SPECIAL ORTHOGONAL POLYNOMIAL SYSTEMS MAPPED ONTO EACH OTHER BY THE FOURIER-JACOBI TRANSFORM

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1. Introduction

R. Askey, in his contribution to these proceedings, emphasized unitary mappings

$$L^2(\text{interval}) \to L^2(\text{discrete set})$$

or

$$L^2(\text{discrete set}) \to L^2(\text{discrete set})$$

with hypergeometric orthogonal polynomial kernel. More generally, one might consider unitary mappings

$$L^2(\text{interval}) \to L^2(\text{interval})$$

with hypergeometric function kernel. As an example consider the *Hankel transform* pair

$$\begin{cases} g(\lambda) = \int_0^\infty f(t) J_\alpha(\lambda t) t\, dt, \\ f(t) = \int_0^\infty g(\lambda) J_\alpha(\lambda t) \lambda d\lambda, \end{cases} \tag{1.1}$$

where

$$J_\alpha(x) := (\tfrac{1}{2}x)^\alpha {}_0F_1(\alpha+1; -\tfrac{1}{4}x^2) / \Gamma(\alpha+1) \tag{1.2}$$

denotes a *Bessel function.* A well-known formula (cf. [7, 8.9 (3)]) states that

$$\int_0^\infty t^\alpha e^{-\frac{1}{2}t^2} L_n^\alpha(t^2) J_\alpha(\lambda t) t\, dt \tag{1.3}$$

$$= (-1)^n \lambda^\alpha e^{-\frac{1}{2}\lambda^2} L_n^\alpha(\lambda^2), \alpha > -1, n = 0,1,2,...,$$

where

$$L_n^\alpha(x) := \frac{(\alpha+1)_n}{n!}\, {}_1F_1(-n; \alpha+1; x) \tag{1.4}$$

denotes a *Laguerre polynomial.* The functions $t \mapsto t^\alpha e^{-\frac{1}{2}t^2} L_n^\alpha(t^2)$, $n = 0,1,2,...$, form a complete orthogonal basis of $L^2(\mathbb{R}_+, t\, dt)$ and they are eigenfunctions for the Hankel transform with eigenvalues $(-1)^n$.

Another important, but more complicated example of a unitary transform with hypergeometric function kernel is given by the *Fourier-Jacobi transform*, cf. for instance [10] or the survey [11]. It involves the *Jacobi function*

$$\phi_\lambda^{(\alpha,\beta)}(t) := {}_2F_1(\tfrac{1}{2}(\alpha+\beta+1+i\lambda),\tfrac{1}{2}(\alpha+\beta+1-i\lambda);\alpha+1;-\mathrm{sh}^2 t). \tag{1.5}$$

Note that

$$\begin{aligned}\phi_{i(2n+\alpha+\beta+1)}^{(\alpha,\beta)}(i\theta) &= {}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;\sin^2\theta) = \\ &= \frac{n!}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(\cos 2\theta)\end{aligned} \tag{1.6}$$

is a normalized *Jacobi polynomial;* this explains the terminology. Note the special cases

$$\phi_\lambda^{(-\frac{1}{2},-\frac{1}{2})}(t) = \cos\lambda t, \tag{1.7}$$

$$\phi_\lambda^{(0,0)}(t) = P_{\frac{1}{2}(i\lambda-1)}(\mathrm{ch}2t)\,(\text{Legendre function}), \tag{1.8}$$

and the limit relation

$$\lim_{r\to\infty} \phi_{r\lambda}^{(\alpha,\beta)}(r^{-1}t) = 2^\alpha\Gamma(\alpha+1)(\lambda t)^{-\alpha}J_\alpha(\lambda t). \tag{1.9}$$

Let

$$\Delta_{\alpha,\beta}(t) := (2\mathrm{sh}t)^{2\alpha+1}(2\mathrm{ch}t)^{2\beta+1},\ t>0, \tag{1.10}$$

$$c_{\alpha,\beta}(\lambda) := \frac{2^{\alpha+\beta+1-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+\alpha+\beta+1))\Gamma(\frac{1}{2}(i\lambda+\alpha-\beta+1))}. \tag{1.11}$$

