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NOTE

AN INEQUALITY IN BINARY VECTOR SPACES

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We prove that if an n-dimensional vector space over GF(2) is the irredundant union of k subspaces, and this collection of subspaces has zero intersection, then $n < k$. This settles a conjecture by G. Bruns.

In [1] Ganter posed the following problem: "Let V be a vector space over GF(2) which is the irredundant union of k subspaces which have a trivial global intersection, i.e.,

$$V = \bigcup_{i=1}^{k} U_i, \qquad V \neq \bigcup_{\substack{1 \leq i \leq k \\ i \neq j}} U_i \quad (j = 1, \ldots, k), \qquad \bigcap_{i=1}^{k} U_i = \{0\}.$$

Does this imply that $\dim V < k$?"

Here we answer this question affirmatively. In fact, in order to make the induction work we prove the slightly stronger

Theorem. *Let X be a vector space over* GF(2) *and V, U_i $(1 \leq i \leq k)$ subspaces of X such that for certain vectors $a_i \in X$ we have*

$$V \subset \bigcup_{i=1}^{k} (a_i + U_i), \qquad V \not\subset \bigcup_{\substack{1 \leq i \leq k \\ i \neq j}} (a_i + U_i) \quad (j = 1, \ldots, k).$$

Then, if $W := V \cap \bigcap_{i=1}^{k} U_i$, we have $k \geq \dim V - \dim W + 1$.

Clearly, Ganter's problem is the case $V = X$, $W = \{0\}$, $a_i = 0$ $(1 \leq i \leq k)$.

Proof. Induction on k and for fixed k on decreasing $\sum_{i=1}^{k} \dim(U_i \cap V)$. (Note that if $(a + U) \cap V \neq \emptyset$ then $\dim((a + U) \cap V) = \dim(U \cap V)$, in fact $(a + U) \cap V = b + (U \cap V)$ for some $b \in (a + U) \cap V$.) If $k = 1$, then the statement of the theorem is obvious. Now assume $k > 1$. Let $n := \dim V$. Since the union is irredundant V meets all $a_i + U_i$ and since $k > 1$ it follows that $\dim(U_i \cap V) \leq n - 1$

for all i. If $\dim(U_i \cap V) = n-1$ for all i, then $W = V \cap \bigcap_{i=1}^k U_i$ implies $\dim W \geq \dim V - k$, and we are done unless $\dim W = \dim V - k$. But in the latter case $\dim(V \backslash \bigcup_{i=1}^k (a_i + U_i)) \geq \dim W \geq 0$ so that $V \backslash \bigcup_{i=1}^k (a_i + U_i) \neq \emptyset$, a contradiction.

Consider $W_I := V \cap \bigcap_{i \notin I} U_i$. Then $W_\emptyset = W$.

Lemma. *If* $0 < |I| < k$, *then* $\dim W_I \leq |I| + \dim W - 1$. *In particular* $W_{\{i\}} = W$.

Proof. Induction on $|I|$. $V \backslash \bigcup_{i \notin I} (a_i + U_i)$ is a nonempty union of translates of W_I, so that for some a we have $a + W_i \subset \bigcup_{i \in I} (a_i + U_i)$. If this union is irredundant, then by the theorem (applied with $|I|$ instead of k) we find $\dim W_I \leq |I| + \dim W - 1$ (note that $W_I \cup \bigcap_{i \in I} U_i = W$). On the other hand, if the union is redundant then we may choose $J \subsetneqq I$ such that $a + W_I \subset \bigcup_{i \in J} (a_i + U_i)$ and this latter union is irredundant. By the theorem and the induction hypothesis we find

$$\dim W_I \leq |J| + \dim W_{I \backslash J} - 1 \leq |J| + |I \backslash J| + \dim W - 2 < |I| + \dim W - 1. \quad \square$$

