. Choose $k > 0$ such that $\delta(q_0, 1^k) = q_0$. Then by looking at the left part of the result we conclude that $\lambda(q_0, 1^k)$ only consists of 0's, so $\lambda(q_0, 1) = 0^p$ for some $p \geq 0$. Looking at the right part we see that $\lambda(q_0, 1)$ is a prefix of 10^ω ; combining these observations yields $\lambda(q_0, 1) = \epsilon$. Now consider the permutation transducer $T' = (Q, \delta(q_0, 1), \delta, \lambda)$. Since $T(10^\omega) = 10^\omega$ and $\lambda(q_0, 1) = \epsilon$ we conclude $T'(0^\omega) = 10^\omega$, being non-periodic, contradicting Lemma 4.

Claim 3 is similar to Claim 2, by symmetry.

This concludes the proof of part (B). \square

Corollary 13. *Neither the shift operator*

$$s : \dots \sigma_{-2}\sigma_{-1}.\sigma_0\sigma_1\sigma_2 \dots \mapsto \dots \sigma_{-2}\sigma_{-1}\sigma_0.\sigma_1\sigma_2 \dots$$

nor the reverse operator

$$\tau \mapsto \tau^R$$

on pointed two-sided infinite sequences results from a permutation transducer; that is, there exist pointed two-sided infinite sequences σ such that for no permutation transducer T holds $T(\sigma) = s(\sigma)$ or $T(\sigma) = \sigma^R$.

Proof. This is immediate from $[\omega 0.10^\omega]_p \approx_p [\omega 01.0^\omega]_p$. \square

4. Unpointed two-sided sequences

Informally, an unpointed two-sided sequence is a two-sided sequence in which no position is distinguished (marked by a point, as the zero-position); thus, indexing the elements of such sequence is arbitrary. More precisely, it is an equivalence class of the set $\{\sigma : \mathbb{Z} \rightarrow \Sigma\}$ with respect to the equivalence relation \sim_s defined by

$$\sigma \sim_s \tau \iff \exists n \in \mathbb{Z} \forall i \in \mathbb{Z} : \sigma_i = \tau_{n+i}.$$

As implied by Corollary 13, pointed sequences in different equivalence classes under \sim_p may become equivalent as unpointed sequences. Before we state our main result on unpointed sequences, let us first formally define the order \geq_{ps} as the transitive closure of the union of \sim_s and \geq_p ; that is, $\sigma \geq_{ps} \tau$ if and only if by alternatingly using shifts and permutation transducers we can transform σ into τ . Equivalence under compositions of shifts and permutation transducers will be indicated by \sim_{ps} , which will then be the conjunction of \geq_{ps} and \leq_{ps} .

To simplify arguments using \geq_{ps} we make the following observation.

Lemma 14. *Let $\sigma, \tau : \mathbb{Z} \rightarrow \Sigma$. Then*

$$\sigma \geq_{ps} \tau \iff \exists \text{ permutation transducer } T \text{ and shift } s : s(T(\sigma)) = \tau.$$

Proof. The condition $\sigma \geq_{ps} \tau$ for pointed two-sided infinite sequences means by definition that σ can be transformed to τ by alternatingly using shifts and permutation transducers: thus the implication from right to left is obvious.

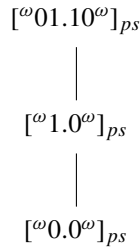
For the other direction, note that for a permutation transducer T and a shift s we have $T \circ s = s' \circ T'$, for some shift s' and a permutation transducer T' that is identical to T except that the initial states may differ: indeed, $T \circ s(\sigma)$ is a pointed sequence that can also be obtained by first traversing the states of T following σ_0 to $\sigma_n = s(\sigma)_0$ and then applying T from this new initial position; the result will differ from $T(\sigma)$ only in the position of the dot. Since a composition of shifts is again a shift, applying [Lemma 3](#) and induction on the number $k + 1$ of alternations of shifts and permutation transducers in τ will finish the proof. \square

It turns out that of the seven classes of pointed ultimately periodic sequences, exactly three equivalence classes of unpointed sequences remain, as stated in the following theorem. To distinguish the equivalence class of σ as unpointed sequence from the class of $[\sigma]_p$ under \sim_p , we will denote it by $[\sigma]_{ps}$.

