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Bifurcation at nonsemisimple 1 : − 1 resonance

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0. Introduction

In this paper we shall give a description of the bifurcation of periodic solutions occurring when a Hamiltonian system of two degrees of freedom passes through nonsemisimple 1 : − 1 resonance at an equilibrium. We will treat the generic case. The bifurcation at nonsemisimple 1 : − 1 resonance is also called a Hamiltonian Hopf bifurcation. This because of the specific behaviour of the eigenvalues of the linearized system.

An example of the above mentioned bifurcation is found in the planar circular restricted problem of three bodies. As is well known, at the Lagrange equilateral equilibrium L_4, for the critical mass ratio of Routh, this system is in nonsemisimple 1 : − 1 resonance. Passage of the mass parameter through the critical value of Routh gives a Hamiltonian Hopf bifurcation.

This paper outlines some of the results of my Ph. D. thesis presented in January 1985 at the University of Utrecht. In the presentation in this paper the restricted problem of three bodies takes a central place. For a more general and more detailed treatment of the subjects in this paper and closely related material the reader is referred to [15].

The paper is organized as follows. In Sect. 1 we show that the Hamiltonian system given by the equations of motion of the restricted three body problem at L_4 belongs to a certain class of parameter dependent Hamiltonian systems passing through nonsemisimple 1 : − 1 resonance. In Sect. 2 we state without proof the main theorems by which we can reduce the analysis of the bifurcation of periodic solutions for this class of systems to the analysis of one simple integrable Hamiltonian system, which is called the standard system for the Hamiltonian Hopf bifurcation. By analyzing the standard system (Sects. 3 and 4) we finally obtain a complete description of the bifurcation of periodic solutions for the generic case of the Hamiltonian Hopf bifurcation.

1. Nonsemisimple 1 : − 1 resonance

In this section we will use the restricted three body problem as an instrument to derive the class of Hamiltonian systems which will be the starting point for our further investigations. At the same time we show that the resticted three body problem at L_4 belongs to this class.

The restricted three body problem describes the movement of a massless body P subject to the attracting forces of two primary bodies P_1 and P_2, of mass $1-\mu$ and μ, rotating in circles about their common center of mass. P is supposed to move in the plane of rotation of the primaries P_1 and P_2.

The equations of motion for P can be formulated as a Hamiltonian system

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x},$$

with Hamiltonian function

$$\begin{gathered} H(x, y) = \frac{1}{2}(y_1^2 + y_2^2) - (x_1 y_2 - x_2 y_1) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \\ r_1^2 = (x_1 + \mu)^2 + x_2^2, \quad r_2^2 = (x_1 - (1-\mu))^2 + x_2^2, \quad 0 < \mu \leqq \tfrac{1}{2}. \end{gathered} \tag{1}$$

Here x_1, x_2 are the coordinates of P in the plane and y_1, y_2 are the corresponding momenta. This problem has five equilibrium points, the Euler equilibria E_1, E_2, E_3, and the Lagrange or equilateral equilibria L_4, L_5 (see Fig. 1). Our interest will be in the movement of P in the neighboorhood of L_4.

According to Deprit [7], after a translation making L_4 the origin, theTaylor series of the Hamiltonian function at the equilibrium is given by

$$H(\xi, \eta) = H_2(\xi, \eta) + \sum_{p+q \geqq 3} \omega_{pq} \xi_1^p \xi_2^q, \tag{2}$$

where

$$H_2(\xi, \eta) = \tfrac{1}{2}(\eta_1^2 + \eta_2^2) - (\xi_1 \eta_2 - \xi_2 \eta_1) - \tfrac{1}{4}(1 - \tfrac{3}{2}\gamma)\xi_1^2 - \tfrac{1}{4}(1 + \tfrac{3}{2}\gamma)\xi_2^2, \tag{3}$$

with $\gamma^2 = 1 - 3(1 - 2\mu)^2$. The characteristic equation for the matrix corresponding to the linearized system is $\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0$. For $\mu = \mu_0 = \frac{1}{2}(1 - \frac{1}{9}\sqrt{69})$ we find the eigenvalues $\pm i\frac{1}{2}\sqrt{2}$, $\pm i\frac{1}{2}\sqrt{2}$. For $0 < \mu < \mu_0$ we find four purely imaginary eigenvalues which are all different. For $\mu_0 < \mu \leqq \frac{1}{2}$

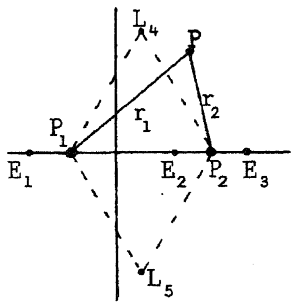


Figure 1
The configuration of the planar circular restricted three body problem in a rotating coordinate frame.

