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SPECIAL FUNCTIONS AS APPROXIMANTS IN UNIFORM ASYMPTOTIC EXPANSIONS OF INTEGRALS; A SURVEY

Summary: *The use of special functions is described in uniform asymptotic expansions. Especially the role of the error function (normal probability function) is explained in various different types of integrals. Furthermore Bessel functions, Airy functions, parabolic cylinder function, and (incomplete) gamma functions are considered, some of them just as an example. The role of critical points is discussed and transformations of integrals into standard forms that show typical phenomena for uniformity parameters. Several open problems are mentioned, especially on the construction of error bounds for the remainders of the expansions.*

1. Introduction.

The higher transcendental functions of mathematical physics occur as approximants in asymptotic expansions of integrals when uniformity parameters disturb simpler expansions in which exponential or circular functions are used. The special functions that take over the role of these simpler functions are error functions, Airy functions, Bessel functions, parabolic cylinder functions, incomplete gamma functions, etc. The first two of this group are functions of one (complex) variable, the other ones are functions of two variables. When both parameters play an essential role, the functions of the second group can be used in problems that are more complicated than where one-pa-

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rameter functions occur as approximants. Examples in the following sections will show these phenomena.

The starting point of the investigations is most frequently proposed in the form

$$(1.1) \qquad I(z) = \int_C f(t)\, e^{-z\phi(t)}\, dt,$$

where C is a contour in the complex t-plane and z is a large complex parameter. The contour and the functions f and ϕ may depend on z and on additional parameters $\alpha, \beta, \dots$ that serve as uniformity parameters. For certain domains of $\alpha, \beta, \dots$ the asymptotic expansion of (1.1) can be given in terms of elementary functions. In this paper we suppose that for certain values of $\alpha, \beta, \dots$ the behaviour of $I(z)$ will change in such a way that special functions are needed to describe the asymptotic behaviour.

As a first example to describe the role of a uniformity parameter and the use of a certain special function we consider the exponential polynomial. Let

$$(1.2) \qquad e_n(x) = e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}, \quad x \geqslant 0, \quad n = 1, 2, \dots$$

We are interested in the asymptotic behaviour of $e_n(x)$ for large values of n. This problem is considered earlier, for instance in WONG [25], TRICOMI [20], and [18]. We have

$$(1.3) \qquad \begin{aligned} &\lim_{n\to\infty} e_n(x) = 1, \ x \ \textit{fixed}, \\ &\lim_{x\to\infty} e_n(x) = 0, \ n \ \textit{fixed}. \end{aligned}$$

So the first limit cannot be uniformly valid when x grows with n. From fig. 1 we observe that, probably, a better formulation of (1.3) is

$$(1.4) \qquad \lim_{n\to\infty} e_n(\lambda n) = \begin{cases} 1, & 0 \leqslant \lambda < 1 \\ 0, & \lambda > 1 \end{cases}$$

and the convergence is good for λ outside an interval around $\lambda = 1$. Furthermore, we observe that the limiting form is the step-function $\theta(1-\lambda)$, $\lambda \geqslant 0$.

To approximate $e_n(\lambda n)$ for large n and $\lambda \in \Lambda$, where Λ is an interval in $[0, \infty)$ containing $\lambda = 1$, we need as basic approximant the

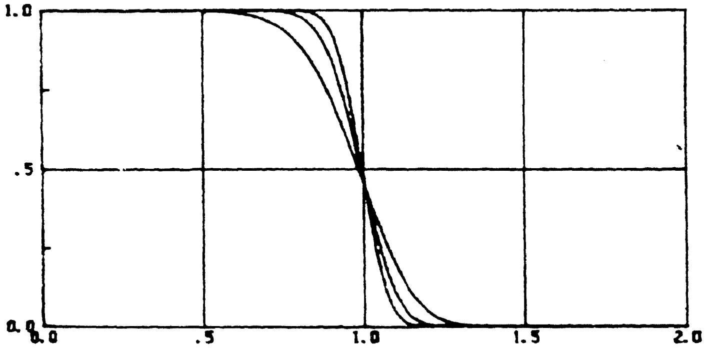

Fig. 1. $e_n(\lambda n)$ for $n = 5, 25, 50$

error function. Incidentally, the central limit theorem for the Poisson distribution already predicts the role of the normal distribution function (error function).

We proceed with an integral representation of $e_n(x)$ in which we can generalize n to be a real (or complex) parameter. We have $e_n(x) = \Gamma(n,x)/\Gamma(n)$, where we use the incomplete gamma function defined by

(1.5) $$\Gamma(a,x) = \int_x^\infty e^{-t}\, t^{a-1}\, dt \;.$$

Here we suppose that $a > 0$, $x \geqslant 0$. A simple transformation gives

$$\Gamma(a,\lambda a) = e^{-a}\, a^a \int_\lambda^\infty e^{-a(\tau-\log\tau-1)}\, \tau^{-1}\, d\tau \;.$$

In the graph of $\tau - \log\tau - 1\,(\tau > 0)$ we recognize the shape of a parabola and the following transformation is appropriate

(1.6) $$\tau - 1 - \log\tau = \tfrac{1}{2}u^2 \;.$$

The mapping $u : R^+ \to R$ is specified further by the condition $\operatorname{sign}(u) = \operatorname{sign}(\tau - 1)$. So we obtain

(1.7) $$\Gamma(a,\lambda a) = e^{-a}\, a^a \int_\alpha^\infty e^{-\frac{1}{2}au^2} f(u)\, du \,,$$

where $\alpha = u(\lambda)$, $f(u) = \tau^{-1}\, d\tau/du$, $f(0) = 1$. The function f is analytic in a domain of the complex t-plane that contains the real axis.

When α is considered as a fixed parameter in (1.7) then we have three different forms of the integral (denoted by I) as $a \to \infty$:

$$\alpha < 0 : \quad I = \frac{-1}{\alpha a}\, e^{-\frac{1}{2} a\alpha^2}\, f(\alpha)\, [1 + O(a^{-1})]\,,$$

$$\alpha = 0 : \quad I = \sqrt{\frac{\pi}{2a}}\, [1 + O(a^{-\frac{1}{2}})]\,,$$

$$\alpha > 0 : \quad I = \sqrt{\frac{2\pi}{a}}\, [1 + O(a^{-1})]\,.$$

Only the error function can describe these three forms in one formula. It is obtained from (1.7) by replacing $f(u)$ by its value at $u = 0$ (where $\exp(-\frac{1}{2} au^2)$ assumes a maximal value). Then the integral can be expressed in terms of the error function which is defined by

$$\text{(1.8)} \qquad \operatorname{erfc} x = 2\pi^{-\frac{1}{2}} \int_x^\infty e^{-t^2}\, dt\,.$$

