

# Computation of the incomplete gamma function for negative values of the argument

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## Abstract

An algorithm for computing the incomplete gamma function  $\gamma^*(a, z)$  for real values of the parameter  $a$  and negative real values of the argument  $z$  is presented. The algorithm combines the use of series expansions, Poincaré-type expansions, uniform asymptotic expansions and recurrence relations, depending on the parameter region. A relative accuracy  $\sim 10^{-13}$  in the parameter region  $(a, z) \in [-500, 500] \times [-500, 0)$  can be obtained when computing the function  $\gamma^*(a, z)$  with the Fortran 90 module **IncgamNEG** implementing the algorithm.

## 1 Introduction

The incomplete gamma function  $\gamma^*(a, z)$  is defined by

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$$\gamma^*(a, z) = \frac{z^{-a}}{\Gamma(a)} \gamma(a, z) = \frac{1}{\Gamma(a)} \int_0^1 t^{a-1} e^{-zt} dt, \quad (1.1)$$

where  $\gamma(a, z)$  is the lower incomplete gamma function [15, Eqn. (8.2.1)].

The function  $\gamma^*(a, z)$  is real for positive and negative values of  $a$  and  $z$ .

Incomplete gamma functions appear in a large number of scientific applications. For positive values of  $z$ , they are related to the central gamma and chi-squared distribution functions (positive  $a$ ) and to exponential integrals (negative  $a$ ). There are numerous application areas for positive  $z$ , for example, [10, 2]. Algorithms and software are available for this parameter regions [5, 4, 7]. For negative  $z$ , the incomplete gamma functions appear, for instance, in the study of Bose plasmas [9, 8] and in the analysis of the Helmholtz equation [12, 11]. However, unlike the positive  $z$  case, software to support this case is very limited. Only recently has an algorithm been constructed for negative  $z$  [18] and this is restricted for half-integer values of  $a$ .

In this paper, we describe an algorithm for computing the function  $\gamma^*(a, z)$  for  $a$  real and  $z < 0$ . Our algorithm improves the range of computation of [18] by allowing real values of  $a$ . The methods of computation used in our algorithm are:

- a) series expansions, recurrence relations, and uniform asymptotic expansions for  $a < 0$ ;
- b) series expansions and Poincaré-type expansions [14, p. 16] for  $a > 0$ .

A Fortran 90 module implementing the algorithm is provided. Numerical tests show that the relative accuracy is close to  $10^{-13}$  in the parameter region  $(a, z) \in [-500, 500] \times [-500, 0)$ . This module complements a previous algorithm for the incomplete gamma function for positive values of the parameters [7].

## 2 Methods of computation

We describe the methods of computation used in the algorithm and in the numerical tests. Details on the region of application of each method are discussed in Section 4.

## 2.1 Recurrence relations

Recurrence relations are useful methods of computation when initial values are available for starting the recursive process. Also, recurrence relations can be used for testing the function values obtained by alternative methods. Usually, the direction of application of the recursion can not be chosen arbitrarily, and the conditioning of the computation of a given solution fixes the direction.

The function  $\gamma^*(a, z)$  satisfies the following inhomogeneous recursion [15, Eqn. (8.8.4)]

$$z\gamma^*(a+1, z) = \gamma^*(a, z) - \frac{e^{-z}}{\Gamma(a+1)} \quad (2.1)$$

When both  $a$  and  $z$  have negative values, replacing  $(a, z)$  by  $(-a-1, -z)$  and using the reflection formula  $\Gamma(a+1)\Gamma(-a) = -\frac{\pi}{\sin(\pi a)}$  in (2.1), we obtain [17, Eqn. (4.1)]

$$\gamma^*(-a-1, -z) + z\gamma^*(-a, -z) = -\frac{1}{\pi} \sin(\pi a) e^z \Gamma(a+1). \quad (2.2)$$

We may also combine two first order recursions of (2.1) to obtain the three-term homogeneous recurrence relation

$$z(a+1)\gamma^*(a+2, z) - (a+1+z)\gamma^*(a+1, z) + \gamma^*(a, z) = 0. \quad (2.3)$$

Starting from (2.2), we obtain

$$\gamma^*(-a-2, -z) + (z+a+1)\gamma^*(-a-1, -z) + z(a+1)\gamma^*(-a, -z) = 0. \quad (2.4)$$

An advantage of using the relation in (2.4) is that possible accuracy problems in the computation of the inhomogeneous term in (2.1) or (2.2) are avoided.