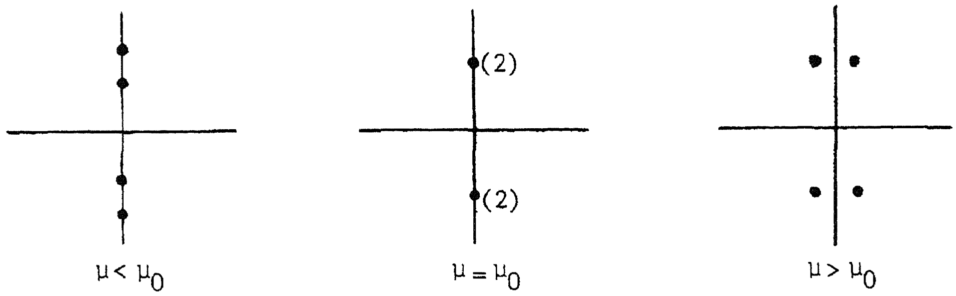


Figure 2
The eigenvalues of the linearized system in the complex plane.

we find four different eigenvalues all having nonzero real and imaginary parts (Fig. 2). Thus for the linearized system we have at μ_0 a transition from a stable to an unstable equilibrium.

The specific character of μ_0 was first recognized by Routh [22] and is therefore called the critical mass ratio of Routh. Our interest is in the behaviour of periodic orbits in the neighborhood of the equilibrium and of period close to $2\pi/\frac{1}{2}\sqrt{2}$ when μ passes through μ_0. This problem is for the first time considered by Brown [1] in 1911.

Our first step will be to write down a normal form for the Hamiltonian (2). As is shown in [4] at $\mu = \mu_0$ there is a linear symplectic change of coordinates which transforms (3) into the normal form

$$H_2(x, y) = \tfrac{1}{2}\sqrt{2}(x_1 y_2 - x_2 y_1) + \tfrac{1}{2}(x_1^2 + x_2^2). \tag{4}$$

The matrix of the linearized system is now

$$A = \begin{pmatrix} 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ -1 & 0 & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & -1 & \frac{1}{2}\sqrt{2} & 0 \end{pmatrix} \tag{5}$$

The matrix A has a trivial Jordan decomposition $A = A_S + A_N$ in a semisimple part A_S and a nilpotent part A_N given by

$$A_S = \begin{pmatrix} 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 0 & \frac{1}{2}\sqrt{2} & 0 \end{pmatrix}, \quad A_N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

We have $A_S A_N = A_N A_S$. The corresponding quadratic Hamiltonian functions are

$$S(x, y) = \tfrac{1}{2}\sqrt{2}(x_1 y_2 - x_2 y_1), \qquad N(x, y) = \tfrac{1}{2}(x_1^2 + x_2^2). \tag{6}$$

The eigenvalues of A_S are $\pm\frac{1}{2}i\sqrt{2}$, $\pm\frac{1}{2}i\sqrt{2}$, that is we have resonant eigenvalues. Because $S(x, y)$ is indefinite we speak of a $1:-1$ resonance. The presence of a nontrivial semisimple and nilpotent part in the Jordan decomposition of A makes that we have a *nonsemisimple* $1:-1$ *resonance.*

At the resonance, that is, $\mu = \mu_0$, the normal form up to order four for the Hamiltonian (2) is given by

$$H(x, y) = \tfrac{1}{2}\sqrt{2}(x_1 y_2 - x_2 y_1) + \tfrac{1}{2}(x_1^2 + x_2^2) + a(y_1^2 + y_2^2)^2 + b(y_1^2 + y_2^2)(x_1 y_2 - x_2 y_1) + c(x_1 y_2 - x_2 y_1)^2 + (\text{higher order terms})$$

where $a > 0$ (see [14]).

In order to study the passage through resonance we introduce a versal deformation of (4). Following [14] we obtain

$$H_2^\delta(x, y) = (\tfrac{1}{2}\sqrt{2} + \delta_1)(x_1 y_2 - x_2 y_1) + \tfrac{1}{2}(x_1^2 + x_2^2) + \tfrac{1}{2}\delta_2(y_1^2 + y_2^2). \quad (7)$$

According to the theory of versal deformations there exists a parameter dependent linear symplectic transformation transforming (3) into (7). This transformation has recently been computed in [6]. The parameter δ_2 can be considered as a detuning parameter. For δ_2 passing through zero we have the behaviour of eigenvalues as in Fig. 2, $\delta_2 = 0$ corresponds to $\mu = \mu_0$, and $\delta_2 > 0$, $\delta_2 < 0$ correspond to $\mu < \mu_0$, $\mu > \mu_0$. The parameter δ_1 turns out not to be important in the following, it only causes a change in the imaginary parts of the eigenvalues.