The Fourier-Jacobi transform $f\mapsto g$ and its inverse are given by

$$\begin{cases} g(\lambda) = \int_0^\infty f(t)\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)dt, \\ f(t) = (2\pi)^{-1}\int_0^\infty g(\lambda)\phi_\lambda^{(\alpha,\beta)}(t)|c_{\alpha,\beta}(\lambda)|^{-2}d\lambda, \end{cases} \tag{1.12}$$

where $\alpha,\beta\in\mathbf{R}$, $|\beta|\leqslant\alpha+1$. The pair (1.12) is certainly valid for $f\in C_c^\infty(\mathbf{R})$ and even. The transform $f\mapsto g$ extends to an isometry of L^2-spaces:

$$\int_0^\infty |f(t)|^2\Delta_{\alpha,\beta}(t)dt = (2\pi)^{-1}\int_0^\infty |g(\lambda)|^2|c_{\alpha,\beta}(\lambda)|^{-2}d\lambda. \tag{1.13}$$

By (1.7), (1.8), (1.9) the Fourier-Jacobi transform becomes the Fourier-cosine transform for $\alpha=\beta=-\frac{1}{2}$, the Mehler-Fock transform for $\alpha=\beta=0$ and converges to the Hankel transform under suitable changes of λ and t.

The present paper will deal with an analogue of (1.3) for the Fourier-Jacobi transform. Since the integral kernel in (1.12) is not symmetric in λ and t, we cannot expect an explicit orthogonal basis consisting of eigenfunctions for the Fourier-Jacobi transform. But we can hope for some nice explicit orthogonal basis in the t-space which is mapped onto another explicit orthogonal basis in the λ-space, and with (1.3) as a limit case. A decisive hind for finding such systems lies in the combination of the two papers Boyer & Ardalan [1] (cf. § 2) and Wilson [16] (cf. § 3). I presented the resulting generalization of (1.3) already in [11, (9.4)], but not yet for the most general values of the parameters. A next advancement was made after Mourad Ismail, in August 1984 in Tunis, called my attention to the papers Diestler [3] and Broad [2], where an analogous problem is considered for the Whittaker transform. This resulted in sections 4 and 5 of this paper.

A more definitive paper will follow later. Because of space restrictions the proofs here will be sketchy or are omitted.

2. A GROUP THEORETIC INTERPRETATION

Let G be the generalized Lorentz group $SO_0(1,p)$ with closed subgroups $K:=SO(p)$ and $H:=S(O_0(1,p-1)\times O(1))$. The groups K and H have a common closed subgroup $M_1:= S(O(p-1)\times O(1))$ and M_1 has connected component $M:=SO(p-1)$. Let $G=KAN$ be an Iwasawa decomposition. Here A is a one-parameter group $\{t\mapsto a_t\}$.

Let π_μ ($\mu\in\mathbb{R}$) be the *unitary spherical principal series representation* of G induced by the one-dimensional representation $ma_t n\mapsto e^{i\mu}$ of the subgroup MAN. Boyer & Ardalan [1] extended the regular representation of H on $L^2(H/M)$ to a realization of the representation π_μ of G. For convenience, let us restrict our discussion to the subspace $L^2(M_1\backslash H/M)$ of M_1-invariant L^2-functions on H/M. These functions are completely determined by their restrictions to a certain one-parameter subgroup $B=\{t\mapsto b_t\}$ of H. In [1], $L^2(M_1\backslash H/M)$ is decomposed with respect to the π_μ-action of either of the subgroups K and H. The action of K yields the orthogonal system

$$M_1 b_t M\mapsto(\mathrm{ch}t)^{-i\mu-\frac{1}{2}p+\frac{1}{2}}P_n^{(\frac{1}{2},-\frac{3}{2},-\frac{1}{2})}(1-2\mathrm{th}^2 t),\, n=0,1,2,..., \tag{2.1}$$

where the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, orthogonal on (-1,1) with respect to the weight function $(1-x)^\alpha(1+x)^\beta$, is defined by (1.6). The action of H yields the generalized orthogonal system

$$M_1 b_t M\mapsto\phi_\lambda^{(\frac{1}{2},-\frac{3}{2},-\frac{1}{2})}(t),\ \lambda\geq 0, \tag{2.2}$$

where the Jacobi function ϕ_λ is defined by (1.5). The integral transform which maps basis (2.1) onto basis (2.2) has kernel

$$K(\lambda,n) = \int_0^\infty (\mathrm{ch}t)^{i\mu-\frac{1}{2}p+\frac{1}{2}}P_n^{(\frac{1}{2},-\frac{3}{2},-\frac{1}{2})}(1-2\mathrm{th}^2 t) \tag{2.3}$$

$$\cdot\phi_\lambda^{(\frac{1}{2},-\frac{3}{2},-\frac{1}{2})}(t)(2\mathrm{sh}t)^{p-2}dt.$$

Boyer & Ardalan [1] evaluated (2.3) as a ${}_4F_3$ hypergeometric function of unit argument which has the form of a Wilson polynomial of degree n in $\frac{1}{4}\lambda^2$, thus giving a group theoretic interpretation avant la lettre of Wilson polynomials.

