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On Orthogonal Polynomials with Positive Zeros and the Associated Kernel Polynomials

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I. INTRODUCTION

Let $\mathscr{P} \equiv \{P_n(x)\}_{n=0}^{\infty}$ be a sequence of monic polynomials satisfying the recurrence relation

$$\begin{aligned} &P_n(x) = (x - c_n)\, P_{n-1}(x) - \lambda_n P_{n-2}(x), \qquad (n > 1) \\ &P_0(x) = 1, \qquad P_1(x) = x - c_1, \end{aligned} \tag{1.1}$$

where c_n is real and $\lambda_{n+1} > 0$ $(n > 0)$. A recurrence of this type is a necessary and sufficient condition for $\mathscr{P}$ to constitute an orthogonal sequence. Specifically, there is a mass distribution $d\psi$ on the real line (with total mass 1 and infinite support) such that

$$\int_{-\infty}^{\infty} P_n(x)\, P_m(x)\, d\psi(x) = \delta_{nm} \prod_{j=1}^{n} \lambda_{j+1}, \tag{1.2}$$

where the empty product is interpreted as unity (cf. Chihara [2, 4] for these and subsequent preliminary results).

$P_n(x)$ has n real, simple zeros $x_{n1}(\mathscr{P}) < x_{n2}(\mathscr{P}) < \cdots < x_{nn}(\mathscr{P})$. Moreover, the zeros of $P_n(x)$ and $P_{n+1}(x)$ separate each other, that is,

$$x_{n+1,i}(\mathscr{P}) < x_{ni}(\mathscr{P}) < x_{n+1,i+1}(\mathscr{P}) \qquad (i = 1, 2, \ldots, n), \tag{1.3}$$

so that the limits

$$\xi_i(\mathscr{P}) \equiv \lim_{n \to \infty} x_{ni}(\mathscr{P}) \qquad (i \geqslant 1)$$

exist, and

$$-\infty \leqslant \xi_1(\mathscr{P}) \leqslant \xi_2(\mathscr{P}) \leqslant \cdots.$$

Throughout this paper we shall assume that the zeros of $P_n(x)$ are positive, that is, $\xi_1(\mathscr{P}) \geqslant 0$.

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We write

$$\sigma(\mathscr{P}) \equiv \lim_{i \to \infty} \xi_i(\mathscr{P}),$$

and remark that for $i \geq 1$

$$\xi_i(\mathscr{P}) = \xi_{i+1}(\mathscr{P}) \Rightarrow \xi_i(\mathscr{P}) = \sigma(\mathscr{P}). \tag{1.4}$$

The distribution $d\psi$ of (1.2) is uniquely determined by $\{c_n, \lambda_{n+1}\}$ if and only if the Hamburger moment problem (Hmp) associated with $\mathscr{P}$ is determined. In this case we have

$$\operatorname{supp}(d\psi) \cap (-\infty, \sigma(\mathscr{P})] = \overline{\Xi(\mathscr{P})} \qquad \text{if} \quad \sigma(\mathscr{P}) < \infty \tag{1.5}$$

and

$$\operatorname{supp}(d\psi) = \Xi(\mathscr{P}) \qquad \text{if} \quad \sigma(\mathscr{P}) = \infty, \tag{1.6}$$

where $\Xi(\mathscr{P}) \equiv \{\xi_1(\mathscr{P}), \xi_2(\mathscr{P}),...\}$ and a bar denotes closure. If the Hmp for $\mathscr{P}$ is indeterminate, then $\sigma(\mathscr{P}) = \infty$ and there is exactly one distribution $d\psi$ satisfying (1.2) and $\operatorname{supp}(d\psi) = \Xi(\mathscr{P})$; any other distribution satisfying (1.2) has at least one supporting point smaller than $\xi_1(\mathscr{P})$.

