TRACES TO TRICOMI IN RECENT WORK ON SPECIAL FUNCTIONS AND ASYMPTOTICS OF INTEGRALS

by

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Summary. This tribute to Tricomi contains a selection of examples which refer to his work on special functions and asymptotics of integrals. The examples include: a class of polynomials introduced by Tricomi and which is used in uniform expansions of Laplace integrals; Watson's lemma for loop integrals; uniform asymptotic expansions of incomplete gamma functions; computational aspects of Tricomi's Ψ function (confluent hypergeometric function).

1. A CLASS OF POLYNOMIALS

In TRICOMI (1951) a set of polynomials related to Laguerre polynomials is introduced. The definition is

$$(1.1) \qquad \ell_n(x) = (-1)^n L_n^{(x-n)}(x), \qquad n = 0,1,2,\ldots,$$

where $L_n^{(\alpha)}(z)$ is the classical orthogonal Laguerre polynomial. Observe that in (1.1) the parameter α depends on x, giving a polynomial essentially different from the classical case. For instance, the degree of ℓ_n is not n but the greatest integer [n/2] in n/2.

Tricomi gives a first attempt at a systematic study of the polynomials. The motivation for the investigations is the occurrence in several situations. For instance in an expansion of a confluent hypergeometric function in terms of Bessel functions, and in an asymptotic expansion of the incomplete gamma functions. In this section we mention two later applications in asymptotic expansions of Laplace type integrals.

Before doing so we remark that in an interesting paper by CARLITZ (1958) an orthogonality relation is given for polynomials related to $\ell_n(x)$. Carlitz defined

$$f_n(x) = -(n+2)\, x^{n+2}\, \ell_{n+2}(x),$$

which is a polynomial of degree n. He proved

$$\int_{-\infty}^{\infty} f_n(x) f_m(x)\, d\psi(x) = \frac{\delta_{mn}}{(n+1)!},$$

where $\psi(x)$ is a step function with the jump

$$\tfrac{1}{2} j^{j-1} e^{-j}/j! \quad \text{at the point} \quad \pm j^{-\frac{1}{2}}, \qquad j = 1,2,\ldots .$$

1.1. Uniform asymptotic expansions

The following integral is important in this connection

$$P_n(\lambda) = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda-1} e^{-t} (t-\lambda)^n dt. \tag{1.2}$$

By expanding $(t-\lambda)^n$ it easily follows that

$$P_n(\lambda) = \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} (-\lambda)^{n-m}.$$

By comparing this with the known representation

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!}$$

it follows from (1.1) that

$$\ell_n(x) = P_n(-x)/n!, \qquad n = 0,1,\ldots .$$

We have

$$P_0(\lambda) = 1, \quad P_1(\lambda) = 0, \quad P_2(\lambda) = \lambda,$$

$$P_3(\lambda) = 2\lambda, \quad P_4(\lambda) = 3\lambda(\lambda+2),$$

and there is a recursion

$$P_{n+1}(\lambda) = n[P_n(\lambda)+\lambda P_{n-1}(\lambda)], \quad n \geq 0.$$

This follows from (1.2) or from a well-known relation for $L_n^{(\alpha)}(x)$.

The polynomials $P_n(\lambda)$ are used by the present author to obtain an asymptotic expansion for the Laplace integral

$$(1.3) \qquad F_\lambda(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} f(t)\,dt,$$

Re $z > 0$, Re $\lambda > 0$, z large, and where λ may be large as well. It is shown in TEMME (1982a) that an expansion is possible, uniform with respect to the parameter $\mu := \lambda/z$, $\mu \in [0,\infty)$, with extension to complex values of μ.

