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An Algorithm for Komlós Conjecture Matching Banaszczyk's Bound

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***Abstract*—We consider the problem of finding a low discrepancy coloring for sparse set systems where each element lies in at most t sets. We give an efficient algorithm that finds a coloring with discrepancy $O((t \log n)^{1/2})$, matching the best known non-constructive bound for the problem due to Banaszczyk. The previous algorithms only achieved an $O(t^{1/2} \log n)$ bound. Our result also extends to the more general Komlós setting and gives an algorithmic $O(\log^{1/2} n)$ bound.**

I. Introduction

Let $(V, \mathcal{S})$ be a finite set system, with $V = \{1, \ldots, n\}$ and $\mathcal{S} = \{S_1, \ldots, S_m\}$ a collection of subsets of V. For a two-coloring $\chi : V \to \{-1, 1\}$, the discrepancy of χ for a set S is defined as $\chi(S) = |\sum_{i \in S} \chi(i)|$ and measures the imbalance from an even-split for S. The discrepancy of the system $(V, \mathcal{S})$ is defined as

$$\mathrm{disc}(\mathcal{S}) = \min_{\chi : V \to \{-1,1\}} \max_{S \in \mathcal{S}} \chi(S).$$

That is, it is the minimum imbalance for all sets in $\mathcal{S}$, over all possible two-colorings χ.

Discrepancy is a widely studied topic and has applications to many areas in mathematics and computer science. For more background we refer the reader to the books [1]–[3]. In particular, discrepancy is closely related to the problem of rounding fractional solutions of a linear system of equations to integral ones [4], [5], and is widely studied in approximation algorithms and optimization.

Until recently, most of the results in discrepancy were based on non-algorithmic approaches and hence were not directly useful for algorithmic applications. However, in the last few years there has been remarkable progress in our understanding of the algorithmic aspects of discrepancy [6]–[12]. In particular, we can now match or even improve upon all known applications of the widely used partial-coloring method [2], [13] in discrepancy. This has, for example, led to several other new results in approximation algorithms [14]–[17].

Sparse Set Systems: Despite the algorithmic progress, one prominent question that has remained open is to match the known non-constructive bounds on discrepancy for low degree or sparse set systems. These systems are parametrized by t, that denotes the maximum number of sets that contain any element. Beck and Fiala [18] proved, using an algorithmic iterated rounding approach, that any such set system has discrepancy at most $2t - 1$. They also conjectured that the discrepancy in this case is $O(t^{1/2})$, and settling this has been an elusive open problem.

The best known result in this direction is due to Banaszczyk [19], which implies an $O(\sqrt{t \log n})$ discrepancy bound for the problem[1]. Unlike most results in discrepancy that are based on the partial-coloring method, Banaszczyk's proof is based on a very different and elegant convex geometric argument, and it is not at all clear how to make it algorithmic. Prior to Banaszczyk's result, the best known non-algorithmic bound was $O(t^{1/2} \log n)$ [20], based on the partial-coloring method. This bound was first made algorithmic in [6], and by now there are several different ways known to obtain this result [8]–[11], [21]. However the question of matching Banaszczyk's bound algorithmically for the problem and its variants has been open despite a lot of attention in recent years [11], [21]–[23]. In particular, as we discuss in Section I-B there is a natural algorithmic barrier to improving the $O(t^{1/2} \log n)$ bound.

A substantial generalization of the Beck-Fiala conjecture is the following:

Komlós Conjecture: Given any collection of vectors $v_1, \ldots, v_n \in \mathbb{R}^m$ such that $\|v_i\|_2 \le 1$ for each $i \in [n]$[2], there exist signs $x_1, \ldots, x_n \in \{-1, 1\}$ such that $\|\sum_{i=1}^n x_i v_i\|_\infty = O(1)$.

This implies the Beck-Fiala conjecture by choosing each v_i as the column corresponding to element i in

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[1] We assume here that $t \ge \log n$, otherwise the $O(t)$ bound is better.
[2] We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$.

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the incidence matrix of the set system scaled by $t^{-1/2}$. Again, the best known non-constructive bound here is $O(\sqrt{\log n})$ due to Banaszczyk and the previous algorithmic techniques can also be adapted to achieve $O(\log n)$ constructively for the Komlós setting.

A. Our Results

In this paper we give the following algorithmic result for the Beck-Fiala problem, which matches the non-constructive bound due to Banaszczyk.

Theorem 1. *Given a set system $(V, \mathcal{S})$ with $|V| = n$ such that each element $i \in V$ lies in at most t sets in $\mathcal{S}$, there is an efficient randomized algorithm that finds an $O(\sqrt{t \log n})$ discrepancy coloring with high probability.*

Our result also extends to the Komlós setting with some minor modifications.

Theorem 2. *Given an $m \times n$ matrix A with all columns of ℓ_2-norm at most 1, there is an efficient randomized algorithm that finds $x \in \{-1, 1\}^n$ such that $\|Ax\|_\infty = O(\sqrt{\log n})$ with high probability.*

Our algorithm gives a new constructive proof of Banaszczyk's result for the Beck-Fiala and Komlós setting. While Theorem 2 implies Theorem 1, for better clarity we first present the algorithm for the Beck-Fiala problem in Sections II and III and then discuss the extension to Theorem 2 in Section IV.

B. High-level Overview

The algorithm has a similar structure to the previous random walk based approaches [6], [8], [10]. It starts with the coloring $x_0 = 0^n$ at time 0, and at each time step k, updates the color of element i by adding a small increment to its coloring at time $k-1$, i.e. $x_k(i) = x_{k-1}(i) + \Delta x_k(i)$. If a variable reaches -1 or 1 it is frozen, and its value is not updated any more. The increment is determined by solving an appropriate SDP and projecting the resulting vectors in a random direction.

However, all the previous approaches get stuck at the $O(t^{1/2} \log n)$ barrier, and it is instructive to understand why this happens before we present our algorithm.

The $O(t^{1/2} \log n)$ barrier: Roughly speaking, the execution of the previous algorithms can be divided into $O(\log n)$ phases (either implicitly or explicitly), where in each phase about half the variables get frozen and each set incurs an expected discrepancy of $O(t^{1/2})$. This gives an overall discrepancy bound of $O(t^{1/2} \log n)$.

Intuitively however, for a fixed set S, one should expect an $O(t^{1/2})$ discrepancy over *all* the phases for the following reason. Assume that all sets are of size $O(t)$. This can be ensured using a standard linear algebraic argument to ensure that sets incur zero discrepancy as long as they have at least $2t$ uncolored elements[3]. After i phases of partial coloring, one would expect that S has about 2^{-i} fraction of its elements left uncolored, and hence it should incur about $O((2^{-i}t)^{1/2})$ discrepancy in the next phase, giving a total discrepancy of $O(\sum_i 2^{-i/2} t^{1/2}) = O(t^{1/2})$.

However, the problem is that the size of sets may not evolve in this ideal manner, as the partial coloring phase does not give us a fine-grained control over how the elements of each set get colored. For example, even though half of the variables (globally) get colored during a phase, it is possible that half the sets get almost completely colored, while the other half only get $t/\log n$ of their elements colored (while still incurring an $\Omega(t^{1/2})$ discrepancy). This imbalance between the discrepancy incurred and "progress" made for *each* set is the fundamental barrier in overcoming the $O(t^{1/2} \log n)$ bound.

Our approach: The key idea of our algorithm is to ensure that during the coloring updates the *squared* discrepancy we add to a set is proportional to the "progress" elements of that set make towards geting colored. More formally, the updates $\Delta x_k(i)$ that we choose at time k satisfy the following properties:

1) *Zero Discrepancy for large sets:* If a set S has more than at unfrozen (alive) elements at time k, for some constant a, we ensure that $\sum_{i \in S} \Delta x_k(i) = 0$. This is similar to the previous approaches and allows us to not worry about the discrepancy of a set until its size falls below at.
2) *Proportional Discrepancy Property:* This is the key new property and (roughly speaking) ensures that the squared discrepancy added to a set is proportional to the "energy" injected into the set. That is,

$$\left(\sum_{i \in S} \Delta x_k(i) \right)^2 \leq 2 \left(\sum_{i \in S} \Delta x_k(i)^2 \right).$$

Note that the left hand side is the square of the discrepancy increment for set S, and the right hand side is the sum of squares of the increments of the elements of S.