So we obtain

$$\text{(1.9)} \qquad \Gamma(a, \lambda a) = \tfrac{1}{2}(2\pi/a)^{\frac{1}{2}}\, e^{-a}\, a^a \operatorname{erfc}(\alpha\sqrt{a/2}) - R_a\,,$$

where R_a is the right-hand side of (1.7) with $f(u)$ replaced by $f(0) - f(u)$; α is given by

$$\text{(1.10)} \qquad \alpha = [2(\lambda - 1 - \log\lambda)]^{\frac{1}{2}}\,, \ \operatorname{sign}(\alpha) = \operatorname{sign}(\lambda - 1)\,.$$

For $e_n(x)$ we now obtain

$$\text{(1.11)} \qquad e_n(\lambda n) = \tfrac{1}{2} A_n \operatorname{erfc}(\alpha\sqrt{n/2}) - R_n/\Gamma(n)\,,$$

$n = 1, 2, \ldots$; α is given in (1.10) and

$$A_n = \sqrt{\frac{2\pi}{n}}\, e^{-n}\, n^n/\Gamma(n) = 1 + O(n^{-1})$$

as $n \to \infty$ (Stirling). For a first approximation we can replace A_n by unity and R_n by zero. In fig. 2 we show the graphs of the "error" curves

$$\Delta e_n(\lambda n) = e_n(\lambda n) - \tilde{e}_n ,$$

where $\tilde{e}_n$ is (1.11) with $A_n = 1, R_n = 0$. They show that the approximations e_n are rather uniform with respect to λ.

A bound for R_a in (1.9) is obtained by verifying numerically

$$0 \leqslant \frac{f(0) - f(u)}{u} \leqslant 1 , \quad u \in R .$$

Then we have

(1.12) $$0 \leqslant R_a / \Gamma(a) \leqslant \frac{e^{-\frac{1}{2} a \alpha^2}}{\sqrt{2\pi a}}$$

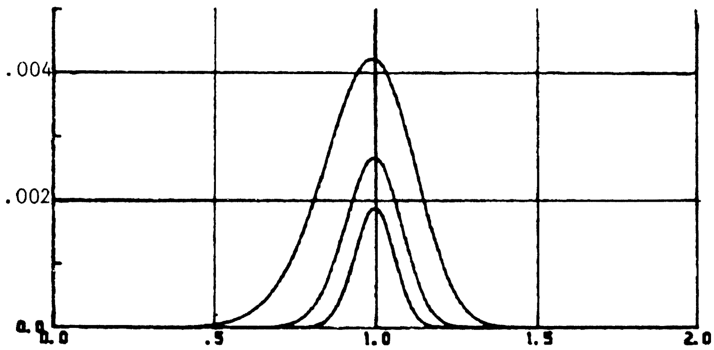

Fig. 2. Error curves $\Delta e_n(\lambda n)$, $n = 5, 25, 50$.

which holds for all $a > 0 , x \geqslant 0$.

In section 5.5 we give the complete expansion for $\Gamma(a,x)$, together with several other examples in which the error function is used as approximant.

An interesting question is whether a simpler function can take over the role of the error function in (1.9). The question is not of academic nature, since a positive answer would have important consequences in the theory of probability functions. As a possible candidate, consider

$$(1.13)\qquad 1-\tanh x=\int_x^{\infty}\frac{dt}{\cosh^2 t}\,.$$

Both integrals (1.8) and (1.13) have a bell-shaped integrand. Also, $\frac{1}{2}[1-\tanh n(\lambda-1)]$ has as limiting form $\theta(1-\lambda)$ as $n\to\infty$. So $e_n(\lambda n)$ may have this hyperbolic function as an approximant, or a similar function with a more complicated argument. In order to bring (1.7) into an integral of the form (1.13) we consider the mapping

$$(1.14)\qquad \tfrac{1}{2}au^2=2\log\cosh at\ ,\ \operatorname{sign}(u)=\operatorname{sign}(t)\,.$$

So now the mapping depends on the large parameter a and

$$\frac{du}{du}=\frac{\tanh at}{u}$$

which is regular at $u=0$. The limit at $u=0$ equals $\sqrt{a}$. The main problem in the mapping defined in (1.14) is that singularities in the complex t-plane (where $\cosh at$ vanishes) approach zero when $a\to\infty$. Therefore an asymptotic expansion, which has to be obtained from values of du/dt and higher derivatives at $t=0$, cannot be obtained in this way. We expect that any approximant in terms of elementary functions will show these features.

To conclude this section we remark that the familiar expansion

$$(1.15)\qquad \Gamma(a,x)\sim x^{a-1}e^{-x}\left[1+\frac{a-1}{x}+\frac{(a-1)(a-2)}{x^2}+\ldots\right]$$

$x\to\infty$, cannot be used when a and x are of the same order. A variant due to TRICOMI [20] is

$$(1.16)\qquad \Gamma(a+1,x)\sim\frac{e^{-x}x^{a+1}}{x-a}\sum_{n=0}^{\infty}\frac{(-1)^n n!}{(x-a)^n}\,\ell_n(a)\ ,$$

$(x-a)/\sqrt{a}\to\infty$. The coefficients $\ell_n(a)$ are polynomials of order $[n/2]$ in a ; $\ell_0(a)=1$, $\ell_1(a)=0$ and the other ones follow from the recursion

$$(n+1)\,\ell_{n+1}(a)=n\,\ell_n(a)-a\,\ell_{n-1}(a)\ .$$

This expansion is stronger than (1.15). For instance it can be used when $a = \mu x$, with $0 \leqslant \mu < 1$ (μ fixed), but the case $\mu = 1$ cannot be covered.

2. General aspects of uniform asymptotic expansions

To obtain uniform expansions for (1.1) the following major steps can be distinguished:

(*i*) Trace the points on C or near C that significantly contribute to $I(z)$

(*ii*) Transform the integral into a standard form

(*iii*) Construct a formal uniform expansion

(*iv*) Investigate the asymptotic properties of the expansion

(*v*) Construct error bounds

(*vi*) Extend the results to wider domains of the parameters.

The first three are most frequenlty the only possibilities to investigate in practical problems. In applications this formal approach is usually accepted. Often the contributions in the expansion have a physical meaning and then just the form of the expansion is the ultimate requirement. In a systematic study of uniform asymptotic expansions the remaining steps should be incorporated. Also, in numerical applications efficient error bounds are particularly important and in this area point (*v*) cannot be forgotten.

The above points are not the only problems to be investigated. Several problems arising in physics (for instance in optics and in scattering theory) yield integrals which are generalizations of Airy-type integrals. Then the approximants are higher transcendental functions which fall outside the classical ones. The computational problems for these generalizations are not easy to solve.

In this paper we discuss several aspects of the steps enumerated above. We give definitions of asymptotic expansions, we consider critical points and various methods and techniques to construct the coefficients and, for some cases, error bounds. Several unsolved problems are mentioned.

