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Proportional green time scheduling for traffic lights

Péter Kovács · Tung Le · Rudesindo Núñez-Queija · Hai L. Vu · Neil Walton

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Abstract We consider the decentralized scheduling of a large number of urban traffic lights. We investigate factors determining system performance, in particular, the length of the traffic light cycle and the proportion of green time allocated to each junction. We study the effect of the length of the traffic cycle on the stability region a urban traffic network. We derive a simple square-root cycle length rule which is optimal for certain road traffic junctions. We prove the maximal stability of a road network under a proportional fair or P0 control scheme. Further, we support of analysis through a simulation analysis of our policy on the Melbourne CBD urban road network.

Keywords Traffic light control · Optimal Cycle Length · Stability · P0 policy · Proportional Fairness.

1 Introduction

Traffic congestion is one of the most apparent problems of modern society. Aside from the dissatisfaction of the people who are stuck in traffic jams, congestion also results in loss of productivity for the economy, has negative environmental effects and also increases the probability of accidents due to

Corresponding author

P. Kovács, R. Núñez-Queija
Korteweg de Vries Institute for Mathematics, University of Amsterdam, Science Park 105-107, 1098 XG, Amsterdam, The Netherlands
E-mail: {p.kovacs, r.nunezqueija}@uva.nl

T. Le, H. L. Vu
Faculty of Science and Technology, Swinburne University of Technology, Australia.
E-mail: {tmle, hvu}@swin.edu.au

N.S. Walton
School of Mathematics, University of Manchester, United Kingdom.
E-mail: neil.walton@manchester.ac.uk

unsafe driving behaviour. The growing population and the advancing economy create densely populated urban areas with increasing traffic demand and little space to build more roads. Solving this problem by increasing network capacity by extending the infrastructure is not always possible and can have undesirable environmental consequences. The other possible solution which presents itself is to provide more efficient utilization of the capacity of the existing networks. Thus one of the major challenges of the study of intelligent transport systems is reaching this goal.

One of the key tools of traffic management is traffic signal control. Efficiently controlled traffic lights can positively influence the traffic flows at intersections, which are often the bottlenecks of the road network. Studying such systems are of interest to both the queueing theory and the traffic management community. These works aimed to design and optimize isolated or coordinated signals that reactively resolve congestion in the urban networks.

In this paper, we consider stability of the decentralized control schemes in urban traffic networks. Decentralized schemes are simple, scalable and can be implemented in a modular form. Further, they can adapt implicitly using local information to provide congestion relief in different traffic scenarios. These provides redunancy that may not be present in centrally planned and fixed time traffic control schemes. However, decentralized schemes do not alway provide good stability properties or require (unavailable) information on traffic route choice. The rule chosen by the local control is important.

We propose a policy based on the proportional fairness criterion. This policy is completely decentralized, using only information from the queues present at each intersection. In terms of division of timing, our model combines a pre-timed approach with a vehicle-actuated approach since we use a set of cycle times that are fixed a priori, whereas their split amongst the service phases, which allow for more lanes to be served simultaneously, are decided on a cycle-to-cycle basis upon the measured traffic by proportionally fair optimality criteria. We form a stronger connection between the results of queueing theory and traffic management, by using the formalism and methods of the former whilst considering the cyclic nature and the specific features of the latter. In summary, our contributions include

- developing a novel optimal traffic signal control policy for urban networks based on proportionally fair allocation,
- introducing and analysing a polling-type model to determine a waiting time-optimal time plan for the cycle lengths in said policy,
- conducting simulations to numerically validate the analytical results of the polling model and to compare the performance of our proposed policy with the performance of the P_0 policy,
- providing a formal proof of stability for the underlying stochastic system that is controlled by our scheme by applying a fluid limit approach.

This paper bridges key queueing theoretic concepts with policies and simulation software familiar to the practioners in urban traffic control. The proportional allocation scheme is a natural generalization of the P0 policy, a well known traffic control scheme Smith (1980). In order to investigate the new policy important queueing theoretic ideas such as polling models and fluid analysis are applied. The work here rigorizes and generalizes the work of Walton (2014), where a fluid analysis is provided, only in this paper, we provide a rigorous fluid limit proof and a proof of positive recurrence. This is provided for a more general model framework, which accounts for the acceleration of vehicles, the cyclic nature of traffic times and the impact on the stability region of bounded cycle lengths. The results are analysed through the SUMO simulation enviroment, an urban mobility simulator regularly used in the analysis and simulation of traffic management systems.[1]

As we show in Section 2 this work fits within a growing literature considering the applicability of the decentralized control paradigm, typically used in communications systems, to the setting of urban mobility. See Lämmer and Helbing (2008); Wunderlich et al (2008); Varaiya (2013); Wongpiromsarn et al (2012); Savla et al (2013); Le et al (2013a); Savla et al (2014) for important recent references. As mentioned above, in this paper we highlight key characteristics found in queueing theoretic literature that are particularly applicable to the urban road traffic setting.

The rest of the paper is organized as follows. In Section 2, we review the relevant literature. In Section 3 we introduce the notation and dynamics of a traffic network. In Section 4 we discuss the proportionally fair allocations of splits of the cycle times and the effects of different choices for the length of the cycles. We introduce a polling-type model to formulate optimality criteria for the cycle times. We provide a set of limiting fluid equations to which the dynamics of the stochastic system converges. We also formulate the stability of the system through the stability of the fluid limit. We validate these results numerically in Section 5. In Section 6 we give a formal proof of convergence to the fluid limit and of the stability of the latter.

2 Literature Review

In this section we provide a literature review on traffic signal control. We discuss several of the most widely applied implementation and different optimization formulations which have been applied to the problem. We then discuss recent work on decentralized optimal control. Similar literature is discussed in the earlier work Le et al (2013a), though we enlist more recent references and also provide a more in-depth discussion of proportional fairness.

Broadly, there are two types of control that have been used for signal control: *pre-timed* and *vehicle-actuated* controls; see Hamilton et al (2013) for a comprehensive review. Pre-timed controls provide a repetitive cycle and a fixed time division among the conflicting movements at the intersection. The optimization of cycle times can be done in isolation or in a coordinated manner, see Gartner et al (1975). Various approximations an analysis on queue lengths

[1] For information on the SUMO package visit
http://www.dlr.de/ts/en/desktopdefault.aspx/tabid-9883/16931_read-41000/

of fixed cycle traffic control can be found in Webster (1958), Miller (1963), Heidemann (1994), van Leeuwaarden (2006) and van den Broek et al (2006).

Vehicle-actuated (or traffic-actuated) controls differ in that their signal timings are not fixed, but assigned to the various service phases based on the actual traffic present in the system. This requires real time measurements which can be done by inductive loops, cameras, etc. The optimal signal timings are then calculated on a cycle-to-cycle basis. Studies of these vehicle actuated controls include Darroch et al (1964), Newell (1969), and Daganzo (1990). Commonly used implementations of this type are SCOOT, see Hunt et al (1981); UTOPIA, see Mauro and Taranto (1990); and the hierarchical scheme RHODES , see Mirchandani and Head (2001). Combinations of both the pre-timed and traffic-actuated control also exist such as the SCATS system from Lowrie (1982). In the paper we consider a scheme with both pre-timed and traffic-actuated controls. In particular, the length of traffic cycles will be determined by a closed-loop optimization which operates on a slower time-scale to the traffic-actuated phases within a cycle.

Another possible way of categorizing the control policies is based on the approaches that have been proposed to optimize the signal plans. Examples include Mixed-Integer Linear Programming problems, see Dujardin et al (2011) and Gartner et al (1975); dynamic programming, see Gartner (1983) and Mirchandani and Head (2001); Model Predictive Control (MPC) optimization Tettamanti and Varga (2010), Le et al (2013b) and Shu et al (2011). However, for a network with many intersections, most of the above methods represent a centralized approach that require exponential complexity computations for a global optimal solution and thus are not scalable. As noted in Papageorgiou et al (2003), control strategies such as OPAC in Gartner (1983) and RHODES in Mirchandani and Head (2001) are not real-time feasible for more than one intersection. In fact, they actually became a decentralized scheme via forced implementation at individual intersections that are heuristically coordinated over the network, see N. H. Gartner and Andrews (2001).

Decentralized schemes on the other hand have the advantage of scalability and simplicity of implementation. However, due to the fact that they mostly use detailed real-time local information for which the technology was in most cases not available until the emergence of wireless technology, they have been studied to a lesser extent in the traffic management literature. The information used to form optimum criteria in these cases can be such as the difference of the number of vehicles queueing up in neighbouring intersections or the expected number of vehicles to enter the intersection during the next cycle. Most of these schemes have strong roots in packet scheduling for communication networks. For example the policies of the first approach, which include work by Wunderlich et al (2008); Varaiya (2013); Wongpiromsarn et al (2012); Le et al (2013a) are based on the max-pressure/back-pressure class of algorithms introduced in Tassiulas and Ephremides (1992). However, stability results that hold for max-pressure do not naturally extend to road traffic scheduling. The policy must be aware of the routes or turning ratios of cars and must communicate queue size information between junctions. Further, scheduling decisions are integral and so do not naturally lend to a fixed traffic cycle. Progress alleviating these specific issues can be found in Le et al (2013a) and Savla et al (2014). However, in this paper, we fully address each of these issues by considering a new decentralized scheme outside of the max-pressure family of controls.

Our approach can be connected to the notion of *proportional fairness*, as was first described by Kelly (1997). It is studied widely, for example in models of highway traffic, see Gibbens and Kelly (2011), and Kelly and Williams (2010); bandwidth sharing, see Massoulié (2007); or switch networks, see Walton (2014). In the context of queueing systems the policy was first mentioned by Schweitzer (1979). In terms of urban road traffic early work of Smith (1980), a so-called P_0 local control policy and its variants Smith (2011) belong to this category of control. More recently Lämmer and Helbing (2008) extended further the P_0 policy to include the switching cost between phases. Here a heuristic mechanism was proposed to prevent possible instability, albeit without a formal proof of stability for the distributed control strategy. Another recent noteworthy result similar to ours was presented in Savla et al (2013). Here a fluid model is considered where one lane is served at each time and a set of monotonic distributed controls are considered based only on the occupancy levels of incoming lanes. In this setting, the maximal stablility of signal controls are presented.

3 Model description

We give a description of the main parameters of our model in this section and the policy investigated in the next section. A table containing a list of notations is given in Table 1.

3.1 Notations

We consider a network of traffic *junctions*, indexed by j. Each junction $j \in \mathcal{J}$ consists of a number of *in-roads*, indexed by $i \in \mathcal{I}$, which represent one or more lanes of traffic. We denote an in-road being present at junction j by the inclusion $i \in j$, whereas the junction which contains i by $j(i)$. The in-roads of a junction receive green time in a *cyclic* manner, when phases of service are enacted. A *service phase* is a set of in-roads receiving green lights simultaneously. Service phases are indexed by σ, and we let Σ_j is the set of phases that are used at junction j. The inclusion $i \in \sigma$ denotes the in-road being served during phase σ, and $\Sigma_i = \{\sigma : i \in \sigma\}$ denotes the set of phases during which in-road i receives service. A link is a pair of in-roads: after the service is completed, the vehicles from in-road $i \in j$ may join the queue at in-road $i' \in j'$ if the link ii' is included amongst the set of possible links $\mathcal{L} \subseteq \mathcal{I} \times \mathcal{I}$. We let the vehicles follow a predefined, albeit unknown *route*. Let us denote the set of all possible routes by $\mathcal{R}$, and the routes themselves by $r \in \mathcal{R}$. A route is the sequence of in-roads that a vehicle visits as it navigates through the network. We will denote by $i \in r$ if route r goes through in-road i. If i is the first in-road on route r, we will denote that by $i = i_0^r$, and if i is the last in-road on route r, we will denote that by $i = i_l^r$. We will use the notations i_-^r and i_+^r for the in-roads that are, respectively, preceding and following in-road i on route r.