Given a coloring x_k, let us define the energy of set S at time k as $\sum_{i \in S} x_k(i)^2$. Clearly, the energy of a set can never exceed its size $|S|$. As we can assume that $|S| = O(t)$ (by the Zero Discrepancy Property above), this property suggests that if the total energy injected $(\sum_k (\sum_{i \in S} \Delta x_k(i)^2))$ into S was comparable to its final energy (which is $O(t)$)

[3]The reader may observe that if all sets were of size $O(t)$, a simple application of the Lovász Local Lemma already gives an $O(\sqrt{t \log t})$ discrepancy coloring, so this should an imply an $O(\sqrt{t \log t})$ discrepancy in general. However, the problem is that the Lovász Local Lemma does not combine with the linear algebraic argument.

and the increments were mean 0 random variables, the squared discrepancy should be $O(t)$.

3) *Approximate Orthogonality Constraints* to relate the injected energy to actual energy: One big problem with the above idea is that the total injected energy into a constraint may be unrelated to its final energy. For example, even for a single variable i if the coloring $x_k(i)$ "fluctuates" a lot around 0 over time, the injected energy $\sum_k \Delta x_k(i)^2$ could be arbitrarily large, while the final energy for i is at most 1. For general sets S, other problems can arise beyond just fluctuations due to correlations between the updates of different elements of S.
To fix this we use the following idea. Suppose we could ensure that for each set S the coloring update at time k was orthogonal to the coloring at time $k-1$, i.e. $\sum_{i \in S} x_{k-1}(i)\Delta x_k(i) = 0$. Then, by Pythagoras theorem, the increase in energy of S would satisfy
$$\begin{aligned} & \sum_{i \in S} x_k(i)^2 - \sum_{i \in S} x_{k-1}(i)^2 \\ = & \sum_{i \in S} \left((x_{k-1}(i) + \Delta x_k(i))^2 - x_{k-1}(i)^2\right) \\ = & 2\sum_{i \in S} x_{k-1}(i)\Delta x_k(i) + \sum_{i \in S} \Delta x_k(i)^2 \\ = & \sum_{i \in S} \Delta x_k(i)^2 \qquad (1) \end{aligned}$$
where the last equality follows from the orthogonality constraint. As the expression in (1) is the injected energy at time k, this would precisely make the total injected equal to the final energy as desired. However, we cannot add such constraints directly for each small set as there might be too many of them. So the idea is to add a weaker version of these orthogonality constraints, where we only require that
$$\left(\sum_{i \in S} x_{k-1}(i)\Delta x_k(i)\right)^2 \leq 2\left(\sum_{i \in S} \Delta x_k(i)^2\right)$$
and show that these suffice for our purpose.

4) *Sufficient Progress Property*: Of course, all the properties above can be trivially satisfied by setting $\Delta x_k(i) = 0$ for each i. So the final step is to ensure that a non-trivial update exists. To this end, we show that there exist updates with the (unnormalized) sum $\sum_i \Delta x_k(i)^2 = \Omega(A_k)$, where A_k is the number of alive variables at time k.
For this purpose, we write an SDP that captures the above constraints and use duality to show the existence of a large feasible solution.

A weaker version of these properties was used in the unpublished manuscript [24] to get a more size-sensitive discrepancy bound for each set, but it still only achieved an $O(t^{1/2}\log n)$ discrepancy in the worst case.

We now describe the algorithm and the SDP we use in Section II. The analysis consists of two main parts. In Section III-A we show the sufficient progress property mentioned above, and in Section III-B we show how this gives an overall discrepancy bound of $O((t \log n)^{1/2})$.

II. Algorithm for the Beck-Fiala Problem

We will index time by k. Let $x_k \in [-1, 1]^n$ denote the coloring at the *end* of time step k. During the algorithm, variables which get set to at least $(1 - 1/n)$ in absolute value are called frozen and their values are not changed anymore. The remaining variables are called alive. We denote by A_k the set of alive variables at the *beginning* of time step k. Initially all variables are alive. Let $\gamma = 1/(n^2 \log n)$, $T = (12/\gamma^2)\log n$ and $a = 6$.

We will call a set $S \in \mathcal{S}$ *big* at time k if it has at least at variables alive at time k, i.e. $|S \cap A_k| \geq at$ and *small* otherwise. We will use $\mathcal{B}_k$ to denote the collection of big sets at time k and $\mathcal{L}_k$ to denote the collection of small (little) sets.

Algorithm:

1) Initialize $x_0(i) = 0$ for all $i \in [n]$ and $A_1 = \{1, 2, ..., n\}$.
2) For each time step $k = 1, 2, \ldots, T$ repeat the following:
   a) Find a solution to the following semidefinite optimization problem:
   $$\text{Maximize} \sum_{i \in A_k} \|u_i\|_2^2 \qquad \text{subject to}$$
   $$\|\sum_{i \in S \cap A_k} u_i\|_2^2 = 0 \quad \forall S \in \mathcal{B}_k \qquad (2)$$
   $$\|\sum_{i \in S \cap A_k} u_i\|_2^2 \leq 2\sum_{i \in S \cap A_k} \|u_i\|_2^2 \quad \forall S \in \mathcal{L}_k \qquad (3)$$
   $$\|\sum_{i \in S \cap A_k} x_{k-1}(i)u_i\|_2^2 \leq 2\sum_{i \in S \cap A_k} \|u_i\|_2^2 \quad \forall S \in \mathcal{L}_k \qquad (4)$$
   $$\|u_i\|_2^2 \leq 1 \quad \forall i \in A_k$$
   b) Let $r_k \in \mathbb{R}^n$ be a random ± 1 vector, obtained by setting each coordinate $r_k(i)$ independently to -1 or 1 with probability $1/2$.
   For each $i \in A_k$, update $x_k(i) = x_{k-1}(i) + \gamma\langle r_k, u_i\rangle$. For each $i \notin A_k$, set $x_k(i) = x_{k-1}(i)$.
   c) Initialize $A_{k+1} = A_k$.
   For each i, if $|x_k(i)| \geq 1 - 1/n$, update $A_{k+1} = A_{k+1} \setminus \{i\}$.

3) Generate the final coloring as follows. For the frozen elements $i \notin A_{T+1}$, set $x_T(i) = 1$ if $x_T(i) \geq 1 - 1/n$ and $x_T(i) = -1$ otherwise. For the alive elements $i \in A_{T+1}$, set them arbitrarily to ± 1.

Note that the SDP at time k uses the vectors u_i to generate the update $\Delta x_k(i)$ by projecting u_i to the random vector r_k and scaling this by γ. If we think of u_i as one dimensional vectors (so $\Delta x_k(i) = \gamma r u_i$ where r is randomly ± 1), constraints (2) will ensure that a set incurs zero discrepancy as long as it is big. Constraints (3) require the updates to satisfy the proportional discrepancy property mentioned earlier. Constraints (4) require the updates to satisfy the approximate orthogonality property mentioned earlier.

III. Analysis

We begin with some simple observations.

Lemma 3. *For any vector $u \in \mathbb{R}^n$ and a random vector $r \in \{\pm 1\}^n$, $\mathbb{E}[\langle r, u\rangle^2] = \|u\|_2^2$ and $|\langle r, u\rangle| \leq \sqrt{n}\|u\|_2$.*

Proof: Writing u in terms of its coordinates $u = (u(1), \ldots, u(n))$,

$$\mathbb{E}[\langle r, u\rangle^2] = \mathbb{E}[(\sum_i r(i)u(i))^2]$$
$$= \sum_{i,j} \mathbb{E}[r(i)r(j)]u(i)u(j) = \|u\|_2^2$$

where the last equality uses that $\mathbb{E}[r(i)r(j)] = 0$ for $i \neq j$ and $\mathbb{E}[r(i)^2] = 1$.