## 2.2 Series expansion

A series expansion for  $\gamma^*(a, z)$  is given by [15, Eqn. (8.7.1)]

$$\gamma^*(a, z) = \frac{1}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-z)^k}{k!(a+k)}. \quad (2.5)$$

As pointed out in [1] and discussed later (see Section 3), this series proves to be very useful computationally. In this form the series cannot be applied when  $a = -n$ ,  $n = 1, 2, \dots$  and special care needs to be exercised when  $a = -n + \epsilon$  and  $\epsilon$  is small. In this case, it is convenient to rewrite the series as

$$\gamma^*(-n + \epsilon, z) = z^n \frac{\Gamma(1 + n - \epsilon) \sin \pi \epsilon}{n! \pi \epsilon} + \frac{1}{\Gamma(-n + \epsilon)} \sum_{k=0, k \neq n}^{\infty} \frac{(-z)^k}{k!(-n + \epsilon + k)}. \quad (2.6)$$

Using (2.6) the series can be computed as  $\epsilon \rightarrow 0$  and we obtain in the limit the result [15, Eqn. (8.4.12)]

$$\gamma^*(-n, z) = z^n. \quad (2.7)$$

### 2.3 Uniform asymptotic expansion for $a < 0$

When  $a$  and  $z$  have large negative values, it is convenient to use the uniform asymptotic expansion described in [17], where the error function is used as main approximant. Replacing  $(a, z)$  with  $(-a, -z)$  we have

$$\gamma^*(-a, -z) = z^a \left\{ \cos(\pi a) - \sqrt{\frac{2a}{\pi}} e^{\frac{1}{2}a\eta^2} \sin(\pi a) \left[ \sqrt{\frac{2}{a}} F\left(\eta\sqrt{\frac{a}{2}}\right) + \frac{1}{a} T_a(\eta) \right] \right\}, \quad (2.8)$$

where  $\eta$  is defined by

$$\frac{1}{2}\eta^2 = \lambda - 1 - \log(\lambda), \quad \lambda = \frac{z}{a}, \quad \text{sign}(\eta) = \text{sign}(\lambda - 1). \quad (2.9)$$

The choice of the sign is based on the similarity of the graphs of the  $\eta$ -function (a parabola) and of the  $\lambda$ -function (a convex function for  $\lambda > 0$ , with its zero-minimum at  $\lambda = 1$ , and with the shape of a parabola).

As commented in [17], it is also useful to consider the normalized function  $\tilde{\gamma}_a(z)$  defined by the relation

$$\gamma^*(-a, -z) = z^a \cos(\pi a) + \sin(\pi a) \Gamma(a) e^z \tilde{\gamma}_a(z), \quad (2.10)$$

giving

$$\tilde{\gamma}_a(z) = -\frac{a}{\pi \Gamma^*(a)} \left[ \sqrt{\frac{2}{a}} F\left(\eta\sqrt{\frac{2}{a}}\right) + \frac{1}{a} T_a(\eta) \right]. \quad (2.11)$$

Using (2.10) in the inhomogeneous recursion, (2.2), we obtain

$$-\tilde{\gamma}_{a+1}(z) + \frac{z}{a}\tilde{\gamma}_a(z) + \frac{1}{\pi} = 0. \quad (2.12)$$

In (2.8) and (2.11),  $F(z)$  is Dawson's integral

$$F(z) = e^{-z^2} \int_0^z e^{t^2} dt = -\frac{1}{2}i\sqrt{\pi}e^{-z^2} \operatorname{erf} iz,$$

where  $\operatorname{erf}$  is the error function.

