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Paradoxes in social networks with multiple products

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Abstract We show that various paradoxes can arise in a natural class of social networks. They demonstrate that more services or products may have adverse consequences for all members of the network and conversely that restricting the number of choices may be beneficial for every member of the network. These phenomena have been confirmed by a number of empirical studies. In our analysis we use a simple threshold model of social networks introduced in Apt and Markakis (2011), and more fully in Apt and Markakis (2014). In this model the agents, influenced by their neighbours, can adopt one out of several alternatives. We identify and analyze here four types of paradoxes that can arise in these networks. These paradoxes shed light on possible inefficiencies arising when one modifies the sets of products available to the agents forming a social network or the network structure. One of the paradoxes corresponds to the well-known Braess paradox in congestion games and shows that by adding more choices to a node, the network may end up in a situation that is worse for everybody. We exhibit a dual version of this, according to which removing a product available to an agent can eventually make everybody better off. The other paradoxes that we identify show that by adding or removing a product from the choice set of an agent may lead to permanent instability. Finally, we also identify conditions under which some of these paradoxes cannot arise.

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1 Introduction

Social networks have developed over the years into a large interdisciplinary research area with important links to sociology, economics, epidemiology, computer science, and mathematics. An important aspect of social networks in the modern days is that they help accelerate the spread of new products and technologies through word-of-mouth effects, as people tend to be influenced by their social circle. The diffusion of such product adoptions, and more generally of behavioral patterns over social networks, was studied in a flurry of numerous articles, notably the influential Morris (2000), and books, see Rogers (2003), Chamley (2004), Goyal (2007), Vega-Redondo (2007), Jackson (2008), Easley and Kleinberg (2010). This helped to delineate better this area and to appreciate the importance of the diffusion process.

However, the fact that more services or products may become available within a social network can also have some adverse consequences. As an example, it was noticed in a number of empirical studies that an abundance of choices may sometimes lead to non-optimal decisions. To quote from (Gigerenzer 2008, p. 38):

> *The freedom-of-choice paradox*. The more options one has, the more possibilities for experiencing conflict arise, and the more difficult it becomes to compare the options. There is a point where more options, products, and choices hurt both seller and consumer.

This phenomenon is sometimes called 'more is less'. This suggests that the spread of a new product over a network may not always lead to a better outcome for the community.

Our main contribution is to demonstrate that such paradoxes can indeed arise in social networks and to provide a formal framework for studying the emergence of these phenomena. The general setup in which we study these paradoxes makes it possible to interpret them as phenomena that can take place in any community the members of which make choices by taking into account the choices of others. An example is a 'bubble' in a financial market, where a decision of a trader to switch to some new financial product triggers a sequence of transactions, as a result of which all traders involved become worse off.

In order to study formally these issues, we use a model of social networks introduced in Apt and Markakis (2011) and more fully in Apt and Markakis (2014). In these networks the agents (players), influenced by their neighbours, can adopt one out of several alternatives. An example of such a network is a group of people who choose providers of mobile phones by taking into account the choice of their friends. This model belongs to the family of *threshold* models where each agent is associated with a threshold number that can be viewed as his resistance for adopting a product (the threshold in this setting can also be viewed as a price one has to pay for the product). We believe this model has the essential ingredients for capturing interactions and influence over social networks.

To analyze the dynamics of such networks, a natural class of *social network games* was introduced in Simon and Apt (2012), and more fully in Simon and Apt (2015). These are strategic games in which the payoff of each player weakly increases when more players choose the same product (strategy) as him. This property, that we call 'join the crowd', captures an essential aspect of social networks. There have been other game-theoretic approaches in social networks, where the players correspond to companies that are trying to promote their products over the network, see e.g., Goyal and Kearns (2012), Tzoumas et al. (2012). However, the models of Goyal and Kearns (2012), Tzoumas et al. (2012) are not suitable for studying strategic considerations of individual nodes within a network (the only strategic players in these models are competing companies that try to infect a network), contrary to the focus of our work.

Given the game-theoretic setup of Simon and Apt (2015), we can now describe our main findings. We start by illustrating that in some games, an addition of a new strategy (a new product) can trigger a sequence of changes (an improvement path) that brings the players from an initial Nash equilibrium to a new one with worse payoff for each player. This is similar in flavour to the Braess paradox in congestion games, one of the most striking paradoxes in game theory, e.g., see (Nisan et al. 2007, pp. 464–465). We also exhibit a natural 'dual' version of this paradox, concerning the removal of a product. Namely, there exist games in which the removal of a product can trigger a sequence of changes bringing the players from an initial Nash equilibrium to a new one with a better payoff for each player. This is in analogy to the dual version of Braess paradox, studied in Fotakis et al. (2012), Fotakis et al. (2012).

However, the analogy with the paradoxes in congestion games is not precise. One of the reasons is that all improvement paths in congestion games always terminate (Rosenthal 1973), in contrast to the social network games, which may even fail to have Nash equilibria (Simon and Apt 2012). Consequently, the set up we consider here is more complex. In particular, it is possible that an addition of a new product to (respectively, a removal of a product from) the choice set of a player results in a permanent instability, in the sense that the sequence of triggered changes may fail to terminate.

Furthermore, we analyze variants of these paradoxes that are obtained by stipulating that the corresponding 'new situation' is inevitable instead of only being possible.

Finally, we also show that the same types of paradoxes arise when instead of adding a new product or removing an existing product, we add an edge or remove an edge. This suggests that the growth of a social network may have an adverse effect on the optimality of the choices made by its members. Given the use of social networks as a platform for advertising and promoting products, and as a natural vehicle for spreading certain patterns of social behaviour, we feel it is important to identify such paradoxes in this context and to clarify that, in contrast to the congestion games, they can take several forms.

Apart from exhibiting these paradoxes, it is also natural to try to identify classes of social networks in which the introduced paradoxes cannot arise. The last part of our work (Sects. 7 and 8) is devoted to such an analysis.

The paper is organized as follows. In the next section we introduce the background material. In Sects. 3, 4, 5, and 6, using the social network games, we define formally and analyze four types of paradoxes. Then, in Sect. 7 we consider the case of networks where the underlying graph has no source nodes and provide sufficient conditions ensuring that one of the main paradoxes cannot arise. Subsequently, we utilize this result in Sect. 8, in which we study the special case where the underlying graph is a simple cycle. Finally, in Sect. 9 we discuss future research directions.

2 Preliminaries

2.1 Social networks

We consider here a specific model of social networks, originally introduced in Apt and Markakis (2011), which we recall first.

Let $V = \{1, \ldots, n\}$ be a finite set of ***agents*** and $G = (V, E, w)$ a weighted directed graph with $w_{ij} \in [0, 1]$ being the weight of the edge (i, j). Given a node i of G, we denote by $N(i)$ the set of nodes from which there is an incoming edge to i. We call each $j \in N(i)$ a ***neighbour*** of i in G. We assume that for each node i such that $N(i) \neq \emptyset$, $\sum_{j \in N(i)} w_{ji} \leq 1$. An agent $i \in V$ is said to be a ***source node*** in G if $N(i) = \emptyset$.

By a ***social network*** (from now on, just ***network***) we mean a tuple $\mathcal{S} = (G, \mathcal{P}, P, \theta)$, where

- G is a weighted directed graph,
- $\mathcal{P}$ is a finite set of alternatives or ***products***,
- P is a function that assigns to each agent i a non-empty set of products $P(i)$ from which it can make a choice,
- θ is a ***threshold function*** that for each $i \in V$ and $t \in P(i)$ yields a value $\theta(i, t) \in (0, 1]$.

A threshold value $\theta(i, t)$ can be viewed as a resistance of node i to adopt product t. Alternatively, it can be seen as the price charged to node i for the acquisition of product t or as the measure of node i's prior preference for product t.

Given such a network $\mathcal{S}$, we denote by $source(\mathcal{S})$ the set of source nodes in the underlying graph G.

Example 1 Figure 1 shows an example of a network. Let the threshold be 0.3 for all nodes and all products. The set of products $\mathcal{P}$ is $\{t_1, t_2, t_3\}$, the product set of each agent is marked next to the node denoting it and the weights are labels on the edges.

Given two social networks $\mathcal{S}$ and $\mathcal{S}'$ we say that $\mathcal{S}'$ is an ***expansion*** of $\mathcal{S}$ if it results from adding a product to the product set of a node in $\mathcal{S}$. We say then also that $\mathcal{S}$ is a ***contraction*** of $\mathcal{S}'$.

2.2 Strategic games

To analyze paradoxes in our model of social network we use strategic games. Recall that a ***strategic game*** for $n > 1$ players, written as $(S_1, \ldots, S_n, p_1, \ldots, p_n)$, consists of a non-empty set S_i of ***strategies*** and a ***payoff function*** $p_i : S_1 \times \cdots \times S_n \rightarrow \mathbb{R}$, for each player i.