The second part follows by Cauchy-Schwarz inequality, as $|\langle r, u\rangle| \leq \|r\|_2\|u\|_2 = \sqrt{n}\|u\|_2$. ■

This implies the following.

Observation 4. *The rounding of frozen elements in step 3 of the algorithm affects the discrepancy of any set by at most $n \cdot (1/n) = 1$. So we can ignore this rounding error. Moreover, as $\|u_i\|_2 \leq 1$, $|\gamma\langle r, u_i\rangle| \leq \gamma\sqrt{n}\|u_i\|_2 \leq 1/n$, which implies that no $x_k(i)$ goes out of the range $[-1, 1]$ during any step of the algorithm.*

The rest of the analysis is divided into three parts. In Section III-A, we show that the SDP is feasible and has value at least $|A_k|/3$ at each time step k. In Section III-B, we use the properties of the SDP to show that each set in $\mathcal{S}$ has discrepancy $O((t \log n)^{1/2})$ after T steps with high probability. Finally, in Section III-C we show that there are no alive elements after T steps with high probability. Together these will imply Theorem 1.

A. SDP is feasible and has value $\Omega(|A_k|)$

To show that the SDP has value at least $|A_k|/3$ at any time step k, we will consider the dual and show that no solution with objective value less than $|A_k|/3$ can be feasible. By strong duality, this suffices as if the optimum (primal) SDP solution was less than $|A_k|/3$, there would also be some feasible dual solution with that value (provided Slater's conditions are satisfied).

It might be useful to point out here that the feasibility of our SDP is incomparable to the main result in [21]; we can ensure a zero discrepancy to a few rows, which was not possible in the approach used in [21] but we can only ensure a partial colouring ($\sum_i \|u_i\|_2^2 \geq |A_k|/3$), whereas the SDP in [21] was feasible with the stronger constraint $\|u_i\|_2 = 1$ for all i.

To make it easier to write the dual, we rewrite the SDP in the following matrix notation by setting X to be the Gram matrix of vectors corresponding to alive elements i.e. $X_{ij} = \langle u_i, u_j\rangle$ for $i, j \in A_k$.

$$\begin{array}{rcll} \text{Maximize } I \bullet X & & \text{subject to} & \\ v_S v_S^T \bullet X & = & 0 & \text{for each } S \in \mathcal{B}_k \\ (v_S v_S^T - 2I_S) \bullet X & \leq & 0 & \text{for each } S \in \mathcal{L}_k \\ (x_S x_S^T - 2I_S) \bullet X & \leq & 0 & \text{for each } S \in \mathcal{L}_k \\ (e_i e_i^T) \bullet X & \leq & 1 & \forall i \in A_k \\ X & \succeq & 0 & \end{array}$$

Here v_S is the indicator vector of set $S \cap A_k$, x_S is the vector with i^{th} entry equal to $x_{k-1}(i)$ if $i \in S \cap A_k$ and 0 otherwise and I_S is the identity matrix restricted to set $S \cap A_k$, i.e. $(I_S)_{ii} = 1$ if $i \in S \cap A_k$ and 0 otherwise. $\bullet$ denotes the usual inner product on matrices $A \bullet B = \mathrm{Tr}(A^T B) = \sum_{ij} A_{ij}B_{ij}$.

We can write the dual of the above SDP (for reference, see [25]), which is given by:

$$\text{Minimize } \sum_{i \in A_k} b_i \quad \text{subject to}$$

$$\sum_{i \in A_k} b_i e_i e_i^T + \sum_{S \in \mathcal{B}_k} \alpha_S v_S v_S^T + \sum_{S \in \mathcal{L}_k} \left(\beta_S(v_S v_S^T - 2I_S) + \beta_S^x(x_S x_S^T - 2I_S)\right) \succeq I \quad (5)$$

$$b_i \geq 0 \qquad \forall i \in A_k \quad (6)$$

$$\alpha_S \in \mathbb{R} \qquad \forall S \in \mathcal{B}_k \quad (7)$$

$$\beta_S, \beta_S^x \geq 0 \qquad \forall S \in \mathcal{L}_k \quad (8)$$

Here $A \succeq B$ denotes that the matrix $A - B$ is positive semi-definite. To show strong duality we use the following result.

Theorem 5 (Theorem 4.7.1, [25])**.** *If the primal program (P) is feasible, has a finite optimum value η and has an interior point $\tilde{x}$, then the dual program (D) is also feasible and has the same finite optimum value η.*

Lemma 6. *The SDP described above is feasible and has value equal to its dual program.*

Proof: We apply Theorem 5, with P equal to the dual of the SDP. This would suffice as the dual D of P

is our SDP.

We claim that $b_i = 1+\epsilon$ for $\epsilon > 0$ for all $i \in A_k$, $\alpha_S = 0$ for all $S \in \mathcal{B}_k$ and $\beta_S = \beta_S^x = \epsilon/(8n^2)$ for all $S \in \mathcal{L}_k$ is a feasible interior point for P. Clearly, this solution is strictly in the interior of the constraints (6)-(8). That (5) is satisfied and has slack in every direction follows as the the number of sets S can be at most $t|A_k| \le tn \le n^2$, and that for any vector v, vv^T is a rank one PSD matrix with eigenvalue $\|v\|_2^2 \le n$, and thus all eigenvalues of $v_S v_S^T - 2I_S$ and $x_S x_S^T - 2I_S$ lie in the range $[-2, n]$.

As this point has objective value at most $(1+\epsilon)n$ and since b_i are non-negative, P has a finite optimum value. ■

We wish to show that any feasible solution to the dual must satisfy $\sum_i b_i \ge |A_k|/3$. To do this, we will show that there is a large subspace W of dimension at least $|A_k|/3$ where the operator

$$\sum_{S \in \mathcal{B}_k} \alpha_S v_S v_S^T + \sum_{S \in \mathcal{L}_k} \left(\beta_S (v_S v_S^T - 2I_S) + \beta_S^x (x_S x_S^T - 2I_S) \right)$$

is negative semidefinite. This would imply that to satisfy (5), b_i's have to be quite large on average. We first give two general lemmas.

Lemma 7. *Given an $h \times n$ matrix M with columns $z_1, z_2, \ldots, z_n$. If $\|z_i\|_2 \le 1$ for all $i \in [n]$, then there exists a subspace W of $\mathbb{R}^n$ satisfying:*

i) $dim(W) \ge \frac{n}{2}$, and
ii) $\forall y \in W$, $\|My\|_2^2 \le 2\|y\|_2^2$

Proof: Let the singular value decomposition of M be given by $M = \sum_{i=1}^n \sigma_i p_i q_i^T$, where $0 \le \sigma_1 \le \cdots \le \sigma_n$ are the singular values of M and $\{p_i : i \in [n]\}, \{q_i : i \in [n]\}$ are two sets of orthonormal vectors (if $h < n$, some p_i's and the corresponding σ_i's will be zero). Then,

$$\sum_{i=1}^n \sigma_i^2 = \mathrm{Tr}[\sum_{i=1}^n \sigma_i^2 q_i q_i^T] = \mathrm{Tr}[M^T M] = \sum_{i=1}^n \|z_i\|_2^2 \le n$$

So at least $\lceil \frac{n}{2} \rceil$ of the squared singular values σ_i^2s have value at most 2, and thus $\sigma_1 \le \cdots \le \sigma_{\lceil \frac{n}{2} \rceil} \le \sqrt{2}$. Let $W = \mathrm{span}\{q_1, \ldots, q_{\lceil \frac{n}{2} \rceil}\}$. For $y \in W$,

$$\begin{aligned}
\|My\|_2^2 &= \|\sum_{i=1}^n \sigma_i p_i q_i^T y\|_2^2 = \|\sum_{i=1}^{\lceil \frac{n}{2} \rceil} \sigma_i p_i q_i^T y\|_2^2 \\
&\le \sum_{i=1}^{\lceil \frac{n}{2} \rceil} \sigma_i^2 (q_i^T y)^2 \quad \text{(since } p_i \text{ are orthonormal)} \\
&\le 2\sum_{i=1}^{\lceil \frac{n}{2} \rceil} (q_i^T y)^2 \\
&= 2\|y\|_2^2 \qquad \text{(since } q_i \text{ are orthonormal)}
\end{aligned}$$

This implies the following result. ■

Theorem 8. *Let $\mathcal{V}$ be any finite collection of vectors $v_1, \ldots, v_h$ in $\mathbb{R}^n$, and for each $v \in \mathcal{V}$, there is some non-negative multiplier $\beta_v \ge 0$. Consider the operator*

$$B = \sum_{v \in \mathcal{V}} \beta_v \left(vv^T - 2\sum_{i=1}^n \langle v, e_i \rangle^2 e_i e_i^T \right)$$

where e_i are the standard basis of $\mathbb{R}^n$. Then there exists a subspace W of dimension at least $n/2$ such that $\langle y, By \rangle \le 0$ for every $y \in W$, or equivalently $y^T By \le 0$ for every $y \in W$.