We introduce the following notation to address the cyclic nature of the system. Denote by $\left\{T_n^j\right\}_{j\in\mathcal{J},n\in\mathbb{N}}$ the sequence of *cycle lengths* for each junction and define the sequence $0 = t_0^j < t_1^j < t_2^j < \ldots$ such that

$$t_{n+1}^j = t_n^j + T_{n+1}^j.$$

We will also use the notation

$$N_j(t) = \max\{n \in \mathbb{N} : t_n^j \leq t\}, \tag{1}$$

to count the number of cycles completed by time t. Thus the end of the last finished cycle is $t^j_{N_j(t)}$, and its length was $T^j_{N_j(t)}$. Under a fixed-time plan the length of the cycle at each junction, T^j will be fixed and we will denote the vector of cycle length by $T = (T^j)_{j\in\mathcal{J}}$.

At each cycle some time is spent switching between phases, during which the vehicles do not receive service. We assume that this requires a fixed amount of time T_{switch} for every switch. We require every phase to be enacted in every cycle. The *effective service time* E_n^j – the period during which vehicles actually receive service – is thus given by

$$E_n^j = T_n^j - |\Sigma_j| \cdot T_{\text{switch}} \tag{2}$$

for $j \in \mathcal{J}$ and $n \in \mathbb{N}$.

The traffic controller's job is to determine the cycle lengths and to allocate proportions of the effective service time to the separate phases. We will denote these proportions by $p = (p_\sigma)_{\sigma\in\Sigma_j, j\in\mathcal{J}}$. In order to utilize the given resources fully, we have

$$\sum_{\sigma\in\Sigma_j} p_\sigma = 1, \tag{3}$$

for each junction $j \in \mathcal{J}$. We introduce the *allocation vector* $y(p) = (y_i(p))_{i\in\mathcal{I}}$ which shows the time proportions during which the in-roads receive service, thus

$$y_i(p) = \sum_{\sigma\in\Sigma_i} p_\sigma. \tag{4}$$

Consequently the set of allocation vectors lie in the following convex set

$$\mathcal{Y} = \Big\{ (y_i)_{i\in\mathcal{I}} \in (0,1)^{\mathcal{I}} : y_i = \sum_{\sigma\in\Sigma_i} p_\sigma, \sum_{\sigma\in\Sigma_j} p_\sigma = 1 \;\forall j \Big\}. \tag{5}$$

Observe that with this notation in-road i will get green light for $y_i E_n^{j(i)}$ amount of time in the given cycle.

3.2 Queue size processes

The cars in a traffic light network form a queueing network, where the vehicles waiting at every in-road form the queues. We assume each junction has knowledge of the queue sizes present at each of its in-roads. We let $Q_i(t)$ denote the queue size in-road i at time t. Under vehicle actuated control these queue sizes are known to the controller. We use the notations $Q^j(t) = (Q_i(t))_{i\in j}$ for the queue sizes at a junction, and $Q(t) = (Q_i(t))_{i\in\mathcal{I}}$ for the overall queue size vector. These change due to the received service, the traffic joining in from queues already served and from external arrivals. We introduce the route-wise queue sizes, $X_{ir}(t)$, i.e. the number of vehicles queueing at in-road$i$ which are following route r. Unlike $Q_i(t)$, the queue sizes $X_{ir}(t)$ are unknown to the controller. We denote the route-wise queue size vector by $X(t) = (X_{ir}(t))_{i\in\mathcal{I},r\in\mathcal{R}}$. By definition

$$Q_i(t) = \sum_{r:i\in r} X_{ir}(t). \tag{6}$$

We work with the assumption that at every in-road the cars following different routes are distributed homogeneously in the queue.

We define $\{A_{ir}(t)\}_{i\in\mathcal{I},r\in\mathcal{R}}$ as the route-wise arrival processes and similarly $\{D_{ir}(t)\}_{i\in\mathcal{I},r\in\mathcal{R}}$ as the route-wise departure processes. Thus the queue sizes develop as follows,

$$X_{ir}(t) = X_{ir}(0) + A_{ir}(t) - D_{ir}(t), \tag{7}$$

where $t \in \mathbb{R}^+$. We note, that the route-wise arrivals described by $A_{ir}(t)$ can be external arrivals, if $i = i_0^r$, or internal arrivals if $i \neq i_0^r$. In the latter case they equal the departures of the previous in-road along route r. The external arrivals follow the process $A_r(t)$ for each route $r \in \mathcal{R}$. We assume, for simplicity, that these are independent Poisson processes and use the vector notation $(a_r)_{r\in\mathcal{R}}$ for the arrival rates. Thus, by definition

$$A_{ir}(t) = \begin{cases} A_r(t) & \text{if } i = i_0^r, \\ D_{i_-^r r}(t) & \text{if } i \neq i_0^r. \end{cases}$$

It follows that the long run arrival rate for in-road i is given by

$$a_i = \sum_{r:i\in r} a_r.$$

The departures $D_{ir}(t)$ are positive counting processes. They depend on both the properties of the junctions and the allocated green times. To model these effects we define the random variables S_{ir}^n as the number of cars served at in-road i during its junction's n^{th} cycle. Thus we have

$$D_{ir}(t_n^{j(i)}) = \sum_{m=1}^{n} S_{ir}^m \tag{8}$$

for each $r \in \mathcal{R}$, $i \in \mathcal{I}$, and $n \in \mathbb{N}$.

We assume that for every in-road there exists a maximum rate with which a steady stream of vehicles can be served, which is determined by the speed limits on the road segments. We will denote the vector of maximum rates by $\mu^{\max} = (\mu_i^{\max})_{i\in\mathcal{I}}$. When y_i is allocated to in-road i, this puts the following bound on the received service

$$\mu_i^{\max} y_i E_n^{j(i)} \geq \sum_{r:i\in r} S_{ir}^n,$$

for each $i \in \mathcal{I}$ and $n \in \mathbb{N}$. Further, we introduce the service function $s(\cdot) = (s_i(\cdot))_{i\in\mathcal{I}} : \mathbb{R}_+ \mapsto \mathbb{R}_+^{\mathcal{I}}$, where $s(t)$ represents the expected number of cars that can be served from each in-road whilst receiving a green light for t units of time. We assume $s(\cdot)$ to be continuous, almost everywhere twice differentiable, increasing and to have the following asymptotic behavior

$$s_i(t) \sim \mu_i^{\max} t, \tag{9}$$

for each in-road i. This represents the idea that after an initial set-up phase, needed for the vehicles in the queue to speed up, the cars can move without interruption. With our assumption on the homogeneous distribution of vehicles taking different routes among queues, we have, when $Q_i(t_n^{j(i)}) > 0$,

$$\mathbb{E}[S_{ir}^n | X(t_{n-1}^{j(i)}), Q^j(t_{n-1}^{j(i)})] = \frac{X_{ir}(t_{n-1}^{j(i)})}{Q_i(t_{n-1}^{j(i)})} s_i(y_i E_n^{j(i)}) \tag{10}$$

for $r \in \mathcal{R}, i \in \mathcal{I}$ and $n \in \mathbb{N}$. We define the average rate at which vehicles are served from each route-wise queue as

$$\mu_{ir}(t) = \frac{\mathbb{E}[S_{ir}^n | X(t_{n-1}^{j(i)}), Q^j(t_{n-1}^{j(i)})]}{T_n^{j(i)}}, \tag{11}$$

when $t_{n-1}^{j(i)} \leq t < t_n^{j(i)}$, thus $\mu_{ir}(t)$ is a piecewise constant function. We also let $\mu_i(t)$ be the rate at which cars leave the in-roads. From (10) and (11) we have

$$\mu_i(t) = \sum_{r:i\in r} \mu_{ir}(t) = \frac{s_i(y_i E_n^{j(i)})}{T_n^{j(i)}}, \tag{12}$$

for $t_{n-1}^{j(i)} \leq t < t_n^{j(i)}$. To ease our notation we will refer to $\mu_{ir}(t_{n-1}^{j(i)})$ as μ_{ir}^n. When we wish to make dependence of y and $T = T_n^{j(i)}$ explicit, we will write $\mu_i(y, T)$ for expression (12). We assume the actual number of cars leaving is close to its expectation in that there exists a constant κ such that

$$\mathrm{Var}(S_{ir}^n) \leq \kappa T_n^{j(i)} \tag{13}$$

for $r \in \mathcal{R}, i \in \mathcal{I}$ and $n \in \mathbb{N}$. This is a property which holds for numerous kinds of stochastic processes such as Poisson-processes and renewal processes with renewal epochs of finite variance. In essence, for (13) to be violated we require the gaps between one car and the next to have infinite variance, while in practice we imagine the time gap between cars to be of finite order.

Notation	Description
$\mathcal{A}(T)$	Capacity set under cycle length T
a_r	Route r arrival rate
A_{ir}	Cummulative route r arrivals at in-road i
c_j	Cycle length control parameter
$\mathcal{C}$	Capacity set
D_{ir}	Cummulative route r departures at in-road i
E^j_n	Effective service of nth cycle at j
i, $\mathcal{I}$	In-roads
ii', $\mathcal{L}$	Links
i^r_0, i^r_l	First and last road on route r
i^r_-, i^r_+	Road before and after i on route r
j, $\mathcal{J}$	Junctions
$j(i)$	Junction of in-road i
μ^n_{ir}	service rate n cycle, route r, junction i
$N_j(t)$	Number of cycles completed by time t
p_σ	Proportion of service for phase σ
Q_i	Queue size of in-road i
r, $\mathcal{R}$	Routes
σ, Σ	Phases of service
$s_i(t)$	Service rate of road i after t units of green time
S^n_{ir}	Number served cars in nth cycle, route r, road i
t^j_n	Time of the nth cycle at junction j
T^j_n	Cycle length of nth cycle at junction j
X_{ir}	Number of route r cars at in-road i
$y_i(p)$	Service to i under proportion p

Table 1: Key notations and descriptions alphabetically ordered.

4 Scheduling

The task of the traffic controller is to determine the cycle lengths and to allocate proportions of them to the service phases and switching. In this section we first present an algorithm which schedules the green times in a fair way. Secondly we discuss the network's capacity given the lengths of cycles. Thirdly we investigate two polling models, which allocate service times in a similar manner to the proportional fiar scheme, in order to determine which choice of cycle lengths would minimize the average waiting time in the system. Finally we summarize our suggested policy.