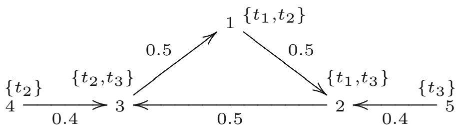


Fig. 1 A social network

Fix a strategic game $G := (S_1, \dots, S_n, p_1, \dots, p_n)$. We denote $S_1 \times \dots \times S_n$ by S, call each element $s \in S$ a ***joint strategy***, denote the ith element of s by s_i, and abbreviate the sequence $(s_j)_{j \neq i}$ to s_{-i}. Occasionally we write (s_i, s_{-i}) instead of s.

We call a strategy s_i of player i a ***best response*** to a joint strategy s_{-i} of his opponents if $\forall s'_i \in S_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i})$. We call a joint strategy s a ***Nash equilibrium*** if each s_i is a best response to s_{-i}. Further, we call a strategy s'_i of player i a ***better response*** given a joint strategy s if $p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i})$.

By a ***profitable deviation*** we mean a pair (s, s') of joint strategies such that $s' = (s'_i, s_{-i})$ for some s'_i and $p_i(s') > p_i(s)$. Following Monderer and Shapley (1996), an ***improvement path*** is a maximal sequence (i.e., a sequence that cannot be extended) of joint strategies such that each consecutive pair is a profitable deviation. Clearly, if an improvement path is finite, then its last element is a Nash equilibrium. Moreover, if s is a Nash equilibrium, then s is also a (trivial) improvement path.

If the initial joint strategy of an improvement path ξ is not a Nash equilibrium, then we call ξ a ***non-trivial improvement path***. When each profitable deviation in an improvement path results from a switch to a best response, we say that it is a ***best response improvement path***.

Given two joint strategies s and s' we write

- $s >_w s'$ if for all i, $p_i(s) \geq p_i(s')$ and for some i, $p_i(s) > p_i(s')$,
- $s >_{st} s'$ if for all i, $p_i(s) > p_i(s')$.

When $s >_w s'$ (respectively, $s >_{st} s'$) holds we say that s' is ***weakly worse*** (respectively, ***strictly worse***) than s.

2.3 Social network games

Next, we recall the strategic games over the social networks here considered, introduced in Simon and Apt (2012). Fix a network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$. Each agent can adopt a product from his product set or choose not to adopt any product. We denote the latter choice by t_0.

With each network $\mathcal{S}$ we associate a strategic game $\mathcal{G}(\mathcal{S})$. The idea is that the agents simultaneously choose a product or abstain from choosing any. Subsequently, each node assesses his choice by comparing it with the choices made by his neighbours. Formally, we define the game as follows:

- the players are the agents (i.e., the nodes),
- the set of strategies for player i is $S_i := P(i) \cup \{t_0\}$,
- For $i \in V$, $t \in P(i)$ and a joint strategy s, let $\mathcal{N}_i^t(s) := \{j \in N(i) \mid s_j = t\}$, i.e., $\mathcal{N}_i^t(s)$ is the set of neighbours of i who adopted in s the product t.

  The payoff function is defined as follows, where c_0 is some given in advance positive constant:
  - for $i \in source(\mathcal{S})$,

$$p_i(s) := \begin{cases} 0 & \text{if } s_i = t_0 \\ c_0 & \text{if } s_i \in P(i) \end{cases}$$

- for $i \notin source(\mathcal{S})$,

$$p_i(s) := \begin{cases} 0 & \text{if } s_i = t_0 \\ \sum_{j \in \mathcal{N}_i^t(s)} w_{ji} - \theta(i,t) & \text{if } s_i = t, \text{ for some } t \in P(i) \end{cases}$$

In the first entry we assume that the payoff function for the source nodes is constant only for simplicity. The second entry in the payoff definition is motivated by the following considerations. When agent i is not a source node, his 'satisfaction' from a joint strategy depends positively from the accumulated weight (read: 'influence') of his neighbours who made the same choice as him, and negatively from his threshold level (read: 'resistance') to adopt this product. The assumption that $\theta(i,t) > 0$ reflects the view that there is always some resistance to adopt a product. Strategy t_0 represents the possibility that an agent refrains from choosing a product.

Example 2 Consider the network given in Example 1 and the joint strategy s where each source node chooses the unique product in its product set and nodes 1, 2 and 3 choose t_2, t_3 and t_2 respectively. The payoffs are then given as follows:

- $p_1(s) = 0.5 - 0.3 = 0.2$,
- $p_2(s) = 0.4 - 0.3 = 0.1$,
- $p_3(s) = 0.4 - 0.3 = 0.1$,
- $p_4(s) = c_0$,
- $p_5(s) = c_0$.

Let s' be the joint strategy in which player 3 chooses t_3 and the remaining players make the same choice as given in s. Then (s, s') is a profitable deviation since $p_3(s') > p_3(s)$. In what follows, we represent each profitable deviation by a node and a strategy it switches to, e.g., $3 : t_3$. Starting at s, the sequence of profitable deviations $3 : t_3$, $1 : t_0$ cannot be extended, so it is an improvement path which results in the joint strategy in which nodes 1, 2 and 3 choose t_0, t_3 and t_3 respectively, and each source node chooses the unique product in its product set.

By definition, the payoff of each player depends only on the strategies chosen by his neighbours, so the social network games are related to graphical games of Kearns et al. (2001). However, the underlying dependence structure of a social network game is a directed graph and the presence of the special strategy t_0 available to each player makes these games more specific. Finally, note that these games satisfy the ***join the crowd*** property that we define as follows:

> Each payoff function p_i depends only on the strategy chosen by player i and the set of players who also chose his strategy. Moreover, the dependence on this set is monotonic.

The last qualification is exactly opposite to the definition of congestion games with player-specific payoff functions of Milchtaich (1996), in which the dependence on the above set is antimonotonic. That is, when more players choose the strategy of player i, then his payoff weakly decreases.

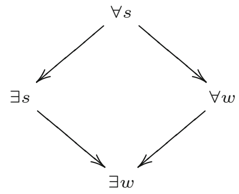


Fig. 2 Dependencies between the notions of vulnerability

3 Vulnerable networks

In what follows we introduce and analyze four types of networks. In each of them an addition (respectively, a removal) of a product to (respectively, from) a product set of a node can trigger a sequence of changes with counterintuitive outcomes. In this section we focus on the following notions.

We say that a social network $\mathcal{S}$ is $\exists w$-***vulnerable*** if for some Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, an expansion $\mathcal{S}'$ of $\mathcal{S}$ exists such that some improvement path in $\mathcal{G}(\mathcal{S}')$ leads from s to a Nash equilibrium s' in $\mathcal{G}(\mathcal{S}')$ such that $s >_w s'$. In general we have four notions of vulnerability, that correspond to the combinations XY, where $X \in \{\exists, \forall\}$ and $Y \in \{w, s\}$. For example, we say that $\mathcal{S}$ is $\forall s$-***vulnerable*** if for some Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, an expansion $\mathcal{S}'$ of $\mathcal{S}$ exists such that each improvement path in $\mathcal{G}(\mathcal{S}')$ leads from s to a Nash equilibrium s' in $\mathcal{G}(\mathcal{S})$ such that $s >_{st} s'$.

First note that there are some obvious implications between the four notions of vulnerability and inefficiency that we exhibit in Fig. 2.

We show now that these implications are the only ones that hold between these four notions.

Example 3 ($\forall w$) In Fig. 3 we exhibit an example of a $\forall w$-vulnerable network that is not $\exists s$-vulnerable. The product set of each node is marked next to it and the weights are labels on the edges. We assume that each threshold is a constant θ, where $0 < \theta < 0.1$. Here and elsewhere the relevant expansion is depicted by means of a product and the dotted arrow pointing to the relevant node. In this case product t_1 is added to node 4.

The initial Nash equilibrium s is the joint strategy formed by the underlined products, i.e., $(t_2, t_3, t_3, t_3, t_1, t_1, t_3, t_3)$. Now consider what happens after product t_1 is added to the product set of node 4. Then s ceases to be a Nash equilibrium since $p_4(t_1, s_{-4}) = 0.4 - \theta > 0.3 - \theta = p_4(s)$. Addition of t_1 triggers the unique best response improvement path

$$4 : t_1, 3 : t_2, 5 : t_2, 6 : t_0, 4 : t_3, 3 : t_3, 5 : t_0$$

resulting in the Nash equilibrium $(t_2, t_3, t_3, t_3, t_0, t_0, t_3, t_3)$. Note that at each step of any improvement path starting in s triggered by the addition of product t_1 to node 4 there is a unique node not playing its best response. For instance, in the second step of the above improvement path, node 3 is such a unique node.

So every such improvement path can be transformed to a unique best response path by deleting profitable deviations that do not result in a switch to a best response.