Returning to the proof of the theorem: we shall carry out the induction by either enlarging some U_i or reducing the number of subspaces k. We may suppose that $\dim(U_g \cap V) < n-1$ for some g $(1 \leq g \leq k)$. Set $U'_g = U_g \cup (a + U_g)$ and $U'_i = U_i$ for $1 \leq i \leq k$, $i \neq g$, where a is chosen such that $\dim((a_g + U'_g) \cap V) > \dim((a_g + U_g) \cap V)$. Now $V \subset \bigcup_{i=1}^k (a_i + U'_i)$ and $W' := V \cap \bigcap_{i=1}^k U'_i = W$ (for: $W \subset W' \subset W_{\{g\}} = W$) so if the union is irredundant we succeeded in reducing the problem to one with larger U_g. On the other hand, if the union is redundant, then we may choose I such that $g \notin I$ and $V \subset \bigcup_{i \notin I} (a_i + U'_i)$ is irredundant. Since $\dim(U'_g \cap V) < n$ we have $|I| < k-1$ so that by the lemma $\dim W' = \dim(U'_g \cap W_{I \cup \{g\}}) \leq \dim W_{I \cup \{g\}} \leq |I| + \dim W$. By the theorem (applied with $k - |I|$ instead of k) we find

$$\dim V \leq k - |I| + |I| + \dim W - 1 = k + \dim W - 1. \quad \square$$

Remark. It is natural to ask what happens for vector spaces over GF(q) with $q > 2$. It is easy to see that there are examples with $k = (n-1)(q-1) + 2$ where $n = \dim V$. We have seen that $k \geq (n-1)(q-1) + 2$ for $q = 2$, and it is trivial to prove the same inequality for $n = 2$. But already for $n = 3$ smaller k occur: First rephrase the problem as a projective problem, and then dualize. Now our problem is:

> "Let V be a projective space of dimension $n+1$ over GF(q) which is spanned by k subspaces U_i $(1 \leq i \leq k)$ such that any hyperplane contains at least one of the U_i, and where there are hyperplanes H_i such that H_i does not contain any U_j $(j \neq i,\ 1 \leq i \leq k)$. Find a lower bound for k."

In the special case $n = 3$ we get $\dim V = 2$ and ask for a minimal blocking set (with less than $2q$ elements). If q is a square then a Baer subplane will do—it provides us with an example with $q + \sqrt{q} + 1$ elements. Also when q is not a square one may have $k < 2q$. For example, if $q = 5$ one may take 4 points on a line and 5 points forming a transversal of the remaining two parallel classes. This gives $k = 9$. (See Hirschfeld [2, Ch. 13] for a discussion of blocking sets.)

Note that for $q = 2$, $n = 3$ we have a blocking family $\{U_i\}_i$ consisting of two points and two lines, but a blocking set consisting of points only does not exist. It is easily seen that for $q \geqslant 3$ we may restrict attention to blocking sets, and thus $k \geqslant q + \sqrt{q} + 1$, with equality precisely in case of a Baer subplane.

The case $n > 3$ remains open. (But see [3].)

Postscript

It turns out that Ganter's question is a slight generalization of a conjecture by G. Bruns on the covering of Boolean algebras by subalgebras. Thus, our result settles Bruns' conjecture. (See also [5].)

References

[1] B. Ganter, letter to J.A. Thas, dated 23.6.80.
[2] J.W.P. Hirschfeld, Projective geometries over finite fields (Clarendon Press, Oxford, 1979).
[3] A.A. Bruen, Blocking sets and skew subspaces of projective space, Canad. J. Math. 32 (1980) 628–630.
[4] G. Bruns, Covering Boolean algebras by subalgebras, Discrete Math., Mathematisches Forschungsinst. Oberwolfach, Tagungsbericht 23 (1980).
[5] G. Bruns and R. Greechie, Orthomodular lattices which can be covered by finitely many blocks, Canad. J. Math. 34 (1982) 696–699.