Dawson's integral can be computed using a continued fraction representation. In our algorithm, we use the representation given in [3, Eqn. (13.1.13b)]. This continued fraction works very well for small and large values of  $z$ .

The function  $T_a(\eta)$  in (2.8) and (2.11) has an asymptotic expansion in negative powers of  $a$

$$T_a(\eta) \sim \sum_{n=0}^{\infty} (-1)^n \frac{C_n(\eta)}{a^n}, \quad (2.13)$$

where the coefficients,  $C_n(\eta)$ , may be obtained starting from the differential equation satisfied by  $T_a(\eta)$ :

$$\frac{d}{d\eta} T_a(\eta) + a\eta T_a(\eta) = a(f(\eta)\Gamma^*(a) - 1), \quad (2.14)$$

with  $f(\eta)$  and  $\Gamma^*(a)$  given by

$$f(\eta) = \frac{\eta}{\lambda - 1}, \quad \Gamma^*(a) = \sqrt{a/(2\pi)}e^a a^{-a}\Gamma(a). \quad (2.15)$$

Substituting the asymptotic expansion (2.13) into (2.14) and using the expansion of the reciprocal gamma function

$$\frac{1}{\Gamma^*(a)} \sim \sum_{n=0}^{\infty} \frac{\gamma_n}{a^n}, \quad a \rightarrow \infty, \quad (2.16)$$

it is possible to find the following relations for the coefficients  $C_n(\eta)$

$$C_0(\eta) = \frac{1}{\lambda - 1} - \frac{1}{\eta}, \quad \eta C_n(\eta) = \frac{d}{d\eta} C_{n-1}(\eta) + \gamma_n f(\eta), \quad n \geq 1. \quad (2.17)$$

When  $|\eta|$  is small ( $\lambda \rightarrow 1$ ) the removable singularities in the representations of the coefficients  $C_n$  can be a source of problems in numerical

computations. In [17] Maclaurin expansions for the coefficients  $C_0, \dots, C_6$  were used to generate the values given in Table 4.1 in that reference. In the present algorithm we use a different approach. Instead of expanding each coefficient,  $C_n(\eta)$ , we expand the function  $T_a(\eta)$  of (2.13) in powers of  $\eta$ :

$$T_a(\eta) = \sum_{n=0}^{\infty} \omega_n \eta^n. \quad (2.18)$$

To compute the coefficients,  $\omega_n$ , we use the differential equation for  $T_a(\eta)$  given in (2.14). Substituting the expansion (2.18) into (2.14) and using the coefficients  $d_n$  in the expansion

$$\frac{\eta}{\lambda - 1} = \sum_{n=0}^{\infty} d_n \eta^n, \quad d_0 = 1, \quad d_1 = -\frac{1}{3}, \quad d_2 = \frac{1}{12}, \quad (2.19)$$

we obtain

$$\omega_1 = a(\Gamma^*(a) - 1), \quad (2.20)$$

and, for general  $\omega_n$ , the recursion relation

$$\omega_n = -\frac{n+2}{a}\omega_{n+2} + d_{n+1}\Gamma^*(a), \quad n = 0, 1, 2, \dots \quad (2.21)$$

If we write

$$\omega_n = \alpha_n \Gamma^*(a), \quad n = 0, 1, 2, \dots, \quad (2.22)$$

we have the recursion

$$\alpha_n = -\frac{n+2}{a}\alpha_{n+2} + d_{n+1}, \quad n = 0, 1, 2, \dots \quad (2.23)$$

Then, we choose a positive integer  $N$ , put  $\alpha_{N+2} = \alpha_{N+1} = 0$ , and compute the sequence

$$\alpha_N, \alpha_{N-1}, \dots, \alpha_1, \alpha_0 \quad (2.24)$$

from the recurrence relation (2.23).