3. A connection between Jacobi polynomials, Jacobi functions and Wilson polynomials

Wilson polynomials were introduced by Wilson [15], [16]. In the notation of J. Labelle's poster [12] they are given by

$$W_n(x^2;a,b,c,d):=(a+b)_n(a+c)_n(a+d)_n \tag{3.1}$$
$$\cdot {}_4F_3\left[\begin{matrix}-n,n+a+b+c+d-1,a+ix,a-ix\\ a+b,a+c,a+d\end{matrix}\Big|1\right],$$

where $n=0,1,2,\ldots$. They are symmetric in the four parameters a,b,c,d. If $a,b,c,d\in\mathbb{R}$ or $a,b\in\mathbb{R}$, $c=\bar{d}$ or $a=\bar{b}$, $c=\bar{d}$ and if $x\in\mathbb{R}$ then W_n is real-valued. If, moreover, a,b,c,d have positive real parts then the functions $x\mapsto W_n(x^2)$ are complete and orthogonal on $\mathbb{R}_+$ with respect to the weight function

$$\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2 \tag{3.2}$$

The key formula of this paper reads as follows:

$$\int_0^\infty (\mathrm{ch}t)^{-\alpha-\beta-\delta-\mu-2}P_n^{(\alpha,\delta)}(1-2\mathrm{th}^2t)\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)dt \tag{3.3}$$
$$=\frac{2^{2\alpha+2\beta+1}\Gamma(\alpha+1)(-1)^n\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda))\Gamma(\frac{1}{2}(\delta+\mu+1-i\lambda))}{n!\Gamma(\frac{1}{2}(\alpha+\beta+\delta+\mu+2)+n)\Gamma(\frac{1}{2}(\alpha-\beta+\delta+\mu+2)+n)}$$
$$\cdot W_n(\tfrac{1}{4}\lambda^2;\tfrac{1}{2}(\delta+\mu+1),\tfrac{1}{2}(\delta-\mu+1),\tfrac{1}{2}(\alpha+\beta+1),\tfrac{1}{2}(\alpha-\beta+1)),$$

where $\beta,\delta,\lambda\in\mathbb{R}$, $\alpha,\delta>-1$, $\delta+\mathrm{Re}\mu>-1$. Then the integral in (3.3) certainly absolutely converges. In order to prove (3.3), first compute $g(\lambda)$ in (1.12) when $f(t)$ is some complex power of $\mathrm{ch}t$. This can be done by rewriting (1.5) in terms of a ${}_2F_1$ of argument th^2t and by termwise integration of the power series. Next expand $P_n^{(\alpha,\delta)}(1-2\mathrm{th}^2t)$ as a power series in $(\mathrm{ch}t)^{-2}$.

The right hand side of (2.3) is the special case $\alpha=\frac{1}{2}p-\frac{3}{2}$, $\beta=\delta=-\frac{1}{2}$, $\mu\in i\mathbb{R}$ of the left hand side of (3.3).

By the orthogonality relations for Jacobi polynomials, the functions

$$t\mapsto(\mathrm{ch}t)^{-\alpha-\beta-\delta-\mu-2}P_n^{(\alpha,\delta)}(1-2\mathrm{th}^2t),\ n=0,1,2,\ldots$$

form, for $\mu\in i\mathbb{R}$, a complete orthogonal system in $L^2(\mathbb{R}_+;\Delta_{\alpha,\beta}(t)dt)$. For $\mu\in\mathbb{R}$ they form a complete system biorthogonal to similar functions with μ replaced by $-\mu$. This, together with (1.13), implies the orthogonality relations for the Wilson polynomials occurring at the right hand side of (3.3) ($\alpha\pm\beta+1>0$). Conversely, the Plancherel formula (1.13) for the Fourier-Jacobi transform follows from (3.3) and the orthogonality relations for Jacobi and Wilson polynomials.

