NIEUW ARCHIEF VOOR WISKUNDE (4), Vol.3, (1985) 237-243

Some Historical and other Notes about the Mertens Conjecture and its Recent Disproof

Herman J.J. te Riele
Centre for Mathematics and Computer Science
P.O. Box 4079
1009 AB Amsterdam
The Netherlands

Summary: This paper answers some questions about the disproof of the Mertens conjecture by Odlyzko and te Riele. In particular, the roles of Stieltjes and Mertens are sketched in their historical perspective and some comments are given on the electronic communication between Amsterdam and Murray Hill (U.S.A.) and on the publicity around the disproof.

1. INTRODUCTION[1)]

On October 18, 1983, the author of this paper sent an electronic letter to Andrew Odlyzko at Bell Labs, Murray Hill, U.S.A., announcing the *disproof* of the Mertens conjecture, as a result of their joint efforts. About eight years ago, Odlyzko and the present author independently started to work on the Mertens conjecture. At the 1978 International Congress of Mathematicians, held in Helsinki, Finland, they became aware of each others work after the announcement of te Riele's results, published later in [10].

In this paper some historical and other notes will be given, in order to answer several questions which have been posed concerning the Mertens conjecture and its disproof. In Section 2 we formulate the mathematical content of the Mertens conjecture, together with its main consequence. In Section 3 we will sketch the roles of Stieltjes and Mertens in the "Mertens story", with citations from original sources. Section 4 is devoted to the electronic communication between Murray Hill (U.S.A.) and Amsterdam (CWI, The Netherlands) during this joint research project, and to the publicity around this project.

1. This note was written at the request of the editors of the Section "Around Mathematics".

2. THE MERTENS CONJECTURE

Let $\mu(n)$ denote the Möbius function, i.e.,

$$\mu(n) = \begin{cases} 1, & \text{if } n=1, \\ 0, & \text{if } n \text{ is divisible by a square } >1, \\ (-1)^k, & \text{if } n \text{ is the product of } k \text{ distinct } primes, \end{cases}$$

and let $M(x) := \Sigma_{n \leqslant x}\, \mu(n)$, $x > 1$. Then $M(x)$ is the difference between the number of *squarefree* integers $\leqslant x$ with an even number of prime factors and those with an odd number of prime factors. The Mertens conjecture, published in 1897 ([8]) by the Polish/Austrian mathematician Franz Mertens, states that

$$|M(x)| < x^{\frac{1}{2}},\ x>1. \tag{1}$$

An important consequence of the truth of the Mertens conjecture would have been the truth of the Riemann Hypothesis, and this undoubtedly has been the main reason for the substantial amount of interest in the Mertens conjecture since its "birth" in 1897. It should be added that already in 1885 the Dutch mathematician T.J. Stieltjes, in a letter to his friend Hermite ([11]), claimed to have a proof of the boundedness of the function $M(x)/x^{\frac{1}{2}}$, which is also sufficient to prove the Riemann Hypothesis. This letter was published in [1] in 1905. Therefore, Mertens should be given the (now dubious) credit of being the first one to have expressed his strong "belief" in the truth of (1).

For a survey of the mathematical work concerning the Mertens conjecture and for details of the computations leading to the disproof of (1), the reader is referred to the "official" publication [9].

3. THE ROLES OF STIELTJES AND MERTENS

3.1 T.J. Stieltjes

T.J. Stieltjes was born in Zwolle, The Netherlands, in 1856, and died in Toulouse, France, in 1894. He became a Professor of Mathematics in Toulouse in 1886. He was a friend of Ch. Hermite, as is shown by their extensive correspondence ([1]). In one of his letters to Hermite, dated July 11, 1885 ([11]), Stieltjes claimed to have a proof of the boundedness of the function $M(x)/x^{\frac{1}{2}}$. After some preliminary remarks, Stieltjes writes (for the Möbius function $\mu(n)$ he used the notation $f(n)$):