Because (see (2.20) and (2.22))

$$\frac{1}{\Gamma^*(a)} = 1 - \frac{1}{a}\alpha_1, \quad (2.25)$$

we have

$$T_a(\eta) \approx \frac{a}{a - \alpha_1} \sum_{n=0}^N \alpha_n \eta^n \quad (2.26)$$

as an approximation for  $T_a(\eta)$ .

## 2.4 Poincaré-type expansion for $a > 0$

A Poincaré-type expansion that is useful for large  $|z|$  and valid for all  $a$  bounded can be obtained using the relation of  $\gamma^*(a, z)$  to the Kummer function  $M(a, b, z)$ ,

$$\gamma^*(a, z) = \frac{1}{\Gamma(a+1)} M(a, 1+a, -z), \quad (2.27)$$

and the expansion given in [13, Eqn. (13.7.1)].

The resulting expression is given by

$$\gamma^*(a, -z) \sim \frac{e^z}{z\Gamma(a)} \sum_{n=0}^{\infty} \frac{(1-a)_n}{z^n}. \quad (2.28)$$

## 2.5 Numerical quadrature

For  $a > 0$ , it is also possible to use numerical quadrature to compute the function  $\gamma^*(a, z)$ . Starting from (1.1) we replace  $z$  by  $-z$

$$\gamma^*(a, -z) = \frac{1}{\Gamma(a)} \int_0^1 y^{a-1} e^{zy} dy,$$

We can then use a quadrature rule to compute this integral to the desired accuracy. One approach is to consider a change of variable that transforms this integral into one that may be computed effectively using the trapezoidal rule. A suitable case for this is when the integrand decays as a double exponential in the real line (see [16] and [6, §5.4]).

We can obtain such an integral representation by using the change of variables  $r = \log\left(\frac{y}{1-y}\right)$ . Then,

$$\gamma^*(a, -z) = \frac{1}{\Gamma(a)} \int_{-\infty}^{\infty} (1+e^{-r})^{-(a+1)} e^{z(1+e^{-r})^{-1}} e^{-r} dr, \quad (2.29)$$

and writing  $r = \sinh(t)$ , we arrive at

$$\gamma^*(a, -z) = \frac{1}{\Gamma(a)} \int_{-\infty}^{\infty} \phi(t)^{a+1} e^{z\phi(t)} e^{-r(t)} \cosh(t) dt, \quad (2.30)$$

where  $\phi(t) = (1+e^{-r(t)})^{-1}$ .

The integrand of (2.30) then has double exponential behaviour as  $|t| \rightarrow +\infty$  which is suitable for the application of the trapezoidal rule. For numerical use, the integral should conveniently be truncated by choosing only a finite interval of integration.

We note here that this quadrature approach is not used in the final version of the algorithm as faster methods are available. However, it does provide a useful method for testing purposes.

### 3 Numerical testing and performance

For  $a < 0$ , we tested the performance of the uniform asymptotic expansion over a wide range of parameters using the normalized gamma function  $\tilde{\gamma}_a(z)$  defined in (2.11) and the recurrence relation given in (2.12). Using  $10^8$  random points over the region of the  $(z, a)$ -plane  $[-1000, 0] \times [-1000, 0]$ , we obtained an accuracy  $\sim 10^{-14}$  in the whole region with the exception of the strips  $|a| < 4.5$  and  $|z| < 1.5$ . The range of computation of the  $\gamma^*(a, z)$  is more limited due to overflow/underflow problems in double precision arithmetic, as can be seen in Figure 1. Function values underflow (overflow) in standard IEEE double precision arithmetic for large positive (negative) values of  $a$ . For that reason, we have limited the rest of the tests to the region of the  $(z, a)$ -plane  $[-500, 0] \times [-500, 500]$ .