By $\mathscr{P}^* = \{P_n^*(x)\}_{n=0}^{\infty}$ we denote the sequence of *kernel polynomials* (with parameter 0) associated with $\mathscr{P}$, that is,

$$xP_n^*(x) = P_{n+1}(x) - \frac{P_{n+1}(0)}{P_n(0)} P_n(x). \tag{1.7}$$

$\mathscr{P}^*$ constitutes a sequence of monic, orthogonal polynomials and therefore there exist (unique) real numbers c_n^* and positive numbers λ_{n+1}^* $(n > 0)$ determining a recurrence of the type (1.1) for $\mathscr{P}^*$. Furthermore, if $\mathscr{P}$ is orthogonal with respect to $d\psi$, then $\mathscr{P}^*$ is orthogonal with respect to the distribution $d\psi^*$ defined by

$$\int_{-\infty}^{x} d\psi^*(t) = c_1^{-1} \int_{-\infty}^{x} t\, d\psi(t) \tag{1.8}$$

(the factor c_1^{-1} ensuring that $d\psi^*$ has total mass 1).

In what follows we write

$$x_{ni} \equiv x_{ni}(\mathscr{P}) \qquad \text{and} \qquad x_{ni}^* \equiv x_{ni}(\mathscr{P}^*).$$

The quantities ξ_i, ξ_i^*, σ, and σ^*, and the sets Ξ and Ξ^* are defined similarly. We note that as a consequence of the separation theorem

$$x_{ni} < x_{ni}^* < x_{n+1,i+1} \qquad (i = 1, 2,..., n) \tag{1.9}$$

[4, Theorem 1.7.2], we have

$$\xi_i \leqslant \xi_i^* \leqslant \xi_{i+1} \qquad (i \geqslant 1). \tag{1.10}$$

After these introductory remarks we are prepared to give the plan of the paper. Exploiting properties of kernel polynomials we establish some results relating Ξ to the parameters $\{c_n, \lambda_{n+1}\}$ and to asymptotic properties of $\mathscr{P}$ in Section II. In Section III the precise conditions for an equality to hold in (1.10) are derived. Using our findings of Sections II and III we prove some further limit theorems for $\mathscr{P}$ in Section IV.

The results of this paper are of particular interest in the analysis of birth–death processes, where sets of orthogonal polynomials satisfying $\xi_1 \geqslant 0$ play a key role (cf. [7]). We hope to report on these applications in subsequent publications.

II. Properties of Ξ

Because of the assumption $\xi_1 \geqslant 0$, we can invoke a result of Chihara [4, Theorem 1.9.1] stating that $\xi_1 \geqslant 0$ is a necessary and sufficient condition for the existence of a unique sequence $\{\gamma_n\}_{n=2}^{\infty}$ of positive numbers such that

$$c_n = \gamma_{2n-1} + \gamma_{2n}, \qquad \lambda_{n+1} = \gamma_{2n}\gamma_{2n+1} \qquad (n > 0), \tag{2.1}$$

where $\gamma_1 \equiv 0$. Clearly, $\{\gamma_n\}$ can be determined recursively from $\{c_n, \lambda_{n+1}\}$. It is convenient to introduce the quantities

$$G_n \equiv \prod_{i=1}^{n} \frac{\gamma_{2i}}{\gamma_{2i+1}}, \qquad H_n \equiv \frac{1}{\gamma_2} \prod_{i=1}^{n} \frac{\gamma_{2i+1}}{\gamma_{2i+2}} \qquad (n \geqslant 0) \tag{2.2}$$

(where we deviate slightly from Chihara's [5] notation), maintaining the convention that the empty product denotes unity. In addition we let

$$K_n \equiv \sum_{i=0}^{n} G_i, \qquad L_n \equiv \sum_{i=0}^{n} H_i \qquad (n \geqslant 0). \tag{2.3}$$

A basic result for which we have later use is the following:

Lemma 1. [5, Theorem 3]. *The Hmp for $\mathscr{P}$ is determined if and only if $\sum_{n=0}^{\infty} G_{n+1} L_n^2 = \infty$.*