A standard reference work for asymptotic expansions is OLVER [10], also for special functions; see also OLVER [11] for uniform expansions for special functions. WONG [27] gives a survey with recent results on error bounds for asymptotic expansions of integrals.

3. Some definitions.

In this section we give definitions of (uniform) asymptotic expansions and we consider critical points.

3.1 Definitions of asymptotic expansions

We use the terminology of generalized asymptotic expansions. First we introduce the concept of asymptotic scale:

a sequence of functions $\{\phi_n(x)\}$ is called an *asymptotic sequence* or *scale* when $\phi_{n+1}(x) = o\,[\phi_n(x)]$ as $x \to \infty$.

Then we have the definition:

the formal series $\Sigma_{n=0}^{\infty} f_n(x)$ is said to be an *asymptotic expansion* of $f(x)$ *with respect to the scale* $\{\phi_n\}$ if

$$(3.1) \qquad f(x) - \sum_{n=0}^{N} f_n(x) = o\,[\phi_N(x)] \quad \textit{as} \quad x \to \infty\,, \quad N = 0, 1, \dots ;$$

in this case we write

$$f(x) \sim \sum_{n=0}^{\infty} f_n(x)\ ;\ \{\phi_n(x)\}\ \textit{as}\quad x \to \infty .$$

In uniform expansions it is required that the "o" sign holds uniformly (with respect to $\alpha \in A$, say).

This general set up is extensively described in ERDELYI & WYMAN [4].

When $f_n = \phi_n$ we have a Poincaré type asymptotic expansion; when $f_n = \phi_n = x^{-n}$ we obtain the definition of Poincaré and Stieltjes, who both introduced the definition of this kind in 1886.

Observe that in (3.1) no requirements are put on $\{f_n\}$: it need not be an asymptotic scale. Rather useless expansions may arise (from an asymptotic point of view) when it is not. Also, we can take the scale too rough to measure the error in (3.1). Both phenomena occur in the following example.

Example 2.1. Take $f_n(x) = (x+n)^{-2}$, and $\phi_n(x) = \log^{-n} x$, $x > 1$, $n = 0, 1, 2, \dots$. Then we have

$$\sum_{n=N+1}^{\infty} (x+n)^{-2} = \mathcal{O}(x^{-1}) = o\,[\phi_m(x)] \qquad \textit{as} \quad x \to \infty$$

for all N, m. So we can write

$$f(x) \sim \sum_{n=0}^{\infty} (x+n)^{-2} \quad ; \quad \{\log^{-n} x\} \quad as \quad x \to \infty$$

where for f we can take the convergent sum, which represent $d^2 \ln \Gamma(x)/dx^2$ (Γ is the Euler gamma function).

Some expansions are provided with a "thin" scale in which successive terms become more and more indistinguishable. The following example is in WIMP [24], a survey on uniform scale functions and asymptotic expansion of integrals.

Example 2.2. The coefficients a_n of the expansion $\Gamma(1-t) = \Sigma_{n=0} a_n t^n$, $|t| < 1$, satisfy the expansion

$$a_n \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+1)^{n+1}} \quad ; \quad \{\phi_k(n)\} \; as \quad n \to \infty,$$

where $\phi_k(n) = (k+1)^{-n}$. The series converges rather fast. However, the scale satisfies $\phi_{k-1}(n)/\phi_k(n) = (1+1/k)^{-n}$, which indeed is $o(1)$ as $n \to \infty$, $k \geqslant 1$, k fixed. But as k increases this ratio tends to unity (n fixed).

For some functions we need a *compound asymptotic expansion*. That is, we have a decomposition

$$(3.2) \qquad \begin{aligned} f(x) &= A_1(x) f_1(x) + \ldots + A_n(x) f_n(x) \\ f_k(x) &\sim \sum_{j=0}^{\infty} f_{jk}(x) \quad ; \quad \{\phi_{jk}\} \quad as \quad x \to \infty, \end{aligned}$$

where, for each k, $\{\phi_{jk}(x)\}$ is an asymptotic scale. In complicated problems the f_k are not known a priori. The functions $A_k(x)$ are usually well-known special functions.

It may be rather difficult to investigate whether an expansion is uniform with respect to a parameter α. A non-uniformity may be recognized when in (3.2) $A_k(x)$, $f_{jk}(x)$ or $\phi_{jk}(x)$ are singular at certain values of the uniformity parameter α, whereas $f(x)$ remains regular fot these values. A good example is the expansion for $e_n(x) = \Gamma(n, x)/\Gamma(n)$ based on (1.16) ($x \to \infty$). The function $e_n(x)$ is regular at $x = n-1$, the coefficients in expansion (1.16) are not.

3.2. Critical points

There is a systematic approach to obtain the asymptotic expansion of (1.1). We have to look for certain distinguished points whose immediate neighbourhoods determine completely the asymptotic behaviour of the integral. Such points are called *critical points* by VAN DER CORPUT [3]. Possible candidates are:

- the end points of the contour
- singular points of the integrand
- stationary or saddle points of ϕ (i.e., where $\partial\phi/\partial z$ vanishes)

The contribution of a signle critical point to the asymptotic value of $I(z)$ is known for a great variety of critical points. We mention some key words in this respect: Watson's lemma, the method of Laplace, the method of steepest descent, the method of saddle points, the principle of stationary phase and the method of Darboux. We give a formulation of one of the most important tools.

Lemma (Watson). *Consider the Laplace integral*

$$I(z) = \int_0^\infty e^{-zt} f(t)\, dt \quad . \tag{3.3}$$

Assume that

(*i*) f *is locally integrable on* $[0, \infty)$;

(*ii*) $f(t) \sim \sum_{s=0}^{\infty} a_s t^{(s+\lambda-\mu)/\mu}$ *as* $t \to 0^+$, μ, λ *fixed,* $\mu > 0$, $\mathrm{Re}\lambda > 0$;

(*iii*) *the abscissa of convergence of* (3.3) *is not* $+\infty$.

Then,

$$I(z) \sim \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right) a_s\, z^{-(s+\lambda)/\mu} \tag{3.4}$$

as $z \to \infty$ *in the sector* $|\arg z| \leqslant \frac{1}{2}\pi - \delta \ (< \frac{1}{2}\pi)$ *where* $z^{(s+\lambda)/\mu}$ *has its principal value.*

Proof. See OLVER [10, p. 113]. □

Observe that (3.4) is obtained by substituting (*ii*) into (3.3) and by interchanging the order of summation and integration. In (*ii*) λ and μ are fixed. When $\lambda = \mathcal{O}(z)$ (or larger) the expansion (3.4) has no meaning. A modification of the lemma is needed then to give a uniform expansion, see [19].

When in (1.1) the additional parameters range over a domain A the critical points may be variable. For certain values in A two or more critical points may coalesce. Usually, the form of the expansions changes and it is unlikely that the sum of the contributions of each critical point will be uniformly valid. For instance, coefficients of the several expansions may become singular when the uniformity parameters take these distinguished values.

