TIME DEPENDENT ANALYSIS OF A QUEUEING MODEL BY FORMULATING A BOUNDARY VALUE PROBLEM

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Abstract

The analysis of queueing models which can be characterized as a random walk in the first quadrant of the plane often leads to the problem of solving a functional equation for a bivariate generating function. Recently, a method has been developed by which a rather general class of such functional equations related to stationary distributions can be solved with the aid of the theory of boundary value problems, see [1],[3],[4],[5],[6],[7],[10]. In the present study we shall show that the same method can be applied in the analysis of the time dependent behaviour of this class of queueing models. For this discussion a relatively simple model with two types of customers, Poissonian arrival streams, paired services and a general service time distribution will be considered. The generating function of the joint queue length distribution at the nth departure instant will be determined. This function forms the starting point for the analysis of the asymptotic behaviour of the process as $n \to \infty$.

Key words: queueing system, two-dimensional state space, time dependent behaviour, functional equation, boundary value problem

1. Introduction, the model

In [3] the stationary M/G/1 queueing system with alternating service has been studied. The functional equation for the generating function of the joint queue length distribution at departure epochs has been reduced to two Riemann-Hilbert boundary value problems. In the present study it will be shown that this technique of solving functional equations by formulating Riemann-Hilbert problems, or by formulating a related Hilbert problem (see [9] for this terminology) can also be applied in the time dependent analysis of queueing models with a two-dimensional state space. To show this we shall consider a queueing system with two types of customers and paired services, of which the functional equation for the generating function of the joint distribution of the number of type 1 and of type 2 customers left behind in the system at the nth departure instant has the same structure as the functional equation analysed in [3], but is of a simpler form.

This model is as follows. Two types of customers arrive independently at a single service facility. For type j customers the interarrival times are independent random variables with a common negative exponential distribution with mean α_j $(j=1,2)$. An arriving customer who finds the system empty is immediately taken into service; otherwise he joins queue 1 or 2 depending on his type. As soon as a service has been completed, a new service is started if any customer is present. In general a couple of two customers of different type is simultaneously served. If after the completion of a service only customers of one type are present, a customer of this type is individually served. In each queue customers are served in order of their arrival. Successive service times are independent random variables with a common distribution function $B(t)$, for paired services as well as for individual services.

Denote by $\underline{x}_j(n)$, $n=0,1,2,..$, $j=1,2$, the number of type j customers left behind in the system at the nth service completion instant. It is assumed that the process starts for $n=0$ with an empty system (this assumption is not essential, see [1]). It is readily seen that the process $\{(\underline{x}_1(n),\underline{x}_2(n)),n=0,1,..\}$ is an irreducible, aperiodic, discrete time Markov chain with state space $\{0,1,2,..\}\times\{0,1,2,..\}$. In the sequel this Markov chain will be analysed. For this we introduce the generating function: for $|r|<1$, $|p_1|\leqslant 1$, $|p_2|\leqslant 1$,

$$(1)\qquad \Phi(r;p_1,p_2) := \sum_{n=0}^{\infty} r^n\, E\{p_1^{\underline{x}_1(n)}\, p_2^{\underline{x}_2(n)} \mid \underline{x}_1(0)=0,\ \underline{x}_2(0)=0\}.$$

Further we define

$$(2)\qquad \frac{1}{\alpha} := \frac{1}{\alpha_1}+\frac{1}{\alpha_2}; \qquad c_j := \alpha/\alpha_j, \quad j=1,2;$$

$$(3)\qquad \beta(s) := \int_0^{\infty} e^{-st}\, dB(t), \quad \mathrm{Re}\ s \geqslant 0;$$

(4) $$\beta_k := \int_0^\infty t^k \, dB(t), \qquad k=1,2,..;$$

(5) $$a := \beta_1/\alpha.$$

2. The functional equation

From the definition of the queueing process it follows that for $j=1,2$, the series $\{\underline{x}_j(n), n=0,1,..\}$ satisfies the relations

(6) $$\underline{x}_j(n) = [\underline{x}_j(n-1) - 1]^+ + \underline{\xi}_j(n), \qquad n=1,2,..; \qquad \underline{x}_j(0) = 0;$$

here $\underline{\xi}_j(n)$, $n=1,2,..$, stands for the number of type j customers who arrive during the *n*th service. The generating function of the distribution of $(\underline{\xi}_1(n), \underline{\xi}_2(n))$ is given by: for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$,

(7) $$E\left\{p_1^{\underline{\xi}_1(n)} \, p_2^{\underline{\xi}_2(n)}\right\} = \beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right), \qquad n=1,2,.. .$$

From the relations (6) the following functional equation for the generating function $\Phi(r;p_1,p_2)$ is deduced by straightforward calculations: for $|r|<1$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$,

(8) $$\left[p_1p_2 - r\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)\right] \Phi(r;p_1,p_2) = p_1p_2 + r\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right) \times$$
$$\times [(p_2-1)\,\Phi(r;p_1,0) + (p_1-1)\,\Phi(r;0,p_2) + (p_1-1)(p_2-1)\,\Phi(r;0,0)].$$

Because the generating function $\Phi(r;p_1,p_2)$ is uniquely determined by the relations (6), this functional equation must have at least one solution with the properties of a generating function.