4.1 Proportionally fair allocation

We aim to allocate proportions of the cycle length to the service phases at each junction in a way that maximizes throughput whereas maintaining service for all in-roads with vehicles on them. A well-regarded way to do so is using a proportional fair scheme. To fully utilize the resources and reach optimal throughput we have to maximize $s_i(t)$ for all $i \in \mathcal{I}$, however (3) and (4) put constraints on this problem. For the $(n+1)$th traffic cycle, the proportionally fair schedule is given by the solution to the following optimization problem.

$$\begin{aligned} \text{maximize} \quad & \sum_{i\in j} Q_i(t_n^j)\log(s_i(E_{n+1}^j y_i)), \\ \text{subject to} \quad & \sum_{\sigma\in\Sigma_i} p_\sigma = y_i, \quad \forall i \in j, \\ & \sum_{\sigma\in\Sigma_j} p_\sigma = 1, \\ \text{over} \quad & y \geq 0, \;\; p_\sigma \geq 0, \; \forall\sigma\in\Sigma^j. \end{aligned} \tag{14}$$

This needs to be considered for each junction $j \in \mathcal{J}$. Solving the problem only requires local information, i.e. the knowledge of queue sizes present at the in-roads of said junction. An advantage of using such a scheme is that the optimization problem can be solved separately for each junction, thus requiring significantly less effort.

We denote the solution to the optimization problem by $p^* = (p^*_\sigma)_{\sigma\in\Sigma_j, j\in\mathcal{J}}$, $y^* = (y_i(p^*))_{i\in\mathcal{I}}$, $s^* = (s_i(y^*, T))_{i\in\mathcal{I}}$ and $\mu^* = (\mu_i(y^*, T))_{i\in\mathcal{I}}$ respectively for further purposes. In general (14) does not have an explicit solution, although if s is given, it can be solved numerically. Since we need (14) to have a single optimal solution, $\log(s_i(E_j y_i(p)))$ has to be concave. This imposes the condition

$$(s_i'(t))^2 > s_i(t)s_i''(t) \quad \forall t > 0, \; \forall i \in \mathcal{I} \tag{15}$$

on the service function, which can be derived from the necessary condition of concavity,

$$(\log(s_i(t)))'' = \left(\frac{s_i'(t)}{s_i(t)}\right)' = -\left(\frac{s_i'(t)}{s_i(t)}\right)^2 + \frac{s_i''(t)}{s_i(t)} < 0, \;\; \forall i \in \mathcal{I}$$

and the fact that s is positive and increasing. Any subexponential function is sufficient for (15) such as polynomial, piecewise linear or similar functions. In special cases (14) can even be solved explicitly, for example if $s_i(t) = vt$ $\forall i \in \mathcal{I}$, i.e. the received service is a linear function of time with the same rate for all in-roads, and all in-roads are served during exactly one service phase, i.e. $|\Sigma_i| = 1 \; \forall i \in \mathcal{I}$. Then the optimal schedule is given by

$$p^*_\sigma = \frac{\sum_{i\in\sigma} Q_i}{\sum_{i\in j} Q_i} \quad \forall \sigma \in \Sigma_j, \tag{16}$$

which demonstrates the proportional nature of the proposed algorithm. We note that the above scheme, (16), coincides with the P0 policy which was previously studied in the context of road traffic congestion, see Smith (1980).

4.2 Network capacity

The maximal throughput is constrained by the physical parameters of the network, which we represent by the vector $\mu^{\max}$. Thus the capacity can be defined as

$$\mathcal{C} = \{(y_i \mu_i^{\max})_{i\in\mathcal{I}} : y \in \mathcal{Y}\},$$

which is convex due to $\mu^{\max}$ being constant and the convexity of $\mathcal{Y}$. The maximal service that could be allocated to the in-roads is given by the case when they receive uninterrupted green time, thus they form the extreme points of $\mathcal{C}$. These rates cannot be reached however, as we require switching in every cycle, which alone interrupts service and also invokes a slower setup phase in realistic service. To represent this we define the set of admissible service rates as

$$\mathcal{A}(T) = \{(\mu_i(y, T))_{i\in\mathcal{I}} : y \in \mathcal{Y}\},$$

which is clearly a subset of the capacity set. It depends on the cycle lengths since by its definition in (12), the vector of service rates $\mu = (\mu_i)_{i\in\mathcal{I}}$ is cycle-length dependent. This dependency is present not only due to switching times, but could also show in the service function as shown in the example in Appendix B. The problem however can be solved by a sufficient choice of cycle lengths as stated in the following proposition. In the statement we use $\mathcal{M}^\circ$ to denote the interior of a set $\mathcal{M}$.

Proposition 1 *In the limit as $T \to \infty$, the set of admissible service rates $\mathcal{A}(T)$ reaches the capacity set $\mathcal{C}$, i.e. $\forall a \in \mathcal{C}^\circ$ there exists a vector of cycle lengths τ, such that if $T_j > \tau_j \;\; \forall j \in \mathcal{J}$, then*

$$a \in \mathcal{A}^\circ(T).$$

A proof of Proposition 1 can be found in Section 6.1.

We have not addressed the possibility of eliminating phases, and the extra switching they require, as another possible solution. However, from the traffic's point of view, this is a similar solution to increasing the cycle lengths since a longer cycle can be viewed as consecutive shorter cycles, which are all serving just one phase.

When choosing the cycle lengths we also have to consider the fact that longer cycles and thus longer service phases produce longer waiting periods for the vehicles on the in-roads receiving red light. There is clearly a trade-off between the network's stability and the average time vehicles spend in the system depending on the cycle lengths. In Section 4.3 we discuss how long the cycles should be to minimize the average waiting times.

4.3 Optimal cycle length

In this section we investigate a polling model which represents a single junction. This provides an example where the optimal scaling of cycle lengths can be determined. The schedule in the model does not necessarily follow strictly the proportional fair allocation given by (14), instead it allocates service times based on the average queue lengths. We chose average queue lengths to enable our results to translate to our policy, where expceted queue lengths are available by taking averages on historic data- However, due to similarities in the allocation method, the model gives us an opportunity to develop a rule for optimal cycle length setting.

Consider a single junction with N competing in-roads. In each schedule, only one in-road can be served. Let each in-road have Poisson arrivals with rates $a_1, a_2, \ldots, a_N$ and exponentially distributed random green times $S_1, S_2, \ldots, S_N$. Let vehicles leave with rate $\mu_1, \mu_2, \ldots, \mu_N$ once receiving service, and define $\rho_i = a_i/\mu_i$. In order to have similar behaviour to the proportionally fair policy the rates $\nu_1, \nu_2, \ldots, \nu_N$ of the green times are chosen to fulfill the following equation.

$$\frac{\frac{1}{\nu_i}}{\sum_{i=1}^N \frac{1}{\nu_i}} = \frac{\bar{Q}_i}{\sum_{i=1}^N \bar{Q}_i}, \tag{17}$$

where $\bar{Q}_1, \bar{Q}_2, \ldots, \bar{Q}_N$ are the respective average queue sizes. Let us denote the expected cycle length, i.e. the sum of the green times and switching times by τ. Thus

$$\tau = \mathbb{E}T = \mathbb{E}\left[\sum_{i=1}^N S_i + NT_{\text{switch}}\right] = \sum_{i=1}^N \frac{1}{\nu_i} + NT_{\text{switch}}. \tag{18}$$

To gain insight on the relation between the average queue lengths and the cycle time let us deduct a system of equations using Little's Law and PASTA (Poisson Arrivals See Time Averages), cf. Wolff (1982). We follow the mean value approach described on page 33 of Adan and Resing (2002). If we denote the average waiting times by $\bar{W}_1, \bar{W}_2, \ldots, \bar{W}_N$, then (by Little's Law) we have the following equations for $i = 1, 2, \ldots, N$,

$$\bar{Q}_i = a_i \bar{W}_i,$$

and, using PASTA in combination with the FCFS discipline in each queue,

$$\begin{aligned} \bar{W}_i =& \frac{1}{\mu_i}\left(1+\bar{Q}_i\right)\left(1+\nu_i\left(\sum_{j\neq i}\frac{1}{\nu_j} + NT_{\text{switch}}\right)\right) \\ &+ \frac{T_{\text{switch}}\left(NT_{\text{switch}} + \sum_{j\neq i}\frac{1}{\nu_j}\right)}{\sum_{j=1}^N \frac{1}{\nu_j} + NT_{\text{switch}}} + \frac{\sum_{j\neq i}\frac{1}{\nu_j}\left(\frac{1}{\nu_j} + T_{\text{switch}}\right)}{\sum_{j=1}^N \frac{1}{\nu_j} + NT_{\text{switch}}}. \end{aligned}$$

From these we can derive the following expressions for the expected queue lengths given as a function of expected cycle length,

$$\bar{Q}_i(\tau) = \frac{1}{1-\rho_i\nu_i\tau} \times \left[\rho_i\nu_i\tau + a_i \cdot \frac{{T_{\text{switch}}}^2 + \sum_{j\neq i}\left(\frac{1}{\nu_j} + T_{\text{switch}}\right)^2}{\tau}\right], \tag{19}$$

for $i = 1, 2, \ldots, N$. From (19) it is easily seen that for stability we need the cycle length to fulfill

$$\tau > \frac{NT_{\text{switch}}}{1 - \sum_{i=1}^{N} \rho_i}. \tag{20}$$

One would expect that T should be 'fairly close' to this limit in order to keep the expected queue lengths and subsequently the waiting times low. This would be the case if both the arrivals and the service times were deterministic as shown in Appendix C. In this model however an optimal cycle length cannot be determined in the general case, only if for all $i = 1, 2, \ldots, N$ we have $a_i = a$ and $\mu_i = \mu$ and thus $\nu_i = \nu$ and $\rho_i = \rho$, and even then the calculations do not yield an explicit formula, see Appendix D for the $N = 2$ case. On the other hand in this case we can derive the proper scaling for the cycle lengths as the system is in its heavy traffic limit as presented by the following proposition.

Proposition 2 *When $\rho \to 1/N$ the cycle length minimizing the average waiting times has the following asymptotic behaviour*

$$T^2 \sim \sum_{i=1}^{N} \bar{Q}_i. \tag{21}$$

Proposition 2 is proven in Section 6.2.

Obtaining a similar exact result for a complex network setting or for a junction controled by the propotional fair optimization would pose a much harder problem. However, as we show in Section 5 setting the cycle lengths on a scale according to this "square root rule" provides pleasant results even when the actual service time allocation is determined by the proportional fair scheduling scheme. Thus we are ready to formulate our proposed traffic light control policy.

4.4 Policy

Given the described optimization in (14) and the results in Proposition 1 and 2, the proposed algorithm for traffic light setting can be summarized as follows.

- Form an unbiased estimate of the expected queue sizes, $\widetilde{Q}$.[2]
- For a given time period[3] a priori set up a sequence of cycle lengths for each junction according to

$$T_{n+1}^{j} = c_j \sqrt{\sum_{i \in j} \widetilde{Q}_i(t_n^j)}, \tag{22}$$

  where c_j is a control parameter determined by the traffic controller.

[2] For a suggested estimation method see Appendix A
[3] This can be a day or a few hours for example.

– At the beginning of each cycle at all the junctions allocate green times to each service phase according to the solution of (14) based on the queue sizes present.

One of our main results is proving maximum stability of this scheduling algorithm. Namely we are going to prove that the capacity set $\mathcal{C}$ coincides with the stability region of the the ensuing stochastic network, i.e. that for any set of arrival rates $a = (a_i)_{i\in\mathcal{I}} \in \mathcal{C}^\circ$ there is a set of allocations y, under which the stochastic system is positive recurrent. Informally, this means that any demand that does not exceed the physical parameters of the network can be satisfied when the service is allocated by our policy. The stability proof is following the fluid limit approach of Dai (1995). First we determine the fluid limits of the processes described in Section 3.2. To prove positive recurrence it suffices to show stability of these fluid limits. We are going to formalize these steps in Section 4.5 and give a full proof in Section 6.