> En effet, si, au lieu de $1{:}\zeta(s) = \Pi(1-p^{-s})$, je considère $1{:}\zeta(s) = 1-2^{-s}-3^{-s}-5^{-s}+6^{-s}\cdots = \Sigma_1^\infty f(n)n^{-s}$, il y a cette différence capitale, entre le produit infini et la série, que la dernière est convergente pour $s>\frac{1}{2}$, tandis que, dans le produit, il faut supposer $s>1$. Voici comment je le démontre: La fonction $f(n)$ est égale à zéro lorsque n est divisible par un carré et pour les autres

> valeurs de n, égale à $(-1)^k$, k étant le nombre des facteurs premiers de n. Or, je trouve que dans la somme
>
> $$g(n) = f(1)+f(2)+ \cdots +f(n),$$
>
> les termes ± 1 se compensent assez bien pour que $g(n)/n^{1/2}$ reste toujours comprise entre deux limites fixes, quelque grand que soit n (probablement on peut prendre pour ces limites $+1$ et -1).

After deriving from the last statement that $\Sigma f(n)n^{-s}$ is convergent for $s>\frac{1}{2}$, Stieltjes proceeds as follows:

> Vous voyez que tout dépend d'une recherche arithmétique sur cette somme $f(1)+f(2)+ \cdots +f(n)$. Ma démonstration est bien pénible; je tâcherai, lorsque je reprendrai ces recherches, de la simplifier encore.

Two days later, Hermite presented a note by Stieltjes to the French Academy of Sciences ([12]) in which Stieltjes "proves" the truth of the Riemann Hypothesis via his claim of the analyticity of the function $1:\zeta(z)$ for all z having real part $>\frac{1}{2}$. We quote:

> Riemann a annoncé comme très probable que toutes ces racines imaginaires[1] sont de la forme $\frac{1}{2}+ai$, a étant réel. Je suis parvenu à mettre cette proposition hors de doute par une démonstration rigoureuse. Je vais indiquer la voie qui m'a conduit à ce résultat. D'après une remarque due à Euler
>
> $$1:\zeta(z) = \Pi(1-p^{-z}),$$
>
> p représentant tous les nombres premiers, ou encore
>
> $$1:\zeta(z) = 1-2^{-z}-3^{-z}-5^{-z}+6^{-z}-7^{-z}+10^{-z} \cdots$$
>
> C'est l'étude plus approfondie de la série qui figure ici dans le second membre qui conduit au but désiré. On peut démontrer, en effet, que cette série est convergente et définit une fonction analytique tant que la partie réelle de z surpasse $\frac{1}{2}$.

Here, Stieltjes does not refer to the boundedness of $M(x)/x^{\frac{1}{2}}$ as a step in his "proof", but in view of his letter to Hermite, he certainly had this in mind when writing the above lines. This also appears from a comment on [12] (page 599 of volume 2 of [14]):

> Dans son raisonnement sur la convergence de la série $1:\zeta(s) = \Sigma_1^\infty f(n)/n^s$ pour $s>\frac{1}{2}$ Stieltjes admet sans

1. I.e., the complex roots of $\zeta(z)$.

démonstration, d'après la lettre 79, que $g(n)/n^{\frac{1}{2}} = \{f(1)+f(2)+ \cdots +f(n)\}/n^{\frac{1}{2}}$ reste comprise entre deux limites fixes. On trouve dans les papiers laissés par Stieltjes un table de $g(n)$ pour les valeurs 1 jusqu'à 1200, 2000 jusqu'à 2100 et 6000 jusqu'à 6100. Apparemment l'hypothèse de Stieltjes s'est fondée sur l'examen de cette table.