As a first investigation we take $p_2 = 1$ in equation (8). This leads to: for $|r|<1$, $|p_1| \leqslant 1$,

(9) $$\left[p_1 - r\,\beta\left(\frac{1-p_1}{\alpha_1}\right)\right] \Phi(r;p_1,1) = p_1 + r\,\beta\left(\frac{1-p_1}{\alpha_1}\right)(p_1-1)\,\Phi(r;0,1).$$

This is a well-known equation from the theory of the M/G/1 queueing system, cf. [2], p.240. Hence, for $|r|<1$,

(10) $$\Phi(r;0,1) = \frac{1}{1-\mu_1(r)},$$

here $p_1 = \mu_1(r)$ is the unique solution inside the unit circle of the equation

$$(11)\qquad p_1 - r\,\beta\left(\frac{1-p_1}{\alpha_1}\right) = 0.$$

An analogous result can be obtained by taking $p_1 = 1$ in equation (8). But for obtaining the complete solution of this equation more powerful techniques are required.

3. Analysis

Throughout this section, r is fixed and real, $0 < r < 1$.
Equation (8) relates the bivariate function $\Phi(r;p_1,p_2)$ to two univariate functions $\Phi(r;p_1,0)$, $\Phi(r;0,p_2)$, and a constant $\Phi(r;0,0)$. A central role in the analysis is played by the *kernel*

$$(12)\qquad p_1p_2 - r\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right),$$

because if for a pair (p_1,p_2), $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, this kernel vanishes, then the righthand side of equation (8) must also vanish. The existence of such pairs (p_1,p_2) can be shown with Rouché's theorem, cf. [1], p.49. This provides us with a relation between the functions $\Phi(r;p_1,0)$ and $\Phi(r;0,p_2)$, which can be written in the following form:

$$(13)\qquad \frac{\Phi(r;p_1,0)}{1-p_1} + \frac{\Phi(r;0,p_2)}{1-p_2} = \Phi(r;0,0) + \frac{1}{(1-p_1)(1-p_2)},$$

$$(14)\qquad \text{if}\quad p_1p_2 = r\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right),\qquad |p_1|\leqslant 1,\ p_1\neq 1,\ |p_2|\leqslant 1,\ p_2\neq 1.$$

Note that the cases $p_1 = 1$ and $p_2 = 1$ have already been discussed in section 2.
From the above functional relation the functions $\Phi(r;p_1,0)$ and $\Phi(r;0,p_2)$ have to be determined. For this purpose we shall first introduce a parameter δ in order to describe the zeros (p_1,p_2) of the kernel (12) as functions of this parameter, cf. [3]. Hence, let

$$(15)\qquad \delta := c_1p_1 + c_2p_2,\qquad\qquad w := 2c_1p_1.$$

Substitution of (15) in equation (14) leads to the equation

$$(16)\qquad w^2 - 2\delta w + 4c_1c_2\,r\,\beta\left(\frac{1-\delta}{\alpha}\right) = 0.$$

This equation defines a two-valued function $w(r;\delta)$ of δ which is given by

$$(17)\qquad w(r;\delta) = \delta \pm \sqrt{\delta^2 - 4c_1c_2\, r\,\beta\left(\frac{1-\delta}{\alpha}\right)}.$$

LEMMA 1. *In the domain* $\mathrm{Re}\ \delta < 1$ *the function* $w(r;\delta)$ *is a two-valued analytic function with exactly two branch points, say* $\delta_1(r)$ *and* $\delta_2(r)$, *which are the roots in the domain* $\mathrm{Re}\ \delta < 1$ *of the equation*

$$(18)\qquad \delta^2 - 4c_1c_2\, r\,\beta\left(\frac{1-\delta}{\alpha}\right) = 0.$$

PROOF. Because $\beta(.)$ is the Laplace-Stieltjes transform of a positive random variable, the function $\beta((1-\delta)/\alpha)$ is regular for $\mathrm{Re}\ \delta < 1$ and bounded in absolute value by one for $\mathrm{Re}\ \delta \leqslant 1$. Hence, the function $w(r;\delta)$ is analytic in $\mathrm{Re}\ \delta < 1$, except at points where the discriminant, cf. (18), of equation (16) vanishes. Because $c_1 + c_2 = 1$, cf. (2), we have for $\mathrm{Re}\ \delta = 1$ as well as for $|\delta| \to \infty$, $\mathrm{Re}\ \delta < 1$, the inequalities

$$(19)\qquad |\delta^2| \geqslant 1 > |r| \geqslant \left|4c_1c_2\, r\,\beta\left(\frac{1-\delta}{\alpha}\right)\right|.$$