4.5 Fluid stability

We can associate a fluid model with the network if we introduce the terms

$$q(t) = (q_i(t))_{i\in\mathcal{I}},$$

and

$$x(t) = (x_{ir}(t))_{i\in\mathcal{I},r\in\mathcal{R}}$$

as the fluid limits of $Q(t)$ and $X(t)$ respectively. If we introduce the notation

$$x_{i^r_- r} = \begin{cases} x_{i'r} & \text{if } i' = i^r_-, \\ x_{0r} & \text{if } i = i^r_0, \end{cases}$$

and similarly

$$x_{i^r_+ r} = \begin{cases} x_{i'r} & \text{if } i' = i^r_+, \\ x_{lr} & \text{if } i = i^r_l, \end{cases}$$

and for every route $r \in \mathcal{R}$, we can define auxiliary variables q_0, q_l and μ^*_0, μ^*_l to have

$$x_{0r} = \frac{q_0}{\mu^*_0} \cdot a_r, \tag{23}$$

and

$$x_{lr} = \frac{q_l}{\mu^*_l} \cdot a_r. \tag{24}$$

The fluid limit of the system is then governed by the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} x_{ir}(t) = \frac{x_{i^r_- r}(t)}{q_{i^r_-}(t)} \mu^*_{i^r_-}(q) - \frac{x_{ir}(t)}{q_i(t)} \mu^*_i(q), \quad \text{if } q_i(t), q_{i^r_-}(t) > 0, \tag{25}$$

with

$$q_i(t) = \sum_{r:i\in r} x_{ir}(t). \tag{26}$$

Remark 1 We note that the derivatives in the fluid model equations are not defined when the queue size components $q_i(t)$ are zero. However, whenever it exists, the derivative of $q_i(t)$ at zero must be zero. Suppose that $q_i(t) = 0$. Using the fact that we know that the derivative exists and that $q_i(t+h)$ is positive

$$\frac{dq_i}{dt} = \lim_{h\searrow 0} \frac{q_i(t+h) - 0}{h} \geq 0 \quad \text{and} \quad \frac{dq_i}{dt} = \lim_{h\nearrow 0} \frac{q_i(t+h) - 0}{h} \leq 0.$$

Thus $\frac{dq_i}{dt} = 0$. Further, note that the deviative of $q_i(t)$ exists almost everywhere. (This same argument holds for x_{ir}.) As we see below in Proposition 3, the fluid limit process is Lipschitz continuous. This implies, that the fluid limit is differentiable except for a set of zero lebesgue measure and is the integral of its derivative (Dudley, 2002, Chatper 7.2).

We formalize our result as follows. Let $\{X_{ir}^{(c)}\}_{c\in\mathbb{N}}$ be a sequence of versions of our route-wise queue size processes, where $\|X_{ir}(0)\|_1 = c$. We define

$$\bar{X}_{ir}^{(c)}(t) = \frac{X_{ir}^{(c)}(ct)}{c}. \tag{27}$$

Our next proposition formally states that the only possible limit for $\bar{X}_{ir}^{(c)}$ is given by x_{ir}.

Proposition 3 *The sequence of stochastic processes $\{\bar{X}^{(c)}\}_{c\in\mathbb{N}}$ are tight with respect to the topology of uniform convergence on compact time intervals. Moreover, any weakly convergent subsequence of $\{\bar{X}^{(c)}\}_{c\in\mathbb{N}}$ converges to a Lipschitz continuous process almost everywhere satisfying fluid equations* (25), (26).

A proof of Proposition 3 can be found in Section 6.3.

Now we are ready to formalize the statement considering the stability of the system. The result is given by the following theorem.

Theorem 1 *If the set of arrival rates are such that*

$$a \in \mathcal{C}^\circ, \tag{28}$$

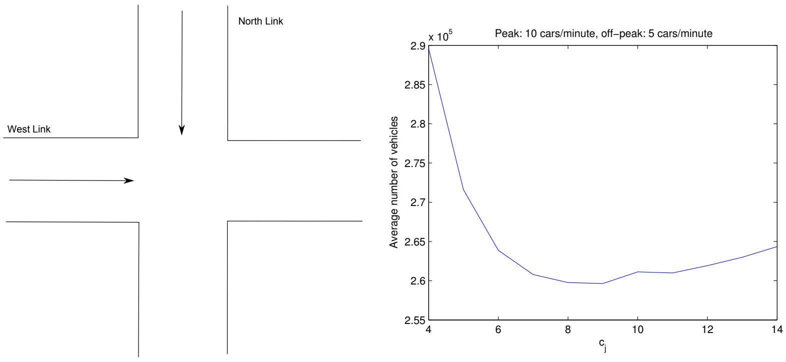
then the fluid limit in (25) *is stable, i.e. there exists a time $\tau > 0$ such that for every fluid model $\{x(t)\}_{t\in\mathbb{R}_+}$ satisfying* (25) *with $\|x(0)\|_1 = 1$,*

$$x_{ir}(t) = 0,$$

for all $t \geq \tau$ for each $i \in r$, and $r \in \mathcal{R}$.

As discussed in Proposition 1, cycle lengths can be chosen for any interior point of the capacity set such that the set of admissible rates will also contain that point. We work with the assumption that such a T is in place, which combined with (28) means that

$$a \in \mathcal{A}^\circ(T). \tag{29}$$



(a) Network topology. (b) Optimal input parameter c_j.

Fig. 1: Single Intersection Network.

This can be ensured by the traffic controller by choosing the control parameters c_j in (22) correctly, as an optimal choice provides a set of admissible rates which is large enough, whilst maintainging shorter cycles to reduce waiting times. Thus satisfying (29) also shows the correct traffic setting to follow.

5 Numerical results

In this section we investigate the optimal value of c_j, which was introduced in (22) and evaluate the performance of our proposed policy by comparing it with fixed cycle proportional policies.

5.1 Parameter validation

Based on our results, Proposition 2, a good estimation for the optimal value of c_j is given by

$$c_j = N \cdot \sqrt[2]{T_{\text{switch}}/\mu^{\max}}, \tag{30}$$

where N denotes the number of competing in-roads at junction j, T_{switch} denotes the fixed switching time and $\mu^{\max}$ denotes the maximal service rate of the in-roads.

The control parameter c_j plays a crucial role in determining the cycle lengths. A suboptimal value of c_j may shrink the network's capacity region as discussed in Section 4.2, which can result increasing congestion. In this section, we validate via simulation the optimal value of c_j as given in (30).

We have considered a single intersection network topology as shown in Figure 1a. Both the west and the east link were single lanes with length of 2000 meters. The long link length ensured that the arriving vehicles were always able to enter the network, even in a congested period. The service rate when receiving green traffic light was capped at 20 cars per minute. We run the simulation with different values of c_j. For the demand, we considered the vehicles entering the network from the west and the north link, passing through the intersection before exiting the network. The arrivals followed a Poisson process with symmetric arrival rates. In each run we have simulated 10 hours of traffic including 10 peak periods and 10 off-peak periods alternately. The length for these periods was set as 30 minutes. The peak periods were considered to have 10 cars per minute arriving on average, whereas the arrival rate in off-peak period was 5 cars per minute.

The results are shown in Figure 1b. Note that the network parameters were $N = 2$, $T_{\text{switch}} = 6$, $\mu_{max} = 20$, hence according to (30) the estimated optimal value of c_j was $c_j = 8.5$. The figure plots the average number of vehicles present in the network throughout the whole run against different values of c_j. Since vehicles cannot disappear from the network through any other mean, but exiting, a lower average number of vehicles can only occur due to shorter travel times. It can be observed that the optimal value for c_j is between 8 and 9 which is consistent with our estimate. For smaller values of c_j, the traffic flows are more frequently disrupted by switching, whereas at greater values of c_j, vehicles must wait longer for green traffic light. One can see form the results in 1b, that this indeed produces longer queues.

5.2 Performance study for varied cycle lengths

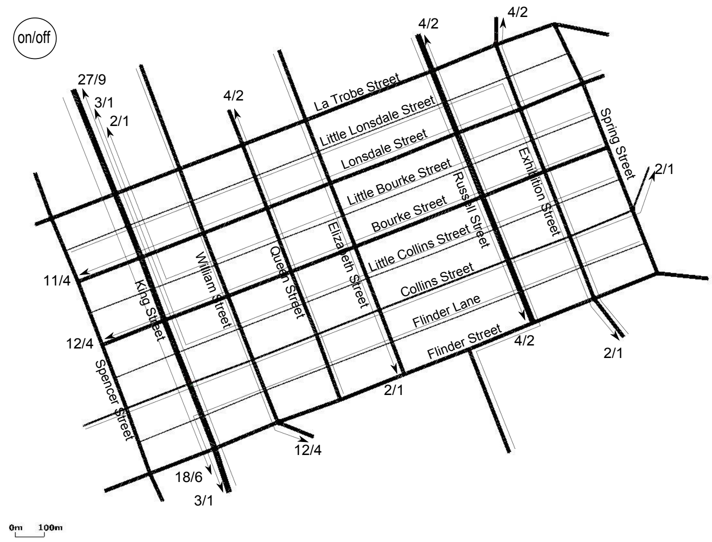


Fig. 2: Large CBD network with demands.

This section studies the performance of our proposed proportional fair policy by comparing it with the fixed cycle proportional policy in Le et al (2013a)

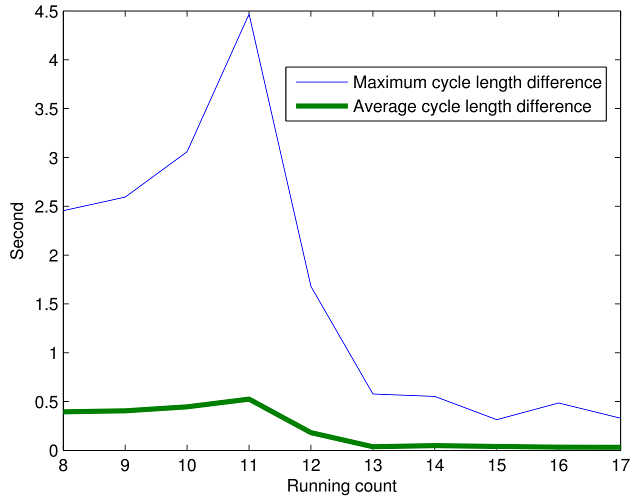


Fig. 3: Convergence.