Proof: Let v_i denote $\langle v, e_i \rangle$. We can express $y^T By$ as

$$\begin{aligned}
&B \bullet yy^T \\
&= \sum_v \beta_v \left(vv^T \bullet yy^T - 2(\sum_i v_i^2 e_i e_i^T) \bullet yy^T \right) \\
&= \sum_v \beta_v \left((\sum_i v_i y_i)^2 - 2\sum_i v_i^2 y_i^2 \right)
\end{aligned}$$

Construct a matrix M with rows indexed by v for each $v \in \mathcal{V}$ and columns indexed by $i \in [n]$. The entries of M are given by $M_{v,i} = \beta_v^{1/2} v_i$. Then, we can write

$$\sum_v \beta_v \left(\sum_i v_i y_i \right)^2 = \|My\|_2^2.$$

For each $i \in [n]$, define $\beta_i^2 = \sum_v \beta_v v_i^2$ as the squared ℓ_2-norm of column i of M, and let D be an $n \times n$ diagonal matrix with entries $D_{ii} = \beta_i$. Then,

$$\begin{aligned}
&\sum_v \beta_v ((\sum_i v_i y_i)^2 - 2\sum_i v_i^2 y_i^2) \\
&= \|My\|_2^2 - 2\|Dy\|_2^2 \qquad (9)
\end{aligned}$$

Let $N \subseteq [n]$ be the set of coordinates with $\beta_i > 0$. We claim that it suffices to focus on the coordinates in N. Let us first observe that if $i \notin N$, i.e. $\beta_i^2 = 0$, then we can set y_i arbitrarily as (9) is unaffected. As the directions e_i for $i \in N$ are orthogonal to the directions in $[n] \setminus N$, it suffices to show that there is a $|N|/2$ dimensional subspace W in $\mathrm{span}\{e_i : i \in N\}$ such that $\|My\|_2^2 - 2\|Dy\|_2^2 \le 0$ for each $y \in W$. The overall subspace we desire is simply $W \oplus \mathrm{span}\{e_i : i \in [n] \setminus N\}$ which has dimension $|N|/2 + (n - |N|) \ge n/2$.

So, let us assume that $N = [n]$ (or equivalently restrict M and D to columns in N), which gives us that $\beta_i > 0$ for all $i \in N$ and hence that D is invertible.

Let $M' = MD^{-1}$. The squared ℓ_2-norm of each column in M' is $\sum_v \beta_v v_i^2 / D_{ii}^2$ which equals 1, and by Lemma 7, there is a subspace W' of dimension at least

$|N|/2$ such that $\|M'y'\|_2^2 \le 2\|y'\|_2^2$ for each $y' \in W'$. Setting $y = D^{-1}y'$ gives

$$\|My\|_2^2 = \|M'y'\|_2^2 \le 2\|y'\|_2^2 = 2\|Dy\|_2^2,$$

and thus $W = \{D^{-1}y' : y' \in W'\}$ gives the desired subspace since $\dim(W) = \dim(W')$. ■

Going back to the dual SDP, this gives the following.

Lemma 9. *Let* $B_k = \sum_{S\in\mathcal{L}_k} \left(\beta_S(v_S v_S^T - 2I_S) + \beta_S^x(x_S x_S^T - 2I_S)\right)$. *Then, there exists a subspace* $W \subseteq \mathbb{R}^{|A_k|}$ *of dimension at least* $|A_k|/2$ *such that for all* $y \in W$, $y^T B_k y \le 0$.

Proof: We apply Theorem 8 with vectors v as v_S and x_S for each small set $S \in \mathcal{L}_k$, with the multipliers β_S and β_S^x. Then,

$$\begin{aligned} B &= \sum_{S\in\mathcal{L}_k} [\beta_S(v_S v_S^T - 2\sum_{i\in A_k} \langle v_S, e_i\rangle^2 e_i e_i^T) + \\ &\beta_S^x(x_S x_S^T - 2\sum_{i\in A_k} \langle x_S, e_i\rangle^2 e_i e_i^T)] \\ &= \sum_{S\in\mathcal{L}_k} [\beta_S(v_S v_S^T - 2I_S) + \\ &\beta_S^x(x_S x_S^T - 2\sum_{i\in S\cap A_k} x_{k-1}(i)^2 e_i e_i^T)] \\ &\succeq \sum_{S\in\mathcal{L}_k} \left(\beta_S(v_S v_S^T - 2I_S) + \beta_S^x(x_S x_S^T - 2I_S)\right) \\ &= B_k \end{aligned}$$

Here we use that v_S is the indicator vector for set $S \cap A_K$ with entries $\langle v_S, e_i\rangle = 1$ iff $i \in S \cap A_k$ and thus, $\sum_{i\in A_k} \langle v_S, e_i\rangle^2 e_i e_i^T = I_S$. Similarly for the vectors x_S, $\langle x_S, e_i\rangle = x_{k-1}(i)$ for $i \in S \cap A_k$ and 0 otherwise. The last step uses that $x_{k-1}(i)^2 \le 1$ and thus

$$-2\sum_{i\in S\cap A_k} x_{k-1}(i)^2 e_i e_i^T \succeq -2I_S.$$

By Theorem 8, there is a subspace W with $\dim(W) \ge |A_k|/2$ such that $y^T B y \le 0$ for each $y \in W$. As $B \succeq B_k$, it also holds that $y^T B_k y \le 0$ for each $y \in W$. ■

We now come to the main theorem of this subsection.

Theorem 10. *At time step* k*, the dual program has value at least* $|A_k|/3$.

Proof: As element i in A_k appears in at most t sets, the number of big sets $|\mathcal{B}_k|$ at time step k is at most $|A_k|t/at = |A_k|/a$. Let W_1 be the subspace orthogonal to $C = \text{span}\{v_S : S \in \mathcal{B}_k\}$. Clearly, $\dim(C) \le |\mathcal{B}_k| \le |A_k|/a$.