With Rouché's theorem it follows that equation (18) has exactly two zeros in the domain $\mathrm{Re}\ \delta < 1$. □

By considering equation (18) for real δ on $(-\infty,1]$ it is seen that the two roots of this equation in $\mathrm{Re}\ \delta < 1$ are real (since r has been chosen to be real, positive), and that they can be chosen such that

$$(20)\qquad -1 < \delta_1(r) < 0 < \delta_2(r) < 1.$$

Now we can say that equation (13) holds if, cf. (15),

$$(21)\qquad p_1 = \frac{1}{2c_1}\, w(r;\delta), \qquad\qquad p_2 = \frac{1}{2c_2}\,[2\delta - w(r;\delta)],$$

for one of the two branches of the function $w(r;\delta)$, cf. (17), and for δ such that $|p_1| \leqslant 1$ and $|p_2| \leqslant 1$. If p_1 and p_2 are given by (21), then for every δ, $\mathrm{Re}\ \delta \leqslant 1$, and for each branch of the function $w(r;\delta)$ the following inequality follows from equation (16):

$$(22)\qquad |p_1p_2| = \frac{1}{4c_1c_2}\,|w(r;\delta)\,[2\delta - w(r;\delta)]| = \left|r\,\beta\left(\frac{1-\delta}{\alpha}\right)\right| < 1.$$

Hence, because either $|p_1| < 1$ or $|p_2| < 1$ for every δ, $\mathrm{Re}\ \delta \leqslant 1$, and for each branch of the analytic function $w(r;\delta)$ if p_1 and p_2 are given by (21), the relation (13) together with (21) can be continued analytically to the domain Re $\delta < 1$ (principle of permanence).

LEMMA 2. *The functions* $\Phi(r;w(r;\delta)/2c_1,0)$ *and* $\Phi(r;0,\ 2\delta-w(r;\delta)\ /2c_2)$ *each possess a continuation as a two-valued analytic function into the domain* Re $\delta < 1$, *with no other branch points than* $\delta_1(r)$ *and* $\delta_2(r)$.

PROOF. The assertion will be proved for the first function, for the second one the proof is similar.
By lemma 1 and the properties of the generating function $\Phi(r;p_1,p_2)$ the function $\Phi(r;w(r;\delta)/2c_1,0)$ is regular for those δ and branches of $w(r;\delta)$ for which $|w(r;\delta)| < 2c_1$. Into a subregion of Re $\delta < 1$ where $|w(r;\delta)| \geqslant 2c_1$ for one of the branches of $w(r;\delta)$ the function $\Phi(r;w(r;\delta)/2c_1,0)$ can be continued analytically by means of relation (13) together with (21), since by (22) then $|2\delta - w(r;\delta)| < 2c_2$ holds. From lemma 1 it is further clear that the only branch points, which the function $\Phi(r;w(r;\delta)/2c_1,0)$ can have in the domain Re $\delta < 1$, are $\delta_1(r)$ and $\delta_2(r)$. □

Next, relation (13) together with (21) will be considered for δ on the real interval between the branch points $\delta_1(r)$ and $\delta_2(r)$, cf. (20). For $\delta \in [\delta_1(r),\delta_2(r)]$ the discriminant of equation (16) is non-positive, so that, cf. (17), for δ on this interval the two branches of the function $w(r;\delta)$ are complex conjungate and lie on the contour

(23) $$L(r) := \{w;\ |w|^2 = 4c_1c_2\, r\, \beta\Big(\frac{1-\text{Re } w}{\alpha}\Big),\quad \text{Re } w < 1\}.$$

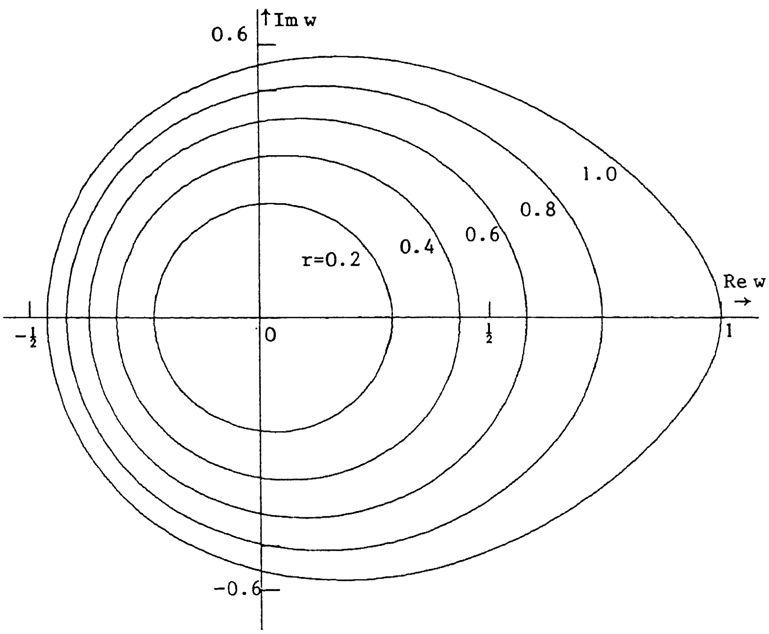

Figure 1

The contour $L(r)$ in the case that $c_1=c_2=\frac{1}{2}$, $\alpha = 1.6$, $\beta(s) = (1+\frac{1}{2}\beta_1 s)^{-2}$, for different values of r

The interior of the contour $L(r)$ will be denoted by $L^+(r)$, its exterior by $L^-(r)$.