So finally we have shown that the Hamiltonian of our original problem belongs to the following class of parameter dependent Hamiltonians

$$H^\delta(x, y) = \alpha(x_1 y_2 - x_2 y_1) + \tfrac{1}{2}(x_1^2 + x_2^2) + \tfrac{1}{2}\delta(y_1^2 + y_2^2) + a(y_1^2 + y_2^2)^2 + b(y_1^2 + y_2^2)(x_1 y_2 - x_2 y_1) + c(x_1 y_2 - x_2 y_1)^2 + (\text{h.o.t}) \quad (8)$$

where $\alpha > 0$, $a \neq 0$.

In the following we will study the bifurcation of periodic solutions for δ passing through zero for systems corresponding to Hamiltonians like (8). The periodic solutions we are interested in have period close to the period of the solutions of the vector field corresponding to the semisimple part $\alpha(x_1 y_2 - x_2 y_1)$ of H_2^0. In the following the word periodic solution stands for these particular periodic solutions.

2. Derivation of a standard system

In this section we state a number of theorems the proofs of which can be found in [15]. These theorems are used to obtain a standard Hamiltonian function for the Hamiltonian Hopf bifurcation.

Let X_H denote the Hamiltonian vector field corresponding to the C^∞ Hamiltonian H, and let $\{.,.\}$ be the usual Poisson bracket. Furthermore let H^δ be as in (8), and let H^δ_m denote the homogeneous term of degree m in the Taylor expansion of H^δ. From the theory of normal forms for Hamiltonian systems we have

Theorem 1. Let $k \in \mathbb{N}$, $k \geqq 3$, then there exists for δ sufficiently small a symplectic C^∞ diffeomorphism φ_δ, depending C^∞ on δ, such that for $\tilde{H}^\delta = H^\delta \circ \varphi_\delta$ we have $\{\tilde{H}^\delta_m, S\} = 0$, $3 \leqq m \leqq k$.

Here and in the following $S(x, y) = x_1 y_2 - x_2 y_1$. We say $\tilde{H}^\delta$ is in S-normal form up to order k. For $\delta = 0$ theorem 1 is just the ordinary normal form theorem (see for instance [18]), with the difference that in our case H^0_2 is nonsemisimple, so instead of H^0_2 we take its semisimple part S.

It can be shown that each polynomial P with $\{P, S\} = 0$ must be a polynomial in the following four homogeneous quadratic functions

$$S(x, y) = x_1 y_2 - x_2 y_1, \quad X(x, y) = \tfrac{1}{2}(x_1^2 + x_2^2), \quad Y(x, y) = \tfrac{1}{2}(y_1^2 + y_2^2), \quad Z(x, y) = x_1 y_1 + x_2 y_2, \tag{9}$$

where we have the relation

$$4XY = Z^2 + S^2. \tag{10}$$

The next theorem is based on theory developped by Moser [19] and Weinstein [24] concerning the existence of periodic solutions for Hamiltonian systems. In this theorem the set of periodic solutions of a nonintegrable system is related to the set of periodic solutions of an integrable system.

Theorem 2. Suppose $\tilde{H}^\delta$ is in S-normal form up to order k, then there exists a C^k function $\hat{H}^\delta$ such that:

a) $\{\hat{H}^\delta, S\} = 0$.
b) $\hat{H}^\delta_m = \tilde{H}^\delta_m$ for $2 \leqq m \leqq k$.
c) The sets of periodic solutions of $X_{\hat{H}^\delta}$ and $X_{\tilde{H}^\delta}$ are C^k diffeomorphic, the diffeomorphism depends C^∞ on δ.

Note that the Hamiltonian systems corresponding to $\hat{H}^\delta$ and $\tilde{H}^\delta$ are only the same (up to C^k diffeomorphism) on this particular set of periodic solutions. Outside the set of periodic solutions the correspondence is only of an assymptotic nature because of b).

The vector field $X_{\hat{H}^\delta}$ is integrable because of theorem 2a). Therefore its set of periodic solutions, or relative equilibria, is determined by the set of points where $d\hat{H}^\delta = \lambda dS$ for some nonzero λ. This set is precisely the critical locus of the energy-momentum map $\hat{H}^\delta \times S: \mathbb{R}^4 \to \mathbb{R}^2; (x, y) \mapsto (\hat{H}^\delta(x, y), S(x, y))$.

Let $\mathscr{S}$ be the one parameter group corresponding to the flow of X_S. Because the flow of X_S is periodic $\mathscr{S}$ is a compact group. Furthermore $\hat{H}^\delta \times S$ is $\mathscr{S}$-invariant. We may now apply equivariant singularity theory to show that $\hat{H}^\delta \times S$ is finitely determined. This is formulated in a precise way in theorem 3. The fact that the energy-momentum map is finitely determined means that in order to determine its critical locus (up to diffeomorphism) we only need to consider a finite part of its Taylor expansion. As a consequence we can describe the critical locus explicitly because we know $\hat{H}^\delta$ explicitly up to arbitrary order in its Taylor series by theorems 1 and 2.