The expansion is obtained by expanding $f(t)$ at $t = \mu$, at which point the dominant part of the integrand of (1.3), that is, $t^\lambda e^{-zt}$, attains its maximal value (considering only real parameters, for the moment). We write

$$(1.4) \qquad f(t) = \sum_{s=0}^{\infty} a_s(\mu)(t-\mu)^s$$

and obtain by substituting this in (1.3) the formal result

$$(1.5) \qquad F_\lambda(z) \sim \sum_{s=0}^{\infty} a_s(\mu)\, P_s(\lambda)\, z^{-s-\lambda}, \quad z \to \infty.$$

To investigate the nature of this expansion we suppose that f is holomorphic in a connected domain Ω of the complex plane with the following conditions satisfied:

(i) the boundary $\partial\Omega$ is bounded away from $[0,\infty)$;

(ii) Ω contains a sector $S_{\alpha,\beta}$, with vertex at $t = 0$, defined by

$$S_{\alpha,\beta} = \{t \in \mathbb{C} \mid -\alpha < \text{phase}(t) < \beta\},$$

where α and β are positive numbers;

(iii) $f(t) = \mathcal{O}(t^p)$ as $t \to \infty$ in $S_{\alpha,\beta}$, where p is a real number.

The uniformity of the expansion (1.5) holds with respect to μ in a closed sector, with vertex at $t = 0$, properly inside $S_{\alpha,\beta}$ and containing $t = 0$ as a boundary point. This is proved in TEMME (1982a). Moreover, error bounds for the remainders are given.

A simple example is $f(t) = 1/(1+t)$, in which event (1.3) is an exponential integral and $a_s(\mu) = (-1)^s/(1+\mu)^{s+1}$. The sector $S_{\alpha,\beta}$ is defined with $\alpha = \beta = \pi-\varepsilon$ (ε small). We have

$$e^z E_\lambda(z) \sim \sum_{s=0}^{\infty} \frac{(-1)^s P_s(\lambda)}{(z+\lambda)^{s+1}}, \tag{1.6}$$

where $E_\lambda(z)$ is the well-known exponential integral.

Expansion (1.5) is very useful when, apart from large values of z, also large values of λ are used. In fact it is useful too, when $\lambda \to \infty$, uniformly with respect to z, $z \geq z_0 > 0$. The example (1.6) shows quite well why the uniformity in λ (or in μ) holds. The degree of the Tricomi polynomial $P_s(\lambda)$ is $[s/2]$, which is amply absorbed by the denominator.

An important feature of the uniform expansion (1.5) is that it modifies Watson's lemma for Laplace integrals in which f is expanded at $t = 0$, giving

$$F_\lambda(z) \sim \sum_{s=0}^{\infty} a_s(0) \frac{\Gamma(\lambda+s)}{\Gamma(\lambda)} z^{-s-\lambda}, \quad z \to \infty.$$

This expansion is useless when λ is large, say $\lambda = \mathcal{O}(z)$. The conditions on f in Watson's lemma are less restrictive, however, than for the uniform case.

The relation between the polynomials $P_n(\lambda)$ in (1.2) and (1.5) and the $\ell_n(x)$ introduced by Tricomi was kindly brought to my attention by Prof. E. Riekstiņš (Riga, Russia).

1.2. Expansions and estimations for remainders in asymptotic series

The polynomials $\ell_n(x)$ of (1.1) or $P_n(\lambda)$ are used for so called converging factors in asymptotic expansions. In this technique the remainder of an expansion is re-expanded. For information on this point, with some historical details, we refer to OLVER (1974, Ch.14). The use of the polynomials $\{P_n\}$ is pointed out by BERG (1977) and quite recently by RIEKSTIŅŠ (1982).

To describe the idea, we follow the latter paper and consider

$$F_\lambda(z) = \int_0^\infty t^{\lambda-1}e^{-zt}f(t)\,dt.$$

We suppose that there is an expansion

$$f(t) = \sum_{s=0}^{n-1} a_s t^{\lambda_s} + R_n(t)t^{\lambda_n},$$

which approximates f at t = 0. That is, $R_n(t) = \mathcal{O}(1)$ as $t \to 0$, $\operatorname{Re}\lambda_{s+1} > \operatorname{Re}\lambda_s$. Furthermore it is assumed that $\operatorname{Re}(\lambda+\lambda_0) > 0$. Then

$$F_\lambda(z) = \sum_{s=0}^{n-1} a_s \frac{\Gamma(\lambda+\lambda_s)}{z^{\lambda+\lambda_s}} + E_n(z),$$

$$E_n(z) = \int_0^\infty t^{\lambda+\lambda_n-1}e^{-zt}R_n(t)\,dt.$$

We suppose that the integrals $F_\lambda(z)$ and $E_n(z)$ are defined when Re z is large enough. Re-expanding $R_n(t)$ as in (1.4) with $\mu = (\lambda+\lambda_n)/z$ gives again an expansion involving the polynomials $P_n(\lambda)$. Riekstiņš' expansion is somewhat different; he takes $\mu = 1$ and supposes $\lambda + \lambda_n - z \sim 0$ (when λ, λ_n and z are real), assuming that there exists certain relation between z and the index n of the remainder. As an example the confluent hypergeometric function of the second kind (also called Tricomi's Ψ function) is treated.