LEMMA 3. *For* $w \in L(r)$,

$$(24)\qquad \frac{\Phi(r;w/2c_1,0)}{1-w/2c_1} + \frac{\Phi(r;0,\bar{w}/2c_2)}{1-\bar{w}/2c_2} = \Phi(r;0,0) + \frac{1}{(1-w/2c_1)(1-\bar{w}/2c_2)}.$$

PROOF. From (17) and (23) it is clear that for every $w \in L(r)$ there exists a δ on the interval $[\delta_1(r),\delta_2(r)]$, namely $\delta = \text{Re}\, w$, such that $w = w(r;\delta)$ for one of the branches of the function $w(r;\delta)$. Moreover, then $2\delta - w(r;\delta) = \bar{w}$. Hence, equation (24) follows from (13) together with (21) by taking $\delta \in [\delta_1(r),\delta_2(r)]$, which is allowed by lemma 2. □

Equation (24) will be the basis for the formulation of a boundary value problem, which will be discussed in section 4. This section will be concluded with the proof that the functions $\Phi(r;w/2c_1,0)$ and $\Phi(r;0,w/2c_2)$ are regular for $w \in L^+(r)$. For simplicity the discussion will be confined to the case $c_2 \leqslant \frac{1}{2} \leqslant c_1$, cf. (2). This is of course no restriction.

First we consider the question whether the point $2c_2$ lies inside, on or outside the contour $L(r)$. Since this contour crosses the real axis only at $w = \delta_1(r)$ and at $w = \delta_2(r)$, cf. lemma 1, we have $2c_2 \in L(r)$ if and only if $2c_2 = \delta_2(r)$, cf. (20). Hence, we insert $\delta = 2c_2$ in equation (18), which leads to the equation (using $c_1 + c_2 = 1$):

$$(25)\qquad c_2 = (1-c_2)\, r\, \beta\left(\frac{1-2c_2}{\alpha}\right).$$

This equation inspires us to define the function

$$(26)\qquad R(s) := \frac{s}{1-s}\left[\beta\left(\frac{1-2s}{\alpha}\right)\right]^{-1}, \qquad \text{Re}\, s \leqslant \tfrac{1}{2}.$$

LEMMA 4. *If* $a \leqslant 2$ *then* $R(c_2) < 1$ *for* $c_2 < \frac{1}{2}$ *and* $R(\frac{1}{2}) = 1$. *If* $a > 2$ *then there exists a constant* $c(a)$, $0 < c(a) < \frac{1}{2}$, *such that* $R(c_2) < \frac{1}{2}$ *for* $c_2 < c(a)$ *and* $R(c_2) \geqslant 1$ *for* $c(a) \leqslant c_2 \leqslant \frac{1}{2}$.

PROOF. Rewrite equation (25) as

$$(27)\qquad \frac{s}{1-s} = r\, \beta\left(\frac{1-2s}{\alpha}\right), \qquad \text{Re}\, s \leqslant \tfrac{1}{2}.$$

On the line $\text{Re}\, s = \frac{1}{2}$ as well as for $|s| \to \infty$, $\text{Re}\, s < \frac{1}{2}$, the inequality

(28) $\quad |\frac{s}{1-s}| > r \geqslant r|\beta\left(\frac{1-2s}{\alpha}\right)|$,

holds for every r, $0 < r < 1$. Hence, by Rouché's theorem equation (27) has for every r, $0 < r < 1$, exactly one root $s = s_0(r)$ in the region Re $s \leqslant \frac{1}{2}$. From the properties of the Laplace-Stieltjes transform $\beta(.)$ it is clear that this root $s_0(r)$ must be real and that $0 < s_0(r) < \frac{1}{2}$. Further, $s_0(r)$ is the inverse of the function R(s) for $0 < r < 1$. This implies that the function R(s) increases strictly from zero to one on the interval $0 < s < s_0(1)$, and that $R(s) \geqslant 1$ for $s_0(1) \leqslant s \leqslant \frac{1}{2}$. Whether $s_0(1) < \frac{1}{2}$ or $s_0(1) = \frac{1}{2}$ depends on the derivative of R(s) at $s = \frac{1}{2}$. From (26) we obtain

(29) $\quad R'(\frac{1}{2}) = 4(1-\frac{1}{2}a)$.