Theorem 3. If the coefficient a of the term $(y_1^2 + y_2^2)^2$ in $\hat{H}^\delta$ (which is the same coefficient as in H^δ) is nonzero then there exist a C^∞ map χ between the parameter spaces with $\chi(\delta) = \nu$, $\chi(0) = 0$, a smooth δ-dependent $\mathscr{S}$-equivariant diffeomorphism $\varphi: \mathbb{R}^4 \to \mathbb{R}^4$, and a smooth diffeomorphism $\psi: \mathbb{R}^2 \to \mathbb{R}^2$ such that $\psi \circ \chi^*(G_\nu \times S) \circ \varphi^{-1} = \hat{H}^\delta \times S$ with

$$G_\nu(x, y) = X(x, y) + \nu\, Y(x, y) + a\, Y(x, y)^2. \tag{11}$$

Here smooth means C^k because $\hat{H}^\delta \times S$ is a C^k map as a consequence of theorem 2. Theorem 3 is in fact obtained by applying the equivariant singularity theory to the map $\hat{H}^0 \times S$, that is, $\delta = 0$. It is shown that $\hat{H}^0 \times S$ is right-left equivalent to $G_0 \times S$. Unfolding this map gives $G_\nu \times S$, which, as a consequence, is right-left equivalent to $\hat{H}^\delta \times S$. Note that the diffeomorphism φ in theorem 3 is not necessarily symplectic, that is, the information about the flow on the fibers of $\hat{H}^\delta \times S$ does not carry over to $G_\nu \times S$. However, the fibration of $\hat{H}^\delta \times S$ and $G_\nu \times S$ is the same up to C^k diffeomorphism. Consequently the critical loci of $\hat{H}^\delta \times S$ and $G_\nu \times S$ are C^k diffeomorphic. Recall that the critical locus of $\hat{H}^\delta \times S$ describes the bifurcation of periodic solutions.

We call the Hamiltonian system corresponding to the Hamiltonian (11) a *standard system for the bifurcation at nonsemisimple* $1:-1$ *resonance* (*or Hamiltonian Hopf bifurcation*).

Note that the condition $a \neq 0$ singles out an algebraic variety in the space of coefficients of Taylor series of systems in nonsemisimple $1:-1$ resonance. This variety has an open and dense complement. Consequently the standard Hamiltonian (11) represents the generic case of the Hamiltonian Hopf bifurcation.

3. Reduction of the standard system

In this section we will study the fibration of the map $G_\nu \times S$. Recall that $G_\nu \times S$ is $\mathscr{S}$-invariant, that is, we may consider $G_\nu \times S$ as the energy-momentum map for the vector field X_{G_ν} with integral S. Thus the fibers of $G_\nu \times S$ are invariant surfaces for X_{G_ν} as well as for X_S. Because of the S^1 symmetry due to the $\mathscr{S}$-action the fibers of $G_\nu \times S$ must be diffeomorphic to points, circles, tori or cylinders. Consequently the critical locus consists of points and circles. The fibers in the critical locus which are points must be critical points for the $\mathscr{S}$-action. In our case only the origin is a critical point for the $\mathscr{S}$-action. Therefore the critical locus minus the origin consists of fibers diffeomorphic to circles. These are precisely the periodic solutions we are looking for.

We will analyse the fibration of the map $G_\nu \times S$ by factorizing $G_\nu \times S$ through the orbit map ϱ for the $\mathscr{S}$-action. Recall that the functions Z, Y, X, and S given by (9) are invariants for the $\mathscr{S}$-action. These functions are precisely the generators for the space of $\mathscr{S}$-invariant polynomials. Instead of Z, Y, X, S we may take Z, Y, G_ν, S. We obtain the orbit map

$$\varrho\colon \mathbb{R}^4 \to \mathbb{R}^4; (x, y) \to (Z, Y, G_\nu, S).$$

Each $\mathscr{S}$-orbit or X_S trajectory is mapped precisely to one point in the image of ϱ. Using the relation (10) we obtain that $\varrho(\mathbb{R}^4)$ is determined by

$$Z^2 + S^2 + 4\nu Y^2 + 4a Y^3 - 4YG_\nu = 0, \quad Y \geqq 0. \tag{12}$$

Now $\varrho(\mathbb{R}^4)$ is homeomorphic to $\mathbb{R}^4/\mathscr{S}$ and one can proof that $\varrho(S^{-1}(s))$ is symplectomorphic to $S^{-1}(s)/\mathscr{S}$. Consequently $\varrho(S^{-1}(s))$ is precisely the reduced phase space as in [13] for X_{G_ν} with respect to the symmetry group $\mathscr{S}$.