Let W_0 be the subspace guaranteed by Lemma 9 for matrix B_k such that $\dim(W_0) \ge |A_k|/2$ and for all $y \in W_0$, $y^T B_k y \le 0$. Define the subspace $W = W_1 \cap W_0$. Then,

$$\begin{aligned} \dim(W) &\ge \dim(W_0) - \dim(C) \\ &\ge |A_k|/2 - |A_k|/a \\ &= |A_k|/3 \end{aligned}$$

Let P_W be the projection operator on the subspace W. Projecting the dual constraint (5) on to W, we get

$$P_W\left(\sum_{i\in A_k} b_i e_i e_i^T + \sum_{S\in\mathcal{B}_k} \alpha_S v_S v_S^T + B_k\right) \succeq P_W$$

By linearity of P_W and as $P_W(v_S v_S^T) = 0$ for each $S \in \mathcal{B}_k$, this implies

$$P_W\left(\sum_{i\in A_k} b_i e_i e_i^T\right) + P_W(B_k) \succeq P_W$$

Taking trace on both the sides and noting that $\text{Tr}[P_W(B_k)] \le 0$ since $y^T B_k y \le 0$ for all $y \in W$, we get

$$\text{Tr}\left[P_W\left(\sum_{i\in A_k} b_i e_i e_i^T\right)\right] \ge \text{Tr}[P_W] = \dim(W)$$

As $b_i \ge 0$ for all $i \in A_k$, the operator $\sum_{i\in A_k} b_i e_i e_i^T$ is positive semi-definite. As taking the projection of a positive semi-definite operator can only decrease its trace, we can lower bound the dual objective as

$$\begin{aligned} \sum_{i\in A_k} b_i &= \text{Tr}\left[\sum_{i\in A_k} b_i e_i e_i^T\right] \\ &\ge \text{Tr}\left[P_W\left(\sum_{i\in A_k} b_i e_i e_i^T\right)\right] \\ &\ge \text{Tr}[P_W] = \dim(W) \ge |A_k|/3 \end{aligned}$$

which completes the proof. ■

B. Bounding the discrepancy

Let $D_S(k)$ denote the signed discrepancy of set $S \in \mathcal{S}$ at end of time step k i.e. $D_S(k) = \sum_{i\in S} x_k(i)$.
We now show the following key result.

Theorem 11. *Fix a set* $S \in \mathcal{S}$. *Then, for any* $\lambda \ge 0$, *the discrepancy of* S *at time step* T *satisfies*

$$\Pr\left[|D_S(T)| \ge \lambda\sqrt{t}\right] \le 8\exp(-\lambda^2/(100a)).$$

Setting $\lambda = O(\log^{1/2} n)$ would imply that with high probability every set has discrepancy $O((t\log n)^{1/2})$ at time T.

Among other things, the proof of Theorem 11 will use a powerful concentration inequality for martingales due to Freedman that we describe below.

Martingales and Freedman's inequality: Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables on some probability space, and let Y_k be a function of $X_1, \ldots, X_k$. The sequence $Y_0, Y_1, Y_2, \ldots, Y_n$ is called a martingale with respect to the sequence $X_1, \ldots, X_n$ if for all $k \in [n]$, $\mathbb{E}[|Y_k|]$ is finite and $\mathbb{E}[Y_k | X_1, X_2, ..., X_{k-1}] = Y_{k-1}$. We will use $\mathbb{E}_{k-1}[Z]$ to denote $\mathbb{E}[Z|X_1, X_2, ..., X_{k-1}]$ where Z is any random variable.

Theorem 12 (Freedman [26])**.** *Let $Y_0, \ldots, Y_n$ be a martingale with respect to $X_1, \ldots, X_n$ such that $|Y_k - Y_{k-1}| \leq M$ for all k, and let*

$$W_k = \sum_{j=1}^{k} \mathbb{E}_{j-1}[(Y_j - Y_{j-1})^2] = \sum_{j=1}^{k} \mathit{Var}[Y_j | X_1, \ldots, X_{j-1}].$$

Then for all $\lambda \geq 0$ and $\sigma^2 \geq 0$, we have

$$\Pr[|Y_n - Y_0| \geq \lambda \text{ and } W_n \leq \sigma^2] \leq 2\exp\left(-\frac{\lambda^2}{2(\sigma^2 + M\lambda/3)}\right).$$

Observe crucially that the above inequality is much more powerful than the related Azuma-Hoeffding or Bernstein's inequality. In particular, the term W_n is the variance encountered by the martingale on the particular sample path it took, as opposed to a worst case bound on the variance over all possible paths.

Simple Observations: We now get back to the proof of Theorem 11 and begin with a few simple observations.

Fix a set $S \in \mathcal{S}$. Let the vector solution returned by the SDP at time k be given by vectors u_i^k for $i \in [n]$ where we take $u_i^k = 0$ if $i \notin A_k$. We say that S becomes *active* at time k if k is the first time step when $|S \cap A_k| \leq at$.

Observation 13. *Before a set $S \in \mathcal{S}$ is active, it incurs zero discrepancy.*

Proof: Suppose S becomes active at time k_S. Then, $D_S(k_S - 1) = \sum_{k=1}^{k_S-1} \gamma\langle r_k, \sum_{i\in S\cap A_k} u_i^k\rangle = 0$, since by SDP constraint (2), $\sum_{i\in S\cap A_k} u_i^k = 0$ for $k < k_S$. ■

Observation 14. *As a set has no more than at alive variables when it becomes active, Observation 13 implies that the maximum discrepancy any set can have is $2at$, which gives Theorem 11 for $\lambda > 2at^{1/2}$. So henceforth we can assume that $\lambda \leq 2at^{1/2}$.*

Define the *energy* of set S at end of time step k as $E_S(k) = \sum_{i\in S} x_k(i)^2$ and change in energy of S at time step k as $\Delta_k E_S = E_S(k) - E_S(k-1)$. Then,

$$\begin{aligned} \Delta_k E_S &= \sum_{i\in S} x_k(i)^2 - \sum_{i\in S} x_{k-1}(i)^2 \\ &= \sum_{i\in S} \left((x_{k-1}(i) + \gamma\langle r_k, u_i^k\rangle)^2 - x_{k-1}(i)^2\right) \\ &= \gamma^2 \sum_{i\in S} \langle r_k, u_i^k\rangle^2 + 2\gamma\langle r_k, \sum_{i\in S} x_{k-1}(i) u_i^k\rangle \end{aligned} \tag{10}$$

The following is a simple but crucial observation.

Observation 15. *Once a set $S \in \mathcal{S}$ becomes active, its energy can increase overall by at most at.*

Proof: When S becomes active, it has at most at alive variables. Moreover, a frozen variable is never updated by the algorithm and can never become alive again. As the energy of a single variable is bounded by 1, the energy of S can increase by at most at after it becomes active. ■

Remark: Note that the energy of a set S does not necessarily increase monotonically over time. It evolves randomly and can also decrease. So, even though the overall increase is at most at, the total energy "injected" $\sum_{k\geq k_S} |\Delta_k E_S|$ can be arbitrarily larger than at. Here k_S denotes the time when S becomes active.

By Observation 13, we only need to bound the discrepancy of S after it becomes active. For notational convenience, let us call the time S becomes active as time 0. So, $D_S(k)$ and $E_S(k)$ will be the signed discrepancy and energy of S respectively, k time steps after it becomes active.

Observation 16. *After S becomes active, $D_S(k)$ behaves like a martingale with variance of increment at time step k bounded by*

$$\mathbb{E}_{k-1}[(D_S(k) - D_S(k-1))^2] \leq 2\gamma^2 \sum_{i\in S\cap A_k} \|u_i^k\|_2^2$$

Proof: The discrepancy update of S at time k is $\gamma\langle r_k, \sum_{i\in S\cap A_k} u_i^k\rangle$. This has expectation 0 (averaging of r_k) and by Lemma 3 this variance is exactly $\gamma^2 \|\sum_{i\in S\cap A_k} u_i^k\|_2^2$, which by SDP constraint (3) is upper bounded by $2\gamma^2 \sum_{i\in S\cap A_k} \|u_i^k\|_2^2$. ■

Proof of Theorem 11: The plan of the proof is the following. Freedman's inequality allows us to bound the discrepancy at time T as a function of the variance $\sum_{k=1}^{T} \mathbb{E}_{k-1}[(D_S(k) - D_S(k-1))^2]$ which is at most $2\gamma^2 \sum_{k=1}^{T} \sum_{i\in S\cap A_k} \|u_i^k\|_2^2$ by Observation 16. As we will see, this term is expected total energy injected into S.

As the overall energy increase of S can be at most at (Observation 15), it would suffice to show that the total injected energy into S is comparable to at. To do this, we will use the approximate orthogonality constraints (4) and apply Freedman's inequality again to show that the injected energy is tightly concentrated around the energy increase. We now give the details.