Since

$$P_n(x)/P_n(0) = \prod_{i=1}^{n} (1 - x_{ni}^{-1} x),$$

the quantity $-\sum_{i=1}^{n} x_{ni}^{-1}$ equals the coefficient of x in the polynomial $P_n(x)/P_n(0)$. To calculate this coefficient we observe from (1.1) and (2.1) that

$$P_n(0) = (-1)^n \prod_{i=1}^{n} \gamma_{2i}. \tag{2.4}$$

Using (1.1), (2.1), and (2.4) we can write down a recurrence formula for $P_n(x)/P_n(0)$ in terms of $\{\gamma_n\}$, from which the result

$$\sum_{i=1}^{n} x_{ni}^{-1} = \sum_{k=0}^{n-1} H_k K_k \tag{2.5}$$

follows readily by induction. Letting n tend to infinity in (2.5) subsequently yields

$$\sum_{i=1}^{\infty} \xi_i^{-1} = \sum_{k=0}^{\infty} H_k K_k, \tag{2.6}$$

where the left-hand side is interpreted as infinity if $\xi_1 = 0$.

THEOREM 1. *The following statements are equivalent*:

(i) $\sum_{i=1}^{\infty} \xi_i^{-1} < \infty$,

(ii) $\sum_{n=0}^{\infty} H_n K_n < \infty$,

(iii) $\{P_n(x)/P_n(0)\}_n$ *converges uniformly on bounded sets to an entire function whose zeros are simple and are precisely the points* ξ_i $(i \geq 1)$,

(iv) $\{P_n(x)/P_n(0)\}_n$ *is bounded as* $n \to \infty$ *for at least one* $x < 0$.

Proof. The equivalence of (i) and (ii) is obvious from (2.6). Statement (iii) implies $\xi_1 > 0$, so that the equivalence of (i) and (iii) follows from [3, Theorem 2].

Finally, a proof for the equivalence of (ii) and (iv) can be found in [7, Lemma 4]. The results of the latter paper are stated in the context of birth–death processes and can be translated to our present notation in the manner indicated by Chihara [5, pp. 335–336]. ∎

This theorem raises two interesting problems. The first, which will be addressed in Section IV, is to obtain more information on the limiting behaviour of $\{P_n(x)/P_n(0)\}_n$ for $x > 0$ when $\sum H_n K_n = \infty$. The second problem comes from the observation that for $\sum_{i=1}^{\infty} \xi_i^{-1}$ to be finite it is necessary and sufficient that both $\xi_1 > 0$ and $\xi_i \to \infty$ sufficiently fast as $i \to \infty$. Since the first event is merely a matter of translation, while the second is a more intrinsic property of $\mathscr{P}$, it is of interest to separate the factors in statement (ii) which are responsible for each of the events individually. Therefore (cf. (1.4)), we now look for a criterion for convergence of the series $\sum_{i=2}^{\infty} \xi_i^{-1}$. To solve this problem we turn our attention to the set $\mathscr{P}^*$ of kernel polynomials associated with $\mathscr{P}$.

By [4, Theorem 1.9.1] the parameters c_n^* and λ_{n+1}^* $(n>0)$ in the recurrence for $\mathscr{P}^*$ satisfy

$$c_n^* = \gamma_{2n} + \gamma_{2n+1}, \qquad \lambda_{n+1}^* = \gamma_{2n+1}\gamma_{2n+2} \qquad (n>0), \tag{2.7}$$

where $\{\gamma_n\}_{n=2}^{\infty}$ is the sequence of positive numbers which is uniquely determined by $\{c_n, \lambda_{n+1}\}$ through (2.1). Note that c_1^* is written in (2.7) as the sum of two positive numbers, which makes the representation (2.7) essentially different from (2.1). However, in view of (1.11), our assumption $\xi_1 \geqslant 0$ implies that $\xi_1^* \geqslant 0$, so that Theorem 1 is valid for $\mathscr{P}^*$ as well. It can easily be verified that the appropriate quantities γ_n^* satisfy

$$\gamma_{2n}^* = \gamma_{2n+1} K_{n-1}^{-1} K_n, \qquad \gamma_{2n+1}^* = \gamma_{2n+2} K_{n-1} K_n^{-1}, \qquad (n>0). \tag{2.8}$$