An important aspect in Riekstiņš' approach is that an exact error bound is given for the remainder in the expansion for the original remainder $E_n(z)$.

2. LOOP INTEGRALS FOR SPECIAL FUNCTIONS

In TRICOMI & ERDÉLYI (1951) an expansion is given for a ratio of gamma functions. The asymptotic expansion is

$$\text{(2.1)} \qquad \frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\beta-\alpha+j)}{\Gamma(\beta-\alpha) j!} B_j^{(\alpha-\beta+1)}(\alpha) z^{\alpha-\beta-j},$$

as $z \to \infty$, $|\arg(z+\alpha)| < \pi$, and the $B_j^{(\mu)}(z)$ are the generalised Bernoulli polynomials defined by

$$e^{zt}(t/(e^t-1))^{\mu} = \sum_{j=0}^{\infty} \frac{t^j}{j!} B_j^{(\mu)}(z), \qquad |t| < 2\pi.$$

The proof is based on a loop integral for the Beta function, viz.

$$\text{(2.2)} \qquad \begin{aligned} \frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} &= \frac{\Gamma(\alpha+1-\beta)}{2\pi i} \int_{-\infty}^{(0^+)} e^{vz} f(v)\, dv, \\ f(v) &= e^{\alpha v}(e^v-1)^{\beta-\alpha-1}. \end{aligned}$$

The notation $\int_{-\infty}^{(0^+)}$ is a loop integral where the path of integration starts at $t = -\infty$ $(\arg t = -\pi)$, encircles the origin once in the counterclockwise direction, and returns to $-\infty$ $(\arg t = \pi)$. FIELDS (1966) has shown that a shift of z in the integral (2.2) produces an expansion in negative powers of $(z+\frac{1}{2}(\alpha+\beta-1))^2$, this expansion essentially being an even one. This is interesting from a computational viewpoint.

These results are summarized in LUKE (1969). OLVER (1974) has generalized the analysis leading to (2.1) into a useful result known as *Watson's lemma for loop integrals*, by considering (2.2) with more general function f. In this section we point out that for several special functions a loop integral representation exist, which is in some sense reciprocal to a Laplace type integral for a related function.

A beautiful example is the pair

$$
(2.3)\qquad
\begin{aligned}
\Gamma(z) &= \int_0^\infty e^{-t} t^{z-1} dt, \qquad \mathrm{Re}\ z > 0 \\
1/\Gamma(z) &= \frac{1}{2\pi i} \int_{-\infty}^{(0^+)} e^{t} t^{-z} dt, \qquad z \in \mathbb{C}.
\end{aligned}
$$

The second integral is known as Hankel's loop integral for the reciprocal of the gamma function. TRICOMI & ERDELYI (1951) used it for (2.1) by expanding f of (2.2) in powers of v. OLVER (1974, p.120) used it for the general case. Observe that the second of (2.3) has no restriction with respect to z; in this way it is more powerful than the first one.

Another example is the pair representing Riemann's zeta function

$$
(2.4)\qquad
\begin{aligned}
\zeta(z) &= \frac{1}{\Gamma(z)} \int_0^\infty [e^{t}-1]^{-1}\, t^{z-1} dt, \qquad \mathrm{Re}\ z > 1 \\
\zeta(1-z) &= \frac{\Gamma(z)}{2\pi i} \int_{-\infty}^{(0^+)} t^{-z}[e^{-t}-1]^{-1} dt, \qquad z \neq 0.
\end{aligned}
$$