The series expansions of Section 2.2 have been tested against a Maple implementation using 30 digits accuracy for  $a < 0$  in the regions  $(z, a) \in [-500, 0] \times [-5, 0]$  and  $(z, a) \in [-1.5, 0] \times [-500, 0]$ . The maximum relative error obtained was  $\sim 10^{-13}$ , although a large number of terms are needed for computing the series when  $|z|$  is large. In this case, a more efficient method of computation is to combine the use of recurrence relations and uniform asymptotic expansions. In particular, we compute first the normalized gamma function  $\tilde{\gamma}_{\tilde{a}}(z)$  for a value of the parameter  $\tilde{a}$  within the range of validity of the uniform asymptotic expansion and then take few steps in the backward direction of the recursion (2.12). The function  $\gamma^*(a, z)$  is finally computed using (2.10).

As already mentioned in Section 2.2, we need to be careful in the computation when  $a$  is close to an integer i.e.,  $a = -n + \epsilon$  where  $\epsilon$  is small. To avoid loss of accuracy both in the series expansions and when computing the coefficients with the trigonometric functions in (2.8), the input argument,  $a$ , is defined as a quadruple precision real variable in our implementation.

For  $a$  positive, testing is made by comparing the available methods of computation: series expansions (2.5), numerical quadrature (2.30) and

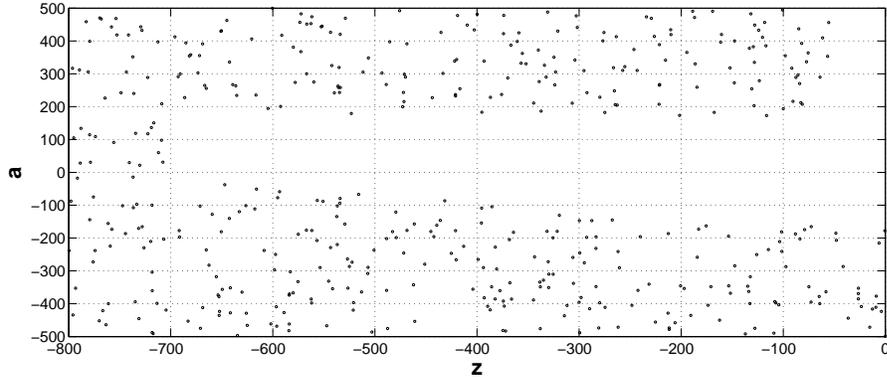


Figure 1: Overflow/underflow limitations in double precision arithmetic in the region  $(z, a) \in (-800, 0) \times (-500, 500)$  when computing the function  $\gamma^*(a, z)$ . The points correspond to values of the variables for which the computation either overflows or underflows.

Poincaré-type expansions (2.28). An accuracy close to  $10^{-14}$  is obtained in the region  $(z, a) \in [-500, 0) \times [0, 500]$  using the series expansion. Numerical quadrature works also accurate over the whole region with the exception of  $a$ -values close to zero, where there is some loss of accuracy in the computed function values. As in the case of  $a < 0$ , the series expansion needs a large number of terms when  $|z|$  is large, which makes the use of the Poincaré-type expansion more efficient for  $|z| > 50$ .

Figure 2 shows the CPU time used by the Fortran version of the algorithm in evaluating the function at 50,000 values of  $a$  and  $z$  on a 2GHz Intel Core i5-43100 under Windows 7 Professional. As we can see, the times are quite uniform across the whole range.