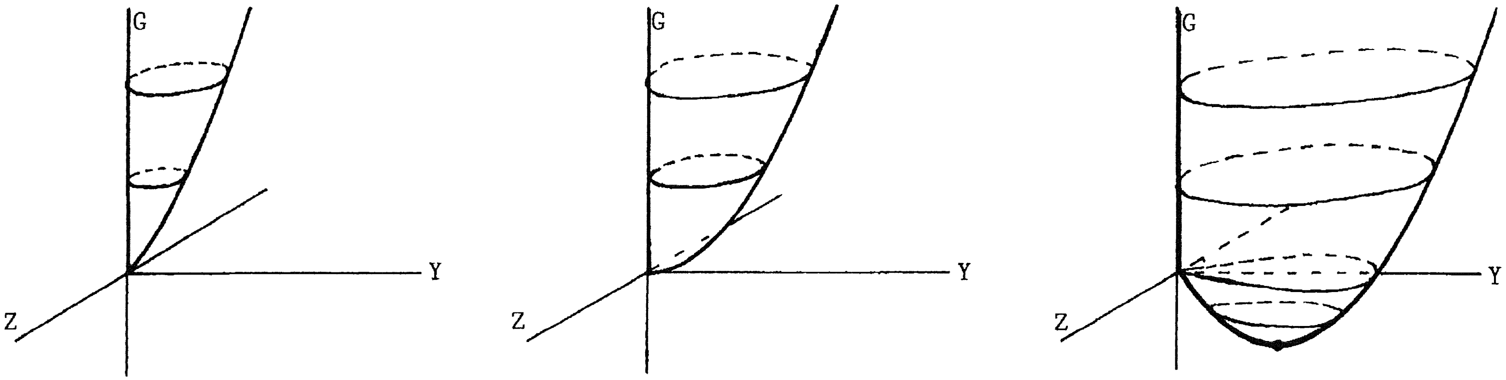
Write $G_\nu \times S = \pi \circ \varrho$ where π is the projection

$$\pi\colon \mathbb{R}^4 \to \mathbb{R}^2; (Z, Y, G_\nu, S) \to (G_\nu, S).$$

Let Σ be the critical locus of $G_\nu \times S$. Then the reduced critical locus $\varrho(\Sigma)$ is precisely the set of stationary points (or relative equilibria) of the reduced systems corresponding to X_{G_ν} (we have a reduced system for each S-value s). The critical values of $G_\nu \times S$ are obtained as $\pi(\varrho(\Sigma))$. In the following section we will describe these critical values along these lines. The result is a description of the bifurcation of periodic solutions in the space of natural parameters given by energy, integral and detuning.

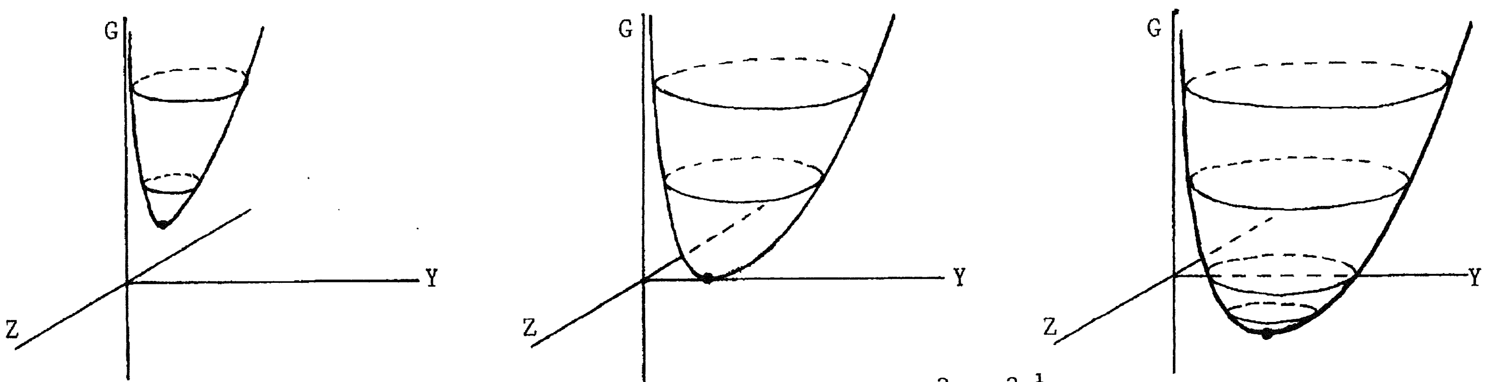
In order to give some insight in the behaviour of the relative equilibria we will sketch the reduced phase spaces (Fig. 3, 4, 5, 6). There are several cases according to the different values of the parameters, which are the coefficient a, the detuning ν, and the S-value s.