In the example for $e_n(x)$ considered in section 1 the critical pionts for the integral in (1.7) are $u = 0$ (saddle point), $u = \alpha$ (end point of interval of integration) and singularities of $f(u)$. The latter are bounded away from the first two, but these in turn may coalesce (recall that $\alpha = 0$ when $\lambda = 1$).

The systematic approach of van der Corput to add several contributions from the critical points was an important step to take away part of the mystery of asymptotics. In uniform problems it is also important to systematize. We can single out the following possibilities for (1.1):

- singularity coincides with stationary point
- end-point of contour coincides with stationary point
- two stationary points coincide.

In VAN DER WAERDEN [22], CHESTER, FRIEDMAN & URSELL [2] and BLEISTEIN [1] important contributions are given for these cases.

By introducing several auxiliary parameters much more situations can occur. Some of them correspond with important physical applications or with problems for the well-known special functions of mathematical physics. A survey is given by OLVER [11].

The approximants in uniform expansions are usually more complicated than the elementary functions used in earlier days. Now we use error functions, Airy functions, Bessel functions, parabolic cylinder functions, etc. The computational problem has been solved for most of these functions, and now they are accepted as approximants.

In classifying relevant cases of coalescing critical points it is instructive to look at approaches via the WKB or Liouville-Green methods for differential equations. Most functions from mathematical physics can be investigated in both directions: they have an integral representation and they satisfy a differential equation. See again [11] for more details on this point.

4. Examples of standard forms.

When the critical points are located and their influence is examined it is time to compare the phenomena with those occurring in integral representations of special functions. Some of these representations show standard type features fo coalescing critical points. Therefore it is important to classify and to consider standard forms of integrals that show typical aspects of uniformity problems.

In the following table we give standard forms of integrals for which well-known special functions are used as approximants; z is a large parameter, α a uniformity parameter. We give the critical points, the coalescence of which causes uniformity problems, and references to the literature.

	Standard form	Approximant	Critical points	References
(4.1)	$\int_{-\infty}^{\infty} e^{-zt^2} \frac{f(t)}{t-i\alpha} dt$	Error function	$t=0$, $t=i\alpha$	[22]
(4.2)	$\int_{-\infty}^{\infty} e^{-zt^2} \frac{f(t)}{(t-i\alpha)^{\mu}} dt$	Parabolic cylinder function	$t=0$, $t=i\alpha$	[1]
(4.3)	$\int_{-\infty}^{\alpha} e^{-zt^2} f(t)\, dt$	Error function	$t=0$, $t=\alpha$	[17]
(4.4)	$\int_{0}^{\infty} t^{\beta-1} e^{-z(\frac{1}{2}t^2-\alpha t)} f(t) dt$	Function cylinder	$t=0$, $t=\alpha$	[1],[5] [12], [25]
(4.5)	$\int_{C} e^{z(\frac{1}{3}t^3-\alpha t)} f(t) dt$	Airy function	$t=\pm\sqrt{\alpha}$	[2],[7]
(4.6)	$\int_{0}^{\infty} t^{\alpha-1} e^{-zt} f(t) dt$	Gamma function	$t=0$, $t=\alpha/z$	[19]

	Standard form	Approximation	Critical points	References
(4.7)	$\int_{\alpha}^{\infty} t^{\beta-1} e^{-zt} f(t)\,dt$	Incomplete gamma function	$t=0$, $t=\alpha$	[6], [14] [16], [28]
(4.8)	$\int_{0}^{\infty} t^{\beta-1} e^{-z(t+\alpha/t)} f(t)\,dt$	Bessel function	$t=0$, $t=\pm\sqrt{\alpha}$	[17]
(4.9)	$\int_{\alpha}^{\infty} f(\sqrt{t^2-\alpha^2}) \sin t\,dt$	Bessel function	$t=\pm\alpha$	[21],[26]
(4.10)	$\int_{0}^{\infty} \frac{\sin z(t-\alpha)}{t-\alpha} f(t)\,dt$	Sine integral	$t=0$, $t=\alpha$	[29]

Remarks

1) Functions f are supposed to be regular in neighbourhoods of the critical points.
2) The integrals reduce to their approximants when $f=1$, except in (4.9) where it occurs for $f(t)=t^{\beta}$.
3) Quite different integrals may have the same approximants.
4) Different intervals of integration are investigated too.
5) In (4.5), (4.8) two saddle points coalesce with each other when $\alpha=0$; both cases are different, however. In (4.8) we have an additional critical point at $t=0$ (end point and singularity).
6) In all cases elementary approximants can be used for fixed values of the uniformity parameter α .
7) Several of the examples need further investigations with respect to the construction of error bounds and the determination of maximal regions of validity.
8) (4.3) is considered in section 1.
9) For all examples it is rather easy to construct a (formal) uniform expansion. Investigation of the steps (*iv*), (*v*), (*vi*) of section 2 is much more difficult.

10) Usually a non-trivial mapping is needed to bring the integral into onє the standard forms. Then f may be rather complicated. For examp see (1.7) and section 6. The construction of error bounds may be rat difficult for such cases.

5. Error functions as approximants.

In this section we discuss five different cases in which the error funct occur as approximants. Although they are not entirely different, it seems w wile to pay attention to each separate case.

5.1. Standard form (4.3)

This type is frequently encountered in functions related with cumul distribution functions. The incomplete gamma functions (including χ^2 tributions and Poisson distributions), incomplete beta functions (inclu Student's distribution), Pearson type IV distributions, the Sievert integral, the incomplete probability integral all can be transformed to the stan form (4.3). For the incomplete gamma function this transformation (b on a quadratic function) is shown in section 1. More information is giv [18]. The asymptotic phenomenon is that for small values of α two cri points 0 and α are close together.

In section 1 we showed how to obtain a first approximation to complete with error bound. Higher order approximations follow fror integration by parts procedure, in which contributions from both cr points 0 and α appear.