## 4 Computational scheme

From the results obtained in the previous section we may state a stable computational scheme for evaluating the function  $\gamma^*(a, z)$  as follows

1. For  $a > 0$ ,
  - If  $z < -50$ , compute the function using the Poincaré-type expansion (2.28).

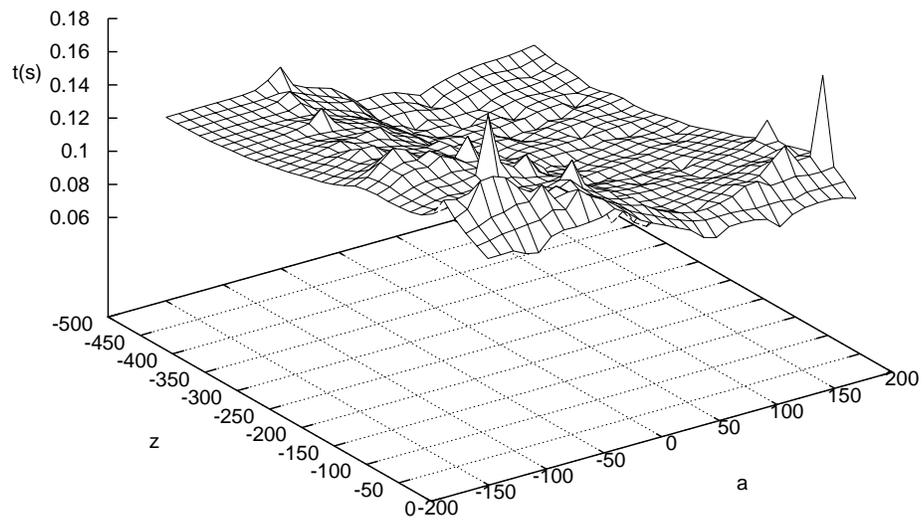


Figure 2: CPU time spent by the algorithm as a function of the variables  $a$  and  $z$ . The times shown correspond to 50000 function evaluations.

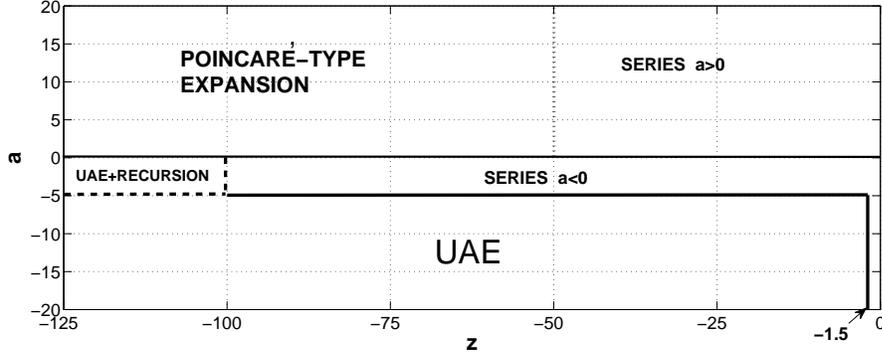


Figure 3: Methods for computation of the  $\gamma^*(a, z)$  function used in the final algorithm. UAE is the uniform asymptotic expansion of Section 2.3. The recursion relation is given in Eq. (2.12).

Otherwise, compute using the series expansion (2.5).

2. For  $a < 0$ ,

If  $a = -n$ ,  $n \in \mathbb{N}$ , use the expression given in (2.7).

Otherwise,

If  $a > -5$  or  $z > -1.5$ ,

If  $z > -100$ , use the series expansion (2.5) or the expression (2.6) if  $a = -n + \epsilon$  and  $\epsilon$  is small.

Otherwise, use the uniform asymptotic expansion (2.8) and the recursion relation given in (2.12).

Otherwise, compute the function using the uniform asymptotic expansion (2.8).

The different methods of computation used in the algorithm, with the exception of the method for  $a = -n$ ,  $n \in \mathbb{N}$ , are shown in Figure 3. The domains of computation are established following a compromise between efficiency and accuracy: we choose the most accurate method, and where two methods are equally accurate in a certain parameter region, we choose the fastest.