Theorem 15. *The ultimately periodic unpointed two-sided sequences consist of the following three classes under \sim_{ps} , being unions of equivalence classes under \sim_p as indicated:*

- $[\omega 0.0^\omega]_{ps} = [\omega 0.0^\omega]_p$
- $[\omega 1.0^\omega]_{ps} = [\omega 1.0^\omega]_p \cup [\omega 1.10^\omega]_p \cup [\omega 01.1^\omega]_p$
- $[\omega 01.10^\omega]_{ps} = [\omega 0.10^\omega]_p \cup [\omega 01.0^\omega]_p \cup [\omega 01.10^\omega]_p$.

The Hasse diagram for these classes is as follows.



Proof. When looking at the Hasse diagram for pointed sequences, it is clear that $\omega 0.10^\omega$ and $\omega 01.0^\omega$ are shifts of each other, that is, equivalent when we ignore the dot; also, $\omega 01.10^\omega \sim_s \omega 0.110^\omega$, which in turn is equivalent under \sim_p to $\omega 0.10^\omega$ as we saw in [Theorem 6](#). Similarly, $\omega 1.10^\omega$ and $\omega 1.0^\omega$ are clearly shift-equivalent. But also $\omega 01.1^\omega$ is a shift of $\omega 0.1^\omega$, which in turn is equivalent under \sim_p (by the one-state permutation transducer interchanging the two symbols) to $\omega 1.0^\omega$.

Remains to prove that these three classes will not become equivalent under shifting. Suppose that an element σ from $[\omega 1.0^\omega]_{ps}$ maps under some permutation transducer T to an element τ of $[\omega 01.10^\omega]_{ps}$. Then σ can be represented by a shifted version σ' of an element from $B = [\omega 1.0^\omega]_p \cup [\omega 1.10^\omega]_p \cup [\omega 01.1^\omega]_p$ while $T(\sigma') = \tau'$ is a shifted version of an element τ from $C = [\omega 01.10^\omega]_p = [\omega 0.10^\omega]_p \cup [\omega 01.0^\omega]_p \cup [\omega 01.10^\omega]_p$. The only observation to finish the proof in this case (and analogously, for the other case), is that any shift of an element in B (or C) is in B (or C) again, and thus the permutation transducer would identify two inequivalent classes under \sim_p , a contradiction. \square

5. Further results and problems

Now that we have completely determined that part of the hierarchy coming from (ultimately) periodic bi-infinite sequences, we take some preliminary steps to look further. Many problems present themselves from analogy with the one-sided case—the search for more atoms, or for

infinitely descending or ascending chains of inequivalent classes, for example. In some cases it is not even clear yet what the two-sided analogue for a one-sided notion should be.

In the remaining part of this section, we resume our study in Section 2, of how the additional requirement on permutation transducers affects the equivalence relation on one-sided sequences, by looking in particular at sequences coming from functions with some given growth rate, such as linear or quadratic. Note that it is not entirely obvious in what way to generalize such restrictions to two-sided sequences: should one require similar growth in both directions, or periodicity in one direction, for example?

Following [2], for any function $f : \mathbb{N} \rightarrow \mathbb{N}$, we write

$$\langle n \mapsto f(n) \rangle = \langle f \rangle = 1\ 0^{f(0)}\ 1\ 0^{f(1)}\ 1\ 0^{f(2)}\ 1\ \dots$$

For the order \geq on degrees it was proved in [3] that $[\langle n \mapsto n \rangle]$ is an atom, and in [2] it was proved that $[\langle n \mapsto n^2 \rangle]$ is an atom. An important tool consists of looking at periodicity properties of such sequences.

The first question we address is whether or not one-sided sequences σ that are not ultimately periodic can have $\sigma \geq_p 10^\omega$. The next lemma states that the answer is negative for the class of sequences that behaves modularly periodic for every modulus.