The contour does not enclose the poles at $t = \pm 2n\pi i$, $n \in \mathbb{N}$. The restriction $z \neq 0$ in the second integral is due to the pole of $\Gamma(z)$ at $z = 0$; all remaining singularities at $z = -n$, $n \in \mathbb{N}$, are removed by the integral. The relation between the integrals is now given by the functional equation

$$
\zeta(z) = 2(2\pi)^{z-1} \sin(\tfrac{1}{2}\pi z)\Gamma(1-z)\zeta(1-z).
$$

For the ratio of Gamma functions we have the pair

$$\frac{\Gamma(z)}{\Gamma(z+\lambda)} = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-zt}(1-e^{-t})^{\lambda-1} dt,$$

(2.5)

$$\frac{\Gamma(z+\lambda)}{\Gamma(z+1)} = \frac{\Gamma(\lambda)}{2\pi i} \int_{-\infty}^{(0^+)} e^{zt}(1-e^{-t})^{-\lambda} dt$$

where the second one is the same as (2.2).

For modified Bessel functions we have

$$\pi^{-\frac{1}{2}}(\tfrac{1}{2}z)^{-\nu} e^{z} K_\nu(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-zt}[t(t+2)]^{\lambda-1} dt,$$

(2.6)

$$\pi^{\frac{1}{2}}(\tfrac{1}{2}z)^{\nu-1} e^{-z} I_{\nu-1}(z) = \frac{\Gamma(\lambda)}{2\pi i} \int_{-\infty}^{(0^+)} e^{zt}[t(t+2)]^{-\lambda} dt,$$

where $\nu = \lambda + \frac{1}{2}$. The reciprocity between the integrals is now reflected in the Wronskian relation

$$I_{\nu-1}(z)K_\nu(z) + I_\nu(z)K_{\nu-1}(z) = 1/z.$$

A pair of parabolic cylinder functions satisfies

$$z^{-\nu} e^{\frac{1}{4}z^2} D_{-\nu}(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-z^2(t+\frac{1}{2}t^2)} t^{\nu-1} dt,$$

(2.7)

$$(2\pi)^{-\frac{1}{2}} z^{\nu-1} e^{-\frac{1}{4}z^2} D_{-\nu}(-z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{z^2(t+\frac{1}{2}t^2)} t^{-\nu} dt.$$

In the second integral the contour cuts the real t-axis at a positive t-value. A relation between the functions is again given by a Wronskian determinant

$$D_{-\nu}(z)D'_{-\nu}(-z) - D_{-\nu}(z)D'_{-\nu}(-z) = \frac{\sqrt{2\pi}}{\Gamma(\nu)} .$$

Finally we mention the pair

$$
(2.8) \qquad
\begin{aligned}
\Psi(a;b;z) &= \frac{1}{\Gamma(a)}\int_0^{\infty} e^{-zt}t^{a-1}(1+t)^{b-a-1}dt \\
x^{b-1}e^{-x}M(a;b;z) &= \frac{\Gamma(b)}{2\pi i}\int_{-\infty}^{(0^+)} e^{zt}t^{-a}(1+t)^{a-b}dt
\end{aligned}
$$

for Tricomi's Ψ function and the other confluent hypergeometric function (Kummer's function). Again there is a Wronskian relation

$$\Psi(a;b;z)M'(a;b;z) - \Psi'(a;b;z)M(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)} z^{-b}e^{z}.$$

As in TRICOMI & ERDÉLYI (1951), asymptotic expansions for the loop integrals in (2.6), (2.7) and (2.8) follow by expanding part of the integrand at $t = 0$ (for (2.7) an elementary transformation is needed before doing so). This leads to well-known asymptotic expansions of the special functions concerned.

3. INCOMPLETE GAMMA FUNCTIONS

TRICOMI (1950) opens his paper on incomplete gamma funcions with the remark that, for some time past, he used to call the incomplete gamma function $\gamma(a,x)$ as the Cinderella of special functions. In that paper he gives interesting results for the asymptotic behaviour of $\gamma(a,x)$, for instance when a and x are both large and of the same order. He found, among others,

$$(3.1) \qquad \frac{\gamma(a+1,a+y(2a)^{\frac{1}{2}})}{\Gamma(a+1)} = \tfrac{1}{2}\operatorname{erfc}(-y)-1/3(2/a\pi)^{\frac{1}{2}}(1+y^2)e^{-y^2}+O(a^{-1}),$$

y,a real, $a \to +\infty$; the expansion is uniformly valid in y on compact intervals of $\mathbb{R}$. The function erfc is the complementary error function defined by

$$(3.2) \qquad \operatorname{erfc}(x) = 2\pi^{-\frac{1}{2}}\int_x^{\infty} e^{-t^2}dt.$$

Some results of Tricomi are corrected and used by KÖLBIG (1972) for the construction of the zeros of the incomplete gamma functions. DINGLE (1973) has generalized the results of Tricomi, but the same restrictions on y must hold as in (3.1).