The reduced phase spaces are described in (Z, Y, G_ν)-space and are given by (12) taking $S = s$. The horizontal sections $G_\nu = g$ give the trajectories $\gamma_{g,s}$ of the reduced vector field. $\gamma_{g,s}$ corresponds in the original phase space to a fiber $\gamma_{g,s} \times S^1$ of $G_\nu \times S$. The relative equilibria are precisely the extrema of the reduced phase spaces. Because of the symmetry with respect to reflection in the $Z = 0$ plane it is easy to see that all relative equilibria are in this plane. Note that the intersection curve of the reduced phase space with the $Z = 0$ plane can be considered as the potential of the reduced system. Note furthermore



(i) $a > 0,\ \nu > 0,\ S = 0.$ (ii) $a > 0,\ \nu = 0,\ S = 0.$ (iii) $a > 0,\ \nu < 0,\ S = 0.$

Figure 3
The reduced phase spaces for $a > 0$, $S = 0$.

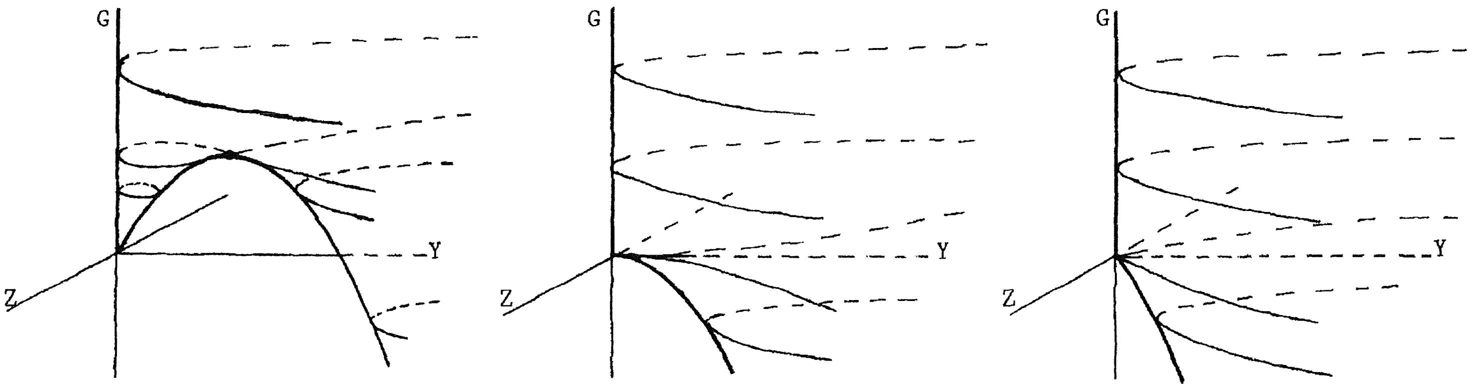


(i) $\nu \geq 0,\ S \neq 0.$
$\nu < 0,\ |S| > (-16\nu^3/27a^2)^{\frac{1}{2}}.$

(ii) $\nu < 0,\ |S| = (-16\nu^3/27a^2)^{\frac{1}{2}}.$

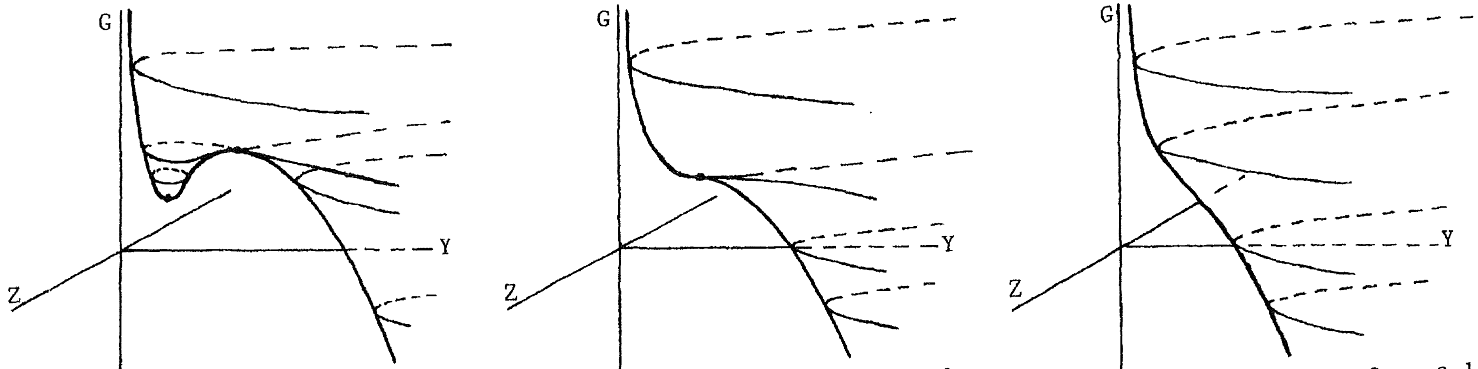
(iii) $\nu < 0,\ 0 < |S| < (-16\nu^3/27a^2)^{\frac{1}{2}}.$

Figure 4
The reduced phase spaces for $a > 0$, $S \neq 0$.