Lemma 16. *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function for which $x \mapsto (f(x) \bmod n)$ is periodic for every $n > 0$. Then no permutation transducer T exists such that $T(\langle f \rangle) = 10^\omega$.*

Proof. Assume such a permutation transducer $T = (Q, q_0, \delta, \lambda)$ does exist, so $T\langle f \rangle = 10^\omega$. Note that, for every symbol $\alpha \in \Sigma$, $q \mapsto \delta(q, \alpha)$ is a permutation of the states; hence there exists a positive integer m (for example, the order of this permutation for $\alpha = 0$) such that $\delta(q, 0^m) = q$ for every state q . Then $\delta(q, 0^{2m}) = \delta(q, 0^m) = q$ and $\lambda(q, 0^{2m}) = \lambda(q, 0^m)\lambda(q, 0^m)$ is a square, for every $q \in Q$, where a finite word $w \in \Sigma$ is called a square if it is of the shape vv for some word v .

For every $u \in \Sigma^*$ and every $\sigma : \mathbb{N} \rightarrow \Sigma$ the sequence $T(u0^{2m}\sigma)$ can be obtained from $T(u\sigma)$ by inserting the square $\lambda(q, 0^{2m})$ at the proper position, for some $q \in Q$. So any insertion of 0^{2m} in the argument of T causes the insertion of a square in the result. The sequence $\langle f \rangle$ is obtained from the periodic sequence $\langle f \bmod 2m \rangle$ by inserting 0^{2m} a (possibly infinite) number of times. As $\langle f \bmod 2m \rangle$ is periodic, by Lemma 4 also $\tau = T(\langle f \bmod 2m \rangle)$ is periodic. So the sequence $T(\langle f \rangle) = 10^\omega$ can be obtained by inserting squares into this periodic sequence τ a (possibly infinite) number of times. If τ contains 1, it will contain it infinitely often, being periodic, which gives a contradiction; so $\tau = 0^\omega$ and the 1 was inserted in a square. But every square that contains 1's, will contain an even number of them, contradiction again. So $T(\langle f \rangle)$ cannot be equal to the sequence 10^ω containing exactly one 1. \square

The simplest non-periodic functions are the linear ones; we turn our attention to those. To transform

$$\langle n \mapsto n \rangle = 11\ 010^2\ 10^3\ 10^4\ 1\ \dots$$

into

$$\langle n \mapsto n + 1 \rangle = 1010^2\ 10^3\ 10^4\ 1\ \dots$$

is easy; the same holds in the other direction for arbitrary finite state transducers, since it is simple to insert the prefix 1. To achieve the same with a permutation transducer is harder; in the proof of the following theorem this is achieved in greater generality.

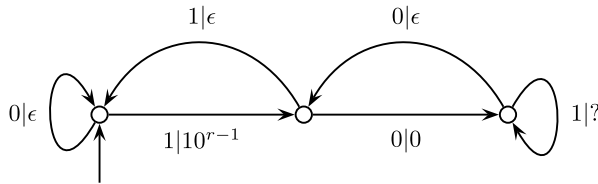
Theorem 17. Let $k, l \in \mathbb{N}$ with $k \geq 1$. Then $\langle n \mapsto kn + l \rangle \sim_p \langle n \mapsto n \rangle$; in other words: all (non-constant) linear functions are equivalent under \sim_p .

Proof. We will prove that $\langle n \mapsto kn + l \rangle \sim_p \langle n \mapsto n \rangle$, for $k, l \in \mathbb{N}$ with $k \geq 1$, in five steps.

The first step deals with the easy direction (from right to left): a simple one-state permutation transducer $T = (\{q_0\}, q_0, \delta, \lambda)$ suffices, if we let $\lambda(q_0, 0) = 0^k$ and $\lambda(q_0, 1) = 10^l$. Indeed, every word 10^n in σ will be replaced by $10^l (0^n)^k = 10^{kn+l}$.