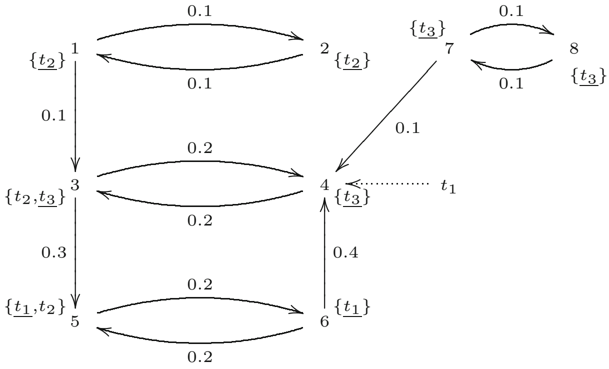


Fig. 3 An $\forall w$-vulnerable network

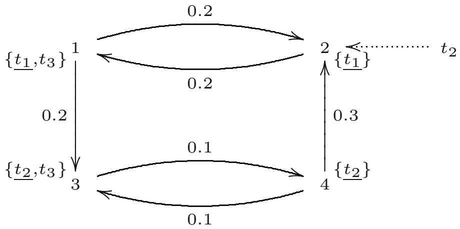


Fig. 4 A $\exists s$-vulnerable network

Therefore to prove that the network depicted in Fig. 3 is $\forall w$-vulnerable, it suffices to consider the outcome of the above best response improvement path.

Now, in the Nash equilibrium $(t_2, t_3, t_3, t_3, t_0, t_0, t_3, t_3)$ the payoffs of players 1, 2, 3, 4, 7 and 8 did not change with respect to the original Nash equilibrium, while the payoffs of players 5 and 6 decreased.

Finally, notice that a network is not $\exists s$-vulnerable since no product addition can cause a change of payoffs of players 1, 2, 7 and 8.

In this specific example the payoff of the player who triggered the change in the end did not change. A slightly more complicated example, that we omit, shows that the initiator's payoff in the final Nash equilibrium can decrease. Also, one can construct examples in which the payoffs in the final Nash equilibrium decrease for an arbitrary large fraction of the players and remain constant for the other players.

Example 4 ($\exists s$) In Fig. 4 we exhibit an example of a $\exists s$-vulnerable network that is not $\forall w$-vulnerable (and hence not $\forall s$-vulnerable). We assume that $\theta(1, t_3) < \theta(1, t_1) < 0.1$ (so that product t_3 is more attractive for node 1 than product t_1) and that on all other arguments the threshold is equal to a constant θ, where $0 < \theta < 0.1$. We underline the strategies that form the initial Nash equilibrium $s = (t_1, t_1, t_2, t_2)$. Note that the payoff of each player in s is > 0.

To see that this network is $\exists s$-vulnerable it suffices to note that starting from s the addition of product t_2 to node 2 triggers an improvement path

$$2 : t_2, 1 : t_3, 3 : t_3, 4 : t_0, 2 : t_0, 1 : t_0, 3 : t_0$$

that ends in a Nash equilibrium in which each strategy equals t_0, and consequently each payoff becomes 0.

To show that this network is not $\forall w$-vulnerable we need to analyze each of the initial Nash equilibria and all possible expansions.

There are in total four Nash equilibria: (t_1, t_1, t_2, t_2), (t_0, t_0, t_2, t_2), (t_1, t_1, t_0, t_0), and (t_0, t_0, t_0, t_0). The only expansions that can trigger a non-trivial improvement path are the ones in which a product is added to a node that has more than one neighbour, so to node 2 or 3.

For each Nash equilibrium and each such expansion we list an improvement path after which the payoff of the node with the modified product set increases or report non-existence of a non-trivial improvement path.

- (t_1, t_1, t_2, t_2).
  - Node 2: addition of t_2 triggers the improvement path $2 : t_2, 1 : t_0$.
  - Node 3: addition of t_1 triggers the improvement path $3 : t_1, 4 : t_0$.
- (t_0, t_0, t_2, t_2).
  - Node 2: addition of t_2 triggers the improvement path $2 : t_2$.
  - Node 3: no non-trivial improvement path is triggered.
- (t_1, t_1, t_0, t_0).
  - Node 2: no non-trivial improvement path is triggered.
  - Node 3: addition of t_1 triggers the improvement path $3 : t_1$.
- (t_0, t_0, t_0, t_0).
  Then no non-trivial improvement path is triggered.

As mentioned earlier, a threshold can be alternatively seen as a price charged to the node for the acquisition of a product. Having this in mind the paradoxes discussed in this and the next section can be also viewed as examples of undesirable consequences of lowering the price. Namely, instead of adding a product to the product set of a node we can simply assume that initially it is already present but is 'expensive' (the threshold equals 1) and the addition of a product to a node amounts to lowering the threshold. In the above example the addition of t_2 to node 2 can thus be simulated by lowering the threshold $\theta(2, t_2)$ from 1 to θ, that we assumed is less than 0.1.

Braess paradox that we mentioned in the introduction can also be interpreted as a statement that a congestion game with a unique Nash equilibrium exists, with the property that an addition of a road segment yields a game with again a unique Nash equilibrium in which every player is strictly worse off.

Such a stronger interpretation of the paradox cannot be reproduced in the setting of social network games. Indeed, to start with, the sought social network cannot have source nodes, as their payoffs in each Nash equilibrium are the same, namely c_0. Now, if the social network has no source nodes, then $\overline{t_0}$ is a Nash equilibrium in which each node has a minimal payoff among all Nash equilibria. So an addition of a product can lead to strictly lower payoffs only if initially also another Nash equilibrium exists.

Example 5 ($\exists w$) Next, we provide an example of a $\exists w$-vulnerable network that is neither $\exists s$-vulnerable nor $\forall w$-vulnerable. It suffices to add to the network given in Fig. 4 a source node 7 with the product set $\{t_1\}$ and connect it to node 1 using an arbitrary threshold and weight. In each Nash equilibrium, node 7 chooses t_1, so its payoff is the same. Further, the choice of this node has no influence on the choices of other nodes in the Nash equilibria in the original and the extended networks. So the conclusion follows from the previous example.

Next, we would like to mention the following intriguing question:

Open problem

Do $\forall s$-vulnerable networks exist?

The following result shows that if they do, they use at least three products.

Theorem 1 *When there are only two products, $\forall w$-vulnerable networks, so a fortiori $\forall s$-vulnerable networks, do not exist.*

Proof Suppose a $\forall w$-vulnerable network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ exists. Let $\mathcal{P} = \{t_1, t_1\}$ be the two products in $\mathcal{S}$. So there exists a Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, a node, say 1, and a product, say t_1, such that for the network expansion $\mathcal{S}'$ obtained by adding t_1 to the product set of node 1 each improvement path that starts in s ends up in a Nash equilibrium s' in $\mathcal{G}(\mathcal{S}')$ such that $s >_w s'$.

Below by an *improvement segment* we mean an initial segment of an improvement path.

For a product $t \in \mathcal{P} \cup \{t_0\}$ and a joint strategy s, we denote by ***t-phase***, a best response improvement segment $\rho' : s = s^0, s^1, s^2, \ldots, s^k$ starting at s such that for all $j \in \{0, \ldots, k-1\}$, for every profitable deviation (s^j, s^{j+1}) in which the deviating player is i, we have $s_i^{j+1} = t$. That is, the deviating player has product t as his best response. Clearly each t-phase is finite since there are only finite number of players in $\mathcal{S}$.

We construct a best response improvement path ρ in $\mathcal{G}(\mathcal{S}')$ by repeatedly concatenating the best response improvement segment obtained from a t_1-phase followed by a t_0-phase. We then prove that ρ is finite and that the last joint strategy in ρ is a Nash equilibrium which is not weakly worse than s.

First consider a best response improvement segment ρ^0 obtained from a t_1-phase starting in the joint strategy s and let s'' be the last joint strategy in ρ^0. Note that for any profitable deviation (s^1, s^2) in ρ^0, if t_1 is a best response for a node i to s^1_{-i}, then t_1 is also a best response for i to s^2_{-i}. Indeed, by the join the crowd property $p_i(t_1, s^2_{-i}) \geq p_i(t_1, s^1_{-i})$ and $p_i(t_2, s^1_{-i}) \geq p_i(t_2, s^2_{-i})$, so $p_i(t_1, s^2_{-i}) \geq p_i(t_2, s^2_{-i})$ since $p_i(t_1, s^1_{-i}) \geq p_i(t_2, s^1_{-i})$. Further, $p_i(t_1, s^1_{-i}) \geq p_i(t_0, s^1_{-i})$, so also $p_i(t_1, s^2_{-i}) \geq p_i(t_0, s^2_{-i})$. Consequently, in s'', for every player i such that $s''_i = t_1$, i is playing a best response.

Now consider the best response improvement segment ρ^1 obtained from a t_0-phase starting in the joint strategy s'' and let s''' be the last joint strategy in ρ^1. Suppose the best response for a player i in s'' is t_0. By the above observation, $s''_i \neq t_1$ and thus $s''_i = t_2$. So, again by the join the crowd property, this deviation does not affect the property that the nodes that selected t_1 in s^0 play a best response. Iterating this reasoning we conclude that in the joint strategy s''', each node that has the strategy t_1 continues to play a best response.

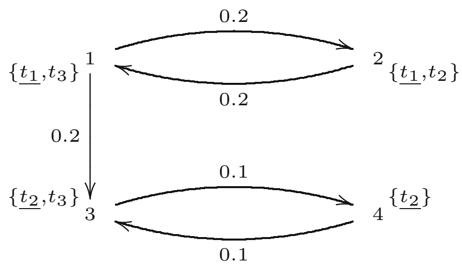


Fig. 5 A modified $\exists s$-vulnerable network

By the same reasoning subsequent t_1 and t_0-phases have the same effect on the set of nodes that have the strategy t_1: each of these nodes continues to play a best response. Moreover, this set continues to weakly increase. Consequently these repeated applications of the t_1-phase followed by the t_0-phase terminate, say in a joint strategy s'.