Thus we can formulate the analogue for $\mathscr{P}^*$ of statement (ii) of Theorem 1 in terms of $\{\gamma_n\}$. A more convenient formulation is obtained, however, if we proceed as before by observing that $-\sum_{i=1}^{n} (x_{ni}^*)^{-1}$ equals the coefficient of x in the polynomial $P_n^*(x)/P_n^*(0)$. From (1.1) and (2.7) we now have

$$P_n^*(0) = (-1)^n K_n \prod_{i=1}^{n} \gamma_{2i+1}. \tag{2.9}$$

Some simple manipulations involving (2.7), (2.9), the recurrence formula for $\mathscr{P}^*$, and an induction argument subsequently yield

$$\sum_{i=1}^{n} (x_{ni}^*)^{-1} = K_n^{-1} \sum_{j=0}^{n} G_j \sum_{i=0}^{j-1} H_i K_i, \tag{2.10}$$

so that

$$\sum_{i=1}^{\infty} (\xi_i^*)^{-1} = \lim_{n\to\infty} K_n^{-1} \sum_{j=0}^{n} G_j \sum_{i=0}^{j-1} H_i K_i. \tag{2.11}$$

The next theorem answers our question regarding the finiteness of $\sum_{i=2}^{\infty} \xi_i^{-1}$.

THEOREM 2. *The following statements are equivalent*:

(i) $\sum_{i=2}^{\infty} \xi_i^{-1} < \infty$,

(ii) $\sum_{i=1}^{\infty} (\xi_i^*)^{-1} < \infty$,

(iii) $\sum_{n=0}^{\infty} G_{n+1} L_n < \infty$ *or* $\sum_{n=0}^{\infty} H_n K_n < \infty$.

Proof. By [6, Lemma A.1] one has $\xi_i^* = \xi_{i+1}$ $(i \geqslant 1)$ if $\xi_1 = 0$. The equivalence of (i) and (ii) now follows readily from (1.10).

Considering (2.11), the equivalence of (ii) and (iii) can be established by proving that for (iii) to be valid it is necessary and sufficient that

$$\lim_{n \to \infty} K_n^{-1} \sum_{j=0}^{n} G_j \sum_{i=0}^{j-1} H_i K_i < \infty. \tag{2.12}$$

Recalling (2.3), the necessity is obvious. Next suppose (2.12) holds. If $K_n < K < \infty$ for all n, it follows immediately that $\sum_{j=0}^{\infty} G_{j+1} L_j < \infty$; if, on the other hand, $K_n \to \infty$ as $n \to \infty$, it is an easy exercise to show that $\sum_{i=0}^{\infty} H_i K_i$ must be finite. This proves the sufficiency. ∎

COROLLARY 1. *One has* $\xi_1 = 0$ *and* $\sum_{i=2}^{\infty} \xi_i^{-1} < \infty$ *if and only if* $\sum_{n=0}^{\infty} G_{n+1} L_n < \infty$ *and* $L_n \to \infty$ $(n \to \infty)$.

Proof. This is an immediate consequence of Theorems 1 and 2. ∎

COROLLARY 2. *If* $\sum_{i=2}^{\infty} \xi_i^{-1} = \infty$, *then the Hmp for* $\mathscr{P}$ *is determined.*

Proof. If $\sum_{i=2}^{\infty} \xi_i^{-1} = \infty$, then, by Theorem 2, $\sum_{n=0}^{\infty} G_{n+1} L_n = \infty$, whence $\sum_{n=0}^{\infty} G_{n+1} L_n^2 = \infty$. The result follows by Lemma 1. ∎

Remarks. (1) By interpreting [7, Lemma 4] in terms of $\mathscr{P}^*$, it follows that these three statements are equivalent:

(i) $\sum_{n=0}^{\infty} G_{n+1} L_n < \infty$,

(ii) $\{K_n P_n^*(x)/P_n^*(0)\}_n$ converges uniformly on bounded sets to an entire function,

(iii) $\{K_n P_n^*(x)/P_n^*(0)\}_n$ is bounded as $n \to \infty$ for at least one $x < 0$.