Hence, $s_0(1) = \frac{1}{2}$ for $a < 2$, and $s_0(1) < \frac{1}{2}$ for $a > 2$. For $a = 2$ we find by considering higher derivatives of the function R(s) at $s = \frac{1}{2}$ that $s_0(1) = \frac{1}{2}$. By taking $c(a) = s_0(1)$ for $a > 2$ the proof has been completed. □

LEMMA 5. *For* $0 < r < \min\{1,R(c_2)\}$ *we have* $2c_2 \in L^-(r)$. *If* $R(c_2) < 1$ *then* $2c_2 \in L(r)$ *for* $r = R(c_2)$, *and* $2c_2 \in L^+(r)$ *for* $R(c_2) < r < 1$.

PROOF. From (25), (26), and the remark above these formulas we have

(30) $\quad 2c_2 \in L(r) \quad \Leftrightarrow \quad 2c_2 = \delta_2(r) \quad \Leftrightarrow \quad r = R(c_2) < 1.$

From equation (18) it is clear that $\delta_2(r)$ is a continuous function of r for $0<r<1$, and that $\delta_2(r) \downarrow 0$ as $r \downarrow 0$, so that $2c_2 \in L^-(r)$ for $r \downarrow 0$. Further, it is seen from (18) that $\delta_2(r)$ as function of r, $0 < r < 1$, has an inverse, so that it must be a strictly increasing function of r, $0 < r < 1$. With (30) this is sufficient to prove the assertion. □

LEMMA 6. *The functions* $\Phi(r;w/2c_1,0)$ *and* $\Phi(r;0,w/2c_2)$ *are regular in the domain* $L^+(r)$ *and continuous up to the boundary* L(r).

PROOF. Because $\Phi(r;p_1,p_2)$ is a bivariate generating function of a probability distribution in p_1 and in p_2, the functions $\Phi(r;p,0)$ and $\Phi(r;0,p)$ are regular for $|p|<1$ and continuous for $|p| \leqslant 1$. Further, it follows from the monotonicity of the function $\beta(.)$ and from the fact that Re $w \leqslant \delta_2(r)$ for $w \in L(r)$, cf. (23) and lemma 1, that for $w \in L(r)$, cf. (23), (18), (20), and therefore also for $w \in L^+(r)$,

(31) $\quad |w| \leqslant 2\sqrt{c_1c_2\, r\, \beta\left(\frac{1-\delta_2(r)}{\alpha}\right)} = \delta_2(r).$

Hence, the assertions for the function $\Phi(r;w/2c_1,0)$ are obvious since we have chosen $c_1 \geqslant \frac{1}{2}$, and $\delta_2(r) < 1$ by (20). Also, the assertions for the function $\Phi(r,0,w/2c_2)$ have been proved by the above in the case $2c_2 > \delta_2(r)$, i.e. for $0 < r < \min\{1,R(c_2)\}$, cf. lemma 5.

Finally, suppose $R(c_2) < 1$, cf. lemma 4, and $R(c_2) \leq r < 1$. In this case we use the analytic continuation of the function $\Phi(r;0,[2\delta-w(r;\delta)]/2c_2)$ discussed in lemma 2. By letting δ tend to $\delta_2(r)$ in equation (13) together with (21) it is seen that $\Phi(r;0,\delta_2(r)/2c_2)$ is finite. Because the function $\Phi(r;0,w/2c_2)$ has a power series expansion at $w=0$ with positive coefficients, cf. (1), it follows that this function is regular in the disk $|w| < \delta_2(r)$ and continuous for $|w| \leq \delta_2(r)$. With (31) this proves the assertions for $\Phi(r;0,w/2c_2)$ in the present case. □

4. Formulation as a Hilbert boundary value problem

Throughout this section r *is fixed and real,* $0 < r < 1$, *and* $c_2 \leq \frac{1}{2} \leq c_1$.

As in [3] equation (24) can be reduced to two Riemann-Hilbert problems on the contour $L(r)$. However, in the following we shall give a slightly different approach by formulating a single Hilbert problem, cf. [9], §§34-37. This method is somewhat simpler, and above it has the advantage that it is still applicable in the analysis of a generalization of the present model in which the duration of individual services has not the same distribution as that of paired services, see [1], §IV.2.

As in [3] equation (24) is transformed into a relation on the unit circle by introducing a conformal mapping.