We now argue that s' is a Nash equilibrium. Suppose a node i does not play a best response to s'_{-i}. If $s'_i = t_0$, then by the construction t_1 is not a best response, so t_2 is a best response. Suppose the initial strategy of node i was also t_0, i.e., $s_i = t_0$. Since s is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$, we have $p_i(t_2, s_{-i}) \leq p_i(t_0, s_{-i})$. By join the crowd property $p_i(t_2, s'_{-i}) \leq p_i(t_2, s_{-i})$, so $p_i(t_2, s'_{-i}) \leq p_i(t_0, s'_{-i})$, which yields a contradiction. Hence node i deviated to t_0 from some intermediate joint strategy s^1 by selecting a best response. So $p_i(t_2, s^1_{-i}) \leq p_i(t_0, s^1_{-i})$. Moreover, by the join the crowd property $p_i(t_2, s'_{-i}) \leq p_i(t_2, s^1_{-i})$, so $p_i(t_2, s'_{-i}) \leq p_i(t_0, s'_{-i})$, which yields a contradiction, as well.

Further, by the construction $s'_i \neq t_1$, so the only alternative is that $s'_i = t_2$. But then either t_0 or t_1 is a best response, which contradicts the construction of s'. We conclude that s' is a Nash equilibrium in $\mathcal{G}(\mathcal{S}')$.

Notice that the payoff of node 1 strictly increased when it switched to t_1 and, on the account of the above arguments, during the remaining steps of the considered improvement path it either increased or remained the same. We conclude that the final Nash equilibrium s' is not weakly worse than the original, which yields a contradiction. $\square$

The definition of vulnerability referred to an expansion of a social network, defined as an addition of a single product to a single node. Another natural definition of an expansion consists of adding a single weighted edge to the social network.

In each of the social networks used in the above examples we added a new product, say t, to a node, say a. In each case there is a single node, say b, that has t in his product set and such that the edge from b to a is present in the considered network.

When we adopt the alternative definition of expansion it suffices to use as the initial network the one in which this edge from b to a is removed and in which the product t is already present in the product set of a. The expansion of the considered network consists then of adding the edge from b to a. For instance, in Example 4 instead of the network depicted in Fig. 4 we consider the social network depicted in Fig. 5 to which we add the edge $4 \rightarrow 2$ with weight 0.3.

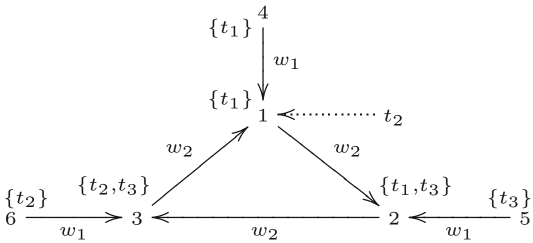


Fig. 6 A fragile network

We leave to the reader the straightforward task of checking that under this alternative definition of expansion (and hence of vulnerability) both the examples and Theorem 1 continue to hold.

4 Fragile networks

Related notions to vulnerable networks are the following ones.

We say that a social network $\mathcal{S}$ is $\exists$-***fragile*** if for some Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, an expansion $\mathcal{S}'$ of $\mathcal{S}$ exists such that some improvement path in $\mathcal{G}(\mathcal{S}')$ that starts in s is infinite. In turn, we say that a social network $\mathcal{S}$ is $\forall$-***fragile*** if for some Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, an expansion $\mathcal{S}'$ of $\mathcal{S}$ exists such that each improvement path in $\mathcal{G}(\mathcal{S}')$ that starts in s is infinite. Finally, we say that a social network $\mathcal{S}$ is ***fragile*** if $\mathcal{G}(\mathcal{S})$ has a Nash equilibrium, while for some expansion $\mathcal{S}'$ of $\mathcal{S}$, $\mathcal{G}(\mathcal{S}')$ does not.

Obviously each fragile network is $\forall$-fragile, while each $\forall$-fragile network is $\exists$-fragile. We now show that these two implications are proper.

Example 6 (Fragile) Consider the network $\mathcal{S}$ given in Fig. 6 where the source nodes are represented by the unique product in their product set. We assume that each threshold is a constant θ such that $\theta < w_1 < w_2$.

Consider the joint strategy s, in which nodes 1, 2, and 3 choose t_1, t_1 and t_2, respectively and the other nodes choose the unique product in their product set. For convenience, we denote s by the choices of nodes 1, 2 and 3, so $s = (t_1, t_1, t_2)$. It is easy to verify that s is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$.

Consider now the expansion $\mathcal{S}'$ of $\mathcal{S}$ in which product t_2 is added to the product set of node 1. In $\mathcal{G}(\mathcal{S}')$ the joint strategy s ceases to remain a Nash equilibrium. In fact, no joint strategy is a Nash equilibrium in $\mathcal{G}(\mathcal{S}')$. Each agent residing on the triangle can secure a payoff of at least $w_1 - \theta > 0$, so it suffices to analyze the joint strategies in which t_0 is not used. There are in total eight such joint strategies. Here is their listing, where in each joint strategy we underline a strategy that is not a best response to the choice of other players: $(\underline{t_1}, t_1, t_2)$, $(t_1, t_1, \underline{t_3})$, $(t_1, t_3, \underline{t_2})$, $(t_1, \underline{t_3}, t_3)$, $(t_2, \underline{t_1}, t_2)$, $(t_2, \underline{t_1}, t_3)$, $(t_2, t_3, \underline{t_2})$, $(\underline{t_2}, t_3, t_3)$. This shows that the initial network $\mathcal{S}$ is fragile.

Example 7 ($\forall$-fragile) Consider the network $\mathcal{S}$ given in Fig. 7. We assume that each threshold is a constant θ, where $\theta < w_1 < w_2$. Consider the joint strategy s, in which the nodes 1, 2, and 3 choose t_1, t_1 and t_2, respectively, and the other nodes choose

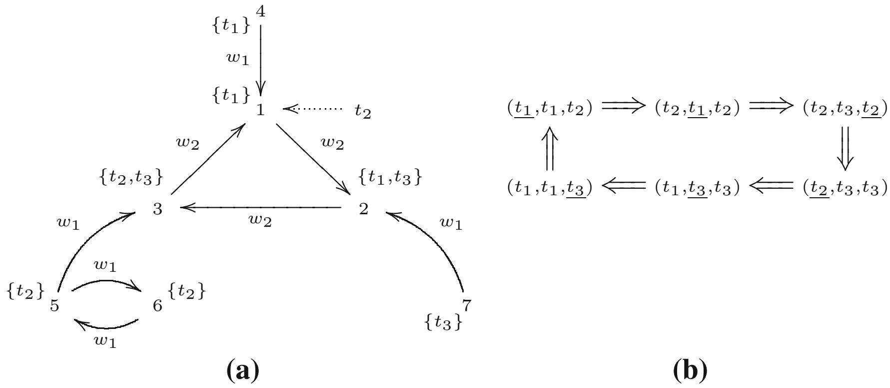


Fig. 7 A ∀-fragile network and an infinite improvement path

the unique product in their product set. As in the previous example we denote s by (t_1, t_1, t_2). It is easy to check that s is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$.

Now consider the expansion $\mathcal{S}'$ obtained by adding the product t_2 to the product set of node 1. Then s ceases to remain a Nash equilibrium in $\mathcal{G}(\mathcal{S}')$. In fact, Fig. 7b shows the unique best response improvement path starting in s which is infinite. We only list strategies selected by nodes 1, 2, and 3 and for each joint strategy in the figure underline the strategy that is not a best response. As in the case of Example 3 at every step of every improvement path starting in s, there is a unique node which is not playing its best response. Therefore each improvement path triggered by the above expansion can be transformed to the best response improvement path depicted in Fig. 7b. So each such improvement path is infinite, which shows that $\mathcal{S}$ is ∀-fragile.

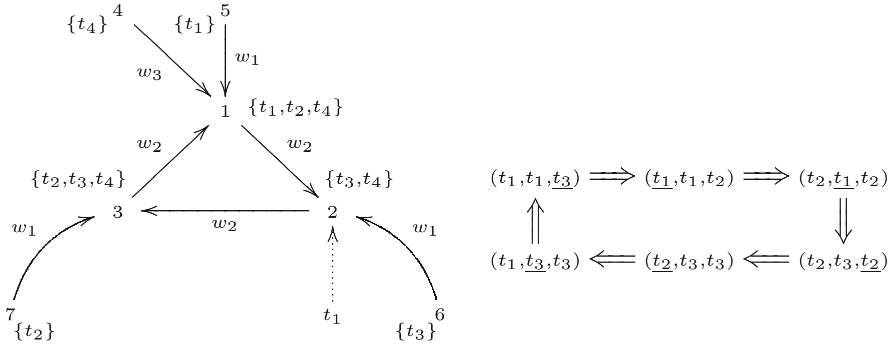
Also note that the game $\mathcal{G}(\mathcal{S}')$ has a Nash equilibrium. Indeed, it is easy to check that $(t_1, t_1, t_0, t_1, t_0, t_0, t_3)$ is a Nash equilibrium. To prove that $\mathcal{S}$ is not fragile we need to check all other expansions and prove that for each of them the underlying game has a Nash equilibrium.