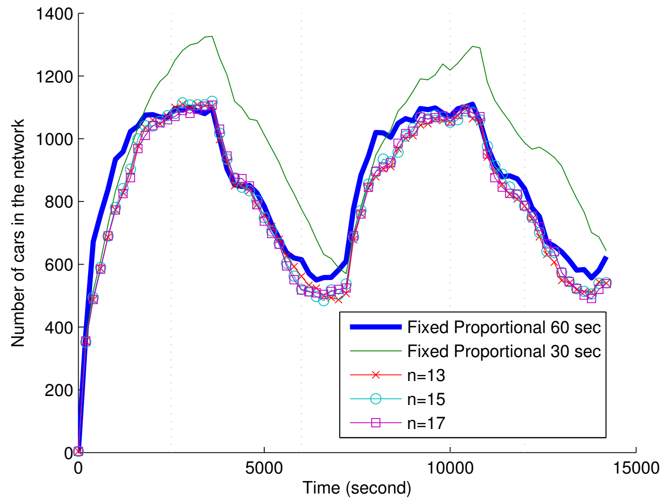


Fig. 4: Total number vehicles in the network.

via simulation in the SUMO environment. We have considered the Melbourne CBD network which is shown in Figure 2. It consists of 73 intersections and 266 links. Most of the roads are bi-directional except for Little Lonsdale Street, Little Bourke Street, Little Collins Street and Flinder Lane which only have a single lane mono-directional traffic. King Street and Russell Street are the

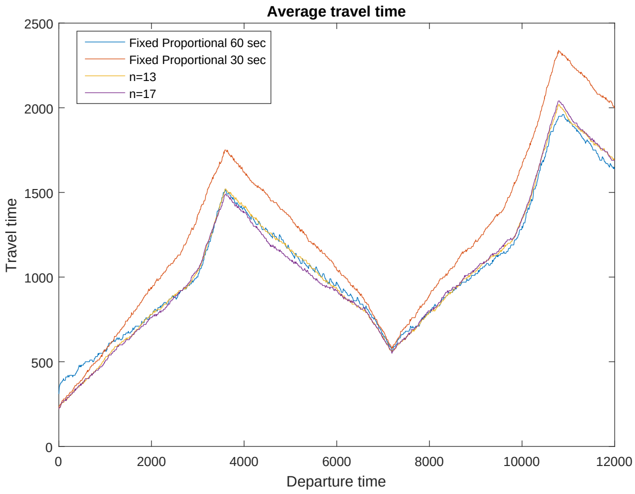


Fig. 5: Average travel times for vehicles in the network.

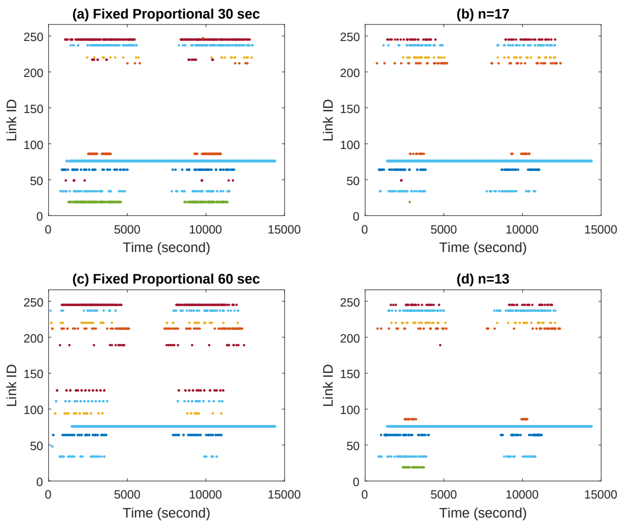


Fig. 6: Congestion measure for different links in the network.

biggest roads in this scenario, each is modeled as 3 lanes each direction. Collins Street has one lane each direction. All other roads have two lanes each direction. The link lengths are varied between 106 meters for the vertical links and 214 meters and 447 meters for the horizontal links except for the ingress links at the edges. Each simulation run consisted of 4 hours including two peak periods and two off-peak periods alternately, each lasting 1 hour. The fix routes are indicated by the arrows in Figure 2 while the following numbers show the arrival rate in peak/off-peak periods.

We have run multiple simulations sequentially and the cycle lengths were pre-calculated before each run based on the ensemble average of queue lengths of previous runs and according to (22). Given the identical demands, the differences in queue evolution widely depended on the cycle lengths mentioned above and, in turn, the cycle lengths would adapt to the new queue evolution records. Naturally, we wanted to evaluate the proposed policy in convergence, that is when the updates of queue lengths no longer changed the cycle lengths. Particularly, we first ran the simulation 5 times with fixed 30 second cycle lengths to obtain the initial queue length records. Then from the 6th time on, the queue lengths were calculated according to (22) based on the ensemble average queue lengths of the 5 previous runs.

We estimated the convergence of our procedure for determining cycle lengths, see Figure 3. We evaluated the difference in cycle lengths at 4 fixed time points: 2500, 6000, 10000 and 12000 seconds respectively (see the vertical lines in Figure 4). At each time of these times we recorded the cycle lengths of each intersection. We then compared the cycle lengths of the current run with the average cycle lengths of the last 3 runs at each intersection and at each time point. The maximum difference and average difference are shown in Figure 3. Intuitively, the smaller values of those two quantities indicate better convergence. The results show strong convergence after the 13th run. For further study we have chosen the estimated cycle lengths we ended with after $n = 13, 17$ runs.

We compared the performance of our proposed policy with other two policies, namely fixed cycle proportional fair policies of cycle lengths with 30 and 60 seconds. Note that 60 seconds was the optimal cycle length for the fixed cycle proportional policy as shown by simulation in Le et al (2013a). The results are presented in Figures 4, 5 and 6 in the form of total number of vehicles in the network, average travel times and congestion measure respectively. The latter represents the amount of time for every link that is spent in congestion, which is defined by the density of the link reaching a threshold. In our case this threshold was at 85% of the density representing a fully congested road, where all the space is occupied by vehicles, thus the traffic cannot move.

As shown in Figure 4, the proposed policy in convergence has significantly less congestion than the fixed cycle proportional fair policy with 30 second cycles and is slightly better than the fixed cycle proportional fair policy with 60 second cycles. This is even more prevalent in Figure 6, where we can see that the number of congested links and the time spent in congestion are lower. In terms of travel times, our proposed policy outperforms the fixed cycle proportional fair policy with 30 second cycles and is similar to the fixed cycle proportional fair policy with 60 second cycles as seen in Figure 5. Thus we can say that the simulation study agrees with our theoretical results, since we aimed for minimizing the queue lengths in the network.

Additionally, we have also repeated the whole simulation with a different a starting point, e.g. the fixed cycle proportional policy with 60 second cycles, and observed similar results in both convergence and performance, which indicates the robustness of our policy. Practically, the network is able to achieve its optimal cycle lengths by updating its cycle lengths based on historical queue length data regardless of the network topology.

6 Proofs

In this section we formally prove our main theoretical results. Firstly, we prove Proposition 1, which claims that the set of admissible service rates reaches the capacity set in the limit as $T \to \infty$. Secondly, we prove Proposition 2 which justifies the square root rule given in (22). Then we prove convergence in the fluid limit as stated in Proposition 3. Finally, we end the section by providing a proof of stability according to Theorem 1.

6.1 Proof of Proposition 1

To prove Proposition 1, we have to look into the behavior of s. By (9) it holds for any i, that

$$\theta_i(t) = \frac{\int_0^t (\mu_i^{\max} - s_i'(z))\mathrm{d}z}{t} \to 0 \quad \text{as } t \to \infty, \tag{31}$$

i.e. for any $\delta > 0$ there exists τ_i such that $\theta_i(t) < \delta$ if $t > \tau_i$. Now if we take the definition (12), we can derive the following bound

$$\begin{aligned}\mu_i(T) =& \frac{s_i(y_i E^{j(i)})}{T^{j(i)}} \\ =& \frac{\int_0^{y_i T^{j(i)}} \mu_i^{\max}\mathrm{d}z - \int_0^{y_i T^{j(i)}} (\mu_i^{\max} - s_i'(z))\mathrm{d}z - \int_{y_i E^{j(i)}}^{y_i T^{j(i)}} s_i'(z)\mathrm{d}z}{T^{j(i)}} \\ \geq& y_i\mu_i^{\max} - \theta_i(T^{j(i)}) - y_i\mu_i^{\max}|\Sigma_j|\frac{T_{\text{switch}}}{T^{j(i)}}.\end{aligned} \tag{32}$$

Now if we choose any $a \in \mathcal{C}^\circ$, then by definition of $\mathcal{C}$ there exists $\epsilon > 0$ such that

$$y_i\mu_i^{\max} - a_i > \epsilon \quad \forall i \in \mathcal{I}$$

for some choice of the allocation vector $y = (y_i)_{i\in\mathcal{I}}$. If we combine (31) with (32) and the fact that T_{switch} is constant, then we see that, for this allocation we can choose τ_i such that

$$\mu_i(\tau_i) > y_i\mu_i^{\max} - \epsilon > a_i.$$

By setting

$$\tau = \left(\tau_j := \max_{i\in j} \tau_i\right)_{j\in\mathcal{J}},$$

then for any $T > \tau$ we have

$$\mu_i(T) - a_i > 0 \quad \forall i \in \mathcal{I},$$

which means that $a \in \mathcal{A}^\circ(T)$, and thus proves Proposition 1. □

6.2 Proof of Proposition 2

Since we are investigating the heavy traffic behaviour of the system, we have $\rho \to 1/N$. Using the symmetry, i.e., that $\tau/N = T_{\text{switch}} + 1/\nu$ by (18), we can rewrite (19) as

$$\bar{Q}_i(\tau) = \frac{\rho\nu\tau + \lambda \frac{{T_{\text{switch}}}^2 + (N-1)(\tau/N)^2}{\tau}}{1 - \rho\nu\tau} = \frac{\rho\nu\tau + \frac{\lambda T_{\text{switch}}^2}{\tau} + \frac{(N-1)\lambda\tau}{N^2}}{1 - N\rho - N\rho\nu T_{\text{switch}}}. \tag{33}$$

If we assume that $\tau \to \infty$, which should be the case by (51), then $\tau \gg T_{\text{switch}}$, and $1/\nu \sim \tau/N$ in the limit. Using this and (33), we have that

$$\bar{Q}_i \sim \frac{N\rho + \frac{\lambda T_{\text{switch}}^2}{\tau} + \frac{(N-1)\lambda}{N\nu}}{1 - N\rho - N\rho\nu T_{\text{switch}}} \sim \frac{(N-1)\lambda}{N} \cdot \frac{1}{(1 - N\rho)\nu - N\rho T_{\text{switch}}\nu^2},$$

which is an expression, for which a minimum in ν can be determined easily, by taking the derivative. This gives

$$\nu^* = \frac{1 - N\rho}{2N\rho T_{\text{switch}}},$$

which by (18) and some algebraic calculations gives

$$\tau^* = \frac{2N^2\rho T_{\text{switch}}}{1 - N\rho} + NT_{\text{switch}} = \frac{N(1 + N\rho)}{1 - N\rho} \cdot T_{\text{switch}}, \tag{34}$$

which clearly goes to infinity as $\rho \to 1/N$. Plugging this into (33) we get

$$\bar{Q_i}^* = \frac{2\lambda T_{\text{switch}}}{N(1 + N\rho)} + \frac{1}{NT_{\text{switch}}} \cdot \tau^* + \frac{2\lambda}{(1 + N\rho)T_{\text{switch}}} \cdot \frac{N-1}{N^3} \cdot \tau^{*2}. \tag{35}$$

Since $\tau^* \to \infty$ on the right side of (35) the dominant term is the last one. Thus by summing over the in-roads we get (21). Furthermore if we consider the coefficient of τ^{*2} and the facts, that $\lambda = \rho\mu \to \mu/N$ and $\sum_{i=1}^N \bar{Q}_i = N\bar{Q}_i$ we justify the formula in (30). □

6.3 Proof of Proposition 3

From Robert (2003) we see that in order to prove the tightness of a sequence $\{Z^{(c)}\}_c$, we must prove for each $\epsilon > 0$ and $t > 0$ that

$$\lim_{\delta \to 0} \mathbb{P}\left(\sup_{\substack{u,v:u,v<t,\\ |u-v|<\delta}} \|Z^{(c)}(u) - Z^{(c)}(v)\|_1 \geq \epsilon \right) = 0. \tag{36}$$