LEMMA 7. *There exists a conformal mapping* $g(r;z)$ *of the unit disk* $|z| < 1$ *onto the domain* $L^+(r)$. *This conformal mapping is uniquely determined by the conditions*

$$(32) \qquad g(r;0) = 0, \qquad g'(r;0) > 0.$$

The conformal mapping $g(r;z)$ *is continuous for* $|z| \leq 1$ *and establishes a one-to-one correspondence between this region and* $L^+(r) \cup L(r)$. *Further it satifies the relation*

$$(33) \qquad g(r;\bar{z}) = \overline{g(r;z)}, \qquad |z| \leq 1.$$

PROOF. Because $L^+(r)$ is a simply connected domain, cf. (23), the existence of the conformal mapping $g(r;z)$ follows from Riemann's mapping theorem, cf. [8], vol.III, §2, theorem 1.2. The uniqueness theorem for conformal mapping, cf. [8], vol. III, §2, theorem 1.3, implies the uniqueness of $g(r;z)$ given the conditions (32). The assertions for $|z| = 1$ follow from the boundary correspondence theorem, cf. [8], vol.III, §8, theorem 2.24. Finally, relation (33) is a consequence of the property that the real axis is an axis of symmetry of the contour $L(r)$, cf. (23), and of the choice made in (32). □

In the sequel the unit circle will be denoted by C.

THEOREM 1. *If* $2c_2 \in L^-(r)$, *then for* $t \in C$,

$$(34)\qquad \frac{\Phi(r;\frac{1}{2c_1}g(r;t),0)}{1-\frac{g(r;t)}{2c_1}}-\Phi(r;0,0)+\frac{\Phi(r;0,\frac{1}{2c_2}g(r;\frac{1}{t}))}{1-\frac{g(r;1/t)}{2c_2}}=\frac{1}{\left[1-\frac{g(r;t)}{2c_1}\right]\left[1-\frac{g(r;1/t)}{2c_2}\right]},$$

and the functions

$$(35)\qquad \frac{\Phi(r;g(r;t)/2c_1,0)}{1-g(r;t)/2c_1}-\Phi(r;0,0),\qquad \text{and}\quad \frac{\Phi(r;0,g(r;t)/2c_2)}{1-g(r;t)/2c_2},$$

are regular for $|t|<1$.
This defines a Hilbert boundary value problem on the unit circle.

PROOF. The boundary condition (34) follows from lemma 3 by inserting $w=g(r;t)$, $t\in C$, cf. lemma 7, and by noting that (33) implies:

$$(36)\qquad \overline{g(r;t)}=g(r;\tfrac{1}{t}),\qquad t\in C.$$

Lemma 6 , the regularity of the conformal mapping $g(r;z)$ for $|z|<1$, and the assumption $2c_2\in L^-(r)$ imply the regularity for $|t|<1$ of the functions in (35). According to the definitions in [9], §37, a Hilbert boundary value problem is defined by the relation (34) for the regular functions in (35) if the known function at the righthand side of (34) satisfies a Hölder condition on C, cf. [9], §3. Such a Hölder condition depends on the boundedness of $\frac{\partial}{\partial z}g(r;z)$ in the region $|z|<1$, which can be proved by using smoothness properties of the contour $L(r)$. For the details of this proof the reader is referred to [1], lemma II.6.2. □

The conformal mapping $g(r;z)$ has an inverse for $|z|\leq 1$, cf. lemma 7. This inverse will be denoted by $g_0(r;w)$, $w\in L^+(r)\cup L(r)$.

THEOREM 2. *If* $2c_2\in L^-(r)$, *i.e.* $0<r<\min\{1,R(c_2)\}$, *then the generating function* $\Phi(r;p_1,p_2)$ *is given by: for* $2c_1p_1\in L^+(r)$, $2c_2p_2\in L^+(r)$,

$$(37)\qquad \Phi(r;p_1,p_2)=\frac{r(1-p_1)(1-p_2)\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{p_1p_2-r\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}\left[\frac{p_1p_2}{r(1-p_1)(1-p_2)\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}-\right.$$
$$-\frac{1}{2\pi i}\int_C\frac{1}{\{1-g(r;t)/2c_1\}\{1-g(r;1/t)/2c_2\}}\frac{t+g_0(r;2c_1p_1)}{t-g_0(r;2c_1p_1)}\frac{dt}{2t}-$$
$$\left.-\frac{1}{2\pi i}\int_C\frac{1}{\{1-g(r;1/t)/2c_1\}\{1-g(r;t)/2c_2\}}\frac{t+g_0(r;2c_2p_2)}{t-g_0(r;2c_2p_2)}\frac{dt}{2t}\right].$$

PROOF. The Hilbert boundary value problem formulated in theorem 1 is of a simple form, cf. [9], §37. It is easily solved by applying the operator

$$(38)\qquad \frac{1}{2\pi i}\int_C \cdots\cdots \frac{dt}{t-z},$$

on both sides of equation (34), for $|z| < 1$ and for $|z| > 1$. By noting that, cf. (32),

$$(39)\qquad \lim_{z\to\infty} \frac{\Phi(r;0,g(r;1/z)/2c_2)}{1-g(r;1/z)/2c_2} = \Phi(r;0,0),$$

the operation (38) on equation (34) leads with the residu theorem to: for $|z| < 1$,

$$(40)\qquad \frac{\Phi(r;g(r;z)/2c_1,0)}{1-g(r;z)/2c_1} = \frac{1}{2\pi i}\int_C \frac{1}{\{1-g(r;t)/2c_1\}\{1-g(r;1/t)/2c_2\}}\frac{dt}{t-z};$$

and similarly to: for $|z| > 1$,

$$(41)\qquad \Phi(r;0,0) - \frac{\Phi(r;0,g(r;1/z)/2c_2)}{1-g(r;1/z)/2c_2} = \frac{1}{2\pi i}\int_C \frac{1}{\{1-g(r;t)/2c_1\}\{1-g(r;1/t)/2c_2\}}\frac{dt}{t-z}.$$