To describe the main steps we write (4.3) in the form

$$F_\alpha(z) = \sqrt{\frac{z}{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{1}{2}zt^2} f(t)\, dt . \tag{5.1}$$

By using (1.8) we obtain

$$F_\alpha(z) = \tfrac{1}{2} f(0)\,\mathrm{erfc}\,(-\alpha\sqrt{z/2}) + \sqrt{\frac{z}{2\pi}}\int_{-\infty}^{\alpha} e^{-\frac{1}{2}zt^2}\,[f(t)-f(0)]\,dt$$

$$= \tfrac{1}{2} f(0)\,\mathrm{erfc}\,(-\alpha\sqrt{z/2}) - (2\pi z)^{-\frac{1}{2}}\int_{-\infty}^{\alpha} \frac{f(t)-f(0)}{t}\,de^{-\frac{1}{2}zt^2}$$

$$= \tfrac{1}{2} f(0)\,\mathrm{erfc}\,(-\alpha\sqrt{z/2}) + (2\pi z)^{-\frac{1}{2}}\,\frac{f(0)-f(\alpha)}{\alpha} + (2\pi z)^{-\frac{1}{2}}\int_{-\infty}^{\alpha} f_1(t)\, e^{-\frac{1}{2}zt^2}\,dt.$$

Repeating this process, we obtain

$$\text{(5.2)}\qquad F_\alpha(z) \sim \tfrac{1}{2}\,\mathrm{erfc}\,(-\alpha\sqrt{z/2}) \sum_{s=0}^{\infty} \frac{a_s}{z^s} + \frac{e^{-\frac{1}{2}z\alpha^2}}{\sqrt{2\pi z}} \sum_{s=0}^{\infty} \frac{b_s(\alpha)}{z^s}\,,$$

where $a_s = f_s(0)$, $b_s(\alpha) = [f_s(0) - f_s(\alpha)]/\alpha$, with

$$\text{(5.3)}\qquad f_s(t) = \frac{d}{dt}\,\frac{f_{s-1}(t) - f_{s-1}(0)}{t}\,,\quad s = 1, 2, \ldots$$

and $f_0(t) = f(t)$. When f is analytic in a domain containing R, then f_s also is, and all coefficients a_s, b_s are well-defined. Observe that (5.2) is a compound asymptotic expansion, as introduced in (3.2). It is valid for $z \to \infty$ and it is uniformly valid with respect to α. The domain of uniformity depends on the function f. See [18].

The "complementary" integral, which is (5.1) with interval of integration $[\alpha, \infty]$ has an expansion as in (5.2). The only changes are the sign of the argument of erfc and that before the second series.

From the construction of (5.2) it follows that a remainder for the expansion is defined in the exact result

$$\text{(5.4)}\qquad F_\alpha(z) = \tfrac{1}{2}\,\mathrm{erfc}\,(-\alpha\sqrt{z/2}) \sum_{s=0}^{n-1} \frac{a_s}{z^s} + \frac{e^{-\frac{1}{2}z\alpha^2}}{\sqrt{2\pi z}} \sum_{s=0}^{n-1} \frac{b_s(\alpha)}{z^s} + \frac{R_n}{z^n}$$

where

$$\text{(5.5)}\qquad R_n = \sqrt{\frac{z}{2\pi}}\int_{-\infty}^{\alpha} e^{-\frac{1}{2}zt^2} f_n(t)\,dt\,,\quad n = 0, 1, \ldots$$

This remainder can be considered for obtaining an error bound; see [18] for more details. The iterated functions (5.3) are not easy to estimate, h ver. This is certainly the case when f is defined by means of a transform as in (1.6).

5.2 Standard form (4.1).

Now the error function appears due to

$$(5.6) \qquad \frac{e^{-\alpha^2}}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-i\alpha}\, dt = \tfrac{1}{2}\,\mathrm{erfc}\,(\alpha) \quad , \ \mathrm{Re}\ \alpha > 0 \ .$$

So, integrals of the form (4.1) can be tackled by writing $f(t) = [f(t) - f($ $+ f\,(i\alpha)$. When f is smooth enough an application of Watson's lemm an asymptotic expansion plus a single term with an error function.

It is not always obvious that a given integral can be transforme standard form (4.1). The incomplete gamma functions, which are know in normalized form as

$$(5.7) \quad P\,(a\,,\,x) = \frac{1}{\Gamma\,(a)}\int_0^x t^{a-1}\, e^{-t}\, dt\,,\ Q\,(a\,,\,x) = \frac{1}{\Gamma\,(a)}\int_x^{\infty} t^{a-1}\, e^{-t}\, dt$$

$(x \geqslant 0\,,\, a > 0)$ have such a representation. To see this we observe t Laplace transform of $dP\,(a, x)/dx$ equals $(s+1)^{-a}$. So we have inversion formula

$$P\,(a\,,\,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{s\,(s+1)^a}\, ds \quad , \quad c > 0 \ .$$

By a simple transformation we obtain

$$(5.8) \qquad P\,(a\,,\,\lambda a) = \frac{e^{-a\phi(\lambda)}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{a\phi(t)}}{t-\lambda}\, dt \quad , \quad c > \lambda \ ,$$

where $\phi\,(t) = t - 1 - \log t$. $Q\,(a\,,\,\lambda a)$ has the same integral with $0 <$ This follows from $P\,(a, x) + Q\,(a\,,\,x) = 1$. The pole at $t = \lambda$ gives residue the value 1, when we shift in (5.8) the path across the pole.

The saddle point follows from $\phi'(t) = 0$. So we have two points: $t = 1$ (saddle point) and $t = \lambda$ (first order pole), which a together when in $P(a, x)$ or $Q(a, x)$ a and x are nearly equal. In [15] we transformed (5.8) into standard form (4.1) by integrating (5.8) first over the path of steepest descent. Observe that for the incomplete gamma function two completely different starting points are available (respectively (4.1) and (4.3)).

Many more distribution functions can be written in the form (4.1). For instance, the incomplete beta functions. See again [15]. The method of splitting off the pole is due to VAN DER WAERDEN [22].

5.3 Standard form (4.4), $\beta = 1$.

The transformation $t - \alpha = u$ changes (4.4) when $\beta = 1$ into the form

$$e^{\frac{1}{2}z\alpha^2} \int_{-\alpha}^{\infty} e^{-\frac{1}{2}zu^2} f(\alpha + u)\, du \tag{5.9}$$

which in fact is (4.3). So the technique outlined in subsection 5.1 can be used for obtaining the asymptotic expansion which is uniformly valid with respect to α, especially for small values of α.

5.4 Standard form (4.7), $\beta = \frac{1}{2}$

In (4.7) it is supposed that $\alpha \geqslant 0$. For general positive β there is a singularity at $t = 0$, and an end point of integration at $t = \alpha$. Both are critical points, although the origin is outside the domain of integration, except when $\alpha = 0$. For fixed α, Watson's lemma can be used by expanding in

$$I(z) = \int_{\alpha}^{\infty} t^{\beta-1} e^{-zt} f(t)\, dt \tag{5.10}$$

the function $t^{\beta-1} f(t)$ in a Taylor series at $t = \alpha$. Due to the singularity at $t = 0$ the coefficients of this series blow up when $\alpha \downarrow 0$. When $\beta = \frac{1}{2}$ the case reduces to the previous one (5.9) or to (4.3). In that event we have

$$I(z) = z \int_{\sqrt{\alpha}}^{\infty} e^{-zu^2} f(u^2)\, du \tag{5.11}$$

and the integration by parts procedure of section 5.1 gives the uniform expansion.

As shown by ERDELYI [5] (cf. also [14], [16], [28]) the general case (5.10) has an expansion in terms of incomplete gamma functions.