The resulting algorithm improves the range of computation of the algorithm presented in [18]. Thompson's algorithm considers the computation

of the lower incomplete gamma function for negative real values of the argument  $z$  and half-integer values of the parameter  $a$  using a function  $S_n(z)$ ,  $n$  integer and  $z > 0$ , related to the lower incomplete gamma function by  $\gamma(n + 1/2, -z) = i(-1)^n e^z z^{n+1} S_n(z)$ . The relation of the function  $S_n(z)$  to the  $\gamma^*(a, z)$  is then given by  $S_n(z) = \Gamma(n + 1/2) e^{-z} \gamma^*(n + 1/2, -z)$ . Precomputed values in Maple to initiate analytic continuation are used in Thompson's algorithm which, in the implementation available in [18], seems to be restricted to  $z$  values in the interval  $[0, 200]$ . Our approach extends the range of computation to real values of the parameter  $a$  and larger negative values of the argument  $z$ , and it does not depend on values precomputed in Maple.

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## References

- [1] D. H. Bailey and J. M. Borwein. Crandall's computation of the incomplete Gamma function and the Hurwitz zeta function, with applications to Dirichlet L-series. *Appl. Math. Comput.*, 268:462–477, 2015.
- [2] G.W. Collins. *Fundamentals of Stellar Astrophysics*. W H Freeman and Co, 1989.
- [3] A. Cuyt, V.B. Petersen, B. Verdonk, H. Waadeland, and W.B. Jones. *Handbook of continued fractions for special functions*. Springer, New York, 2008.
- [4] A.R. Didonato and A.H. Morris. Computation of the incomplete gamma function ratios and their inverse. *ACM Trans. Math. Softw.*, 12:377–393, 1986.
- [5] W. Gautschi. A computational procedure for incomplete gamma functions. *ACM Trans. Math. Softw.*, 5:466–481, 1979.
- [6] A. Gil, J. Segura, and N. M. Temme. *Numerical methods for special functions*. SIAM, Philadelphia, PA, 2007.

- [7] A. Gil, J. Segura, and N. M. Temme. Efficient and accurate algorithms for the computation and inversion of the incomplete gamma function ratios. *SIAM J. Sci. Comput.*, 34(6):A2965–A2981, 2012.
- [8] V. Kowalenko. The modes of an ultra-cold strongly magnetized charged Bose gas. *Eur. Phys. B*, 1:161–168, 1998.
- [9] V. Kowalenko and N. E. Frankel. Asymptotics for the Kummer function of Bose plasmas. *J. Math. Phys.*, 35(11):6178–6198, 1994.
- [10] K. Krishnamoorthy. *Handbook of Statistical Distributions with Applications*. Statistics: A Series of Textbooks and Monographs (Book 188). Chapman and Hall/CRC, 2006.
- [11] C. M. Linton. Lattice sums for the Helmholtz equation. *SIAM Rev.*, 52(4):630–674, 2010.
- [12] A. Moroz. Quasi-periodic Greens functions of the Helmholtz and Laplace equations. *J. Phys. A: Math. Gen.*, 39:11247–11282, 2006.
- [13] A. B. Olde Daalhuis. Confluent hypergeometric functions. In *NIST handbook of mathematical functions*, pages 321–349. U.S. Dept. Commerce, Washington, DC, 2010.
- [14] F. W. J. Olver. *Asymptotics and special functions*. AKP Classics. A K Peters Ltd., Wellesley, MA, 1997. Reprint of the 1974 original [Academic Press, New York].
- [15] R. B. Paris. Incomplete gamma and related functions. In *NIST handbook of mathematical functions*, pages 175–192. U.S. Dept. Commerce, Washington, DC, 2010.
- [16] H. Takahasi and M. Mori. Double exponential formulas for numerical integration. *Publ. Res. Inst. Math. Sci.*, 9:721–741, 1973/74.
- [17] N. M. Temme. Uniform asymptotics for the incomplete gamma functions starting from negative values of the parameters. *Methods Appl. Anal.*, 3:335–344, 1996.
- [18] I. Thompson. Algorithm 926: Incomplete Gamma Functions with Negative Arguments. *ACM Tran Math Soft*, 39(2):Article 14, 2012.