(2) Theorem 1 can be partially traced back to Stieltjes [10, pp. 524–527] (see also [1]) by making the identifications

$$a_{2n+1} = G_n, \qquad a_{2n+2} = H_n$$

and

$$Q_{2n}(x) = P_n(-x)/P_n(0), \qquad Q_{2n+1}(x) = xK_n P_n^*(-x)/P_n^*(0).$$

(3) An unproven statement by Küchler [8, p. 229] amounts to the equivalence of (i) and (iii) in Theorem 2.

III. Relations between Ξ and Ξ^*

We start off to give some relations between the polynomials P_n and P_n^*. From (1.7) and (2.4) we obtain

$$xP_n^*(x) = P_{n+1}(x) + \gamma_{2n+2} P_n(x). \tag{3.1}$$

Combining this result with (1.1) and (2.1) gives us

$$P_n(x) = P_n^*(x) + \gamma_{2n+1} P_{n-1}^*(x). \tag{3.2}$$

Some simple algebra involving (2.9) and (3.2) subsequently yields

$$G_n P_n(x)/P_n(0) = K_n P_n^*(x)/P_n^*(0) - K_{n-1} P_{n-1}^*(x)/P_{n-1}^*(0), \tag{3.3}$$

so that

$$\sum_{k=0}^{n} G_k P_k(x)/P_k(0) = K_n P_n^*(x)/P_n^*(0). \tag{3.4}$$

Invoking Theorem 1 one easily obtains from (3.4) the next lemma, which is essentially due to Stieltjes [10, pp. 525–526].

Lemma 2. *If $\sum_{n=0}^{\infty} H_n K_n < \infty$ and $K_n \to \infty$ as $n \to \infty$, then $P_n(x)/P_n(0)$ and $P_n^*(x)/P_n^*(0)$ tend to the same entire function as $n \to \infty$.*

Remark. One can also prove that if $\sum_{n=0}^{\infty} G_{n+1} L_n < \infty$ and $L_n \to \infty$ as $n \to \infty$, then $xK_n P_n^*(x)/P_n^*(0)$ and $L_n^{-1} P_n(x)/P_n(0)$ tend to the same entire function as $n \to \infty$ (cf. Corollary 1 and Remark 1 in Section II).

We are now in a position to state the main result of this section (cf. Lemma 1).

Theorem 3. (i) *If $\xi_1 = 0$, then $\xi_i^* = \xi_{i+1}$ $(i \geqslant 1)$.*

(ii) *If $\xi_1 > 0$ and the Hmp for $\mathscr{P}$ is determined, then $\xi_i^* = \xi_i$ $(i \geqslant 1)$.*

(iii) *If $\xi_1 > 0$ and the Hmp for $\mathscr{P}$ is indeterminate, then $\xi_i < \xi_i^* < \xi_{i+1}$ $(i \geqslant 1)$.*

Proof. Part (i) has been established in [6, Lemma A.1].

As for (ii) we denote by $d\psi$ the unique distribution with respect to which $\mathscr{P}$ is orthogonal. Then $\mathscr{P}^*$ is orthogonal with respect to the distribution $d\psi^*$ defined by (1.8). Note that

$$\text{supp}(d\psi^*) = \text{supp}(d\psi), \tag{3.5}$$

since $\xi_1 = \min(\text{supp}(d\psi)) > 0$. If the Hmp for $\mathscr{P}^*$ is determined, then $d\psi^*$ is the unique distribution for $\mathscr{P}^*$, so that $\xi_i^* = \xi_i$ $(i \geqslant 1)$ by (1.5), (1.6), and (3.5). Therefore, let us assume that the Hmp for $\mathscr{P}^*$ is indeterminate. From [5, Theorem 3] we then have $\sum_{n=1}^{\infty} H_n(K_n - 1)^2 < \infty$, whence $\sum_{n=0}^{\infty} H_n K_n < \infty$. It follows that at the same time $K_n \to \infty$ as $n \to \infty$, for the opposite would imply indeterminacy of the Hmp for $\mathscr{P}$ by Lemma 1. Thus we can invoke Lemma 2 and apply Theorem 1 to both $\mathscr{P}$ and $\mathscr{P}^*$ to conclude that $\xi_i^* = \xi_i$ $(i \geqslant 1)$.