The limit transition from (3.3) to (1.3) can be done as follows. In (3.3) replace λ by $\lambda\delta^{\frac{1}{2}}$, make the change of integration variable $t\mapsto t\delta^{-\frac{1}{2}}$ in the integral at the left hand side and multiply both sides by $\delta^{\alpha+1}$. Then (3.3) becomes

$$2^{2\alpha+2\beta+2}\int_0^\infty (\mathrm{ch}(\delta^{-\frac{1}{2}}t))^{\alpha+\beta-\mu-\delta}P_n^{(\alpha,\delta)}(1-2\mathrm{th}^2(\delta^{-\frac{1}{2}}t)) \tag{3.4}$$

$$\cdot\phi^{(\alpha,\beta)}_{\lambda\delta^{\frac{1}{2}}}(\delta^{-\frac{1}{2}}t)(\delta^{\frac{1}{2}}\operatorname{th}(\delta^{-\frac{1}{2}}t))^{2\alpha+1}dt = 2^{3\alpha+2\beta+2}\Gamma(\alpha+1)(-1)^n$$

$$\cdot\frac{(\frac{1}{2}\delta)^{2n+\alpha+1}\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda\delta^{\frac{1}{2}}))\Gamma(\frac{1}{2}(\delta+\mu+1-i\lambda\delta^{\frac{1}{2}}))}{\Gamma(\frac{1}{2}(\alpha+\beta+\delta+\mu+2)+n)\Gamma(\frac{1}{2}(\alpha-\beta+\delta+\mu+2+n)}$$

$$\cdot\frac{2^{2n}}{\delta^{2n}\,n!}W_n(\tfrac{1}{4}\delta\lambda^2;\tfrac{1}{2}(\delta+\mu+1),\tfrac{1}{2}(\delta-\mu+1),\tfrac{1}{2}(\alpha+\beta+1),\tfrac{1}{2}(\alpha-\beta+1)).$$

By use of (3.1), (1.4) and Stirling's formula, the right hand side of (3.4) converges, as $\delta\to\infty$, to

$$2^{3\alpha+2\beta+2}\Gamma(\alpha+1)(-1)^n e^{-\frac{1}{2}\lambda^2}L_n^{\alpha}(\lambda^2).$$

By (1.9) and

$$\lim_{\delta\to\infty} P_n^{(\alpha,\delta)}(1-2\delta^{-1}x) = L_n^{\alpha}(x), \tag{3.5}$$

the integrand in (3.4) converges, as $\delta\to\infty$, to

$$e^{-\frac{1}{2}t^2}L_n^{\alpha}(t^2)2^{\alpha}\Gamma(\alpha+1)(\lambda t)^{-\alpha}J_{\alpha}(\lambda t)t^{2\alpha+1}.$$

If $\alpha\geqslant-\frac{1}{2}$ then I can give an integrable upper bound for the absolute value of the integrand in (3.4) which is independent of δ. Then (1.3) follows by the dominated convergence theorem.

Remark 3.1. Flensted-Jensen [9, Appendix 1] extended (1.13) to the case that $\alpha>-1$, $\beta\in\mathbf{R}$. If $|\beta|>\alpha+1$ then there are additional discrete terms $\sum_{\lambda\in D_{\alpha,\beta}} d_{\alpha,\beta}(\lambda)|g(\lambda)|^2$ in the right hand side of (1.13), where $D_{\alpha,\beta}$ is a finite subset of the positive imaginary axis. Because of (3.3) this must correspond to a mixed continuous and discrete orthogonality for the Wilson polynomials if one of their parameters is negative. This is indeed a known phenomenon, cf. Wilson [16, (3.3)].

Remark 3.2. It is tempting to obtain a group theoretic interpretation of Wilson polynomials and of (3.3) which is valid for more general parameter values than the one given by Boyer & Ardalan [1], cf. § 2. In view of the interpretation of Racah polynomials as $6-j$ symbols (cf. Wilson [16, § 5]) it would be natural to look at some noncompact real form of $SL(2,\mathbf{C})\times SL(2,\mathbf{C})\times SL(2,\mathbf{C})$ in order to obtain a similar interpretation for Wilson polynomials. However, I did not succeed until now. A different group theoretic interpretation of Racah polynomials is suggested by Dunkl's [5, Theorem 1.7] observation that orthogonal polynomials on the triangle have three different canonical orthogonal bases mapped onto each other by matrices with Racah polynomials as entries. The three canonical bases have group theoretic interpretations as $O(p)\times O(q)\times O(r)$-invariant spherical harmonics on the unit sphere in $\mathbf{R}^{p+q+r}$, decomposed with respect to one of the three subgroups $O(p+q)\times O(r)$, $O(q+r)\times(p)$, $O(r+p)\times O(q)$. A noncompact analogue of this are the $O(p)\times O(q)\times O(r)$-invariant eigenfunctions of the Laplace-Beltrami operator on the hyperboloid $\{(x,y,z)\in\mathbf{R}^p\times\mathbf{R}^q\times\mathbf{R}^r \mid -|x|^2-|y|^2+|z|^2=1\}$, decomposed with respect to one of the two subgroups $O(p+q)\times O(r)$ and $O(q,r)\times O(p)$. For fixed eigenvalue I get respectively an ordinary and generalized orthogonal basis for the eigenspace. The integral transform mapping the one basis onto the other has a kernel expressed in terms of Wilson polynomials. If, in this expansion, one lets $z\to\infty$ on the hyperboloid, one gets a formula equivalent to (3.3).