Recall that by (10), the energy change at time k is a random variable given by

$$\Delta_k E_S = \gamma^2 \sum_{i \in S} \langle r_k, u_i^k \rangle^2 + 2\gamma \langle r_k, \sum_{i \in S} x_{k-1}(i) u_i^k \rangle$$

Denote the first term above as

$$\Delta_k Q_S = \gamma^2 \sum_{i \in S} \langle r_k, u_i^k \rangle^2$$

which we will call the change in quadratic energy of S at time step k and let $Q_S(k) = \sum_{j=1}^{k} \Delta_j Q_S$, the total quadratic energy of S till time k.

Similarly, denote the second term as

$$\Delta_k L_S = 2\gamma \langle r_k, \sum_{i \in S} x_{k-1}(i) u_i^k \rangle$$

which we will call the change in linear energy of S at time step k, and let $L_S(k) = \sum_{j=1}^{k} \Delta_j L_S$, the total linear energy of S till time k. The energy of S at time k is given by $E_S(k) = Q_S(k) + L_S(k)$.

Define $Q'_S(k)$ as

$$Q'_S(k) = \sum_{j=1}^{k} \mathbb{E}_{j-1}[\Delta_j E_S] = \sum_{j=1}^{k} \mathbb{E}_{j-1}[\Delta_j Q_S].$$

By lemma 3,

$$Q'_S(k) = \sum_{j=1}^{k} \gamma^2 \sum_{i \in S} \|u_i^j\|_2^2$$

We are now ready to prove the tail bound on discrepancy. The probability that discrepancy of S at time T exceeds $\lambda\sqrt{t}$ can be written as

$$\begin{aligned} &\Pr\left[|D_S(T)| \geq \lambda\sqrt{t}\right] \\ &\leq \Pr\left[|D_S(T)| \geq \lambda\sqrt{t}, Q'_S(T) \leq 16at\right] + \\ &\Pr\left[Q'_S(T) > 16at\right] \end{aligned} \quad (11)$$

We now bound each of the terms in (11) separately.

Bounding the first term. Recall that $D_S(k)$ is a martingale. To apply Freedman's inequality(Theorem 12) we bound M and W_k as follows. By Lemma 3,

$$\begin{aligned} M &\leq |D_S(k) - D_S(k-1)| = |\gamma \langle r_k, \sum_{i \in S} u_i^k \rangle| \\ &\leq \gamma\sqrt{n} \|\sum_{i \in S} u_i^k\|_2 \leq \gamma n^{3/2} \end{aligned}$$

Similarly from Lemma 3 and the SDP constraint (3),

$$\begin{aligned} W_k &= \sum_{j=1}^{k} \mathbb{E}_{j-1}[(D_S(j) - D_S(j-1))^2] \\ &= \sum_{j=1}^{k} \mathbb{E}_{j-1}[\gamma^2 \langle r_j, \sum_{i \in S} u_i^j \rangle^2] \\ &= \sum_{j=1}^{k} \gamma^2 \|\sum_{i \in S} u_i^j\|_2^2 \\ &\leq \sum_{j=1}^{k} 2\gamma^2 \sum_{i \in S} \|u_i^j\|_2^2 = 2Q'_S(k) \end{aligned}$$

Freedman's inequality now gives,

$$\begin{aligned} &\Pr\left[|D_S(T)| \geq \lambda\sqrt{t} \text{ and } Q'_S(T) \leq 16at\right] \\ \leq\ &\Pr\left[|D_S(T)| \geq \lambda\sqrt{t} \text{ and } W_T \leq 32at\right] \\ \leq\ &2\exp\left(\frac{-\lambda^2 t}{2[32at + \gamma n^{3/2}\lambda\sqrt{t}/3]}\right) \\ \leq\ &2\exp\left(\frac{-\lambda^2}{100a}\right) \qquad (\text{using } \lambda \leq 2a\sqrt{t}) \end{aligned} \quad (12)$$

Bounding the second term. We can write

$$\begin{aligned} &\Pr\left[Q'_S(T) > 16at\right] \\ &= \sum_{j=0}^{\infty} \Pr\left[2^{j+4}at < Q'_S(T) \leq 2^{j+5}at\right] \\ &\leq \Pr\left[Q_S(T) \leq Q'_S(T) - 8at\right] + \\ &\sum_{j=0}^{\infty} \Pr\left[Q'_S(T) \leq 2^{j+5}at, Q_S(T) \geq 2^{j+3}at\right] \end{aligned} \quad (13)$$

The inequality above holds as the event $\{Q'_S(T) > 16at\}$ is contained in the union of the two events in (13).

As the energy $E_S(T)$ of S cannot exceed at, we have $E_S(T) = L_S(T) + Q_S(T) \leq at$. Thus, $Q_S(T) \geq 2^{j+3}at$ implies $L_S(T) \leq at - 2^{j+3}at \leq -7 \cdot 2^j at$, giving

$$\begin{aligned} &\Pr[Q'_S(T) > 16at] \leq \Pr[Q_S(T) \leq Q'_S(T) - 8at] + \\ &\sum_{j=0}^{\infty} \Pr[L_S(T) \leq -7 \cdot 2^j at, Q'_S(T) \leq 2^{j+5}at] \end{aligned} \quad (14)$$

To bound the second term on the right hand side of (14), we will crucially use the approximate orthogonality constraints in the SDP (4) and use Freedman's inequality. To this end, note that $L_S(k)$ is a martingale whose difference sequence can be bounded by

$$\begin{aligned} M &\leq |L_S(k) - L_S(k-1)| = |2\gamma \langle r_k, \sum_{i \in S} x_{k-1}(i) u_i^k \rangle| \\ &\leq 2\gamma\sqrt{n} \|\sum_{i \in S} x_{k-1}(i) u_i^k\|_2 \leq 2\gamma n^{3/2} \end{aligned} \quad (15)$$

where we used Lemma 3 in the first inequality and the fact that $|x_{k-1}(i)| \le 1$.

By Lemma 3 and SDP constraint (4),

$$\begin{aligned} W_k &= \sum_{j=1}^{k} \mathbb{E}_{j-1}[|L_S(j) - L_S(j-1)|^2] \\ &= \sum_{j=1}^{k} \mathbb{E}_{j-1}[4\gamma^2 \langle r_j, \sum_{i \in S} x_{j-1}(i) u_i^j \rangle^2] \\ &= \sum_{j=1}^{k} 4\gamma^2 \| \sum_{i \in S} x_{j-1}(i) u_i^j \|_2^2 \\ &\le \sum_{j=1}^{k} 8\gamma^2 \sum_{i \in S} \|u_i^j\|_2^2 = 8Q'_S(k) \end{aligned}$$

Applying Freedman's inequality now with these bound on M and W_k, we obtain

$$\begin{aligned} &\Pr\left[|L_S(T)| \ge 7 \cdot 2^j at \text{ and } Q'_S(T) \le 2^{j+5} at\right] \\ &\le \Pr\left[|L_S(T)| \ge 7 \cdot 2^j at \text{ and } W_T \le 2^{j+8} at\right] \\ &\le 2 \exp\left(\frac{-49 \cdot 2^{2j} a^2 t^2}{2[2^{j+8} at + 2\gamma n^{3/2} \cdot 7 \cdot 2^j at/3]} \right) \\ &\le 2 \exp\left(\frac{-2^j at}{20} \right) \end{aligned}$$

Together with $\lambda \le 2a\sqrt{t}$ (by our assumption), this gives

$$\begin{aligned} &\sum_{j=0}^{\infty} \Pr[L_S(T) \le -7 \cdot 2^j at, Q'_S(T) \le 2^{j+5} at] \\ &\le 4 \exp\left(\frac{-at}{20} \right) \le 4 \exp(\frac{-\lambda^2}{100a}) \end{aligned} \tag{16}$$

It remains to bound $\Pr[Q_S(T) \le Q'_S(T) - 8at]$, the first term in (14). We use Freedman's inequality in a simple way (even Azuma-Hoeffding would suffice here).