Form (5.10) may also be considered with interval of integration $[0,\alpha]$. For $\beta = \frac{1}{2}$ then the error function erf is used as approximant. In some problems both cases $[0, \alpha]$, $[\alpha, \infty)$ are complementary, and can be derived from each other. The following example shows that special attention for the interval $[0, \alpha]$ may be important.

We consider the "incomplete" Bessel function, which plays a role in the von Mises distribution,

$$F_\alpha(z) = e^{-z} \int_0^\alpha e^{z\cos\theta}\, d\theta\,, \quad 0 \leqslant \alpha \leqslant \pi\,. \tag{5.12}$$

We have $F_\pi(z) = \pi e^{-z} I_0(z)$, the modified Bessel function. A simple transformation $\sin\frac{1}{2}\theta = u$ gives

$$F_\alpha(z) = 2\int_0^{\sin\frac{1}{2}\alpha} \frac{e^{-2zu^2}}{\sqrt{1-u^2}}\, du\,.$$

For fixed positive α an asymptotic expansion can be obtained by Watson's lemma (expansion of $(1-u^2)^{-1/2}$ at $u=0$ and by replacing $\sin\frac{1}{2}\alpha$ by ∞). For a uniform expansion we let $\sin\frac{1}{2}\alpha$ as it stands. The result is a convergent ($z \in C$) expansion in terms of incomplete gamma functions, of which the first one is an error function. We have

$$F_\alpha(z) = \frac{1}{\sqrt{2z}} \Sigma \left(\frac{-1}{2z}\right)^k \binom{-\frac{1}{2}}{k} \gamma\left(k+\tfrac{1}{2}, 2z\sin^2\tfrac{1}{2}\alpha\right) \tag{5.13}$$

where $\gamma(a, z) = \Gamma(a) - \Gamma(a, z)$ (see (1.5)). Error bounds can be obtained rather easily since much information on this point is available for the binomial expansion of $(1-u^2)^{-\frac{1}{2}}$. As remarked above $\gamma(\frac{1}{2}, x)$ is related to the error function. We have

$$\gamma\left(\tfrac{1}{2}, 2z\sin^2\tfrac{1}{2}\alpha\right) = \sqrt{\pi}\, \mathrm{erf}\left(\sin\tfrac{1}{2}\alpha\sqrt{2z}\right).$$

The remaining terms can be obtained via the recursion relation for $\gamma(a, x)$, i.e.,

$$\gamma(a+1, x) = a\gamma(a, x) - x^a e^{-x}\,,$$

but numerically it is not stable. It has to be used in the so-called backward form.

In HILL [8] an algorithm for the numerical computation of (5.12) is given, in which for large z the asymptotic normality of the distribution function related with (5.12) is used. An approximation is used in terms of the error function. However, we expect that with (5.13) an algorithm can be constructed that is more powerful than that of Hill.

5.5 Another variant

It is of the form

$$F_\alpha(z) = \int_z^\infty e^{-\alpha t} f(t)\, dt . \tag{5.14}$$

So, compared with (4.7), the parameters z and α have interchanged their places. We will not consider it as a standard form since we do not know many applications. We encountered this type of integrals in studying integrals of modified Bessel functions of the form

$$G(x, y) = \int_0^x \int_0^y e^{-\xi-\eta} I_0(2\sqrt{\xi\eta})\, d\xi d\eta . \tag{5.15}$$

Integrals of this type are used in filtration problems. When ξ and η are large and equal, the integrand is $\mathcal{O}(\xi^{-\frac{1}{2}})$, whereas it is exponentially small when ξ and η are not of the same size. So in the (x, y) quarter plane there is a ridge along the diagonal where the function values are relatively large compared with other far parts of the quarter plane. We recognize features of integrals of subsection 5.1 and we expect an error function for a uniform expansion of $G(x, y)$ as $x, y \to \infty$. Due to symmetry in $f(x,y)$ we can restrict ourselves to $x \geqslant y \geqslant 0$. In a future publication we will show that $G(x, y)$ can be expressed in terms of $F_\alpha(z)$ given in (5.14), with

$$\alpha = \frac{\rho - 1}{\sqrt{\rho}}, \quad \rho = \sqrt{x/y}, \; z = 2\sqrt{xy}, \; f(t) = e^{-t} I_0(t) . \tag{5.16}$$

A uniform expansion follows by expanding f at infinity, i.e., by using the well-known expansion

$$(5.17)\qquad f(t) \sim \frac{1}{\sqrt{2\pi t}} \sum_{m=0}^{\infty} \frac{a_m}{t^m}, \quad a_m = \frac{2^{-m}\,\Gamma(m+\frac{1}{2})}{m!\,\Gamma(\frac{1}{2}-m)} \quad .$$

We obtain for (5.14) the expansion

$$(5.18)\qquad F_\alpha(z) \sim \frac{1}{\sqrt{2\pi\alpha}} \sum_{m=0}^{\infty} \alpha^m\, a_m\, \Gamma(\tfrac{1}{2}-m, \alpha z)$$

where the incomplete gamma function appears defined in (1.5). The first term reduces to the erfc-function with argument $\sqrt{\sqrt{x}-\sqrt{y}}$, a simple recursion gives the remaining terms. We have detailed information on error bounds for the remainders of the expansion in (5.17) (see OLVER [10, p. 269]). So we also have such information for the expansion of $F_\alpha(z)$ in (5.18).

More information on the double integral (5.15), together with several series expansions, can be found in LASSEY [9]. In our opinion an efficient algorithm for computing $G(x, y)$ for large values of x and y is only possible by using the expansion as given in (5.18). In [9] no such expansions are given.

In the notation of the definition of section 3.1 we can consider (5.18) as a Poincaré-type expansion with $f_m = \phi_m = \alpha^m\, a_m\, \Gamma(\frac{1}{2}-m, \alpha z)$. It can be proved that the expansion holds for $z \to \infty$ and uniformly with respect to $\alpha \in [0, \infty)$.

6. Modified Bessel functions as approximants

In this section we consider (4.8) and we construct a uniform expansion. We recall that z is a large parameter and α the uniformity parameter. We suppose that $\alpha \geqslant 0$ and also that z is positive. As in § 5.1 we use an integration by parts procedure, but it should not be done in a straightforward way. It is based on a method of BLEISTEIN [1] and it is useful in various types of integrals.

Before giving the integration by parts method we remark that a uniform expansion of a more limited value can be obtained by expanding f in

$$(6.1)\qquad F(z) = \int_0^\infty t^{\beta-1}\, e^{-\alpha/t - zt} f(t)\, dt$$

at $t = 0$. Then the parameter α may range in a compact interval $[0, \alpha_0]$. The procedure below gives an expansion that is, under suitable conditions on f, uniform with respect to $\alpha \in [0, \infty)$.