4. Representation of the Jacobi function differential operator as a tridiagonal matrix

The most remarkable thing about (3.3) is that its right hand side again involves orthogonal polynomials. In particular, the right hand side must satisfy a three term recurrence relation. In analogy to Broad [2, Appendix], where the Whittaker function transform is considered, we can obtain this recurrence from a tridiagonalization of the Jacobi function differential operator $\mathcal{L}_{\alpha,\beta}$.

Let $\Delta_{\alpha,\beta}$ be given by (1.10) and

$$(\mathcal{L}_{\alpha,\beta}f)(t):=(\Delta_{\alpha,\beta}(t))^{-1}\frac{d}{dt}(\Delta_{\alpha,\beta}(t)\frac{df(t)}{dt}),\ t>0. \tag{4.1}$$

Then

$$\mathcal{L}_{\alpha,\beta}\phi_\lambda^{(\alpha,\beta)}=-(\lambda^2+(\alpha+\beta+1)^2)\phi_\lambda^{(\alpha,\beta)} \tag{4.2}$$

and, if g is related to f and G to $F:=\mathcal{L}_{\alpha,\beta}f$ according to (1.12) then $G(\lambda)=-(\lambda^2+(\alpha+\beta+1)^2)g(\lambda)$. Put

$$p_n(t):=(\mathrm{ch}t)^{-\alpha-\beta-\delta-\mu-2}P_n^{(\alpha,\delta)}(1-2\mathrm{th}^2t), \tag{4.3}$$

$$q_n(\lambda):=\frac{(-1)^n(\alpha+1)_n(\frac{1}{2}(\alpha+\beta+\delta-\mu+2))_n}{n!(\frac{1}{2}(\alpha-\beta+\delta+\mu+2))_n} \tag{4.4}$$

$$\cdot{}_4F_3\left[\begin{matrix}-n,\alpha+\delta+1,\frac{1}{2}(\alpha+\beta+1+i\lambda),\frac{1}{2}(\alpha+\beta+1-i\lambda)\\ \alpha+1,\frac{1}{2}(\alpha+\beta+\delta+\mu+1),\frac{1}{2}(\alpha+\beta+\delta-\mu+1)\end{matrix}\ \Big|\ 1\right].$$

It follows from the differential equation [6, 10.8 (14)] for Jacobi polynomials that

$$\begin{aligned}\mathcal{L}_{\alpha,\beta}p_n(t)&=-2(\mu+1)\mathrm{th}t\ p_n'(t)\\&+((\alpha+\beta+\delta+\mu+2)(\alpha-\beta+\delta+\mu)\mathrm{th}^2t-2(\alpha+1)(\alpha+\beta+\delta+\mu+2)\\&-4n(n+\alpha+\delta+1)\mathrm{ch}^{-2}t)p_n(t).\end{aligned} \tag{4.5}$$

By use of the differential recurrence relation [6, 10.8 (15)] and three term recurrence relation [6, 10.8 (11)] for Jacobi polynomials it follows from (4.5) that $\mathcal{L}_{\alpha,\beta}$ becomes *tridiagonal* with respect to the orthogonal basis of functions p_n:

$$-\mathcal{L}_{\alpha,\beta}p_n=A_np_{n+1}+B_np_n+C_np_{n-1}, \tag{4.6}$$

where

$$A_n=\frac{(n+1)(n+\alpha+\delta+1)(2n+\alpha+\beta+\delta+\mu+2)(2n+\alpha-\beta+\delta+\mu+2)}{(2n+\alpha+\delta+1)(2n+\alpha+\delta+2)}$$

$$C_n=\frac{(n+\alpha)(n+\delta)(2n+\alpha+\beta+\delta-\mu)(2n+\alpha-\beta+\delta-\mu)}{(2n+\alpha+\delta)(2n+\alpha+\delta+1)},$$

$$B_n=\frac{n+\alpha+1}{n+1}A_n+\frac{n}{n+\alpha}C_n.$$

It follows from (4.6) that

$$(\lambda^2+(\alpha+\beta+1)^2)q_n(\lambda) = A_n q_{n+1}(\lambda)+B_n q_n(\lambda)+C_n q_{n-1}(\lambda). \tag{4.7}$$

Thus we have obtained the recurrence relation in Wilson [15, (4.40)].