(i) $a < 0,\ \nu > 0,\ S = 0.$ (ii) $a < 0,\ \nu = 0,\ S = 0.$ (iii) $a < 0,\ \nu < 0,\ S = 0.$

Figure 5
The reduced phase spaces for $a < 0$, $S = 0$.



(i) $\nu > 0,\ 0 < |S| < (4\nu^3/27a^2)^{\frac{1}{2}}.$ (ii) $\nu > 0,\ |S| = (4\nu^3/27a^2)^{\frac{1}{2}}.$ (iii) $\nu > 0,\ |S| > (4\nu^3/27a^2)^{\frac{1}{2}},$ $\nu \le 0,\ S \ne 0.$

Figure 6
The reduced phase spaces for $a < 0$, $S \neq 0$.

that the case $S = 0$ is singular because it involves the origin as a stationary point of the $\mathscr{S}$-action. However with exception of the origin the case $S = 0$ can be treated as the other cases.

4. The periodic solutions

Recall that the reduced phase spaces are given by (12) and that for the reduced critical locus $\varrho(\Sigma)$ we have $Z = 0$. Furthermore we have that $\varrho(\Sigma)$ corresponds to the extrema of the reduced phase spaces, that is, at $\varrho(\Sigma)$ the derivative with respect to Y of the left hand side of the equation in (12) vanishes. Consequently we obtain the $\pi(\varrho(\Sigma))$ as the sections ν = constant of the set in (ν, S, G_ν)-space determined by solving Y from the following equations

$$4aY^3 + 4\nu Y^2 - 4YG_\nu + S^2 = 0, \tag{13}$$

$$12aY^2 + 8\nu Y - 4G_\nu = 0, \quad (Y \geqq 0). \tag{14}$$

Hence we have to determine the discriminant locus Δ_{13} of Eq. (13). The surface determined by $\Delta_{13} = 0$ is for $a > 0$ a swallowtail surface as in Fig. 7, for $a < 0$ the swallowtail is upside down.

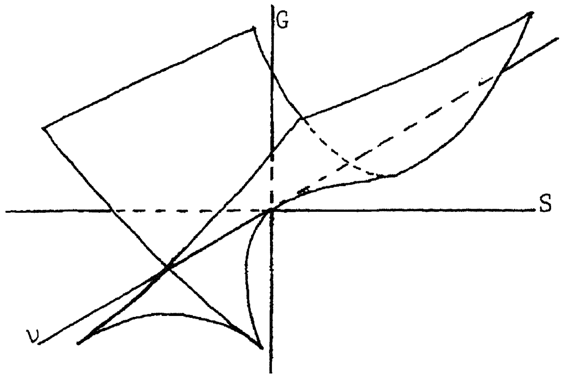


Figure 7
The swallowtail surface $\Delta_{13} = 0$ for $a > 0$, without the restriction $Y \geqq 0$.

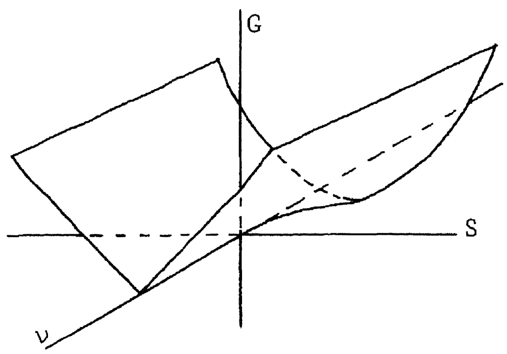


Figure 8
$\Delta_{13} = 0$ for $a > 0$, $Y \geqq 0$.

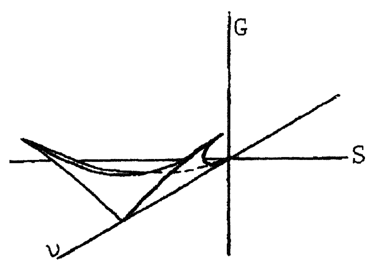


Figure 9
$\Delta_{13} = 0$ for $a < 0$, $Y \geqq 0$.