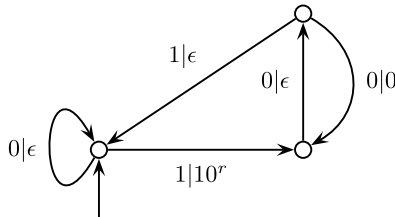
For the other direction we first prove that $\langle n \mapsto n + l \rangle \geq_p \langle n \mapsto n + l - 1 \rangle$ for every $l \geq 1$; we then use the same construction to prove for fixed $k \geq 1$ that $\langle n \mapsto kn + l \rangle \geq_p \langle n \mapsto kn + l - 1 \rangle$. Next we show that $\langle n \mapsto kn \rangle \sim_p \langle n \mapsto n \rangle$, and finally we just apply transitivity.

To see that $\langle n \mapsto n + l \rangle \geq_p \langle n \mapsto n + l - 1 \rangle$ for every $l \geq 1$ we distinguish the cases l is even and l is odd. If l is even write $l = 2r$ with $r > 0$ and consider the following permutation transducer (where the string ? is irrelevant for the proof):



It is easily seen to transform 10^{2r+n} to $10^{r-1}0^{r+n/2} = 10^{2r-1+n/2} = 10^{l-1+n/2}$ for even n and to ϵ for odd n , which is exactly the desired result.

For odd $l > 1$, write $l = 2r + 1$ with $r \geq 0$, and consider the following transducer:



This maps 10^{2r+1+n} to ϵ for odd n and to $10^r 0^{r+n/2} = 10^{2r+n/2} = 10^{l-1+n/2}$ for all even n , again precisely what is required.

Moreover, in both cases, exactly the same transducer applied to $\langle n \mapsto kn + l \rangle$ produces $\langle n \mapsto kn + l - 1 \rangle$.

But also $\langle n \mapsto kn \rangle \geq_p \langle n \mapsto n \rangle$; for this let $T = (Q, q_0, \delta, \lambda)$ be a k -state partial transducer with $\delta(q_0, 1) = q_0$ and $\lambda(q_0, 1) = 1$, with a directed k -cycle $(q_0, q_1, \dots, q_{k-1}, q_0)$ with transitions $\delta(q_i, 0) = q_{i+1 \bmod k}$ and $\lambda(q_0, 0) = 0$ but $\lambda(q_i, 0) = \epsilon$ for $i > 0$. This can be extended to a permutation transducer T' by Lemma 2, that is easily seen to have the desired effect of replacing any block of k consecutive 0s by a single 0.

By the transitivity property of Lemma 3 we can compose the $l + 1$ steps

$$\langle n \mapsto kn + l \rangle \geq_p \dots \geq_p \langle n \mapsto kn \rangle \geq_p \langle n \mapsto n \rangle,$$

for any $l \geq 0$ and $k > 0$, which finishes the proof. \square

We can slightly extend the class of ‘linear functions’ from the previous lemma, by allowing rational coefficients combined with rounding to get a map to natural numbers again, as will be expressed in Corollary 19. The construction with which it can be achieved, however, can be

generalized to the following situation, resembling the ‘spiraling’ functions from [2]. This result may be useful by itself, and could be used to prove [Theorem 17](#) as well.

Theorem 18. *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function for which $n \mapsto (f(n) \bmod m)$ is periodic for every $m > 0$. Then $\langle f \rangle \geq_p \sigma$ for $\sigma : \mathbb{N} \rightarrow \Sigma$ if and only if there exist $k, h > 0$ and $p_0, \dots, p_{k-1}, c_0, \dots, c_{k-1} \in \Sigma^*$ such that*

$$\sigma = \prod_{j=0}^{\infty} \left(\prod_{i=0}^{k-1} p_i c_i^{\lfloor f(i+jk)/h \rfloor} \right) = p_0 c_0^{\lfloor f(0)/h \rfloor} p_1 c_1^{\lfloor f(1)/h \rfloor} \dots$$

Proof. Assume $\langle f \rangle \geq_p \sigma$ for some $\sigma : \mathbb{N} \rightarrow \Sigma$. Then a permutation transducer $T = (Q, q_0, \delta, \lambda)$ exists such that $T(\langle f \rangle) = \sigma$. Let $m = \#Q$ and let h be a positive integer such that $\delta^h(q, x) = q$ for every $q \in Q$ and every permutation $\delta(-, x) : Q \rightarrow Q$, for fixed $x \in \Sigma$. (For instance, h could be taken as $m!$, or as the least common multiple of the orders of the permutations for the various x .) Since $n \mapsto (f(n) \bmod h)$ is periodic, also $\langle f \bmod h \rangle$ is periodic, so by [Lemma 4](#) we have $T(\langle f \bmod h \rangle) = u^\omega$ for some u . Replacing u by u^h if necessary, we may assume $\delta(q_0, u) = q_0$.