Finally turning to part (iii) we note that if $\xi_1 > 0$ and the Hmp for $\mathscr{P}$ is indeterminate, then also the Stieltjes moment problem for $\mathscr{P}$ is indeterminate [2]. By [5, Theorem 2] this is equivalent to $\{K_n + L_n\}_n$ being bounded. Under the latter condition, however, the validity of (iii) was established in [5, p. 340]. ∎

IV Further Limit Theorems

For any sequence $\{a_n\}_{n=0}^{\infty}$ we denote by $S\{a_n\}$ the number of sign changes in the sequence $\{a_n\}$ after deleting all zero terms. By convention, $S\{0\} = -1$. Now let $\mathscr{R} \equiv \{R_n(x)\}_{n=0}^{\infty}$ be any sequence of monic orthogonal polynomials. From [6, Theorem 3] we then have

$$S\{(-1)^n R_n(x)\} = k \Leftrightarrow \xi_k(\mathscr{R}) < x \leqslant \xi_{k+1}(\mathscr{R}), \qquad (k \geqslant 0), \tag{4.1}$$

where $\xi_0(\mathscr{R}) \equiv -\infty$. Assuming that $\xi_1(\mathscr{R}) > -\infty$ we subsequently define

$$I(\mathscr{R}) \equiv \bigcup_{i=1}^{\infty} (-\infty, \xi_i(\mathscr{R})].$$

Note that $I(\mathscr{R}) = (-\infty, \sigma(\mathscr{R})]$ or $I(\mathscr{R}) = (-\infty, \sigma(\mathscr{R}))$, depending on the occurrence of the event $\xi_i(\mathscr{R}) = \xi_{i+1}(\mathscr{R})$ for some i (cf. (1.4)).

Returning to the context of the previous sections we note that $I(\mathscr{P}) = I(\mathscr{P}^*)$ in view of (1.10). Now applying (4.1) to both $\mathscr{P}$ and $\mathscr{P}^*$ one readily sees that for each $x \in I(\mathscr{P})$ there exists an integer $N = N(x)$ such that the sequence $\{P_n(x)/P_n(0)\}_{n=N}^{\infty}$ is monotone and without sign changes. In particular for $x < 0$ $(\leqslant \xi_1)$ it is easily shown that the sequence $\{P_n(x)/P_n(0)\}_{n=0}^{\infty}$ is positive and increasing. Whether $P_n(x)/P_n(0)$ tends to infinity or not as $n \to \infty$ must be decided from Theorem 1. In what follows we restrict our attention to positive x. Theorem 3 enables us to relate the behaviour of $\{P_n(x)/P_n(0)\}$ to the points ξ_i $(i \geqslant 1)$. Indeed, from Theorem 3(ii) and (4.1), applied to both $\mathscr{P}$ and $\mathscr{P}^*$, we easily obtain the following lemma.

Lemma 3. *If the Hmp for* $\mathscr{P}$ *is determined,* $\xi_1 > 0$, $x > 0$ *and*

$\xi_k < x \leqslant \xi_{k+1}$ $(k \geqslant 0)$, *then there exists an integer* $N = N(x)$ *such that the sequence* $\{(-1)^k P_n(x)/P_n(0)\}_{n=N}^{\infty}$ *is positive and decreasing* $(N(x) = 0$ *if* $0 < x \leqslant \xi_1)$.

Under the conditions of this lemma the sequence $\{P_n(x)/P_n(0)\}_n$ tends to a finite limit. The next theorem, which is a generalization of [3, Lemma 3] gives a criterion for this limit to be zero when $x < \sigma$ (cf. Theorem 1).

THEOREM 4. *If* $\xi_1 > 0$, $\sum_{i=2}^{\infty} \xi_i^{-1} = \infty$ *and* $0 < x < \sigma$, *then* $P_n(x)/P_n(0) \to 0$ *as* $n \to \infty$.