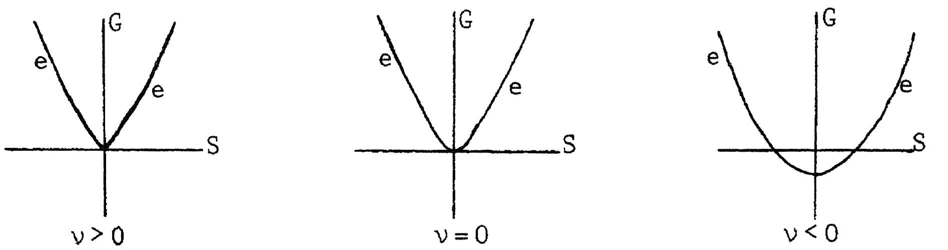


Figure 10
Sections ν = constant of the surface in Fig. 8.

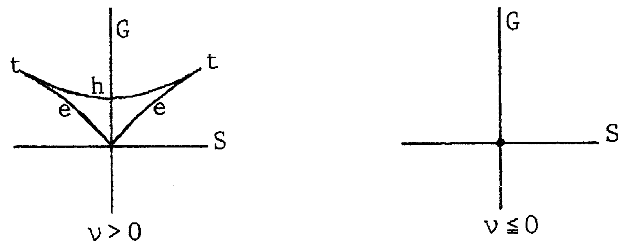


Figure 11
Sections ν = constant of the surface in Fig. 9.

Because of the condition $Y \geqq 0$ we have to restrict to a part of these swallowtail surfaces. Again we have to distinguish between $a > 0$ and $a < 0$. We obtain two qualitatively completely different bifurcations (see Fig. 8, 9). Notice that each point of these surfaces corresponds to a periodic solution, to be precise to an invariant manifold which is diffeomorphic to a circle. In order to illustrate the bifurcation for changing ν we give in Figs. 10 and 11 the sections ν = constant of the surfaces in Figs. 8 and 9. The stability type indicated by e for elliptic, h for hyperbolic, and t for transitional is obtained by considering the reduced phase spaces.

5. Bibliographical notes

The bifurcation of periodic solutions under consideration in this paper has repeatedly been subject of research. The first one studying the bifurcation of periodic solutions at the critical mass ratio of Routh in the restricted problem of three bodies was Brown [1] in 1911. Brown considers series expansions of the vector field up to third order. By theorem 3 we now know that this in fact is sufficient to obtain the right picture. Brown's work was in more detail repeated by Pedersen [20].

In [2] Buchanan (1939) gives a linear normal form for the vector field at $\mu = \mu_0$. In his 1941 paper [3] he computes the families of periodic solutions at $\mu = \mu_0$.

More recently Deprit [8] gave a fourteenth order analysis of the orbits for $\mu > \mu_0$. Deprit and Henrard [9], [10] give an extensive treatment of the periodic solutions in the restricted three body problem including a review of the results up to that time of the bifurcation at the critical mass ratio μ_0. Their description is partially based on numerical results.

The problem is again taken up by Meyer and Schmidt [16] who actually prove the existence of families of periodic solutions at L_4 for mass ratio's near Routh critical value. They also give the limiting behaviour for $\mu \to \mu_0$. Their results are in a different way obtained by Schmidt and Sweet [23] and Roels [21]. In [17] Meyer gives a short survey of the literature on bifurcations near resonant equilibria including the bifurcation at nonsemisimple $1:-1$ resonance.

The paper [5] by Caprino et al. treats the Hamitonian Hopf bifurcation as a special case in a more general class of Hopf bifurcations. This paper only treats existence theorems for periodic solutions.

By gathering the analytical and numerical results in the above cited literature a pretty good picture of the Hamiltonian Hopf bifurcation can be obtained. However this picture remains incomplete in several aspects. The main question that remains is how exactly the bifurcation is organized at the bifurcation point. By giving a geometric description which can be followed through the bifurcating value this question is answered in this paper.

The basic scheme followed in Sect. 2 is based on ideas of Duistermaat [11], improved in [12]. Duistermaat treats the $\ell : k$ resonances with $|k| < \ell \leqq 1$. The theory of this paper generalizes Duistermaat's scheme in such a way that the nonsemisimple case is covered too, and conversely influenced [12] at some points. As said before proofs of the main theorems can be found in [15].

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Abstract

In this paper a description is given of the bifurcation of periodic solutions occurring when a Hamiltonian system of two degrees of freedom passes through nonsemisimple $1:-1$ resonance at an equilibrium. A bifurcation like this is found in the planar circular restricted problem of three bodies at the Lagrange equilibrium L_4 when the mass parameter passes through the critical value of Routh.

Zusammenfassung

Gegenstand dieses Artikels ist die Verzweigung periodischer Lösungen in Hamilton'schen Systemen mit zwei Freiheitsgraden beim Durchgang durch eine nicht-einfache $1:-1$-Resonanz an einem Gleichgewicht. Ein Beispiel ist das ebene restringierte Dreikörperproblem am Lagrange-Punkt L_4, wenn die Masse durch den kritischen Wert von Routh hindurchgeht.

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