Let k be the number of 1’s in u , then $u = u_0 u_1 \dots u_{k-1}$ for $u_i = 1 0^{f(i) \bmod h}$, for $i = 0, 1, \dots, k - 1$, where in this proof we mean by $x \bmod h$ the least positive remainder of the division of x by h . Also define

$$q_{i+1} = \delta(q_i, u_i), \quad p_i = \lambda(q_i, u_i), \quad c_i = \lambda(q_{i+1}, 0^h)$$

for $i = 0, 1, \dots, k - 1$; since $\delta(q_0, u) = q_0$ we have $q_k = q_0$.

Since $f(i) = (f(i) \bmod h) + h \lfloor f(i)/h \rfloor$ for all $i \geq 0$ and $\delta(q_i, 0^h) = q_i$ for $i = 0, 1, \dots, k - 1$, the sequence $T(\langle f \rangle)$ can be obtained from $T(\langle (f \bmod h) \rangle) = (p_0 p_1 \dots p_{k-1})^\omega$ by inserting $\lfloor f(i)/h \rfloor$ copies of $c_i = \lambda(q_{i+1}, 0^h)$ just before the $(i + 1)$ th 1 in $\langle f \rangle$ is read. This yields

$$\sigma = p_0 c_0^{\lfloor f(0)/h \rfloor} p_1 c_1^{\lfloor f(1)/h \rfloor} \dots p_0 c_0^{\lfloor f(k)/h \rfloor} p_1 c_1^{\lfloor f(k+1)/h \rfloor} \dots$$

Conversely, for $\sigma = p_0 c_0^{\lfloor f(0)/h \rfloor} \dots$ we define the partial permutation transducer $T = (Q, q_0, \delta, \lambda)$ with $q_0, q_1, \dots, q_{k-1} \in Q$, by doing the following for $i = 0, 1, \dots, k - 1$. If $f(i) \equiv 0 \pmod h$ then define $\delta(q_i, 1) = q_{(i+1) \bmod k}$ and $\lambda(q_i, 1) = p_i$, and create a cycle from $q_{(i+1) \bmod k}$ to itself of length h , by adding h new states, and all arrows labeled by 0, with exactly one of them having output c_i , all others have output ϵ . If $f(i) \not\equiv 0 \pmod h$ then we also create a fresh cycle from $q_{(i+1) \bmod k}$ to itself of length h in which all arrows are labeled by 0, but now $\delta(q_i, 1)$ is defined to be the $h - (f(i) \bmod h)$ th state in this cycle. The first arrow in the cycle has output c_i , all others have output ϵ . Now by [Lemma 2](#) T can be extended to a permutation transducer, and by construction we have $T(\langle f \rangle) = \sigma$. \square

Corollary 19. *All linear functions $\langle n \mapsto \lfloor rn + s \rfloor \rangle$ from \mathbb{N} to \mathbb{N} with $0 < r \in \mathbb{Q}$ and $0 \leq s \in \mathbb{Q}$, are equivalent under \sim_p .*

Proof. We sketch the proof to show that $\langle n \mapsto \lfloor rn + s \rfloor \rangle \sim_p \langle n \mapsto n \rangle$ for all $r > 0, s \geq 0$, by showing both \leq_p and \geq_p hold, using [Theorem 18](#) for the second part.

There exist positive a, d and a non-negative integer b such that $rn + s = \frac{an+b}{d}$. Then $a = e + fd$ and $b = g + hd$, with $0 \leq e, g < d$; as in the proof of [Theorem 17](#) we can deal with the integral parts of a and b , so we will assume that $rn + s = \frac{en+g}{d}$.

