NOTE

AN INEQUALITY IN BINARY VECTOR SPACES

A.E. BROUWER

Department of Pure Mathematics, Stichting Mathematisch Centrum, Amsterdam, The Netherlands

 Received 30 September 1981
 Revised September 1982

We prove that if an \( n \)-dimensional vector space over GF(2) is the irredundant union of \( k \) subspaces, and this collection of subspaces has zero intersection, then \( n < k \). This settles a conjecture by G. Bruns.

In [1] Ganter posed the following problem: "Let \( V \) be a vector space over GF(2) which is the irredundant union of \( k \) subspaces which have a trivial global intersection, i.e.,

\[
V = \bigcup_{i=1}^{k} U_i, \quad V \neq \bigcup_{1 \leq i < k \atop i \neq j} U_i \quad (j = 1, \ldots, k), \quad \bigcap_{i=1}^{k} U_i = \{0\}.
\]

Does this imply that \( \dim V < k \)?"

Here we answer this question affirmatively. In fact, in order to make the induction work we prove the slightly stronger

**Theorem.** Let \( X \) be a vector space over GF(2) and \( V, U_i \ (1 \leq i \leq k) \) subspaces of \( X \) such that for certain vectors \( a_i \in X \) we have

\[
V \subset \bigcup_{i=1}^{k} (a_i + U_i), \quad V \not\subset \bigcup_{1 \leq i < k \atop i \neq j} (a_i + U_i) \quad (j = 1, \ldots, k).
\]

Then, if \( W := V \cap \bigcap_{i=1}^{k} U_i, \) we have \( k \geq \dim V - \dim W + 1 \).

Clearly, Ganter's problem is the case \( V = X, \ W = \{0\}, \ a_i = 0 \ (1 \leq i \leq k) \).

**Proof.** Induction on \( k \) and for fixed \( k \) on decreasing \( \sum_{i=1}^{k} \dim (U_i \cap V) \). (Note that if \( (a + U) \cap V \neq \emptyset \) then \( \dim ((a + U) \cap V) = \dim (U \cap V) \), in fact \( (a + U) \cap V = b + (U \cap V) \) for some \( b \in (a + U) \cap V \).) If \( k = 1 \), then the statement of the theorem is obvious. Now assume \( k > 1 \). Let \( n := \dim V \). Since the union is irredundant \( V \) meets all \( a_i + U_i \) and since \( k > 1 \) it follows that \( \dim (U_i \cap V) \leq n - 1 \).
for all i. If \( \dim(U_i \cap V) = n - 1 \) for all i, then \( W = V \cap \bigcap_{i=1}^{k} U_i \) implies \( \dim W \geq \dim V - k \), and we are done unless \( \dim W = \dim V - k \). But in the latter case \( \dim(V \setminus \bigcup_{i=1}^{k} (a_i + U_i)) \geq \dim W \geq 0 \) so that \( V \setminus \bigcup_{i=1}^{k} (a_i + U_i) \neq \emptyset \), a contradiction.

Consider \( W_1 := V \cap \bigcap_{i \in I} U_i \). Then \( W_0 = W \).

**Lemma.** If \( 0 < |I| < k \), then \( \dim W_I \leq |I| + \dim W - 1 \). In particular \( W_{(i)} = W \).

**Proof.** Induction on \( |I| \). \( V \setminus \bigcup_{i \in I} (a_i + U_i) \) is a nonempty union of translates of \( W_I \), so that for some \( a \) we have \( a + W_I \subset \bigcup_{i \in I} (a_i + U_i) \). If this union is irredundant, then by the theorem (applied with \( |I| \) instead of \( k \)) we find \( \dim W_I \leq |I| + \dim W - 1 \) (note that \( W_I \cap \bigcap_{i \in I} U_i = W \)). On the other hand, if the union is redundant then we may choose \( J \subseteq I \) such that \( a + W_I \subset \bigcup_{i \in J} (a_i + U_i) \) and this latter union is irredundant. By the theorem and the induction hypothesis we find

\[
\dim W_I \leq |J| + \dim W_I \cup J - 1 \leq |J| + |I \setminus J| + \dim W - 2 < |I| + \dim W - 1. \quad \square
\]