5. A connection between Laguerre polynomials, Whittaker functions and continuous dual Hahn polynomials

In (3.3) replace (α,β) by $(\alpha+\gamma,\beta+\gamma)$, make the change of integration variable $t\mapsto t+\frac{1}{2}\log\gamma$ in the integral at the left hand side and multiply both sides by $\gamma^{\frac{1}{2}(-\alpha+\beta+\delta+\mu)}2^{-(4\gamma+\alpha+\beta+\delta+\mu+2)}(-1)^n$. We obtain

$$\begin{aligned}\int_{-\frac{1}{2}\log\gamma}^{\infty} & P_n^{(\delta,\alpha+\gamma)}(1-2\mathrm{ch}^{-2}(t+\tfrac{1}{2}\log\gamma)) \\ & \cdot 2^{-2\gamma}\gamma^{\beta+\gamma}e^{2\gamma t}\phi_\lambda^{(\alpha+\gamma,\beta+\gamma)}(t+\tfrac{1}{2}\log\gamma) \\ & \cdot e^{-2\gamma t}(2\gamma^{-\frac{1}{2}}\mathrm{sh}(t+\tfrac{1}{2}\log\gamma))^{2\alpha+2\gamma+1} \\ & \cdot(2\gamma^{-\frac{1}{2}}\mathrm{ch}(t+\tfrac{1}{2}\log\gamma))^{-\alpha+\beta-\delta-\mu-1}dt \\ = & \frac{2^{\alpha+\beta-\delta-\mu-1}\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda))\Gamma(\frac{1}{2}(\delta+\mu+1-i\lambda))}{n!\Gamma(\frac{1}{2}(\alpha-\beta+\delta+\mu+2)+n)} \\ & \cdot\frac{\gamma^{\frac{1}{2}(-\alpha+\beta+\delta+\mu)}\Gamma(\alpha+\gamma+1)}{\Gamma(\frac{1}{2}(\alpha+\beta+\delta+\mu+2)+n+\gamma)} \\ & \cdot\gamma^{-n}W_n(\tfrac{1}{4}\lambda^2;\tfrac{1}{2}(\delta+\mu+1),\tfrac{1}{2}(\delta-\mu+1),\tfrac{1}{2}(\alpha+\beta+1)+\gamma,\tfrac{1}{2}(\alpha-\beta+1)).\end{aligned} \tag{5.1}$$

Now let $\gamma\to\infty$. Then, pointwise:

$$P_n^{(\delta,\alpha+\gamma)}(1-2\mathrm{ch}^{-2}(t+\tfrac{1}{2}\log\gamma))\to L_n^{\delta}(4e^{-2t}), \tag{5.2}$$

$$\begin{aligned}&\gamma^{-n}W_n(\tfrac{1}{4}\lambda^2;\tfrac{1}{2}(\delta+\mu+1),\tfrac{1}{2}(\delta-\mu+1),\tfrac{1}{2}(\alpha+\beta+1)+\gamma,\tfrac{1}{2}(\alpha-\beta+1)) \\ &\to S_n(\tfrac{1}{4}\lambda^2;\tfrac{1}{2}(\delta+\mu+1),\tfrac{1}{2}(\delta-\mu+1),\tfrac{1}{2}(\alpha-\beta+1)).\end{aligned} \tag{5.3}$$

Here S_n, in the notation of Labelle [12], is the *continuous dual Hahn polynomial:*

$$S_n(x^2;a,b,c):=(a+b)_n(a+c)_n\ {}_3F_2\left[\begin{matrix}-n,a+ix,a-ix\\ a+b,a+c\end{matrix}\ \Big|\ 1\right]. \tag{5.4}$$

If $c>0$ and $a,b>0$ or $a=\bar{b}$ with $\mathrm{Re}\,a>0$ then the functions $x\mapsto S_n(x^2)$ are complete and orthogonal on $\mathbb{R}_+$ with respect to the weight function

$$\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2, \text{ cf. Wilson [16, (4.4)].} \tag{5.5}$$