It is an easy exercise to construct a permutation transducer to show that $\langle n \mapsto \lfloor \frac{en+g}{d} \rfloor \rangle \geq_p \langle n \mapsto n \rangle$, because for the first d values $n = 0, 1, \dots, d - 1$ there is a jump of 1 in $\frac{en+g}{d}$ fewer than d times, since $e, g < d$, and these jumps are repeated in every consecutive block of d values for n . The permutation transducer now needs not more than d states, and on input $\langle n \mapsto \lfloor \frac{en+g}{d} \rfloor \rangle$ will copy only a one where an increment occurs, together with the corresponding zeroes, and erase symbols at the other positions, thus forming $\langle n \mapsto n \rangle$.

To show that $\langle f : n \mapsto n \rangle \geq_p \langle n \mapsto \lfloor en + g \rfloor \rangle = \sigma$, we apply [Theorem 18](#). It is clear that $f \bmod m$ is periodic for every m ; it remains to show that σ can be written in the right form. This is easily done using the integers d, e, g introduced above, choosing $k = h = d$ and setting $p_i = 10^{\frac{ei+g}{d}}, c_i = 0$ for $i = 0, 1, \dots, d - 1$. \square

Endrullis et al. proved that the class of one-sided linear functions is an atom, together with some results about classes of polynomial functions of higher degree, see [\[2\]](#). The corresponding properties under \sim_p are open for now.

We do make some additional basic observations on the hierarchy under \sim_p .

To begin with, common upper bounds for σ and τ always exist, due to the same construction of ‘interleaving’ as in the case of general transducers; let

$$\text{zip}(\sigma, \tau) = \sigma_0\tau_0\sigma_1\tau_1\sigma_2\tau_2 \dots$$

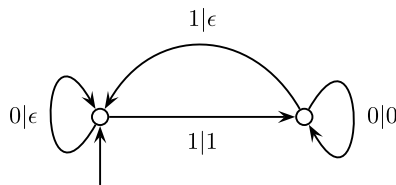
Then a simple 2-state transducer alternating copying and erasing a symbol will prove that both $\text{zip}(\sigma, \tau) \geq_p \sigma$ and $\text{zip}(\sigma, \tau) \geq_p \tau$ by choosing the proper state as initial state.

It is not so easy to determine in what case suprema do exist, for instance because of the existence of infinitely descending chains. But let us first observe that infinitely ascending chains under \geq_p do exist due to the zip-construction: in fact, we can construct an infinitely ascending chain starting with any σ , as follows. Choose τ not in or below the class $[\sigma]_p$, that is, outside $\{\sigma' : \sigma \geq_p \sigma'\}$; this is possible since the set contains at most countably many elements as there are no more transducers. Then $\text{zip}(\sigma, \tau) \geq_p \sigma$ but not conversely.

At least one infinitely descending chain is given by

$$\langle n \mapsto 2^n \rangle \geq_p \langle n \mapsto 4^n \rangle \geq_p \langle n \mapsto 16^n \rangle \geq_p \dots$$

where \geq_p is proved by the transducer



and the converse inequality does not even hold for \geq , see [\[2\]](#).

6. Concluding remarks

The research in this paper grew out of curiosity after attending the talk by Jörg Endrullis at the workshop ‘Automatic sequences, Number theory, Aperiodic order’ in October 2015, about the concept of equivalent infinite sequences. We would like to thank Jörg Endrullis for inspiration, Jeff Shallit for useful conversations, and also Robbert Fokkink for earlier discussions with the first-mentioned author about bi-infinite versions of periodicity. This, together with unhappiness

about the intuitively undesirable fact that in the ordering with respect to ordinary transducers arbitrarily long initial segments of infinite sequences are irrelevant, formed the main motivation for considering two-sided infinite sequences and their natural transducers.

The situation for (ultimately) periodic sequences has now been clarified in all three settings (permutation transducers operating on one-sided and two-sided sequences with or without shift). As we have pointed out, there are many as yet unanswered questions remaining (but the same is true for the original ordering); we mention three examples.

- Apart from the ultimately periodic classes we identified as atoms, what other atoms can be identified in the hierarchies under consideration?
- In what generality do suprema and infima exist?
- Where in the hierarchies are particular automatic sequences (such as the Thue–Morse sequence) to be found?

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