Consider first the expansion resulting from adding the product t_3 to the product set of node 1. Then $(t_3, t_3, t_3, t_1, t_2, t_2, t_3)$ is a Nash equilibrium.

In every other expansion the product set of node 1 remains $\{t_1\}$, so nodes 1, 2 and 4 always play a best response when they select t_1, while nodes 5, 6 and 7 always play a best response when they select respectively t_2, t_2 and t_3. So if product t_1 is added to the product set of node 3, then $(t_1, t_1, t_1, t_1, t_2, t_2, t_3)$ is a Nash equilibrium and otherwise $(t_1, t_1, t_2, t_1, t_2, t_2, t_3)$ is a Nash equilibrium. Hence $\mathcal{S}$ is indeed not fragile.

Example 8 (∃*-fragile*) Consider the network $\mathcal{S}$ given in Fig. 8a. Let the threshold be a constant θ, where $\theta < w_3 < w_1 < w_2$. Assume that each source node selects its unique product. Identify each joint strategy that extends this selection with the selection of the strategies by the nodes 1, 2 and 3. The joint strategy $s = (t_1, t_3, t_3)$ is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$.

Now consider the expansion $\mathcal{S}'$ obtained by adding the product t_1 to the product set of node 2 in $\mathcal{S}$. The joint strategy s ceases to remain a Nash equilibrium in $\mathcal{G}(\mathcal{S}')$ since node 2 can profitably deviate to t_1. Figure 8b shows an infinite improvement path starting in (t_1, t_1, t_3). Therefore $\mathcal{S}$ is ∃-fragile.



(a) **(b)**

Fig. 8 A ∃-fragile network and an infinite improvement path

However, $\mathcal{S}$ is not ∀-fragile. First, one can check that $s = (t_1, t_3, t_3)$ is the only Nash equilibrium in $\mathcal{G}(\mathcal{S})$ for which profitable additions exist. Indeed, the only other Nash equilibrium is (t_4, t_4, t_4) in which nodes 1, 2 and 3 already secure the maximal possible payment, due to the conditions on the weights.

Below we analyze the two profitable additions in the case of the Nash equilibrium (t_1, t_3, t_3).

- Addition of t_1 to node 2 (depicted in Fig. 8a). The following improvement path

$$2 : t_1, 3 : t_2, 1 : t_2, 2 : t_3, 3 : t_3, 1 : t_4, 2 : t_4, 3 : t_4$$

  starting in s terminates in the joint strategy (t_4, t_4, t_4) which is a Nash equilibrium.
- Addition of t_3 to node 1. This triggers a unique one-step improvement path that terminates in a new Nash equilibrium (t_3, t_3, t_3).

5 Inefficient networks

The last two types of deficiency are concerned with product removal. These form the dual versions of the paradoxes we have seen so far. In this section we study the following notions.

We say that a social network $\mathcal{S}$ is $\exists w$-***inefficient*** if for some Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, a contraction $\mathcal{S}'$ of $\mathcal{S}$ exists such that some improvement path in $\mathcal{G}(\mathcal{S}')$ leads from s to a Nash equilibrium s' in $\mathcal{G}(\mathcal{S}')$ such that $s' >_w s$. We note here that if the contraction was created by removing a product from the product set of node i, we impose that any improvement path in $\mathcal{G}(\mathcal{S}')$, given a starting joint strategy from $\mathcal{G}(\mathcal{S})$, begins by having node i making a choice (we allow any choice from his remaining set of products or t_0 as an improvement move). Otherwise the initial payoff of node i in $\mathcal{G}(\mathcal{S}')$ is not well-defined.

As in the case of the vulnerability, we have four notions of inefficiency that correspond to the combinations XY, where $X \in \{\exists, \forall\}$ and $Y \in \{w, s\}$. For example, we say that $\mathcal{S}$ is $\forall s$-***inefficient*** if for some Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, a contraction $\mathcal{S}'$ of

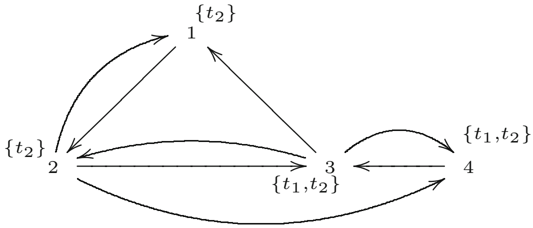


Fig. 9 An example of a ∀s-inefficient network

$\mathcal{S}$ exists such that each improvement path in $\mathcal{G}(\mathcal{S}')$ leads from s to a Nash equilibrium s' in $\mathcal{G}(\mathcal{S}')$ such that $s' >_{st} s$.

We now show that the implications between the various notions shown in Fig. 2 for the case of vulnerable networks are also proper implications for inefficient networks. However, in contrast to the concept of vulnerability, here even when there are only two choices ($|P| = 2$), there exist ∀s-inefficient networks.

Example 9 (∀s) We exhibit in Fig. 9 an example of a ∀s-inefficient network. The weight of each edge is assumed to be w and is omitted, and we also have the same product-independent threshold, θ, for all nodes, with $w > \theta$.

Consider as the initial Nash equilibrium the joint strategy $s = (t_2, t_2, t_1, t_1)$. It is easy to check that this is a Nash equilibrium, with a payoff equal to $w - \theta$ for all nodes. Suppose now that we remove product t_1 from the product set of node 3. We claim that all improvement paths then lead to the Nash equilibrium in which all nodes adopt t_2.

To see this, we note that in all these improvement paths, nodes 1 and 2 do not switch from t_2. Consequently, in all joint strategies in all of these improvement paths, t_2 is a unique best response for both node 3 and node 4. Hence all improvement paths result in all nodes adopting t_2 and producing a payoff of $2w - \theta$ for each node, which is strictly better than the payoff in s.

Example 10 (∀w) We now exhibit a network which is ∀w-inefficient but not ∃s-inefficient (and hence also not ∀s-inefficient). We proceed as in Example 5 and add to the network given in Fig. 9 a source node 5 with the product set $\{t_1\}$ and connect it to node 1, using the same weight w and threshold θ. By the same argument as in Example 5 the conclusion follows by virtue of the previous example.

We also remark that one can construct even simpler networks with three nodes and two products that exhibit the same behaviour.

Example 11 (∃s) Next, we exhibit a network that is ∃s-inefficient but not ∀w-inefficient. The network is shown in Fig. 10. The weight of each edge is assumed to be w and is omitted, and we also have a product-independent threshold θ (with $w > \theta$), that applies to all nodes and products except three cases: $\theta(2, t_3) < \theta$, and $\theta(5, t_2) = \theta(5, t_3) > \theta$. Note that in the underlying graph, each node has exactly two incoming edges, one from the set $\{1, 2\}$ and one from $\{3, 4, 5\}$, which is crucial in order to ensure certain equilibrium outcomes.

To see first that this is a ∃s-inefficient network, consider the Nash equilibrium $(t_2, t_2, t_1, t_1, t_1)$. In this joint strategy each node receives suport from exactly one out

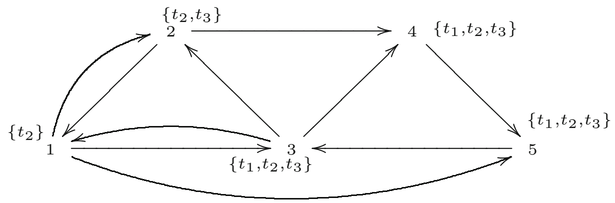


Fig. 10 An example of a $\exists s$-inefficient network

of its two neighbours. If we delete t_1 from the choice set of node 3, then we can see that the following improvement path

$$3 : t_2, 4 : t_2, 5 : t_2$$

converges to all nodes adopting t_2 (by having first node 3 adopt t_2, followed by nodes 4 and 5). Hence in this new Nash equilibrium every node receives support from all its neighbours and the payoff of everyone has strictly increased.

To show that this is not a $\forall w$-inefficient network, we need to consider all Nash equilibria and argue about all the possible contractions from each equilibrium. One can verify that the initial game has four Nash equilibria, namely $(t_2, t_2, t_1, t_1, t_1)$, $(t_2, t_2, t_2, t_2, t_2)$, $(t_0, t_0, t_1, t_1, t_1)$, and $(t_0, t_3, t_3, t_3, t_3)$. Note that we do not have to consider contractions from node 1, who has only a singleton product set. Note also that for the other nodes, the only contraction we have to consider is the removal of the product they have at the equilibrium, otherwise no improvement path arises. We claim now that in any contraction, either node 1 or node 2 or node 5 will get worse off in some improvement path. This will happen either because nodes 1 or 2 will lose support for t_2 or because node 5 will end up with product t_3, which is a worse choice for him than t_1. We analyze below all the different contractions that one needs to examine from these four Nash equilibria.