Define the martingale $Z_k = Q_S(k) - \sum_{j=1}^{k} \mathbb{E}_{j-1}[\Delta_j Q_S] = Q_S(k) - Q'_S(k)$ (this is the standard Doob decomposition of $\Delta_k Q_S$). By Lemma 3,

$$\begin{aligned} &M \le |Z_k - Z_{k-1}| = |\Delta_k Q_S - \mathbb{E}_{k-1}[\Delta_k Q_S]| \\ &\le 2|\Delta_k Q_S| \le 2\gamma^2 n \sum_{i \in S} \|u_i^k\|_2^2 \le 2\gamma^2 n^2 \end{aligned}$$

Using the trivial bound $\mathbb{E}_{j-1}[(Z_j - Z_{j-1})^2] \le M^2$, we obtain that

$$\begin{aligned} W_T &= \sum_{j=1}^{T} \mathbb{E}_{j-1}[(Z_k - Z_{k-1})^2] \\ &\le 4T\gamma^4 n^4 = 48\gamma^2 n^4 \log n. \end{aligned}$$

As $Q_S(T) \le Q'_S(T) - 8at$ is the same as $Z_T \le -8at$, by Freedman's inequality we get

$$\begin{aligned} &\Pr[Q_S(T) \le Q'_S(T) - 8at] \le \Pr[|Z_T| \ge 8at] \\ &\le 2 \exp\left(\frac{-64 a^2 t^2}{2[W_T + 16\gamma^2 n^2 at/3]} \right) \\ &\le 2 \exp\left(\frac{-a^2 t^2}{2\gamma^2 n^4 \log n} \right) \\ &\le 2 \exp(-a^2 t^2) \le 2 \exp(\frac{-\lambda^2}{100a}) \end{aligned} \tag{17}$$

In the last step we use that $\gamma = 1/(n^2 \log n)$, and Observation 14.

Combining equations (11),(12),(14),(16) and (17), we obtain the desired bound

$$\Pr[|D_S(T)| \ge \lambda\sqrt{t}] \quad \le \quad 8 \exp\left(\frac{-\lambda^2}{100a} \right)$$

C. Termination and finishing the proof

To finish the proof, we show that the last rounding step at time $T+1$ does not cause problems.

Theorem 17. *After time T, there are no alive variables left with probability at least $1 - O(n^{-2})$.*

Proof: Given the coloring x_k at time k, define $G_k = \sum_{i \in A_k} (1 - x_k(i)^2)$. Clearly $G_1 \le n$. As $x_k(i) = x_{k-1}(i) + \gamma \langle r_k, u_i^k \rangle$, we have that $\mathbb{E}_{k-1}[x_k(i)^2] = x_{k-1}(i)^2 + \gamma^2 \|u_i^k\|_2^2$. It follows

$$\begin{aligned} \mathbb{E}_{k-1}[G(k)] &= \mathbb{E}_{k-1}\left[\sum_{i \in A_k} (1 - x_k(i)^2) \right] \\ &= \sum_{i \in A_k} \left(1 - x_{k-1}(i)^2\right) - \gamma^2 \sum_{i \in A_k} \|u_i^k\|_2^2 \\ &\le \sum_{i \in A_k} \left(1 - x_{k-1}(i)^2\right) - \gamma^2 |A_k|/3 \\ &\le (1 - \gamma^2/3) \sum_{i \in A_k} (1 - x_{k-1}(i)^2) \\ &\le (1 - \gamma^2/3) \sum_{i \in A_{k-1}} (1 - x_{k-1}(i)^2) \\ &= (1 - \gamma^2/3) G_{k-1} \end{aligned}$$

Thus by induction,

$$\begin{aligned} \mathbb{E}[G_{T+1}] &\le (1 - \gamma^2/3)^T G_1 \le e^{-\gamma^2 T/3} n \\ &= n^{-4} \cdot n = 1/n^3. \end{aligned}$$

Thus by Markov's inequality, $\Pr[G_{T+1} \ge 1/n] \le 1/n^2$. However, $G_{T+1} \le 1/n$ implies that $A_{T+1} = 0$ as each alive variable contributes at least $1 - (1 - 1/n)^2 > 1/n$ to G_{T+1}. ■

Theorem 1 now follows directly. Applying Theorem 11 with $\lambda = c \log^{1/2} n$ for c a large enough constant and taking a union bound over the at most $nt \le n^2$ sets, we get that $|D_S(T)| = O((t \log n)^{1/2})$ with probability at least $1-1/\text{poly}(n)$ for all sets S. By Theorem 17 with probability at least $1-O(n^{-2})$, all variables are frozen by time T and hence at most an additional discrepancy of 1 is added by rounding the frozen variables to ± 1. □

IV. Extension to the Komlós setting

The algorithm also extends to the more general Komlós setting with some additional modifications. Recall that in the Komlós setting, we are given an $m \times n$ matrix B with arbitrary real entries b_{ji} such that for each column i, it holds that $\sum_j b_{ji}^2 \le 1$. Let r_j denote the j-th row of B and let a be the constant as in the previous section. We will show the following result.

Theorem 18. *Fix any row r_j of matrix B. Then, for any $\lambda \ge 0$, the discrepancy of r_j at time step T (the end of the algorithm) satisfies*

$$\Pr\left[|D_T(r_j)| \ge \lambda\right] \le 8\exp(-\lambda^2/(1000a))$$

where $|D_T(r_j)|$ is the discrepancy of row j after time step T.

The previous argument does not work directly when the entries b_{ji} are arbitrary as we may not get strong concentration if some entries b_{ji} are too large. So we consider the following modified algorithm.

Algorithm: Given a matrix B, for any $\lambda > 0$ we denote by r_j^λ the λ-truncation of row j containing only the entries b_{ji} that are at most $4a/\lambda$ in absolute value i.e. r_j^λ only contains those entries i of row j for which $|b_{ji}| \le 4a/\lambda$ and 0 otherwise.

As previously, let A_k denote the set of alive variables at beginning of time step k, and we set $\gamma = 1/n^6$ and $T = (12/\gamma^2)\log n$. A row j is called *big* at time step k if $\sum_{i\in A_k} b_{ji}^2 > a$, and *small* otherwise. As the ℓ_2-norm of columns of B is at most 1, there at most $|A_k|/a$ big rows at any time step k.

The modified SDP: The SDP is modified as follows. Similar to (2) we still require the discrepancy of big rows to be zero. That is,

$$\|\sum_{i\in A_k} b_{ji}u_i\|_2^2 = 0 \qquad \text{for each big row } j \tag{18}$$

For a small row r_j at time k, we add proportional discrepancy and approximate orthogonality constraints *for every λ-truncation r_j^λ of r_j* i.e., for every $\lambda > 0$, we add the proportional discrepancy constraint (3) (same as before, we just multiply the u_i's by b_{ji}'s)

$$\|\sum_{i\in A_k, |b_{ji}|\le 4a/\lambda} b_{ji}u_i\|_2^2 \le 2\sum_{i\in A_k, |b_{ji}|\le 4a/\lambda} b_{ji}^2\|u_i\|_2^2 \tag{19}$$

and the approximate orthogonality constraints (4)

$$\|\sum_{i\in A_k, |b_{ji}|\le 4a/\lambda} b_{ji}^2 x_{k-1}(i)u_i\|_2^2 \le 2\sum_{i\in A_k, |b_{ji}|\le 4a/\lambda} b_{ji}^4\|u_i\|_2^2. \tag{20}$$

Notice that as stated, for each small row we add two SDP constraints for every value of $\lambda > 0$. However it suffices to add at most $2n$ constraints in total for each active row: just sort the entries of a row in increasing order of absolute value and add the proportional discrepancy and orthogonality constraints in the SDP for every prefix of this sorted row (alternatively, one could also consider geometrically increasing values of λ). Thus the SDP has a polynomial number of constraints at any time step.