The evolution of $\bar{X}^{(c)}_{ir}(t)$ is given by (7). Since we have only defined $D_{ir}(t)$ as a counting process, we have to extend this definition to continuous time to have a clean definition in (27). Thus, we set

$$D_{ir}(t) = \sum_{n=1}^{N_{j(i)}(t)} S^n_{ir} + \mu_{ir}(t) \cdot \left(t - t^{j(i)}_{N_{j(i)}(t)}\right) + \Delta_{ir}(t - t^{j(i)}_{N_{j(i)}(t)}), \tag{37}$$

where $\Delta_{ir}(t)$ represents the fluctuation in service during the cycles, which comes from the ordering of the phases and stochastic effects. It is clearly bounded as

$$|\Delta_{ir}(t)| \leq \mu^{\max}_i t. \tag{38}$$

We also point out that, as long as the cycle lengths are preset which is the case in our policy, by the definition in (1) $N_{j(i)}(t)$ is deterministic. Even more so $N_j(t)$ is bounded for all $j \in \mathcal{J}$ since

$$T^j_n \geq |\Sigma_j| \cdot T_{\text{switch}} \;\; \forall n, \; \forall j \in \mathcal{J},$$

and thus

$$N_j(t) \leq \left\lceil \frac{t}{|\Sigma_j| \cdot T_{\text{switch}}} \right\rceil. \tag{39}$$

Now we are ready to demonstrate that (36) holds for $\bar{X}^{(c)}_{ir}(t)$ in the case when $i = i^r_0$. The result then follows for all other cases since the arrivals there are internal, since they equal the previous in-roads departures, for which we show tightness here. Thus, by simple summation and the triangle inequality we could repeat the same arguments as we make here for the departures at source links. By (7), (27) and (37) we have

$$\begin{aligned}
&\bar{X}^{(c)}_{ir}(t) = \bar{X}^{(c)}_{ir}(0) + \frac{1}{c}\left[A_r(ct) - D_{ir}(ct)\right] \\
=&\bar{X}^{(c)}_{ir}(0) + \frac{1}{c}\left[A_r(ct) - \sum_{n=1}^{N_{j(i)}(ct)} S^n_{ir} - \mu_{ir}(ct)\left(ct - t^{j(i)}_{N_{j(i)}(ct)}\right) - \Delta_{ir}\left(ct - t^{j(i)}_{N_{j(i)}(ct)}\right)\right] \\
=&\bar{X}^{(c)}_{ir}(0) + a_r t + \bar{L}^{(c)}_r(t) - \bar{M}^{(c)}_{ir}(t) - \frac{1}{c}\int_0^{ct} \mu_{ir}(s)\mathrm{d}s - \frac{1}{c}\Delta_{ir}\left(ct - t^{j(i)}_{N_{j(i)}(ct)}\right),
\end{aligned} \tag{40}$$

where we define the following terms,

$$\bar{L}_r^{(c)}(t) = \frac{1}{c}(A_r(ct) - a_r ct),$$

$$\bar{M}_{ir}^{(c)}(t) = \frac{1}{c}\sum_{n=1}^{N_{j(i)}(ct)} (S_{ir}^n - T_n^{j(i)}\mu_{ir}^n). \tag{41}$$

By the triangle inequality it suffices to prove (36) holds for each term of the sum in (40) to prove that $\bar{X}_{ir}^{(c)}$ is tight. For $\bar{X}_{ir}^{(c)}(0) + a_r t$ (36) trivially holds.

For the integral we can use the fact that $\mu_{ir}(t)$ is bounded by $\mu_i^{\max}$ for all t, thus we have the Lipschitz-condition

$$\left|\frac{1}{c}\int_0^{cu}\mu_{ir}(s)\mathrm{d}s - \frac{1}{c}\int_0^{cv}\mu_{ir}(s)\mathrm{d}s\right| < \mu_i^{\max}|u-v|.$$

Thus with $\delta < \epsilon/\mu_i^{\max}$ (36) is satisfied.

For Δ_{ir} we use the fact that it is bounded as described in (38), and that we have pre-fixed cycle lengths. Thus

$$\left|\frac{1}{c}\Delta_{ir}(cu - t^{j(i)}_{N_{j(i)}(cu)}) - \frac{1}{c}\Delta_{ir}(cv - t^{j(i)}_{N_{j(i)}(cv)})\right| \leq \frac{2\mu_i^{\max}}{c}\sup_{n\leq N_{j(i)}} T_n^{j(i)} \to 0$$

as $c \to \infty$ from (27), since $\sup_{n\leq N_{j(i)}} T_n^j$ cannot be infinite by definition.

The process $\bar{M}_{ir}^{(c)}(t)$ can also be defined as

$$\bar{M}_{ir}^{(c)}(t) = \frac{1}{c}\sum_{n=1}^{\infty}\mathbb{I}_{\{n\leq N_{j(i)}(ct)\}}(S_{ir}^n - T_n^{j(i)}\mu_{ir}^n),$$

which definition is equivalent to (41) and helps our understanding of $\bar{M}_{ir}^{(c)}(t)$ as a martingale. We can look at S_{ir}^n as a series of random variables in $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$, where $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is a filtration running over the indices of finished cycles n, which is generated by the random variables S_{ir}^n. Then $T_n^{j(i)}\mu_{ir}^n$ are their expected values respectively by (11). Since $N_{j(i)}(ct)$ is deterministic and bounded by (39), by using Doob's Optimal Stopping Theorem $\bar{M}_{ir}^{(c)}(t)$ is a martingale on $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. For $m > n$,

$$\begin{aligned} &\mathbb{E}\left[(S_{ir}^n - T_n^j\mu_{ir}^n)(S_{ir}^n - T_m^j\mu_{ir}^m)\right] \\ =&\mathbb{E}\left[\mathbb{E}[(S_{ir}^n - T_n^j\mu_{ir}^n)(S_{ir}^n - T_m^j\mu_{ir}^m)|\mathcal{F}_n]\right] \\ =&\mathbb{E}\left[(S_{ir}^n - T_n^j\mu_{ir}^n)\mathbb{E}[(S_{ir}^n - T_m^j\mu_{ir}^m)|\mathcal{F}_n]\right] = 0. \end{aligned} \tag{42}$$

Thus we can apply Doob's L_2 inequality and (13) to get

$$\begin{aligned}
&\mathbb{P}\left(\sup_{\substack{u,v:u,v<t,\\|u-v|<\delta}} |\bar{M}_{ir}^{(c)}(u) - \bar{M}_{ir}^{(c)}(v)| \geq \epsilon\right)\\
\leq &\mathbb{P}\left(\sup_{u:u<t} |\bar{M}_{ir}^{(c)}(u)| \geq \frac{\epsilon}{2}\right)\\
\leq &\frac{4}{\epsilon^2}\mathbb{E}\left[\left(\bar{M}_{ir}^{(c)}(t)\right)^2\right] = \frac{4}{c^2\epsilon^2}\sum_{n=0}^{N(ct)} \mathbb{E}\left[\left(S_{ir}^n - T_n^j \mu_{ir}^n\right)^2\right]\\
\leq &\frac{4}{c^2\epsilon^2}\sum_{n=0}^{N(ct)} \kappa T_n^j \leq \frac{4\kappa t}{c\epsilon^2} \to 0 \text{ as } c \to \infty,
\end{aligned} \tag{43}$$

as the crossterms cancel out due to (42).

Since the external arrivals follow a Poisson-process, the process $\bar{L}_r^{(c)}(t)$ is also a martingale, so once again we can use Doob's L_2 inequality,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{\substack{u,v:u,v<t,\\|u-v|<\delta}} |\bar{L}_{r}^{(c)}(u) - \bar{L}_{r}^{(c)}(v)| \geq \epsilon\right)\\
\leq &\mathbb{P}\left(\sup_{u:u<t} |\bar{L}_{r}^{(c)}(u)| \geq \frac{\epsilon}{2}\right)\\
\leq &\frac{4}{\epsilon^2}\mathbb{E}\left[\left(\bar{L}_{r}^{(c)}(t)\right)^2\right] = \frac{4ta_r}{c\epsilon^2} \to 0 \text{ as } c \to \infty.
\end{aligned} \tag{44}$$

We have now established the tightness of each of the respective terms in (40). Thus our sequence of processes $\{X_{ir}^{(c)}\}_{c\in\mathbb{N}}$ are tight.

Therefore each subsequence of $\{X_{ir}^c(ct)/c\}_c$ has a weakly convergent subsequence: $X_{ir}^c(ct)/c \to \tilde{x}_{ir}(t)$, where $\tilde{x}_{ir}$ is some deterministic process. By the Skorohod Representation Theorem we may place these random variables on the same probability space and assume that convergence holds almost surely. Since $Q_i^c(ct)/c = \sum_{r:r\in i} X_{ir}^c(ct)/c$, we can define $\tilde{q}_i(t) = \sum_{r:r\in i} \tilde{x}_{ir}(t)$ to have

$$\frac{X_{ir}^c(ct)/c}{Q_i^c(ct)/c} \to \frac{\tilde{x}_{ir}(t)}{\tilde{q}_i(t)}. \tag{45}$$

Now we are left to prove that the limit is indeed given by the differentiability condition (25). Note that we have also proved that $\bar{L}_r^{(c)}$ and $\bar{M}_{ir}^{(c)}$ converge in distribution to zero for all $r \in \mathcal{R}$ and $i \in \mathcal{I}$ in (44) and (43) respectively. The same holds for Δ_{ir} as

$$\left|\frac{1}{c}\Delta_{ir}(ct - t^{j(i)}_{N_{j(i)}(ct)})\right| \leq \frac{\mu_i^{\max}}{c} \sup_{n\leq N_{j(i)}} T_n^{j(i)} \to 0. \tag{46}$$

By the piecewise constant property of μ_{ir}, we also have

$$\frac{1}{c}\int_0^{ct} \mu_{ir}(s)\mathrm{d}s = \int_0^t \mu_{ir}(cs)\mathrm{d}s.$$

We are going to describe the limit in the $i = i_0^r$ case. In the case where $i \neq i_0^r$ the same holds for both the arrivals and the departures as what is described below for the departures. By the definition (27) the partitioning (40), and the convergences given by (43), (44) and (46) we have

$$\begin{aligned}\lim_{c\to\infty} \bar{X}_{ir}^{(c)}(t) &= \tilde{x}_{ir}(0) + a_r t - \lim_{c\to\infty}\int_0^t \mu_{ir}(cs)\mathrm{d}s \\ &= \tilde{x}_{ir}(0) + a_r t - \int_0^t \lim_{c\to\infty}\mu_{ir}(cs)\mathrm{d}s.\end{aligned} \tag{47}$$

In the latter equation we have used the Skorohod representation theorem. To arrive at the final conclusion we require that

$$\lim_{c\to\infty}\mu_{ir}(Q^c(cs)) \to \frac{\tilde{x}_{ir}(s)}{\tilde{q}_i(s)}\mu_i^*(\tilde{q}(s)). \tag{48}$$

This is demonstrated in Lemma 4, which is contained in Appendix.