In order to find the limit of the Jacobi function in (5.1) as $\gamma\to\infty$, consider

$$\Phi_\lambda^{(\alpha,\beta)}(t):=(2\mathrm{ch}\,t)^{i\lambda-\alpha-\beta-1} \tag{5.6}$$

$$\cdot {}_2F_1(\tfrac{1}{2}(\alpha+\beta+1+i\lambda),\tfrac{1}{2}(\alpha-\beta+1+i\lambda);1-i\lambda;\mathrm{ch}^{-2}t),$$

a second solution of the differential equation (4.2) such that $\Phi_\lambda^{(\alpha,\beta)}(t)=e^{(i\lambda-\alpha-\beta-1)t}(1+o(1))$ as $t\to\infty$. Then

$$\phi_\lambda^{(\alpha,\beta)} = c_{\alpha,\beta}(\lambda)\Phi_\lambda^{(\alpha,\beta)}+c_{\alpha,\beta}(-\lambda)\Phi_{-\lambda}^{(\alpha,\beta)}, \tag{5.7}$$

$c(\lambda)$ being given by (1.11). It follows from (5.6) respectively (1.11) that

$$\begin{aligned}&\lim_{\gamma\to\infty} \gamma^{\frac{1}{2}(-i\lambda+\alpha+\beta+1+2\gamma)} e^{2\gamma t}\,\Phi_\lambda^{(\alpha+\gamma,\beta+\gamma)}(t+\tfrac{1}{2}\log\gamma) \\ &= e^{(i\lambda-\alpha-\beta-1)t}\exp(-2e^{-2t})\,{}_1F_1(\tfrac{1}{2}(\alpha-\beta+1-i\lambda);1-i\lambda;4e^{-2t}),\end{aligned} \tag{5.8}$$

$$\lim_{\gamma\to\infty} 2^{-2\gamma}\gamma^{\frac{1}{2}(i\lambda-\alpha+\beta-1)} c_{\alpha+\gamma,\beta+\gamma}(\lambda) = \frac{2^{\alpha+\beta+1-i\lambda}\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+\alpha-\beta+1))}. \tag{5.9}$$

Hence, by (5.7) and [6, 6.7 (8)]:

$$\begin{aligned}&\lim_{\gamma\to\infty} 2^{-2\gamma}\gamma^{\beta-\gamma}e^{2\gamma t}\,\phi_\lambda^{(\alpha+\gamma,\beta+\gamma)}(t+\tfrac{1}{2}\log\gamma) \\ &= 2^{\alpha+\beta+1-i\lambda}e^{(i\lambda-\alpha-\beta-1)t}\exp(-2e^{-2t}) \\ &\cdot\Psi(\tfrac{1}{2}(\alpha-\beta+1-i\lambda);1-i\lambda;4e^{-2t}) \\ &= 2^{\alpha+\beta}e^{-(\alpha+\beta)t}\,\mathcal{W}_{\frac{1}{2}(\beta-\alpha),\frac{1}{2}i\lambda}(4e^{-2t}),\end{aligned} \tag{5.10}$$

where Ψ is Tricomi's confluent hypergeometric function of the second kind and $\mathcal{W}$ is the *Whittaker function of the second kind* (cf. [6, 6.5 (2), 6.9 (2)]).

The Whittaker function transform and its inverse are given by

$$\begin{cases} g(\lambda) = \int_0^\infty f(x)(2x)^{-\frac{1}{2}}\mathcal{W}_{\kappa,i\lambda}(2x)x^{-1}dx, \\ f(x) = (2\pi)^{-1}\int_0^\infty g(\lambda)(2x)^{-\frac{1}{2}}\mathcal{W}_{\kappa,i\lambda}(2x)\left|\dfrac{\Gamma(2i\lambda)}{\Gamma(\frac{1}{2}+i\lambda-\kappa)}\right|^2 d\lambda, \end{cases} \tag{5.11}$$

where $\kappa\leqslant\frac{1}{2}$ and f is in a suitable function class. The inversion formula follows by spectral theory of ordinary differential operators, cf. Titchmarsh [14, § 4.16], Dunford & Schwartz [4, Exercise XIII.9.I.6] and, in particular, Faraut [8, § IV]. For $\kappa=0$ we get (cf. [6, 6.9 (14)])

$$(2x)^{-\frac{1}{2}}\mathcal{W}_{0,i\lambda}(2x) = \pi^{\frac{1}{2}}K_{i\lambda}(x), \tag{5.12}$$

where K denotes the modified Bessel function of the third kind, and (5.11) then reduces to the *Kontorovich-Lebedev transform pair.*