Analysis: First, exactly as before the SDP is feasible and has a solution with value at least $|A_k|/3$. This follows from Theorem 8, which shows that there is a subspace W of dimension at least $|A_k|/2$ where the corresponding operator is negative semidefinite on W, and then applying the argument in Theorem 10. In fact, this would be true even if (20) was replaced by the stronger constraint

$$\|\sum_{i\in A_k, |b_{ji}|\le 4a/\lambda} b_{ji}^2 x_{k-1}(i)u_i\|_2^2 \le 2\sum_{i\in A_k, |b_{ji}|\le 4a/\lambda} b_{ji}^4 x_{k-1}(i)^2\|u_i\|_2^2.$$

Let $D_k(r_j)$ denote the signed discrepancy of row j at the end of time step k,

$$D_k(r_j) = \sum_{i\in[n]} b_{ji}x_k(i).$$

We also extend this definition to truncations of rows:

$$D_k(r_j^\lambda) = \sum_{i\in[n], |b_{ji}|\le 4a/\lambda} b_{ji}x_k(i).$$

We now show Theorem 18. Fix a row r_j and a $\lambda \ge 0$. Call an entry b_{ji} *large* if $|b_{ji}| > 4a/\lambda$. We first make the following key observation.

Observation 19. *When a row becomes small, the ℓ_1-norm of the alive variables in that row that are large can be at most $\lambda/4$.*

Proof: Each large entry is at least $4a/\lambda$ in absolute value. As a row r_j becomes small when $\sum_{i\in A_k} b_{ji}^2 \le a$, there can be at most $a/(4a/\lambda)^2 = \lambda^2/16a$ alive variables with b_{ji} large. By Cauchy-Schwarz inequality,

$$\left(\sum_{i\in A_k, |b_{ji}|>4a/\lambda} |b_{ji}|\right) \le \left(\sum_{i\in A_k} b_{ji}^2\right)^{1/2}\left(\frac{\lambda^2}{16a}\right)^{1/2} \le \lambda/4.$$

■

The above observation implies that when a row becomes small, the large entries in it can change discrepancy by at most $\lambda/2$. Thus to prove Theorem 18, it suffices to show

$$\Pr\left[|D_T(r_j^\lambda)| \geq \lambda/2\right] \leq 8\exp(-\lambda^2/(1000a)).$$

This follows similarly to the analysis as before, using the proportional discrepancy (19) and approximate orthogonality constraints (20) for r_j^λ and noting that (20) implies that

$$\begin{aligned}
&\| \sum_{i \in A_k, |b_{ji}| \leq 4a/\lambda} b_{ji}^2 x_{k-1}(i) u_i\|_2^2 \\
&\leq 2 \sum_{i \in A_k, |b_{ji}| \leq 4a/\lambda} b_{ji}^4 \|u_i\|_2^2 \\
&\leq \frac{32a^2}{\lambda^2} \sum_{i \in A_k, |b_{ji}| \leq 4a/\lambda} b_{ji}^2 \|u_i\|_2^2 \qquad (21)
\end{aligned}$$

as $|b_{ji}| \leq 4a/\lambda$ for all entries in the truncated row r_j^λ. Let us define the energy of λ-truncation of row j at time k as

$$E_k(r_j^\lambda) = \sum_{i \in [n], |b_{ji}| \leq 4a/\lambda} b_{ji}^2 x_k(i)^2.$$

As previously, once the row becomes small, its energy can rise by at most a.

The analysis in Section III-B had two main ideas:

1) First we showed that the expected squared discrepancy of a set S at time T was $O(1)$ times the energy injected into the set $Q_S(T)$ (using constraints (3)). This argument works exactly as before using constraints (19) and we sketch the details below.

   For ease of notation we will denote the entries of the truncated row r_j^λ as b_{ji} where it is understood that we are setting $b_{ji} = 0$ if b_{ji} was large in the original matrix. The change in energy at time k is a random variable given by

$$\begin{aligned}
&\Delta_k E(r_j^\lambda) \\
&= \gamma^2 \sum_{i \in [n]} b_{ji}^2 \langle r_k, u_i^k \rangle^2 + 2\gamma \langle r_k, \sum_{i \in [n]} b_{ji}^2 x_{k-1}(i) u_i^k \rangle
\end{aligned}$$

   Denote the first term above as $\Delta_k Q(r_j^\lambda)$, the change in quadratic energy of r_j^λ at time step k and let $Q_k(r_j^\lambda) = \sum_{k'=1}^k \Delta_{k'} Q(r_j^\lambda)$, the total quadratic energy of r_j^λ till time k.
   Similarly, denote the second term as $\Delta_k L(r_j^\lambda)$, the change in linear energy of r_j^λ at time step k, and let $L_k(r_j^\lambda) = \sum_{k'=1}^k \Delta_{k'} L(r_j^\lambda)$, the total linear energy of r_j^λ till time k.

   Define $Q'_k(r_j^\lambda) = \sum_{k'=1}^k \mathbb{E}_{k'-1}[\Delta_{k'} E(r_j^\lambda)] = \sum_{k'=1}^k \mathbb{E}_{k'-1}[\Delta_{k'} Q(r_j^\lambda)]$. By lemma 3,

$$Q'_k(r_j^\lambda) = \sum_{k'=1}^k \gamma^2 \sum_{i \in [n]} b_{ji}^2 \|u_i^{k'}\|_2^2$$

   Just as before, discrepancy $D_k(r_j^\lambda)$ behaves as a martingale with the variance W_k bounded by $2Q'_k(r_j^\lambda)$. Freedman's inequality then gives,

$$\begin{aligned}
&\Pr\left[|D_T(r_j^\lambda)| \geq \lambda/2 \text{ and } Q'_T(r_j^\lambda) \leq 16a\right] \\
&\leq 2\exp\left(\frac{-\lambda^2}{1000a}\right) \qquad (22)
\end{aligned}$$

   Next we showed that $Q'_S(T)$ was essentially the same as $Q_S(T)$ (shown in (17)). In fact this difference can be made arbitrarily small by reducing γ and the argument works exactly as before here. In particular, we get

$$\begin{aligned}
&\Pr[Q_T(r_j^\lambda) \leq Q'_T(r_j^\lambda) - 8a] \\
&\leq 2\exp\left(\frac{-\lambda^2}{1000a}\right) \qquad (23)
\end{aligned}$$

2) The second part was to show that the linear term does not cause problems. In particular, the crucial argument was that $Q_S(T)$ cannot be much more than at as (i) the total rise in energy $L_S(T)+Q_S(T)$ cannot exceed at and (ii) $L(T)$ was a martingale with squared deviation comparable to $Q_S(T)$ and hence cannot be much larger than $Q_S^{1/2}(T)$. This step used the constraints (4).
   This argument also works similarly in our setting here. For a truncated row r_j^λ, $Q_T(r_j^\lambda)$ cannot be much more than a as (i) the total rise in energy $L_T(r_j^\lambda) + Q_T(r_j^\lambda)$ cannot exceed a and (ii) $L_T(r_j^\lambda)$ is a martingale with squared deviation comparable to $\frac{32a^2}{\lambda^2} Q'_T(r_j^\lambda)$ (by (20) and (21)). Proceeding exactly as before and applying Freedman's inequality we obtain that,

$$\begin{aligned}
&\sum_{\ell=0}^{\infty} \Pr\left[L_T(r_j^\lambda) \leq -7 \cdot 2^\ell a, \quad Q'_T(r_j^\lambda) \leq 2^{\ell+5} a\right] \\
&\leq 4\exp(\frac{-\lambda^2}{1000a}). \qquad (24)
\end{aligned}$$

Theorem 18 now follows by combining (22),(23),(24) as before and using Observation 19.

Theorem 2 now follows easily, by observing that m can be assumed to be polynomially bounded in n and applying a union bound. Indeed, we can discard all rows of ℓ_1-norm less than $\sqrt{\log n}$ since they can only ever have discrepancy at most $\sqrt{\log n}$. The remaining rows have squared ℓ_2-norm at least $\frac{\log n}{n}$, as by Cauchy-Schwarz inequality

$$\sqrt{\log n} \leq \sum_{i\in[n]} |b_{ji}| \leq \left(\sum_{i\in[n]} b_{ji}^2\right)^{1/2} (n)^{1/2}.$$

As $\sum_{i,j} b_{ji}^2 \leq n$, there can be at most $n^2/\log n$ such rows. We now set $\lambda = O(\sqrt{\log n})$ in Theorem 18 and take a union bound over all these rows.

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