No doubt, the note [12] must have puzzled many mathematicians. Mittag-Leffler immediately asked Stieltjes for details. This appears from four letters of Stieltjes to Mittag-Leffler given in an Appendix to [1]. In the fourth letter, dated April 15, 1887 ([14]), Stieltjes still claims the boundedness of $M(x)/x^{\frac{1}{2}}$:

En désignant par

$$\sum_{1}^{\infty}\lambda(n)n^{-s}$$

la série obtenue par le développement du produit infini

$$1:\zeta(s) = \Pi(1-p^{-s}),$$

la convergence de la série pour $s>\frac{1}{2}$ est une conséquence de ce *lemme*.

L'expression $\{\lambda(1)+\lambda(2)+ \cdots +\lambda(n)\}/n^{\frac{1}{2}}$ reste toujours entre deux limites fixes. *(Voir* la théorie des séries de cette espèce dans la *Théorie des nombres* de Lejeune-Dirichlet, Dedekind.)
Mais la démonstration de ce lemme est purement arithmétique et très difficile et je ne l'obtiens que comme résultat de toute une série de propositions préliminaires. J'espère que cette démonstration pourra encore être simplifiée, mais en 1885 j'ai déjà fait de mon mieux et envisageant encore la question d'une autre manière et en remplaçant ce lemme par un autre, d'une nature pareille toutefois.

In his paper proving the Prime Number Theorem ([3]), Hadamard mentioned that Stieltjes had much stronger results on the zeros of the zeta function, but that Hadamard's new results might still be of interest because of their simpler proofs!

3.2 F. Mertens.
In 1897, F. Mertens published a paper ([8]) with a 50-page table of $\mu(n)$ and $M(n)$, for $n = 1,2,...,10000$. After defining the Möbius function, he states (writing $\sigma(n)$ instead of $M(n)$):

Wird

$$\mu(1)+\mu(2)+\mu(3)+ \cdots +\mu(n) = \sigma(n)$$

> gesetzt, so spielt die zahlentheoretische Function $\sigma(n)$ in vielen auf Primzahlen sich beziehenden asymptotischen Aufgaben eine wichtige Rolle.
>
> In der am Schlusse dieses Aufsatzes beigefügten Tafel findet man die Werthe von $\sigma(n)$ von $n = 1$ bis $n = 10000$ berechnet, und es ergibt sich aus derselben die merkwürdige Thatsache, dass der absolute Werth von $\sigma(n)$ — in dem Spielraum der Tafel mit Ausnahme des Werthes $n=1$ — immer unter $n^{\frac{1}{2}}$ liegt. Leider begegnet der allgemeine Beweis dieser Eigenschaft beinahe unübersteiglichen Schwierigkeiten.

18 pages further, Mertens writes:

> Da die Ungleichung $|\sigma(m)| < m^{1/2}$, wie die Induction lehrt, sehr wahrscheinlich ist, so ist auch die Riemann'sche Behauptung sehr wahrscheinlich, dass die imaginären Wurzeln der Gleichung $\zeta(z) = 0$ alle den reellen Bestandtheil 1/2 haben.

Here, Mertens does not mean: "mathematical or complete induction", but "induction" in the philosophical sense, viz., that from a number of observations of a certain causal relation, the conclusion is drawn that this causal relation holds true in general.

Mertens was a first rate mathematician. From an obituary written by Ph. Furtwängler we cite ([2]):

> Am 5. März 1927 starb in Wien im hohen Alter von 87 Jahren Hofrat Dr. Franz Mertens, emeritierter Professor der Mathematik an der Universität Wien. Er gehörte der Akademie 35 Jahre an, seit dem Jahre 1892 als korrespondierendes Mitglied und seit 1894 als wirkliches Mitglied.
>
> Franz Mertens wurde am 20. März 1840 in Schroda (Posen = Poznan) geboren. Seine Studienzeit verbrachte er in Berlin, wo damals das glänzende Dreigestirn am mathematischen Himmel: Kronecker, Kummer, Weierstrass, die jungen Mathematiker aus ganz Deutschland anzog. Von diesen sind die beiden ersten von entscheidendem Einfluss auf Mertens' wissenschaftliche Tätigkeit gewesen. Im Jahre 1865 promovierte Mertens mit der Dissertation: De functioni potentiali duarum ellipsoidium homogenearum, die an Dirichlet'sche Methoden anknüpft, und wurde dann noch in demselben Jahre als ausserordentlicher Professor der Mathematik an die Universität Krakau berufen. Nach fünf Jahren erhielt er dort eine ordentliche Professur, die er bis zum Jahre 1884 innehatte. In diesem Jahre wurde er an das Polytechnikum in Graz berufen, wo er zehn Jahre tätig war. Im Jahre 1894 kam er