We apply (48) to the integral in (47) to deduce the differentiability properties of $\tilde{x}$. In particular, for any t such that $\tilde{x}_{ir}(t) > 0$ and for h sufficiently small that $\tilde{x}_{ir}(s) > 0$ for all $t \leq s \leq t+h$, we have that

$$\begin{aligned}\frac{\tilde{x}_{ir}(t+h) - \tilde{x}_{ir}(t)}{h} &= a_r + \frac{1}{h}\int_t^{t+h} \lim_{c\to\infty}\mu_{ir}(cs)\mathrm{d}s \\ &= a_r + \frac{1}{h}\int_t^{t+h} \lim_{c\to\infty}\mu_{ir}(Q^c(cs))\mathrm{d}s \\ &= a_r + \frac{1}{h}\int_t^{t+h} \frac{\tilde{x}_{ir}(s)}{\tilde{q}_i(s)}\mu_i^*(\tilde{q}(s))ds,\end{aligned}$$

which gives the differentiability condition (25) as we take the limit $h \to 0$, thus $\tilde{x}_{ir} = x_{ir}$ and $\tilde{q}_i = q_i$ for all indices. □

6.4 Proof of stability

Consider the function

$$H(x) = \sum_{r\in\mathcal{R}}\sum_{i\in r} x_{ir}\log\left(\frac{x_{ir}\mu_i^*(q)}{q_i a_r}\right),$$

where we point out that $\mu_i^*(q)$ is the solution to the optimization problem (14) for junction $j \ni i$, and thus its value depends on the other queue sizes present at j. However, due to the properties of proportionally fair optimization the partial derivatives do not show this dependence, as stated in the following lemma.

Lemma 1

$$\frac{\partial H}{\partial x_{ir}} = \log\left(\frac{x_{ir}\mu_i^*(q)}{q_i a_r}\right).$$

Our next lemma puts a bound on $H(x)$.

Lemma 2 *The function $H(x)$ is positive, bounded when $\|x\|_1 = 1$ and minimized when $x = 0$.*

Furthermore we will use a technical lemma which goes as follows.

Lemma 3 *If u and v are two positive vectors with components indexed by $\mathcal{M}$, such that*

$$\sum_{m\in\mathcal{M}} u_m = \sum_{m\in\mathcal{M}} v_m,$$

then

$$\sum_{m\in\mathcal{M}} u_m \log\left(\frac{u_m}{v_m}\right) \geq \frac{1}{\sum_{m\in\mathcal{M}} u_m} \cdot \sum_{m\in\mathcal{M}} (u_m - v_m)^2.$$

Lemma 1, 2 and 3 are given proofs in the appendix. Now we can prove that $H(x)$ is a Lyapunov function for the fluid system (25). First, we show that the following equalities hold

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \sum_{r\in\mathcal{R}}\sum_{i\in r}\left(\frac{x_{i_-^r r}(t)}{q_{i_-^r}(t)}\mu_{i_-^r}^* - \frac{x_{ir}(t)}{q_i(t)}\mu_i^*\right)\log\left(\frac{x_{ir}\mu_i^*}{q_i a_r}\right)\\ &= -\sum_{r\in\mathcal{R}} a_r \sum_{i\in r}\frac{x_{ir}\mu_i^*}{q_i a_r}\log\left(\frac{\left[\frac{x_{ir}\mu_i^*}{q_i a_r}\right]}{\left[\frac{x_{i_+^r r}\mu_{i_+^r}^*}{q_{i_+^r} a_r}\right]}\right).\end{aligned}$$

In the first equation we included $i = i_0^r$ too, which does not change the sum since by (23)

$$\log\left(\frac{x_{0r}\mu_0^*}{q_0 a_r}\right) = \log 1 = 0.$$

By incrementing the first terms of the summation such that the coefficients would become equal and multiplying and dividing by a_r we gained the final expression. Once again including $i = i_l^r$ does not change the sum, since by (24)

$$\log\left(\frac{x_{lr}\mu_l^*}{q_l a_r}\right) = \log 1 = 0.$$

Now we can bound the derivative of the proposed Lyapunov function as

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &\leq -\sum_{r\in\mathcal{R}}\left(a_r\left[\sum_{i\in r}\frac{x_{ir}\mu_i^*}{q_i a_r}\right]^{-1}\cdot\sum_{i\in r}\left[\frac{x_{ir}\mu_i^*}{q_i a_r} - \frac{x_{i_+^r r}\mu_{i_+^r}^*}{q_{i_+^r} a_r}\right]^2\right)\\ &\leq -\sum_{r\in\mathcal{R}}\left(\frac{a_r^2}{\tilde{\mu}}\sum_{i\in r}\left[\frac{x_{ir}\mu_i^*}{q_i a_r} - \frac{x_{i_+^r r}\mu_{i_+^r}^*}{q_{i_+^r} a_r}\right]^2\right) \leq -\epsilon,\end{aligned}$$

for some $\epsilon > 0$. In the first inequality we applied Lemma 3. In the second inequality we applied that

$$\sum_{i \in r} \frac{x_{ir}\mu_i^*}{q_i a_r} \le \frac{\tilde{\mu}}{a_r},$$

where we introduced

$$\tilde{\mu} = |\mathcal{I}| \max_{i \in \mathcal{I}} \mu_i^{\max}.$$

To prove the third inequality we first look into the conditions under which the sum could equal zero. To have that for each in-road $i \in r$ on each route $r \in \mathcal{R}$ we would need

$$\frac{x_{ir}\mu_i^*}{q_i a_r} - \frac{x_{i_+^r r}\mu_{i_+^r}^*}{q_{i_+^r} a_r} = 0.$$

This would mean that the terms are constant along each in-road i along a route, including the auxiliary i_0, for which the fraction was set as 1. Thus, this requires that

$$a_r = \frac{x_{ir}\mu_i^*}{q_i} \quad \forall i \in r,\ \forall r \in \mathcal{R}.$$

By summing over for each in-road, we get

$$\sum_{i \in \mathcal{I}} a_i = \sum_{r \in \mathcal{R}} \sum_{i : i \in r} a_i = \sum_{r \in \mathcal{R}} a_r = \sum_{i \in \mathcal{I}} \mu_i^*.$$

This contradicts (29), since μ_i^* clearly belongs to the boundary of $\mathcal{A}(T)$. Thus (6.4) cannot hold and $H(x)$ has a strictly negative drift. Since by Lemma 2 $H(x) = 0$ only if $x = 0$, for all $x(0)$ such that $H(x(0)) \le h$ for some positive constant h, we have for all $t \ge h/\epsilon$ that

$$q(t) = 0.$$

By definition this means that the fluid system is stable. □

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A On the estimation of queue lengths

In this section we shortly discuss the estimation of the cycle lengths $\widetilde{Q}$ introduced in Section 4.4. For our proposed policy to be stable the only condition needed on these estimators is that they have to be unbiased, i.e. $\mathbb{E}Q = \mathbb{E}\widetilde{Q}$. However, for practical purposes we suggest to use a "moving average", which is given as follows

$$\widetilde{Q}(U) = \sum_{u=1}^{Z} \alpha_u Q(U-u),$$

where $Q(U)$ denotes the measured queue size at sampling time U, Z denotes the number of samples considered in the estimation process and α_u denotes the weights put on each sample, with $\sum_{u=1}^{Z} \alpha_u = 1$. An example could be the following. To provide the estimation for a certain point of the day consider the samples from the previous days at the same time making a distinction between workdays and holidays and set α to be linearly decreasing, which would mean that the most recent measurements would have the highest weights. We also point out that the cycle length settings described in Section 5 follow a similar method when using the results of previous runs to make the queue length estimations.

B Example showing the dependencies of the set of admissible service rates

Take for all in-roads the following service function (with time units given in minutes),

$$s_i(t) = \begin{cases} 10t, & \text{if } t \leq 0.1 \\ 1 + 30(t-0.1) & \text{if } t > 0.1, \end{cases}$$

which includes the mentioned setup phase. If we take the switching times $T_{\text{switch}} = 0.1$ min, then at a single junction with two competing in-roads, that both have 11 vehicles/min arrivals on average, the queues will build up if $T_j = 1.2$ min. The same junction can however be served, if $T_j = 2.4$ min. Even more interesting is the fact that in the $T_j = 1.2$ min case, if one in-road has 22 vehicles/min arrivals on average, while there is no traffic arriving to the other in-road, the junction can be served again. Thus the arrival vector $(11, 11)$ is outside of the set $\mathcal{A}(T)$, even though both $(22, 0)$ and $(0, 22)$ are in and $(11, 11)$ is their convex combination.

C Example of a polling model with deterministic rates

Consider a junction with N competing in-roads, each having identical, deterministic arrival rates λ and identical constant service rates μ when receiving service. Scheduling green times G_i proportionally to the average queue lengths results in allocating $1/N$ of the effective time to each phase. Thus each in-road will be idle for

$$T - G_i = \frac{N-1}{N} \cdot T + T_{\text{switch}}.$$

During the idle period a queue will build up, which will leave with all new arrivals during the green period. Thus at the end of the idle period, the queue size present will be

$$Q^{\text{idle}} = \lambda \left(\frac{N-1}{N} \cdot T + T_{\text{switch}} \right),$$

which can empty out in

$$T^{\text{empty}} = \frac{\lambda}{\mu - \lambda} \cdot \left(\frac{N-1}{N} \cdot T + T_{\text{switch}} \right).$$

If the green time allocated is shorter than T^{empty}, then the queues never empty out, instead they grow infinitely large and consequently the waiting times become infinitely large too. If $G_i \geq T^{\text{empty}}$, then the queues will receive green light even after they emptied out which will result in all subsequent arriving vehicles leaving immediately with the queue size starting to grow from 0 again during the idle period. The average queue size in this case can be calculated by

$$\begin{aligned}\bar{Q} &= \frac{1}{T} \int_0^T Q(\tau) \mathrm{d}\tau = \frac{1}{T} \left[\frac{1}{2} \cdot \frac{\lambda\mu}{\mu - \lambda} \cdot \left(\frac{N-1}{N} \cdot T + T_{\text{switch}} \right)^2 \right] \\ &= \frac{T}{2} \cdot \frac{\lambda\mu}{\mu - \lambda} \left[\frac{T_{\text{switch}}}{T} + \frac{N-1}{N} \right]^2.\end{aligned}$$

Thus by Little's law the average waiting time is

$$\bar{W}(T) = \frac{\bar{Q}}{\lambda} = \frac{T}{2} \cdot \frac{\mu}{\mu - \lambda} \left[\frac{T_{\text{switch}}}{T} + \frac{N-1}{N} \right]^2, \tag{49}$$

if the system is stable. In order to have this quantity, we need $G_i \geq T^{\text{empty}}$, which happens if

$$T \geq \frac{\mu}{\mu - N\lambda} \cdot 2NT_{\text{switch}}, \tag{50}$$

from where we can also see that $\mu > N\lambda$ is needed for stability. Since the function in (49) has its minimum at $T = \frac{N}{N-1} T_{\text{switch}}$ and is strictly increasing for bigger values of T, the vehicles have minimal average waiting time if there is an equality in (50). Thus in a deterministic system the shortest cycle length which ensures stability is the optimal choice.