Tricomi used ad-hoc methods to derive results as (3.1). Later on, important general methods were introduced to derive uniform asymptotic expansions of integrals, from which his results follow as special cases. In fact by the new methods more powerful results were obtained than by the method of Tricomi.

In TEMME (1975) we used a loop integral for $\gamma(a,x)$ where a saddle point coincides with a pole in the "transition" case $a = x$. The method of VAN DER WAERDEN (1951) was used to give an expansion of $\gamma(a,\lambda a)$ for $a \to \infty$, which holds uniformly in $\lambda \geq 0$ (here we consider real parameters). When comparing this with (3.1) we observe that $\lambda = 1 + y(2/a)^{\frac{1}{2}}$. Since y ranges in a compact set we infer that in Tricomi's expansion λ ranges in an interval around unity of length $\mathcal{O}(a^{-\frac{1}{2}})$. In TEMME (1982b) we used a different method, which yielded error bounds for the remainders and in which complex variables can be used. We shall give a short description of the last method, which can be used for many more probability functions.

It is convenient to consider the normalized function

$$P(a,x) := \frac{\gamma(a,x)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} dt.$$

A simple transformation gives (we consider $a > 0$, $x > 0$)

$$P(a,x) = \frac{e^{-a} a^a}{\Gamma(a)} \int_0^{x/a} e^{-a(t-1-\ell n t)} t^{-1} dt.$$

Next we define the mappings

$$\tfrac{1}{2}\zeta^2 = t-1-\ell n t, \quad \text{sign } \zeta = \text{sign}(t-1)$$

(3.3)

$$\tfrac{1}{2}\eta^2 = \lambda-1-\ell n \lambda, \quad \text{sign } \eta = \text{sign}(\lambda-1), \qquad \lambda = x/a.$$

With the ζ and η transformations the integral is

$$(3.4) \qquad P(a,x) = \left(\frac{a}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\eta} e^{-\frac{1}{2}a\zeta^2} f_a(\zeta)\,d\zeta,$$

a standard form for the integrals considered in TEMME (1982b). The function $f_a(\zeta)$ is given by

$$(3.5) \qquad f_a(\zeta) = \frac{e^{-a}a^a(2\pi/a)^{\frac{1}{2}}}{\Gamma(a)}\,\phi(\zeta), \qquad \phi(\zeta) = \frac{\zeta}{t-1}.$$

The asymptotic problem is to give for the integral in (3.4) an asymptotic expansion for $a \to \infty$, which is uniformly valid with respect to η. For fixed values of η, the integral has three different asymptotic expansions, according whether $\eta < 0$, $\eta = 0$, $\eta > 0$. It is possible to combine these three expansions into one, in which the error function (3.2) describes the transition. An integration by parts procedure is used by taking into account the contributions in the integral (3.4) at $\zeta = 0$ (the saddle point) and at $\zeta = \eta$ (end point of integration.) Since $\phi(0) = 1$ we write

$$\left(\frac{a}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\eta} \phi(\zeta)e^{-\frac{1}{2}a\zeta^2}d\zeta =$$

$$\tfrac{1}{2}\,\mathrm{erfc}(-\eta\sqrt{a/2}) - (2\pi a)^{-\frac{1}{2}} \int_{-\infty}^{\eta} \frac{\phi(\zeta)-1}{\zeta}\,de^{-\frac{1}{2}a\zeta^2} =$$

$$\tfrac{1}{2}\,\mathrm{erfc}(-\eta\sqrt{a/2}) + (2\pi a)^{-\frac{1}{2}}\,\frac{1-\phi(\eta)}{\eta} + (2\pi a)^{-\frac{1}{2}} \int_{-\infty}^{\eta} \phi_1(\zeta)e^{-\frac{1}{2}a\zeta^2}d\zeta,$$