We can now take formal limits in (5.1) as $\gamma\to\infty$ and obtain

$$\begin{aligned}&2^{\alpha+\beta}\int_{-\infty}^{\infty} L_n^\delta(4e^{-2t})\mathcal{W}_{\frac{1}{2}(\beta-\alpha),\frac{1}{2}i\lambda}(4e^{-2t}) \\ &\cdot e^{-(\delta+\mu)t}\exp(-2e^{-2t})dt \\ &= \frac{2^{\alpha+\beta-\delta-\mu-1}\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda))\Gamma(\frac{1}{2}(\delta+\mu+1-i\lambda))}{n!\,\Gamma(\frac{1}{2}(\alpha-\beta+\delta+2)+n)} \\ &\cdot S_n(\tfrac{1}{4}\lambda^2;\tfrac{1}{2}(\delta+\mu+1),\tfrac{1}{2}(\delta-\mu+1),\tfrac{1}{2}(\alpha-\beta+1)).\end{aligned} \tag{5.13}$$

For $\alpha \geqslant -\frac{1}{2}$, $|\beta| \leqslant \max\{\frac{1}{2},\alpha\}$, $\lambda > 0$ I can show that the integrand in (5.1) is in absolute value bounded by

$$\text{const. } e^{-(2n+\delta+\mathrm{Re}\mu+1)t}\exp(-2e^{-2t}),\ t \in \mathbb{R},$$

uniformly in γ, which justifies (5.13) by the dominated convergence theorem. (Recall that $\delta+\mathrm{Re}\mu > -1$.)

We can rewrite (5.13) as

$$\int_0^\infty (2x)^{\frac{1}{2}(\delta+\mu+1)} e^{-x} L_n^\delta(2x)(2x)^{-\frac{1}{2}} \mathcal{W}_{\kappa,i\lambda}(2x)x^{-1}dx \tag{5.14}$$

$$= \frac{\Gamma(\frac{1}{2}(\delta+\mu+1)+i\lambda)\Gamma(\frac{1}{2}(\delta+\mu+1)-i\lambda)}{n!\Gamma(-\kappa+\frac{1}{2}(\delta+\mu)+n+1)}$$

$$\cdot S_n(\lambda^2;\tfrac{1}{2}(\delta+\mu+1),\tfrac{1}{2}(\delta-\mu+1),\tfrac{1}{2}-\kappa),\kappa \leqslant \tfrac{1}{2},\ \delta+\mathrm{Re}\mu > -1,\ \lambda \in \mathbb{R}.$$

The functions $x \mapsto (2x)^{\frac{1}{2}(\delta+\mu+1)} e^{-x} L_n^\delta(2x)$ $(n=0,1,2,\ldots)$ form a complete orthogonal system in $L^2(\mathbb{R}_+;x^{-1}dx)$ and are mapped by the Whittaker function transform onto a similar comlete (bi) orthogonal system in

$$L^2(\mathbb{R}_+;\left|\frac{\Gamma(2i\lambda)}{\Gamma(\frac{1}{2}+i\lambda-\kappa)}\right|^2 d\lambda).$$

Remark 5.1. Formula (5.14) can also be proved independently of (3.3) and it can be continued for $\kappa > \frac{1}{2}$. For $\kappa > \frac{1}{2}$ discrete terms have to be added in the inversion formula in (5.12), cf. Faraut [8, § IV]. Also the polynomials S_n at the right hand side of (5.14) get mixed continuous and discrete orthogonality relations if $\kappa > \frac{1}{2}$.

Remark 5.2. In Broad [2, Appendix] (see also Diestler [3, § 4]) a special case of (5.14) is discussed with $\mu=0$ and $\kappa-\frac{1}{2}\delta \in \mathbb{N}$. Then, for $n=0,1,\ldots,\kappa-\frac{1}{2}\delta-1$, the right hand side of (5.14) vanishes if $\lambda \in \mathbb{R}$, but is nonzero at certain imaginary λ of discrete mass.

Remark 5.3. Formulas (3.3) and (5.14) are not only related to each other by a limit transition, but, for special values of the parameters, also by a Hankel transform connecting Jacobi with Laguerre polynomials and a quadratic transformation connecting Wilson with continuous dual Hahn polynomials, cf. the factorization of Jacobi transform as a composition of Hankel and Kontorovich-Lebedev transform given in Roehner & Valent [13].

Remark 5.4. Labelle's tableau [12] suggests the existence of more limit cases of (3.3). Group theoretic interpretations of such limit cases, in particular of (5.14) should also be found.

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