By taking $z=0$ in (40) we obtain the unknown constant in (41):

$$(42)\qquad \Phi(r;0,0) = \frac{1}{2\pi i}\int_C \frac{1}{\{1-g(r;t)/2c_1\}\{1-g(r;1/t)/2c_2\}}\frac{dt}{t}.$$

Next, by substituting $z=g_0(r;2c_1p_1)$ in (40) and $1/z=g_0(r;2c_2p_2)$ in (41) together with (42), expressions for the functions $\Phi(r;p_1,0)$ and $\Phi(r;0,p_2)$ are obtained for $2c_1p_1 \in L^+(r)$ and $2c_2p_2 \in L^+(r)$ respectively. Finally, by substituting these expressions for $\Phi(r;p_1,0)$ and for $\Phi(r;0,p_2)$ and (42) for $\Phi(r;0,0)$ in the functional equation (8) the relation (37) is obtained after some simple rearrangements. □

REMARK 1. As it has been noted in section 2, the functional equation (8) must have at least one solution which is a generating function of a joint probability distribution in p_1 and p_2, and which is a generating function of a series with coefficients bounded in absolute value by one in r. With our analysis it has been proved that equation (8) possesses at most one solution with these properties, for $0 < r < \min\{1,R(c_2)\}$. This implies that the righthand side of (37) represents the generating function defined in (1) for $2c_1p_1 \in L^+(r)$, $2c_2p_2 \in L^+(r)$, $0 < r < \min\{1,R(c_2)\}$. Moreover, the expression (37) determines the power series expansion of the function $\Phi(r;p_1,p_2)$ at $r=0$, $p_1=0$, $p_2=0$. Hence, by analytic continuation the function $\Phi(r;p_1,p_2)$ has been uniquely determined in theorem 2 for $|r| < 1$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$.

REMARK 2. Explicit formulas for the function $\Phi(r;p_1,p_2)$ for $2c_1p_1 \in L^-(r)$ and/or $2c_2p_2 \in L^-(r)$, $0 < r < \min\{1,R(c_2)\}$, can be obtained by using the analytic continuation of the function $g_0(r;w)$ into $L^-(r)$, and by applying the Plemelj formulas, cf. [9], §17, to the integrals in (37), see [1], theorem II.7.2.

The main interest of the solution of the time dependent distribution of the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),n=0,1,..\}$ is that it forms the basis for the asymptotic analysis of this Markov chain as $n \to \infty$. For this purpose it is important to derive explicit expressions for the function $\Phi(r;p_1,p_2)$ for r in a neighbourhood of one, because e.g. (see [2], p.18 and appendix 1):

$$\lim_{n\to\infty} \Pr\{\underline{x}_1(n)=0,\ \underline{x}_2(n)=0\} = \lim_{r\to 1} (1-r)\ \Phi(r;0,0). \tag{43}$$

However, from lemma 4 and 5 it is seen that theorem 2 does not provide us with such an expression for all values of the parameters α and c_2. Therefore we shall derive below an expression for $\Phi(r;p_1,p_2)$ in the case $R(c_2) < 1$, cf. lemma 4, for $R(c_2) < r < 1$, i.e. for $2c_2 \in L^+(r)$, cf. lemma 5. Hence, suppose $2c_2 \in L^+(r)$. Then theorem 1 is still valid, except that the second function in (35) possesses a single pole in the region $|t| \leq 1$ due to a zero of the denominator at the point

$$z_0(r) := g_0(r;2c_2). \tag{44}$$

With this observation the following result is obtained:

THEOREM 3. *If* $2c_2 \in L^+(r)$, *i.e.* $R(c_2) < r < 1$, *then the generating function* $\Phi(r;p_1,p_2)$ *is given by: for* $2c_1p_1 \in L^+(r)$, $2c_2p_2 \in L^+(r)$,