We consider (6.1) and we write $\mu^2 = \alpha$. Saddle points occur at $t = \pm \mu$, μ is supposed to be positive. The first step is the representation

(6.2) $$f(t) = a_0 + b_0 t + (t - \mu^2/t)\, g(t)$$

where a_0, b_0 follow from substitution of $t = \pm \mu$. We have

$$a_0 = \tfrac{1}{2}[f(\mu) + f(-\mu)],\ b_0 = \frac{1}{2\mu}[f(\mu) - f(-\mu)].$$

So we obtain upon inserting (6.2) into (6.1)

(6.3) $$F(z) = a_0\, \Phi_0 + b_0\, \Phi_1 + F_1(z)$$

where Φ_0, Φ_1 are modified Bessel functions

$$\Phi_j = 2\,(\mu / \sqrt{z})^{\beta+j}\, K_{\beta+j}(2\mu\sqrt{z}),\ j = 0, 1.$$

An integration by parts gives

$$F_1(z) = \int_0^\infty t^{\beta-1}\, e^{-z(t+\mu^2/t)}\, (t - \mu^2/t)\, g(t)\, dt$$

$$= -\frac{1}{z}\int_0^\infty t^{\beta}\, g(t)\, de^{-z(t+\mu^2/t)}$$

$$= \frac{1}{z}\int_0^\infty t^{\beta-1}\, e^{-z(t+\mu^2/t)} f_1(t)\, dt,$$

with $f_1(t) = t^{1-\beta} \dfrac{d}{dt}[t^\beta g(t)] = \beta g(t) + t g'(t)$. We see that $zF_1(z)$ is of the same form as $F(z)$. The above procedure can now be applied to $zF_1(z)$ and we obtain for (6.1) the formal expansion

$$F(z) \sim \Phi_0 \sum_{s=0}^{\infty} \frac{a_s}{z^s} + \Phi_1 \sum_{s=0}^{\infty} \frac{b_s}{z^s},\quad \text{as}\quad z \to \infty,$$

where we define inductively $f_0(t)=f(t)$, $g_0(t)=g(t)$ and for $s=1,2,\ldots$,

$$f_s(t)=t^{1-\beta}\frac{d}{dt}\,[t^{\beta}g_{s-1}(t)]=a_s+b_s\,t+(t-\mu^2/t)\,g_s(t),$$

$$a_s=\tfrac{1}{2}\,[f_s(\mu)+f_s(-\mu)],\; b_s=\frac{1}{2\mu}\,[f_s(\mu)-f_s(-\mu)].$$

Next we show that it is rather easy to obtain an expansion in which β acts as a second uniformity parameter. Then we exploit fully the fact that the Bessel functions in Φ_j are functions of two variables. The form of the new expansion is exactly as in (6.4), with the same Φ_j, but with different coefficients a_s and b_s.

We write $\beta=2\nu z$, $\nu\in R$. The saddle points $t_{\pm}$ are now zeros of $d\,[t+\mu^2/t-2\nu\log t\,]/dt$, which gives $t_{\pm}=\nu\pm(\nu^2+\mu^2)^{1/2}$. The modification of (6.2) is

$$f(t)=c_0+d_0\,t+(t-2\nu-\mu^2/t)\,h_0(t)$$

and we obtain for (6.2) the formal expansion

(6.5) $$F(z)\sim\Phi_0\sum_{s=0}^{\infty}\frac{c_s}{z^s}+\Phi_1\sum_{s=0}^{\infty}\frac{d_s}{z^s},\quad as\; z\to\infty.$$

Now the coefficients follow from

$$\tilde{f}_0(t)=\tilde{f}(t),\;\tilde{f}_s(t)=t\,\frac{d}{dt}\,h_{s-1}(t)=c_s+d_s\,t+(t-2\nu-\mu^2/t)\,h_s(t),$$

$$c_s=\frac{t_+\,\tilde{f}_s(t_-)-t_-\tilde{f}_s(t_+)}{t_+-t_-},\quad d_s=\frac{\tilde{f}_s(t_+)-\tilde{f}_s(t_-)}{t_+-t_-}.$$

When f of (6.1) is analytic, say in the strip $|\mathrm{Im}\,t|<a$, then all $f_s,\tilde{f}_s$ are well defined and analytic in the same strip for all $\nu\in R,\mu\geqslant 0$. An important question is now: is (6.5) a uniform expanion for the same ν,μ values? For ν,μ in compact sets of respectively R, $[0,\infty)$ it is not difficult to give a positive answer. For the full intervals we may follow the method of [19]. The method of proof is too technical to give here. Of course we need suitable conditions of f.

We now consider a related integral

(6.6) $$G(z)=\frac{1}{2\pi i}\int_C t^{-\beta-1}\,e^{\alpha/t+zt}\,g(t)\,dt$$

where C is a contour shown in fig. 6.1. We suppose that $t^{-\beta}$ is real and positive when t is positive, and that $\beta \in R$, $\alpha \geqslant 0$, $z > 0$.

Using the integration by parts method that leads to (6.5) we obtain the formal asymptotic expansion

$$(6.7) \qquad G(z) \sim \psi_0 \sum_{s=0}^{\infty} \frac{(-1)^s c_s}{z^s} + \psi_1 \sum_{s=0}^{\infty} \frac{(-1)^s d_s}{z^s}, \quad z \to \infty,$$

where c_s, d_s are the same as in (6.5) and

$$\psi_j = (\mu/\sqrt{z})^{j-\beta} I_{\beta-j}(2\mu\sqrt{z}).$$

$I_\nu(z)$ is the modified Bessel function of the first kind; we use the well-known integral representation (see WATSON [23, p. 181]). The shape of (6.5) and (6.7) is quite the same, apart from the approximants ϕ_j, ψ_j. This is one example of related expansions for related integrals. A similar relationship occurs for the two independent solutions of second order differential equations. Then the asymptotic expansions of these solutions may be related as well. Many examples can be found in OLVER [10].

An interesting application of the expansions in (6.5), (6.7) can be given for confluent hypergeometric functions. Let us consider

$$(6.9) \qquad U(a, b, x) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} (1+t)^{b-a-1} e^{-xt}\, dt$$