- $(t_2, t_2, t_1, t_1, t_1)$.
  - Node 2: deletion of t_2 triggers the improvement path $2 : t_0, 1 : t_0$, where nodes 1 and 2 are worse off.
  - Node 3: deletion of t_1 triggers the improvement path $3 : t_3, 4 : t_3, 5 : t_3, 2 : t_3, 1 : t_0$. The deviation of node 2 in this path is ensured by our assumption that $\theta(2, t_3) < \theta$. This implies that node 1 is worse off at the end.
  - Node 4: deletion of t_1 triggers the improvement path $4 : t_3, 5 : t_3, 3 : t_3, 2 : t_3, 1 : t_0$. Note that the second move in the improvement path is possible since $\theta(5, t_2) = \theta(5, t_3)$. Hence again, node 1 is worse off at the end.
  - Node 5: deletion of t_1 similarly triggers the improvement path $5 : t_3, 3 : t_3, 4 : t_3, 2 : t_3, 1 : t_0$. It is again, node 1 as above that is worse off at the end.
- $(t_2, t_2, t_2, t_2, t_2)$.
  - Node 2: deletion of t_2 triggers the improvement path $2 : t_0, 1 : t_0$, where nodes 1 and 2 are worse off.
  - Node 3, 4, or 5: deletion of t_2 in these nodes results in the improvement path $i : t_0, i \in \{3, 4, 5\}$. But then the node whose product set was contracted is worse off.

– $(t_0, t_0, t_1, t_1, t_1)$.
  - Node 3: deletion of t_1 triggers the improvement path $3 : t_3, 4 : t_3, 5 : t_3, 2 : t_3$, and at the end node 5 is worse off, since $\theta(5, t_3) > \theta$.
  - Node 4: deletion of t_1 triggers the improvement path $4 : t_3, 5 : t_3, 3 : t_3, 2 : t_3$. Again node 5 is worse off.
  - Node 5: deletion of t_1 makes node 5 worse off in all improvement paths since for the remaining products, $\theta(5, t_2) = \theta(5, t_3) > \theta$.

– $(t_0, t_3, t_3, t_3, t_3)$.
  - Node 2: deletion of t_3 triggers the improvement path $2 : t_0$, with node 2 being worse off.
  - Node 3: deletion of t_3 triggers the improvement path $3 : t_0, 2 : t_0, 4 : t_0, 5 : t_0$, with most nodes being worse off.
  - Node 4: deletion of t_3 triggers the improvement path $4 : t_0, 5 : t_0, 3 : t_0, 2 : t_0$, with most nodes being worse off.
  - Node 5: deletion of t_3 triggers the improvement path $5 : t_0, 3 : t_0, 2 : t_0, 4 : t_0$, again with most nodes being worse off.

This concludes the analysis of the equilibria and hence we can deduce that this is not a $\forall w$-inefficient network.

Example 12 ($\exists w$) Finally, we exhibit a $\exists w$-inefficient network that is neither $\forall w$-inefficient nor $\exists s$-inefficient. We proceed as in Example 10 and simply modify the previous example. We add to the network given in Fig. 10 a source node 6 with the product set $\{t_2\}$ and connect it to node 1, using the same weight w and threshold θ. By the same argument as in Examples 5 and 10 the conclusion follows by virtue of the previous example.

Actually, the argument that this network is not $\forall w$-inefficient becomes now simpler because after the addition of the source node 6 the initial game has only two Nash equilibria, namely $(t_2, t_2, t_1, t_1, t_1, t_2)$ and $(t_2, t_2, t_2, t_2, t_2, t_2)$.

In analogy to Sect. 3 we now consider a modified definition of contraction according to which we remove a single edge. This yields an alternative definition of inefficient social networks. Then the social networks illustrating the types of inefficient networks considered here, continue to exist. However, in contrast to Sect. 3 we cannot modify the exhibited networks in a ‘standard’ way. The reason is that in these two definitions the changes in the product selection are triggered differently.

According to the original definition, in the examples of appropriately inefficient networks we always need to remove the product selected by a node in the initial Nash equilibrium. This necessarily causes a temporary ‘illegal situation’ which is restored by stipulating that this node switches to an arbitrary alternative product in his product set or to t_0. In contrast, when we remove an edge, no ‘illegal situation’ is ever created. Moreover, the initial Nash equilibrium can ‘survive’ such a removal. A sequence of changes will be triggered only if the node at the end of the removed edge can switch to a better alternative.

By way of example we only provide here in Fig. 11 an example of a network that is $\forall s$-inefficient according to the alternative definition. All the weights are w (and are omitted in the drawing) and we use a product-independent threshold θ, where $w > \theta$.

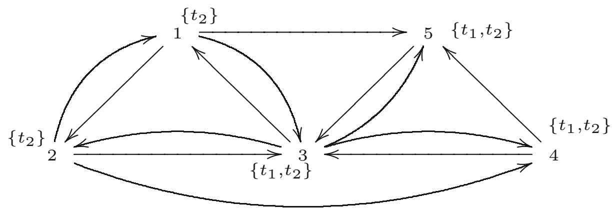


Fig. 11 An alternative $\forall s$-inefficient network

It is easy to check that the removal of the edge $4 \rightarrow 3$ triggers an (not unique) improvement path each of which leads from the initial Nash equilibrium $(t_2, t_2, t_1, t_1, t_1)$ to a Nash equilibrium $(t_2, t_2, t_2, t_2, t_2)$ in which each node has a strictly higher payoff. Consequently, this network is indeed $\forall s$-inefficient.

6 Unsafe networks

Finally, we have three notions that are counterparts of the fragility notions. We say that a social network $\mathcal{S}$ is $\exists$-***unsafe*** (respectively, $\forall$-***unsafe***) if for some Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, a contraction $\mathcal{S}'$ of $\mathcal{S}$ exists such that some (respectively, each) improvement path in $\mathcal{G}(\mathcal{S}')$ that starts in s is infinite. Further, a social network $\mathcal{S}$ is ***unsafe*** if $\mathcal{G}(\mathcal{S})$ has a Nash equilibrium, while for some contraction $\mathcal{S}'$ of $\mathcal{S}$, $\mathcal{G}(\mathcal{S}')$ does not.

Analogously to Sect. 4 each unsafe network is $\forall$-unsafe, while each $\forall$-unsafe network is $\exists$-unsafe. We now prove that these two implications are proper.

Example 13 (Unsafe) Let $\mathcal{S}_1$ be the modification of the network $\mathcal{S}$ given in Fig. 6 where node 1 and the source node marked with $\{t_1\}$ has the product set $\{t_1, t_2\}$. Consider the joint strategy in which this source node along with node 1 choose t_2, nodes 2 and 3 choose t_3 and nodes marked by $\{t_2\}$ and $\{t_3\}$ choose the unique product in their product set. This is a Nash equilibrium in $\mathcal{S}_1$. Now consider the contraction $\mathcal{S}_2$ of $\mathcal{S}_1$ in which the product t_2 is removed from the source node with product set $\{t_1, t_2\}$. Then $\mathcal{S}_2$ is same as the network $\mathcal{S}'$ in Example 6. Following the argument in Example 6 we conclude that the initial network $\mathcal{S}_1$ is unsafe.

Example 14 ($\forall$-unsafe) Let $\mathcal{S}_1$ be the modification of the network $\mathcal{S}$ given in Fig. 7 in which nodes 1 and 4 have now the product set $\{t_1, t_2\}$. We depict it in Fig. 12.

Consider the joint strategy in which nodes 1 and 4 choose t_2, nodes 2 and 3 choose t_3 and the remaining nodes choose the unique product in their product set. This is a Nash equilibrium in $\mathcal{S}_1$. Now consider the contraction $\mathcal{S}_2$ of $\mathcal{S}_1$ in which the product t_2 is removed from the product set $\{t_1, t_2\}$ of node 4. Then $\mathcal{S}_2$ is same as the network $\mathcal{S}'$ in Example 7. Following the argument in Example 7 we conclude that the initial network $\mathcal{S}_1$ is $\forall$-unsafe.

To prove that $\mathcal{S}_1$ is not unsafe for each contraction of it we need to find a Nash equilibrium in the associated game. Below we denote each contraction in a self-explanatory way and list the corresponding Nash equilibria by only giving the selections of nodes 1–4, as in each Nash equilibrium the choice of nodes 5–7 is, respectively, t_0, t_0 and t_3.

1 - t_1: (t_0, t_3, t_3, t_1),

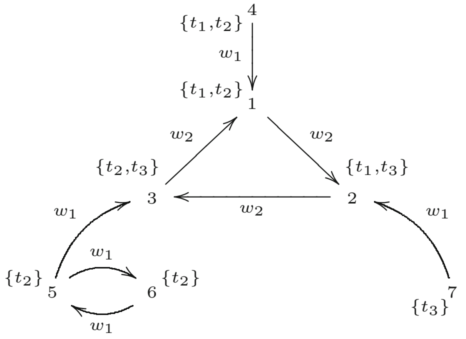


Fig. 12 A ∀-unsafe network that is not unsafe

1 - t_2: (t_0, t_3, t_3, t_2),
2 - t_1: (t_2, t_3, t_3, t_2),
2 - t_3: (t_2, t_0, t_0, t_2),
3 - t_2: (t_2, t_3, t_3, t_2),
3 - t_3: (t_0, t_3, t_0, t_2),
4 - t_1: (t_2, t_3, t_3, t_2),
4 - t_2: (t_1, t_1, t_0, t_1).