D Determining the optimal cycle lengths for the symmetric case of the polling model in Section 4.3

Let us consider the polling model of Section 4.3. Take the $N = 2$ case in symmetry, i.e. assume that the two phases behave in the same way, $\lambda_1 = \lambda_2 = \lambda$, $\mu_1 = \mu_2 = \mu$, . This allows us to give a system of equations with (17), (19) and (18), from which ν can be eliminated. Assuming that τ takes the form of

$$\tau = \frac{2(T_{\text{switch}} + \epsilon)}{1 - 2\rho}, \tag{51}$$

thus assuring (20) with the perturbation ϵ being positive, we can express the expected queue lengths as a function of ϵ as

$$\begin{aligned}\mathbb{E}Q(\epsilon) =& \frac{1}{2\mu(1-2\rho)^2\epsilon(T_{\text{switch}}+\epsilon)}\\ &\times\big[4T_{\text{switch}}^2 + 4\epsilon^2 - 8\mu\rho^2T_{\text{switch}}^3 + 6\mu\rho T_{\text{switch}}^3 + 8T_{\text{switch}}^3\mu\rho^3 - 8\rho T_{\text{switch}}^2\\ &-\mu T_{\text{switch}}^3 + \mu\epsilon^3 - 2\mu\rho\epsilon T_{\text{switch}}^2 - 2\mu\rho\epsilon^2 T_{\text{switch}} + 4\mu\rho^2\epsilon T_{\text{switch}}^2\\ &-16\rho\epsilon T_{\text{switch}} + 4\mu\epsilon^2 T_{\text{switch}} + 3\mu\epsilon T_{\text{switch}}^2 + 8\epsilon T_{\text{switch}} - 8\rho\epsilon^2\big].\end{aligned}$$

Finding the minimum of this amongst positive values for ϵ leads to solving a fourth order polynomial, which is a doable task. We will save the reader from the lengthy symbolic expression. This tells us, that ϵ has a clear positive value, and is not just a small perturbation contrary to what could be expected from the result of Appendix C. Thus in a stochastic system one should do more than find the minimal stable value in order to find optimality. Therefore the approximations used by our policy, which are based on the results of Section 4.3 are well founded.

E Proofs for lemmas used in the stability proof

In the proof of Proposition 2, we required the following lemma which we now prove.

Lemma 4 *In the case of $\tilde{x}_{ir} > 0$ the following holds for the fluid limit of $\mu_{ir}(Q^c(cs))$,*

$$\lim_{c\to\infty} \mu_{ir}(Q^c(cs)) \to \frac{\tilde{x}_{ir}(s)}{\tilde{q}_i(s)}\mu_i^*(\tilde{q}(s)). \tag{52}$$

Proof To prove our statement let us use the definition of μ_{ir}, which is given in (11). Thus

$$\mu_{ir}(Q^c(cs)) = \frac{X_{ir}^c(cs)}{Q^c(cs)}\frac{s_i(E_i y_i(Q^c(cs)))}{T_i}. \tag{53}$$

For the condition that $\tilde{x}_{ir} > 0$, we need $X_{ir}^c(ct) \to \infty$ as $c \to \infty$. Thus $Q_i^c(ct) \to \infty$ as $c \to \infty$, and the same holds for $\sqrt{Q_i^c(ct)}$. By the policy described in Section 4.4, we have $E_i(c) \to \infty$ as $c \to \infty$, since $E_i \propto \sqrt{Q_i}$. Also by the definition in (2), we have that

$$\frac{E_i(c)}{T_i(c)} \to 1, \tag{54}$$

as $c \to \infty$. Furthermore, by the assumption (9), we have that

$$\frac{s_i(ky)}{k} \to \mu_i^{\max}, \tag{55}$$

as $k \to \infty$. Finally, by Lemma A.3 in Kelly and Williams (2004), we have that $y_i(\tilde{q})$ is continuous in $\tilde{q}$ for all indexes i with $\tilde{q}_i > 0$. Thus if $\tilde{x}_{ir} > 0$, we have

$$y_i(Q^c(cs)) = y_i\left(\frac{Q^c(cs)}{c}\right) \to y_i(\tilde{q}(s)), \tag{56}$$

from which we can deduce that

$$\mu_i(Q^c(cs)) = \frac{s_i^*(E_i y_i(Q^c(cs)))}{T_i} \to y_i^*(q(s))\mu_i^{\max} = \mu_i^*(q(s)). \tag{57}$$

If we combine the assumption in (10) with (57), we can conclude that (52) holds.□

In this section we use the relative entropy, which is defined for two probability distributions u and v, which are both defined on the same finite set $\mathcal{M}$, as

$$D(u\|v) = \sum_{m\in\mathcal{M}} u_m \log\left(\frac{u_m}{v_m}\right). \tag{58}$$

We note that $D(u\|v)$ is strictly non-negative and the following bound also holds on it,

Lemma 5 *(Pinsker's Inequality)*

$$D(u\|v) \geq \sum_{m\in\mathcal{M}} |u_m - n_m|. \tag{59}$$

E.1 Proof of lemma 1

We prove this lemma by taking the derivatives from first principles.

We first suppose that $x_{ir} > 0$. Let us use the notation $x_{ir}^h = x_{ir} + h$ and all other components of x remain the same. Naturally $q_i^h = q_i + h$, while $q_{i'}^h = q_{i'}$ for $i' \neq i$. Then for $h > 0$,

$$\begin{aligned}
&\frac{H(x^h) - H(x)}{h} \\
=&\frac{1}{h}\left[\sum_{r\in\mathcal{R}}\sum_{i\in r} x_{ir}^h \log\left(\frac{x_{ir}^h \mu_i^*(q^h)}{q_i^h a_r}\right) - \sum_{r\in\mathcal{R}}\sum_{i\in r} x_{ir}\log\left(\frac{x_{ir}\mu_i^*(q)}{q_i a_r}\right)\right] \\
\geq&\frac{1}{h}\left[\sum_{r\in\mathcal{R}}\sum_{i\in r} x_{ir}^h \log\left(\frac{x_{ir}^h \mu_i^*(q)}{q_i^h a_r}\right) - \sum_{r\in\mathcal{R}}\sum_{i\in r} x_{ir}\log\left(\frac{x_{ir}\mu_i^*(q)}{q_i a_r}\right)\right] \\
=&\frac{1}{h}\left[(x_{ir}^h \log x_{ir}^h - x_{ir}\log x_{ir}) + (q_i^h \log q_i^h - q_i \log q_i)\right] + \log\left(\frac{\mu_i^*(q)}{a_r}\right),
\end{aligned} \tag{60}$$

where the inequality derives from the fact that $\mu_i^*(q)$ is suboptimal for the proportional fair optimization with the parameter choice q^h. On the other hand if we leave the first summation in the first equality the same, but exchange $\mu_i^*(q)$ with $\mu_i^*(q^h)$, we can apply the same logic, thus

$$\begin{aligned}
&\frac{H(x^h) - H(x)}{h} \\
=&\frac{1}{h}\left[\sum_{r\in\mathcal{R}}\sum_{i\in r} x_{ir}^h \log\left(\frac{x_{ir}^h \mu_i^*(q^h)}{q_i^h a_r}\right) - \sum_{r\in\mathcal{R}}\sum_{i\in r} x_{ir}\log\left(\frac{x_{ir}\mu_i^*(q)}{q_i a_r}\right)\right] \\
\leq&\frac{1}{h}\left[\sum_{r\in\mathcal{R}}\sum_{i\in r} x_{ir}^h \log\left(\frac{x_{ir}^h \mu_i^*(q^h)}{q_i^h a_r}\right) - \sum_{r\in\mathcal{R}}\sum_{i\in r} x_{ir}\log\left(\frac{x_{ir}\mu_i^*(q^h)}{q_i a_r}\right)\right] \\
=&\frac{1}{h}\left[(x_{ir}^h \log x_{ir}^h - x_{ir}\log x_{ir}) + (q_i^h \log q_i^h - q_i \log q_i)\right] + \log\left(\frac{\mu_i^*(q)}{a_r}\right).
\end{aligned} \tag{61}$$

These two bounds can be derived the same way when $h < 0$. Since the $q \mapsto \mu^*(q)$ function is continuous by the properties of the service function, taking the limit as $h \to 0$ on these bounds imply that

$$\frac{\partial H(x)}{\partial x_{ir}} = \log x_{ir} - \log q_i + \log\left(\frac{\mu_i^*(q)}{a_r}\right), \tag{62}$$

as stated in Lemma 1.

We now suppose that $x_{ir} = 0$. (We now pursue an argument similar to Lemma 7 in Anselmi et al (2013).) We know that that $\frac{dx_{ir}}{dt} = 0$, recall Remark 1. Take

$$p(t) = \frac{x_{ir}(t)\mu_i^*(q(t))}{q_i(t)a_r}.$$

Note that $p(t)$ is bounded above by $p_{\max}$. Now, note that $x_{ir}(t)\log(p(t)/p_{\max})$ is negative, so we have

$$\frac{dx_{ir}\log(p/p_{\max})}{dt} = \lim_{h\searrow 0}\frac{x_{ir}(t+h)\log(p(t+h)/p_{\max}) - 0}{h} \le 0,$$
$$\frac{dx_{ir}\log(p/p_{\max})}{dt} = \lim_{h\nearrow 0}\frac{x_{ir}(t+h)\log(p(t+h)/p_{\max}) - 0}{h} \ge 0.$$

Therefore

$$\frac{dx_{ir}\log(p/p_{\max})}{dt}$$

and thus

$$\frac{dx_{ir}\log p}{dt} = \frac{dx_{ir}\log(p/p_{\max})}{dt} + \frac{dx_{ir}\log(p_{\max})}{dt} = 0.$$

□

E.2 Proof of lemma 2

Observe that $H(x)$ can be expressed as linear combination of relative entropy terms as

$$\begin{aligned} H(x) &= \sum_{i\in\mathcal{I}} q_i D\left(\left(\frac{x_{ir}}{q_i}\right)_{r\ni i} \| \left(\frac{a_r}{a_i}\right)_{r\ni i}\right) + \sum_{i\in\mathcal{I}} q_i \log\left(\frac{\mu_i^*(q)}{a_i}\right) \\ &\ge \sum_{i\in\mathcal{I}} q_i \log\left(\frac{\mu_i^*(q)}{a_i}\right) \ge 0. \end{aligned} \tag{63}$$

The first inequality is a consequence of the positivity of relative entropy, whereas the second follows by the optimality of $\mu^*(q)$ and (29). From the entropy equality we see that $H(x)$ is continuous for $\|x\|_1 = 1$ and so bounded. Further the inequalities above hold with equality iff $x = 0$. □

E.3 Proof of lemma 3

In order to prove this lemma we define $\hat{u}_m$, and $\hat{v}_m$ respectively, as follows,

$$\hat{u}_m = \frac{u_m}{\sum_{m'\in\mathcal{M}} u_{m'}}. \tag{64}$$

Then by applying the definition of relative entropy on $\hat{u}_m$ and $\hat{v}_m$ and applying Pinsker's Inequality, we get

$$\begin{aligned} \sum_{m\in\mathcal{M}} u_m \log\left(\frac{u_m}{v_m}\right) &= D(\hat{u}\|\hat{v}) \sum_{m\in\mathcal{M}} u_m \\ &\ge \left(\sum_{m\in\mathcal{M}} |\hat{u}_m - \hat{v}_m|\right)^2 \sum_{m\in\mathcal{M}} u_m \\ &= \frac{1}{\sum_{m\in\mathcal{M}} u_m}\left(\sum_{m\in\mathcal{M}} |u_m - v_m|\right)^2 \\ &\ge \frac{1}{\sum_{m\in\mathcal{M}} u_m}\sum_{m\in\mathcal{M}} (u_m - v_m)^2, \end{aligned} \tag{65}$$

by rearranging after the second equality, applying (3) and bounding afterwards. □