Returning to the proof of the theorem: we shall carry out the induction by either enlarging some \( U_i \) or reducing the number of subspaces \( k \). We may suppose that \( \dim(U_g \cap V) < n - 1 \) for some \( g \) \( (1 \leq g \leq k) \). Set \( U_g' = U_g \cup (a + U_g) \) and \( U_i' = U_i \) for \( 1 \leq i \leq k, \ i \neq g \), where \( a \) is chosen such that \( \dim((a_g + U_g') \cap V) > \dim((a_g + U_g) \cap V) \). Now \( V \subset \bigcup_{i=1}^{k} (a_i + U_i) \) and \( W' := V \cap \bigcap_{i=1}^{k} U_i' = W \) (for: \( W \subset W' \subset W_{(g)} = W \)) so if the union is irredundant we succeeded in reducing the problem to one with larger \( U_g \). On the other hand, if the union is redundant, then we may choose \( I \) such that \( g \notin I \) and \( V \subset \bigcup_{i \in I} (a_i + U_i) \) is irredundant. Since \( \dim(U_g' \cap V) < n \) we have \( |I| < k - 1 \) so that by the lemma \( \dim W' = \dim(U_g' \cap W_{I \cup (g)}) \leq \dim W_{I \cup (g)} \leq |I| + \dim W \). By the theorem (applied with \( k - |I| \) instead of \( k \)) we find

\[
\dim V \leq k - |I| + |I| + \dim W - 1 = k + \dim W - 1. \quad \square
\]

**Remark.** It is natural to ask what happens for vector spaces over GF\((q)\) with \( q > 2 \). It is easy to see that there are examples with \( k = (n - 1)(q - 1) + 2 \) where \( n = \dim V \). We have seen that \( k \geq (n - 1)(q - 1) + 2 \) for \( q = 2 \), and it is trivial to prove the same inequality for \( n = 2 \). But already for \( n = 3 \) smaller \( k \) occur: First rephrase the problem as a projective problem, and then dualize. Now our problem is:

"Let \( V \) be a projective space of dimension \( n + 1 \) over GF\((q)\) which is spanned by \( k \) subspaces \( U_i \) \( (1 \leq i \leq k) \) such that any hyperplane contains at least one of the \( U_i \), and where there are hyperplanes \( H_i \) such that \( H_i \) does not contain any \( U_j \) \( (j \neq i, 1 \leq i \leq k) \). Find a lower bound for \( k \)."

In the special case \( n = 3 \) we get \( \dim V = 2 \) and ask for a minimal blocking set (with less than \( 2q \) elements). If \( q \) is a square then a Baer subplane will do—it provides us with an example with \( q + \sqrt{q} \) elements. Also when \( q \) is not a square one may have \( k < 2q \). For example, if \( q = 5 \) one may take 4 points on a
line and 5 points forming a transversal of the remaining two parallel classes. This gives $k = 9$. (See Hirschfeld [2, Ch. 13] for a discussion of blocking sets.)

Note that for $q = 2$, $n = 3$ we have a blocking family $\{U_i\}$, consisting of two points and two lines, but a blocking set consisting of points only does not exist. It is easily seen that for $q \geq 3$ we may restrict attention to blocking sets, and thus $k \geq q + \sqrt{q} + 1$, with equality precisely in case of a Baer subplane.

The case $n > 3$ remains open. (But see [3].)

Postscript

It turns out that Ganter's question is a slight generalization of a conjecture by G. Bruns on the covering of Boolean algebras by subalgebras. Thus, our result settles Bruns' conjecture. (See also [5].)

References