Proof. By Corollary 2 the Hmp for $\mathscr{P}$ is determined when $\sum_{i=2}^{\infty} \xi_i^{-1} = \infty$, so that Lemma 3 applies. Let a be any positive number smaller than σ, and

$$R_a \equiv \max_{0 \leqslant x \leqslant a} \max_{n \geqslant 0} |P_n(x)/P_n(0)|.$$

By Lemma 3, $R_a < \infty$. Moreover, by [3, Lemma 3], $P_n(x)/P_n(0) \to 0$ as $n \to \infty$ for $0 < x < \xi_1$. The result follows by the Stieltjes–Vitali theorem. ∎

Remark. We conjecture that Theorem 4 remains valid when the condition $0 < x < \sigma$ is replaced by the condition $0 < x < \lim_{n \to \infty} x_{nn}$.

Let us now turn to the case $\xi_1 = 0$. Theorem 3(i) and (4.1), applied to both $\mathscr{P}$ and $\mathscr{P}^*$, readily yield the next lemma.

LEMMA 4. *If* $\xi_1 = 0$ *and* $\xi_k < x \leqslant \xi_{k+1}$ $(k \geqslant 1)$, *then there is an integer* $N = N(x)$ *such that the sequence* $\{(-1)^k P_n(x)/P_n(0)\}_{n=N}^{\infty}$ *is positive and increasing.*

Actually, we can show the following.

THEOREM 5. *If* $\xi_1 = 0$ *and* $\xi_k < x < \xi_{k+1}$ $(k \geqslant 1)$, *then* $(-1)^k P_n(x)/P_n(0) \to \infty$ *as* $n \to \infty$.

Proof. First suppose that the Hmp for $\mathscr{P}$ is determined. If $\sigma = 0$ the theorem is vacuously true, therefore we also assume $\sigma > 0$. Denoting by $p_n(x)$ the nth *orthonormalized* polynomial corresponding to $\mathscr{P}$, it is readily seen from (1.2), (2.1), (2.2), and (2.4) that

$$p_n(x) = G_n^{1/2} P_n(x)/P_n(0), \tag{4.2}$$

whence

$$\sum_{n=0}^{\infty} p_n^2(x) = \sum_{n=0}^{\infty} G_n P_n^2(x)/P_n^2(0). \tag{4.3}$$

By (1.5), (1.6) and a well-known result from the theory of moments [9, Corollary 2.6] we have on the one hand $\sum p_n^2(0) < \infty$, whence, by (4.3), $\sum G_n < \infty$, and on the other hand $\sum p_n^2(x) = \infty$ for $\xi_k < x < \xi_{k+1}$. It follows from (4.3) that in the latter case $\{P_n(x)/P_n(0)\}_n$ must be unbounded. The required result follows by Lemma 4.

Next assume that the Hmp for $\mathscr{P}$ is indeterminate, so that, by Corollary 2, $\sum_{i=2}^{\infty} \xi_i^{-1} < \infty$. Let $a > 0$ and $\mathscr{R} \equiv \{R_n(x)\}_{n=0}^{\infty}$, where $R_n(x) = P_n(x-a)$. Clearly, $\xi_i(\mathscr{R}) = \xi_i + a$ (so that $\xi_1(\mathscr{R}) > 0$) and $\sum_{i=1}^{\infty} (\xi_i(\mathscr{R}))^{-1} < \infty$. Applying Theorem 1 to $\mathscr{R}$ yields that $R_n(a)/R_n(0)$ tends to zero, whereas, for $\xi_k < x < \xi_{k+1}$, $R_n(x+a)/R_n(0)$ tends to a non-zero limit as $n \to \infty$. Since

$$\frac{P_n(x)}{P_n(0)} = \frac{R_n(x+a)}{R_n(0)} \Big/ \frac{R_n(a)}{R_n(0)},$$

it follows that $|P_n(x)/P_n(0)| \to \infty$ as $n \to \infty$. Lemma 4 now gives the required result. ∎

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