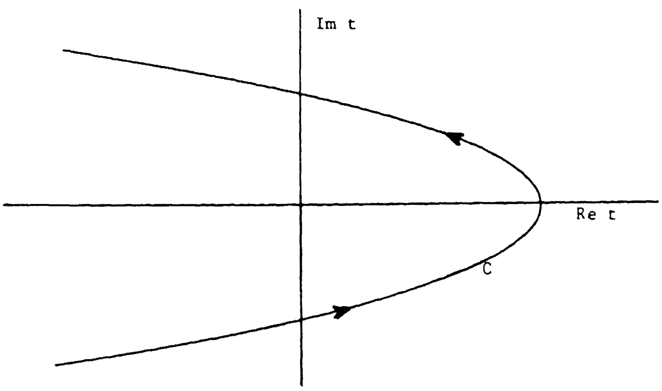

Fig. 6.1. Contour for (6.6)

for $a \to +\infty$, with $x > 0$, $b \in R$. A transformation to the standard form (4.8) is needed, but first we give a simple transformation. The function $[t/(1+t)]^a$ takes its maximal value (on $[0,\infty)$) at $t=+\infty$. This function plays the role of an exponential function. Therefore we take as a new variable of integration τ defined by $t/(1+t) = \exp(-\tau)$. Then (6.9) becomes

$$(6.10) \qquad U(a,b,x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-a\tau - x/(e^\tau - 1)}\, \tau^{-b} f(\tau)\, d\tau$$

where $f(\tau) = [\tau/(1-e^{-\tau})]^b$. The easiest way to arrive at the standard form (4.8) is to write

$$(6.11) \qquad U(a,b,x) = \frac{e^{\frac{1}{2}x}}{\Gamma(a)} \int_0^\infty \tau^{-b}\, e^{-a\tau - x/\tau}\, \tilde{f}(\tau)\, d\tau$$

where

$$\tilde{f}(\tau) = f(\tau) \exp\{x[1/\tau - 1/(e^\tau - 1) - \tfrac{1}{2}]\}.$$

Now we can use the procedure leading to (6.4), with $\beta = 1-b$. The result is an expansion for $a \to \infty$, which holds uniformly with respect to $x \in [0, x_0]$, where x_0 is a fixed positive number. It is not possible here to replace x_0 by ∞; the main reason is that $f(\tau)$ depends on x, in such a way that coefficients a_s and b_s in (6.4) grow too fast when $x \to \infty$.

A more powerful expansion is obtained (with respect to the uniformity domain of x) when we transform (6.10) into (4.8) by using the mapping $u : [0,\infty) \to [0,\infty)$ that is defined by

$$(6.12) \qquad \tau + \frac{\lambda}{e^\tau - 1} = u + \frac{\alpha}{u} + A$$

where $\lambda = x/a$; α and A are to be determined. We compute them by the following condition on the mapping u: the critical points at the left-hand side of (6.12) ($\tau = \pm\gamma$, where γ is the positive number satisfying $\cosh \gamma = 1 + \frac{1}{2}\lambda$) should correspond with those at the right ($u = \pm\mu$, where $\mu^2 = \alpha$).

If follows that

$$A = -\tfrac{1}{2}\lambda, \quad \mu = \tfrac{1}{2}(\gamma + \sinh \gamma).$$

These choices make the mapping u regular with $u(0) = 0$, $u(+\infty) = +\infty$.

We obtain now from (6.12) and (6.10)

$$(6.13)\qquad U(a,b,x)=\frac{e^{-\frac{1}{2}x}}{\Gamma(a)}\int_0^\infty u^{-b}\,e^{-a(u+\alpha/u)}\,f^*(u)\,du$$

where

$$f^*(u)=(u/\tau)^b\,f(\tau)\,\frac{d\tau}{du}\ .$$

We expect that (6.13) as starting point for the construction of an expansion as in (6.5) will give $[0,\infty)$ as uniformity domain for x. Proofs are needed. The first thing to do is to prove the regularity of f in a fixed domain containing $[0,\infty)$ in its interior. Observe that f^* depends on the uniformity parameter α, or x as $\tilde{f}$ in (6.11) does. However, this dependence is completely different for both functions.

An expansion with b as a second uniformity parameter, which eventually should lead to (6.5), should not be based on (6.11) or (6.13). The uniformity domain for b would be rather limited then, due to the way $\tilde{f}$ and f^* depend on b. An optimal domain for b and x can be obtained by using the mapping w: $[0,\infty)\rightarrow[0,\infty)$ defined by

$$(6.14)\qquad -2\nu\log(1-e^{-\tau})+\lambda/(e^\tau-1)+\tau=-2\nu\log w+w+\alpha/w+A$$

where $b=-2\nu a$, $\lambda=x/a$ and α, A have to be determined from the corresponding critical points. After this we obtain for (6.10) the standard form that can be used for deriving (6.5). Various steps have to be verified. A difficult point in (6.14) is to prove the uniform regularity of the mapping for ν and λ, especially for small values of these parameters. When $\nu=\lambda=0$ (6.14) is the identity mapping. For other values of ν and λ there is a removable singularity at $w=0\ (\tau=0)$. We leave this and other open problems for this example as they stand now.

To conclude this section we remark that a related function to (6.9) in the form (6.6) is the other confluent hypergeometric function, ${}_1F_1(a,b,x)$. A suitable contour integral is (see SLATER [13])

$${}_1F_1(a,b,x)=\frac{\Gamma(1+a-b)\,\Gamma(b)}{\Gamma(a)\,2\pi i}\int_C t^{a-1}(t-1)^{b-a-1}\,e^{zt}\,dt$$

where C is taken as the circle $|t-1|=1$. A first transformation $\tau=$

$= i/(t-1)$ transforms this circle into itself. Afterwards we put $\tau = e^w$, which gives

$$ {}_1F_1(a, b, x) = \frac{\Gamma(1+a-b)\,\Gamma(b)}{2\pi i\,\Gamma(a)} \int_L w^{-b}\, e^{aw + x/w}\, f(-w)\, dw $$

where L is as in fig. 6.1 and f is the same as in (6.10). So we can obtain expansion (6.7). Due to the change of sign in the argument of f and a somewhat different role of b , the orders of the Bessel functions in ϕ and ψ are the same (use $K_{-\nu}(z) = K_\nu(z)$) and also the factors $(-1)^s$ in both series in (6.7) disappear. Further details will not be given here. We hope to work out the ideas of this section in a future publication.

7. Concluding remarks

In section 2 we have given 6 steps that are important in the investigations on uniform asymptotic expansions of integrals. In the preceding sections only the first three points are considered in detail for several examples. The examples are selected on the basis of our own experience and taste. Various other examples can be found in the literature and we do not have enough possibilities to treat all interesting examples, neither to give their references.

The remaining three points of the above mentioned list of section 2 are scarcely touched upon in this paper. This is mainly due to the fact that the theory for, say, obtaining error bounds of uniform expansions is not well developed, at least whent the starting point is an integral. Even for well-studied cases, such as Airy-type expansions for the standard form (4.5), no general results are available.

Special functions of mathematical physics are frequently treated as examples to demonstrate the methods of asymptotic analysis. Functions of hypergeometric type satisfy a differential equation and they have integral representations. Error bounds for the remainders in the expansions of special functions are derived most frequently from a differential equation. In Olver's work, see [10], general methods are derived to obtain strict and realistic error bounds. For a survey on error bounds for expansions of integrals we refer to WONG [27], where also a chapter on uniform expansions is included. Wong's conclusion is that the error theory for uniform expansion is still in its infancy. We agree with him that it is important to develop the theory. In many applications there is no choice between integrals and differential equations.

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