Example 15 (∃-unsafe) Let $\mathcal{S}_1$ be the modification of the network $\mathcal{S}$ given in Fig. 8 where node 2 has the product set $\{t_1, t_3, t_4\}$ and the source node marked with $\{t_3\}$ has the product set $\{t_1, t_3\}$. Consider the joint strategy in which this source node along with node 2 choose t_1, nodes 1 and 3 choose t_2 and nodes marked by $\{t_1\}$, $\{t_2\}$ and $\{t_4\}$ choose the unique product in their product set. This is a Nash equilibrium in $\mathcal{S}_1$. Now consider the contraction $\mathcal{S}_2$ of $\mathcal{S}_1$ in which the product t_1 is removed from the source node with product set $\{t_1, t_3\}$. Then $\mathcal{S}_2$ is same as the network $\mathcal{S}'$ in Example 8. Following the argument in Example 8 we conclude that the initial network $\mathcal{S}_1$ is ∃-unsafe but not ∀-unsafe.

7 Networks without source nodes

Given the variety of paradoxes exhibited in the above examples it is natural to investigate the status of selected networks. In this section we focus first on networks where there are no source nodes. This is a reasonable assumption in social networks as everybody usually has some friends who influence his decisions. We first identify a property which ensures the non-existence of $\exists w$-vulnerable networks, when the underlying graph has no source nodes.

For a joint strategy s and product t, let $\mathcal{A}_t(s) := \{i \in V \mid s_i = t\}$, and $prod(s) := \{s_i \mid i \in V\} \setminus \{t_0\}$. Hence, $prod(s)$ is the set of distinct strategies that are used in profile s. We let also $\bar{t}$ denote the joint strategy in which every player selects t. We say that a profile s is a ***multiple product*** profile, if $|prod(s)| > 1$.

Theorem 2 *Consider a network $\mathcal{S}$ whose underlying graph has no source nodes. If $\mathcal{G}(\mathcal{S})$ does not have a multiple product Nash equilibrium, then $\mathcal{S}$ is not $\exists w$-vulnerable.*

To prove this result, we use a specific structural property of Nash equilibria in networks whose underlying graph has no source nodes. Below, we only consider subgraphs that are *induced* and identify each such subgraph with its set of nodes. Recall that (V', E') is an induced subgraph of (V, E) if $V' \subseteq V$ and $E' = E \cap (V' \times V')$. For subgraphs C_1 and C_2, we denote by $C_1 \cap C_2$ the intersection of the nodes of the graphs. We say that a (non-empty) strongly connected subgraph (in short, SCS) C of G is ***self sustaining*** for a product t if for all $i \in C$,

- $t \in P(i)$,
- $\sum_{j \in N(i) \cap C} w_{ji} \geq \theta(i, t)$.

Hence, C is a self sustaining SCS for a product t if assigning this product to every node in C ensures that each node in C gets a non-negative payoff. A self sustaining SCS C is ***minimal*** for product t if no subgraph C' of C is a self sustaining SCS for product t. First we prove the following auxiliary result.

Lemma 1 *Let $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ be a network whose underlying graph has no source nodes. If $s \neq \overline{t_0}$ is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$, then for all products $t \in prod(s)$, there exists a minimal self sustaining SCS C for t such that $C \subseteq \mathcal{A}_t(s)$.*

Proof Suppose $s \neq \overline{t_0}$ is a Nash equilibrium. Take any product $t \neq t_0$ and an agent i such that $s_i = t$ (by assumption, at least one such t and i exists). Recall that $\mathcal{N}_i^t(s)$ denotes the set of neighbours of i who adopted in s the product t. Consider the set of nodes Pred $:= \bigcup_{m \in \mathbb{N}} \text{Pred}_m$, where

- $\text{Pred}_0 := \{i\}$,
- $\text{Pred}_{m+1} := \text{Pred}_m \bigcup \left(\bigcup_{j \in \text{Pred}_m} \mathcal{N}_j^t(s) \right)$.

We argue that some $C_t \subseteq$ Pred is a self sustaining SCS for product t. Consider the node i. Since s is a Nash equilibrium, we have $\sum_{k \in \mathcal{N}_i^t(s)} w_{ki} \geq \theta(i, t)$. Since $\theta(i, t) > 0$, we have $\mathcal{N}_i^t(s) \neq \emptyset$. By the construction of Pred, $\mathcal{N}_i^t(s) \subseteq \text{Pred}_1$. Iterating this reasoning, we get that the following invariant holds:

$$\text{for all } m \geq 0 \text{ and all } j \in \text{Pred}_m, \mathcal{N}_j^t(s) \neq \emptyset \text{ and } \sum_{k \in \text{Pred}_{m+1} \cap \mathcal{N}_j^t(s)} w_{kj} \geq \theta(j, t).$$

Since V is finite, there exists an $l \in \mathbb{N}$ such that $\text{Pred}_l = \text{Pred}_{l+1}$. We conclude that there exists $C_t \subseteq$ Pred which is an SCS and is self sustaining for product t. From C_t, we can then construct a minimal self sustaining SCS C for product t by dropping the appropriate nodes. □

Given a network $\mathcal{S}$ and a product $t \in \mathcal{P}$, let $\mathcal{C}_t(\mathcal{S})$ be the set of all minimal self sustaining SCSs for product t. Let $X_t(\mathcal{S}) = \bigcap_{C \in \mathcal{C}_t(\mathcal{S}), \mathcal{C}_t(\mathcal{S}) \neq \emptyset} C$ and $Y(\mathcal{S}) = \bigcap_{t \in \mathcal{P}, X_t(\mathcal{S}) \neq \emptyset} X_t(\mathcal{S})$.

Proof of Theorem 2: To prove the theorem, we show in fact the following claim. Suppose that for the network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ one of the following conditions holds:

1. for all $t \in \mathcal{P}$, $\mathcal{C}_t(\mathcal{S}) = \emptyset$,

2. $Y(\mathcal{S}) \neq \emptyset$.

Then, $\mathcal{S}$ is not $\exists w$-vulnerable.

In other words, if the network $\mathcal{S}$ does not have any self sustaining SCSs or if the intersection of the set of all minimal self sustaining SCSs is non-empty then $\mathcal{S}$ is not $\exists w$-vulnerable. We will show that condition 1 implies that $\overline{t_0}$ is the only Nash equilibrium in $\mathcal{G}(\mathcal{S})$ and condition 2 implies that $|prod(s)| = 1$ for any Nash equilibrium s in $\mathcal{G}(\mathcal{S})$ that is different from $\overline{t_0}$.

Note that if $\mathcal{S}$ is a network whose underlying graph has no source nodes then the joint strategy $\overline{t_0}$ is always a Nash equilibrium. This is because for all nodes i, and for all products $t \in \mathcal{P}$, $\mathcal{N}_i^t(\overline{t_0}) = \emptyset$. Therefore, no player has a profitable deviation from $\overline{t_0}$. Suppose that for all $t \in \mathcal{P}$, $\mathcal{C}_t(\mathcal{S}) = \emptyset$. From Lemma 1, it follows that $\overline{t_0}$ is the only Nash equilibrium in $\mathcal{S}$. Consider any expansion $\mathcal{S}'$ of $\mathcal{S}$. Then no player has a profitable deviation from $\overline{t_0}$ in $\mathcal{S}'$. Therefore, $\mathcal{S}$ is not $\exists w$-vulnerable.

Now suppose that $Y(\mathcal{S}) \neq \emptyset$. In this case, we first claim that every non-trivial Nash equilibrium s in $\mathcal{S}$ has the property that $prod(s) \subseteq \{t_1\}$ for some $t_1 \in \mathcal{P}$. Suppose this is not the case then for some two different products t_1 and t_2, $\{t_1, t_2\} \subseteq prod(s)$. Since s is a Nash equilibrium, by Lemma 1, there exists a minimal self sustaining SCS $C_1 \subseteq \mathcal{A}_{t_1}(s)$ for t_1 and a minimal self sustaining SCS $C_2 \subseteq \mathcal{A}_{t_2}(s)$ for t_2. By definition, $\mathcal{A}_{t_1}(s) \cap \mathcal{A}_{t_2}(s) = \emptyset$ and therefore, $\mathcal{C}_{t_1}(\mathcal{S}) \cap \mathcal{C}_{t_2}(\mathcal{S}) = \emptyset$. This contradicts the assumption that $Y(\mathcal{S}) \neq \emptyset$.

Consider a Nash equilibrium s in $\mathcal{S}$ and an expansion $\mathcal{S}'$. By the above claim $prod(s) \subseteq \{t\}$ for some $t \in \mathcal{P}$. In the expansion $\mathcal{S}'$, if a new product $t' \neq t$ is added to the product set of a node i, then there is no profitable deviation from s since $\mathcal{N}_i^{t'}(s) = \emptyset$. Consequently, there is no improvement path starting at s in $\mathcal{G}(\mathcal{S}')$.