$$\begin{aligned}\Phi(r;p_1,p_2) = \frac{r(1-p_1)(1-p_2)\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{p_1p_2 - r\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)} \Bigg[&\frac{p_1p_2}{r(1-p_1)(1-p_2)\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)} - \\ &- \frac{1}{2\pi i}\int_C \frac{1}{\{1-g(r;t)/2c_1\}\{1-g(r;1/t)/2c_2\}} \frac{t+g_0(r;2c_1p_1)}{t-g_0(r;2c_1p_1)}\frac{dt}{2t} - \\ &- \frac{1}{2\pi i}\int_C \frac{1}{\{1-g(r;1/t)/2c_1\}\{1-g(r;t)/2c_2\}} \frac{t+g_0(r;2c_2p_2)}{t-g_0(r;2c_2p_2)}\frac{dt}{2t} - \\ &- \frac{2c_2}{g'(r;z_0(r))}\frac{1}{1-\mu_1(r)}\frac{1-g_0(r;2c_1p_1)\,g_0(r;2c_2p_2)}{\{1-z_0(r)g_0(r;2c_1p_1)\}\{z_0(r)-g_0(r;2c_2p_2)\}}\Bigg];\end{aligned} \tag{45}$$

here

$$(46)\qquad g'(r;z_0(r)) := \frac{\partial}{\partial z} g(r;z)\Big|_{z=z_0(r)}.$$

PROOF. By applying - as in theorem 2 - the operator (38) on both sides of equation (34), and by taking into account the pole at $z_0(r)$, cf. (44), the residue theorem leads to: for $|z| < 1$,

$$(47)\qquad \frac{\Phi(r;g(r;z)/2c_1,0)}{1-g(r;z)/2c_1} + \frac{2c_2}{z_0(r)g'(r;z_0(r))}\,\frac{\Phi(r;0,1)}{z_0(r)z-1} =$$

$$= \frac{1}{2\pi i}\int_C \frac{1}{\{1-g(r;t)/2c_1\}\{1-g(r;1/t)/2c_2\}}\,\frac{dt}{t-z},$$

and for $|z| > 1$,

$$(48)\qquad \Phi(r;0,0) + \frac{2c_2}{z_0(r)g'(r;z_0(r))}\,\frac{\Phi(r;0,1)}{z_0(r)z-1} - \frac{\Phi(r;0,g(r;1/z)/2c_2)}{1-g(r;1/z)/2c_2} =$$

$$= \frac{1}{2\pi i}\int_C \frac{1}{\{1-g(r;t)/2c_1\}\{1-g(r;1/t)/2c_2\}}\,\frac{dt}{t-z},$$

here $g'(r;z_0(r))$ is given by (46). The constant $\Phi(r;0,1)$ has been determined in (10). By taking $z=0$ in (47) the last unknown constant is obtained:

$$(49)\qquad \Phi(r;0,0) = \frac{2c_2}{z_0(r)g'(r;z_0(r))}\,\frac{1}{1-\mu_1(r)} +$$

$$+ \frac{1}{2\pi i}\int_C \frac{1}{\{1-g(r;t)/2c_1\}\{1-g(r;1/t)/2c_2\}}\,\frac{dt}{t}.$$

In a similar way as in the proof of theorem 2 the relations (47), (48) and (49) lead to the expression (45). □

REMARK 3. An expression for the function $\Phi(r;p_1,p_2)$ in the case $2c_2 \in L(r)$, i.e. $r = R(c_2) < 1$, can be derived from (37) or (45) by using the continuity of this function, see [1], theorem II.7.5.

5. Concluding remarks

The results of theorem 2 and 3 form the starting point for the analysis of the asymptotic behaviour of the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),n=0,1,..\}$. Especially in the case $c_1 = c_2 = \frac{1}{2}$ this analysis requires an extensive use of theorems on the boundary behaviour of conformal mappings and their derivatives. Here we merely state the main results of this asymptotic analysis, which has been carried through in [1], §II.8. It turns out that the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),n=0,1,..\}$ is transient if

$\max\{c_1, c_2\}a > 1$, that it consists of null states if $\max\{c_1, c_2\}a = 1$, and that it is recurrent if $\max\{c_1, c_2\}a < 1$. In the last case the Markov chain possesses a stationary distribution, and the stationary probability ϕ_0 of the empty state (0,0) is given by (here $c_2 \leq \frac{1}{2} \leq c_1$), cf. (43):

$$(50) \qquad \phi_0 := \lim_{r \to 1} (1-r)\ \Phi(r;0,0) = \frac{2c_2(1-c_1 a)}{\lim_{r \to 1} z_0(r)\ g'(r;z_0(r))}.$$

For $c_1 a < 1$ the limit in the denominator in (50) is finite and non-vanishing, and it can be numerically evaluated, cf. [3], §6.

The technique of solving functional equations by formulating a Hilbert boundary value problem can also be applied in the analysis of the continuous time parameter queueing process. For the present model this leads to a stationary distribution which differs from that of the imbedded discrete time parameter process, see [1], §III.8.

If the present queueing model is generalized by allowing a service time distribution $B_j(t)$ for individual services of type j customers ($j = 1,2$), which may differ from the service time distribution for paired services, the boundary condition as in (34) becomes more intricate, but it still defines a Hilbert boundary value problem which can be solved with the general method given in [9], §§34-37; see [1], §IV.2.

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