with $\phi_1(\zeta) = d/d\zeta[\{\phi(\zeta)-1\}/\zeta]$. Repeating this process we obtain

$$(3.6) \qquad P(a,x) = \tfrac{1}{2}\,\mathrm{erfc}(-\eta(a/2)^{\frac{1}{2}}) + \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}}\left[\sum_{s=0}^{n-1}\frac{B_s(\eta)}{a^s} + \frac{\bar{B}_n(a,\eta)}{a^n}\right]$$

$$B_0(\eta) = \frac{1}{\eta} - \frac{1}{\lambda-1}, \quad B_1(\eta) = \frac{1}{(\lambda-1)^3} + \frac{1}{(\lambda-1)^2} + \frac{1}{12(\lambda-1)} - \frac{1}{\eta^3}.$$

For the remainder $\bar{B}_n$ bounds are available. In (3.6) the variable η is defined in (3.3). In terms of the original variables x and a we can say that (3.6) is a uniform expansion for $a \to \infty$, holding uniformly with respect to x, $x \geq 0$, especially for $x \sim a$. An interesting point is that for the other incomplete gamma function $Q(a,x) :=$ $\Gamma(a,x)/\Gamma(a)$ a similar expansion exists, viz.

$$(3.7)\qquad Q(a,x) = \tfrac{1}{2}\,\mathrm{erf}(\eta\sqrt{a/2}) - \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}}\left[\sum_{s=0}^{n-1}\frac{B_s(\eta)}{a^s} + \frac{\bar{B}_n(a,\eta)}{a^n}\right]$$

where the $B_s(\eta)$ are the same as in (3.6). Hence the addition rule $P(a,x) + Q(a,x) = 1$ is reflected in the expansion for both functions, since $\mathrm{erfc}(z) + \mathrm{erfc}(-z) = 2$.

4. COMPUTATIONAL ASPECTS OF TRICOMI's Ψ FUNCTION

The Ψ-function can be defined by

$$(4.1)\qquad \Psi(a;b;z) = \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}e^{-zt}(1+t)^{b-a-1}dt,$$

this representation being valid for $\mathrm{Re}\ a > 0$, $\mathrm{Re}\ z > 0$, $b \in \mathbb{C}$. In general, i.e., for general a and b, $\Psi(a;b;z)$ is singular at $z = 0$. With respect to a and b, $\Psi(a;b;z)$ is an entire function. For $a = 0,-1,-2,\ldots$, it can be written in terms of Laguerre polynomials

$$\Psi(-n;\alpha+1;z) = (-1)^n n!\, L_n^{(\alpha)}(z),$$

hence Tricomi's polynomial (1.1) is a special case of Tricomi's Ψ function. If $b-a-1 = n$, $n = 0,1,2,\ldots$, it also reduces to an elementary function. It easily follows from (4.1) that

$$\Psi(a;a+n+1;z) = \sum_{k=0}^{n}(a)_k\binom{n}{k}z^{-a-k},$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$, $k = 0,1,2,\ldots$.

Information about the Ψ function can be found in TRICOMI (1954), BUCHHOLZ (1953) and SLATER (1960). Computational aspects are considered by FIELDS & WIMP (1970), WIMP (1974), LUKE (1969) and TEMME (1983). In the latter a computer program is given for computing a set of functions $\{\Psi(a+k;b;z)\}_{k=0}^{K}$, for $b \in \mathbb{R}$, $z > 0$, $a \geq 0$.

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where the $B_s(\eta)$ are the same as in (3.6). Hence the addition rule $P(a,x) + Q(a,x) = 1$ is reflected in the expansion for both functions, since $\mathrm{erfc}(z) + \mathrm{erfc}(-z) = 2$.

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In the publications of Fields, Wimp and Luke the emphasis is on expansions of the Ψ function, which enable computation for a wide range of the parameters. An interesting result is

$$(\omega z)^a \Psi(a;b;\omega z) = \sum_{n=0}^{\infty} C_n(z) T_n^*(1/\omega)$$

where $T_n^*(x)$ is a shifted Chebyshev polynomial and in which the coefficients $C_n(z)$ easily follow from a four-term recurrence relation. The expansion is given in LUKE (1969), together with a device for computing $C_n(z)$ (a Miller algorithm).

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