Thus the relevant case is when the product t is added to a node i (with $s_i = t_0$). Consider any improvement path in $\mathcal{G}(\mathcal{S}')$ that leads to a Nash equilibrium s' in $\mathcal{G}(\mathcal{S}')$. Since s is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$ the first profitable deviation in the improvement path is of the form $i : t_1$. In the improvement path, if a joint strategy s^2 is obtained from s^1 by having some nodes switch to product t_1 and t_1 is a best response for a node j to s^1_{-j}, then t_1 is also a best response for j to s^2_{-j}. Indeed, by the join the crowd property $p_j(t_1, s^2_{-j}) \geq p_j(t_1, s^1_{-j}) \geq p_j(t_1, s^1_{-j}) = 0 = p_j(t_0, s^2_{-j})$. So the only deviations in this improvement path are to t_1. Consequently in s' which is a Nash equilibrium, t_1 is the product selected by node i (i.e., $s'_i = t_1$) and $p_i(s') > p_i(s)$. Therefore, the network is not $\exists w$-vulnerable. □

8 Simple cycle networks

In this section we focus on networks where the underlying graph is a simple cycle, say $1 \to 2 \to \ldots \to n \to 1$. We assume that the counting is done in cyclic order within $\{1, \ldots, n\}$ using the increment operation $i \oplus 1$ and the decrement operation $i \ominus 1$. In particular, $n \oplus 1 = 1$ and $1 \ominus 1 = n$. We start with the following corollary to Theorem 2.

We begin with the following observation.

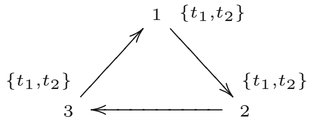


Fig. 13 An $\exists s$-inefficient simple cycle network

Note 1 Consider a simple cycle network $\mathcal{S}$. If s is a Nash equilibrium of the game $\mathcal{G}(\mathcal{S})$ then $s = \bar{t}$ for some $t \in \mathcal{P} \cup \{t_0\}$. Moreover, $\overline{t_0}$ is always a Nash equilibrium.

Proof First note that $\overline{t_0}$ is indeed a Nash equilibrium. Consider now a Nash equilibrium $s \neq \overline{t_0}$. Then there exists a product t and a node i such that $s_i = t$. Since s is a Nash equilibrium, we have $p_i(s) \geq p_i(t_0, s_{-i}) = 0$, so $s_{i \ominus 1} = t$ as well as otherwise the node i would have negative payoff. Iterating this reasoning we conclude that $s = \bar{t}$. □

Corollary 1 *Simple cycle networks are not* $\exists w$*-vulnerable (and a fortiori not* XY*-vulnerable, where* $X \in \{\exists, \forall\}$ *and* $Y \in \{w, s\}$*).*

Proof This is an immediate consequence of Note 1 and Theorem 2. □

The remaining types of deficiency are easy to determine. For the case of fragile networks we prove the following result.

Theorem 3 *Simple cycle networks are not* $\exists$*-fragile (and a fortiori not* $\forall$*-fragile and not fragile).*

Proof Consider a simple cycle network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$, a Nash equilibrium s of $\mathcal{G}(\mathcal{S})$, and an expansion $\mathcal{S}'$ of $\mathcal{S}$. By Note 1 $s = \bar{t}$, where $t \in \mathcal{P} \cup \{t_0\}$. Hence s remains a Nash equilibrium of $\mathcal{G}(\mathcal{S}')$. Consequently $\mathcal{S}$ is not $\exists$-fragile. □

In the case of inefficient networks we have the following result.

Theorem 4 *(i) There exists a simple cycle network* $\mathcal{S}$ *that is* $\exists s$*-inefficient (and a fortiori* $\exists w$*-inefficient).*
(ii) Simple cycle networks are not $\forall w$*-inefficient (and a fortiori not* $\forall s$*-inefficient).*

Proof (i) Consider the network shown in Fig. 13. Suppose that $\theta(i, t_1) > \theta(i, t_2)$ for all nodes $i = 1, 2, 3$ and that $s = \overline{t_1}$ is a Nash equilibrium. Starting from s, suppose we remove t_1 from the product set of node 1. Then there exists a finite improvement path where all the nodes end up adopting t_2, by simply having node 1 adopt t_2 and then having the remaining nodes follow their best response. Since $\theta(i, t_1) > \theta(i, t_2)$, all nodes are strictly better off in this new Nash equilibrium, $\overline{t_2}$.
(ii) Consider a simple cycle network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$, a Nash equilibrium s of $\mathcal{G}(\mathcal{S})$, and a contraction $\mathcal{S}'$ of $\mathcal{S}$. By Note 1 $s = \bar{t}$, where $t \in \mathcal{P} \cup \{t_0\}$. If $s = \overline{t_0}$, then it is impossible that by deleting any product we could make some player better off. Suppose $s = \overline{t_1}$ for some product t_1. If we delete t_1 from some product set, say of node 1, then there is always an improvement path that terminates at the Nash equilibrium $\overline{t_0}$ (simply

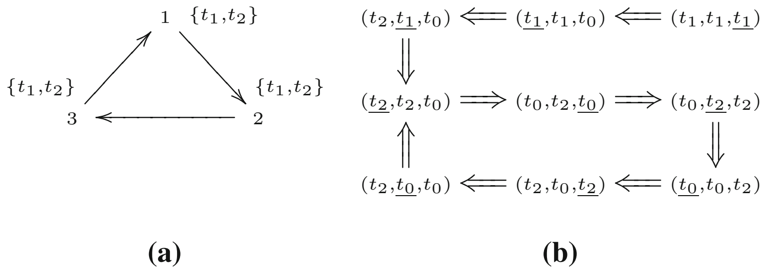


Fig. 14 A simple cycle network and an infinite improvement path

start with node 1 adopting t_0, and proceed clockwise. Then gradually every other node will switch to t_0 since they eventually lose support for t_1). Hence no node is better off in this new Nash equilibrium. In conclusion, there can be no Nash equilibrium from which all improvement paths after the contraction will make the set of nodes weakly better off. □

Finally, we consider the case of unsafe networks.

Theorem 5 *(i) There exists a simple cycle network $\mathcal{S}$ that is ∃-unsafe.*
(ii) Simple cycle networks are not ∀-unsafe (and a fortiori not unsafe).

Proof (i) Consider the network shown in Fig. 14a. The weight of each edge is assumed to be w and is omitted. We also assume that $\theta(1, t_2) < \theta(1, t_1) < w$ and $\theta(2, t_2) < \theta(2, t_1) < w$ (so that product t_2 is more attractive for nodes 1 and 2 than product t_1) and that on all other arguments the threshold is equal to a constant θ, where $0 < \theta < w$. These assumptions imply that $\overline{t_1}$ is a Nash equilibrium.

By removing from the product set of node 3 the product t_1 we get in the resulting game an infinite improvement path depicted in Fig. 14b. (In each joint strategy we underline a strategy that is not a best response to the choice of other players.) So the initial network is ∃-unsafe.

(ii) By Theorem 28 in Simon and Apt (2015) for every simple cycle network $\mathcal{S}$ there exists a finite improvement path in $\mathcal{G}(\mathcal{S})$. This implies both claims. □

The above analysis does not carry through to all strongly connected graphs. Indeed, we showed in particular that simple cycle networks cannot be $\forall w$-vulnerable and also not $\forall s$-inefficient. However, in Example 3 we exhibited a network that is $\forall w$-vulnerable and in Example 9 a network that is $\forall s$-inefficient. The underlying graphs of both networks are strongly connected.

9 Conclusions

In this paper we provided a systematic study of paradoxes that can arise in social networks with multiple products. Such paradoxes allow us to better understand possible undesirable consequences of modifying the choices that are available to the agents within a social network (which can correspond to new products, new technologies or adoption of new ideas). The focus of our work was on identifying these paradoxes and on determining their relative strength.

To analyze them, we used a natural game-theoretic framework in the form of social network games introduced in Simon and Apt (2012) and Simon and Apt (2015). These games do not always admit (pure) Nash equilibria, and as a result, more types of paradoxes can arise (as we exhibited), than in the class of congestion games with its celebrated Braess paradox. Out of all the notions of paradoxes that we introduced and studied, one question still remained open: do $\forall s$-vulnerable networks exist?

In future work we plan to assess the computational complexity of determining the presence of these paradoxes and plan to analyze other selected networks. We also plan to expand our analysis of selected classes of networks, determining which paradoxes can then be present.

Finally, in our analysis we assumed that the agents can refrain from selecting a product. In Apt and Simon (2013) an alternative version of the social network games is studied, in which each agent has to choose a product. This corresponds to natural situations, for instance when pupils have to choose a primary school or when each student has to select a laptop. Such social networks are studied by modifying our framework so that the strategy t_0 is not available. This change leads to a different analysis and different results. In particular, in this setup the authors solved the open problem reported in Sect. 3, by showing that the $\forall s$-vulnerable networks exist.

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