> schliesslich als ordentlicher Professor der Mathematik an die Universität Wien, wo er verehrt und beliebt bei seinen Kollegen und Schülern, bis zu seiner Emeritierung im Jahre 1911 seinen Lehrberuf ausübte. Mertens hat es verstanden, sich bis ins höchste Alter eine ausgezeichnete geistige und Körperliche Frische zu erhalten, die ihn in den Stand setzte, eine ausserordentlich fruchtbare und ausgedehnte wissenschaftliche Tätigkeit zu entfalten. Er hat mehr als 100 Abhandlungen verfasst, die zum grössten Teil in den Berichten der Akademie erschienen sind. Seine letzte Arbeit datiert aus dem Jahre 1926, ist also im Alter von 86 Jahren verfasst.

In his description of the mathematical work of Mertens, Furtwängler then proceeds on page 186:

> Eine andere bedeutende Leistung auf dem Gebiete der analytischen Zahlentheorie ist seine Arbeit über die Riemann'sche Zetafunktion, in der es ihm zuerst gelang, einen wichtigen Schritt zum Beweise der sogenannten "Riemann'schen Vermutung", die heute noch unbewiesen ist, nach vorwärts zu machen.

This seems to point to Mertens's work [8], so that also Furtwängler appeared to believe in the truth of (1). It was only in 1942 that A.E. Ingham cast the first serious doubts on (1) with his important paper [7].

4. Communication between Murray Hill and Amsterdam, and publicity

In this joint research project the bulk of the communication between Murray Hill (U.S.A.) and Amsterdam (CWI, The Netherlands) was maintained via electronic mail. This was transmitted via the UNIX networks EUNET (in Europe) and USENET (in North America). Via these networks, with more than 2500 connected "sites", it is possible to quickly exchange all kinds of private and public information. In the EUNET network, the CWI is a so-called "backbone" site for Europe, which has the task to feed mail/news to smaller sites in Europe. It regularly calls, by means of dial-up, two other backbone sites in North America to exchange mail/news, then feeds this out to other European countries. Detailed information about these networks may be found in [4] and [5].

Quite a lot of publicity has been given to our work, both in the daily and in the popular scientific press. The quality of the newspaper articles was mixed: some were very successful in translating our mathematical results into language understandable to educated laymen, others were just quick extracts of other articles, which negatively influenced their intelligibility. In general, in these articles the Riemann Hypothesis was emphasized too much, and too few attention was paid to the vital role of the lattice basis reduction algorithm in our work and to the fact that our results show that trying to prove the RH via (1) or via some weakened form of (1) is a dead end.

Recently, some interest has been expressed ([6]) in building a collection of newspaper and other articles which translate mathematical results for an audience of laymen. The purpose (and hope) is to clear up certain misconceptions about mathematics which seem to live among laymen, and to influence the quality and quantity of mathematical journalism in the future. We heartily support this initiative !

ACKNOWLEDGEMENT

We like to thank Mrs. B. Amstädter, Head of the Library of the "Österreichische Akademie der Wissenschaften" in Vienna, for helping us so kindly with collecting material about the life and work of F. Mertens.

We are also indebted to H. Jager (Mathematical Institute of the University of Amsterdam) for some pertinent comments on an early version of this paper, and to Dik Winter (CWI) for his assistance with the electronic communication between Amsterdam and Murray Hill.

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