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STABILIZATION OF THE LAX-WENDROFF METHOD AND A GENERALIZED ONE-STEP RUNGE-KUTTA METHOD FOR HYPERBOLIC INITIAL-VALUE PROBLEMS

E.D. DE GOEDE
Centre for Mathematics and Computer Science, 1009AB Amsterdam, The Netherlands

In order to integrate hyperbolic systems we distinguish explicit and implicit time integrators. Implicit methods allow large integration steps, but require more storage and are more difficult to implement than explicit methods. However explicit methods are subject to a restriction on the integration step. This restriction is a drawback if the variation of the solution in time is so small that accuracy considerations would allow a larger integration step. In this report we apply a smoothing technique in order to stabilize the Lax-Wendroff method and a generalized one-step Runge-Kutta method. Using this technique, the integration step is not limited by stability considerations.

1. Introduction

Consider the differential equation

$$\mathbf{u}_t = \mathbf{f}(\mathbf{u},\mathbf{u}_{x_1},\mathbf{u}_{x_2},\ldots,\mathbf{u}_{x_n},\mathbf{x},t), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^n, \quad t > 0, \tag{1.1}$$

where $\mathbf{u} = (u_1(\mathbf{x},t),u_2(\mathbf{x},t),\ldots,u_N(\mathbf{x},t))$, defining a first-order quasi-linear hyperbolic system with N equations [1]. We will assume that the initial condition at $t=0$ and the boundary conditions on $\partial\Omega$ determine a unique solution. In order to perform the time integration for such systems we distinguish explicit and implicit time integrators. Implicit methods allow large integration steps, but require more storage and are more difficult to implement than explicit methods. However, using explicit methods, the integration step Δt is restricted by the Courant-Friedrichs-Levy (C.F.L.) condition. In many problems, this condition restricts the integration step more severely than necessary for accuracy. For instance, in order to represent an irregular geometry a fine space mesh is needed. If the variation of the solution in time is very slow, then one likes to use much larger integration steps than the one allowed by the C.F.L.-condition.

In order to improve the stability condition for explicit methods **smoothing techniques** are often used. However, usually the grid function $\mathbf{u}$ is smoothed. For example, Richtmyer schemes and Strang schemes (see [4]) are famous methods for first-order hyperbolic systems. In all these schemes the numerical solution $\mathbf{u}$ is smoothed in the first stage. This smoothing of $\mathbf{u}$ may only be applied, without danger of loss of accuracy, if $\mathbf{u}$ itself is smooth (i.e. $\mathbf{u}$ has small space derivatives).

Another smoothing technique was introduced by Wubs [5]. Wubs observed that the property of a smooth right-hand side in space can be used effectively to stabilize an explicit time integration method. Using this observation, the right-hand side was smoothed. In this case $\mathbf{u}$ itself may have large space derivatives.

Moreover, it is possible to use a similar smoothing technique, which is incorporated in the time integrator for (1.1). In this paper we will show two examples of methods, which inherently contain this technique, namely the Crank-Nicolson method and the 'box integration' method . In both cases, the method when applied to linear problems, can be rewritten to an explicit method in which an implicit smoothing operator occurs. Omitting the implicit smoothing operator, the first method gives a generalized one-step Runge-Kutta method which is unstable for all integration steps. The second method gives the Lax-Wendroff scheme which is conditionally stable.

Our main purpose is to investigate the stabilization of the generalized one-step Runge-Kutta method and the Lax-Wendroff method by using explicit smoothing operators, because explicit smoothing requires less computational effort than implicit smoothing. Results will be given for a linear and a non-linear test problem. For the non-linear case we will use the second-order two-stage Runge-Kutta scheme and the MacCormack scheme. For linear problems both methods can be rewritten to an explicit method in which an implicit smoothing operator occurs. Omitting the smoothing operator, the Runge-Kutta scheme and the MacCormack scheme are identical to respectively the generalized one-step Runge-Kutta scheme and the Lax-Wendroff scheme.

2. Theory

To illustrate the theory, we will use a simple example. Consider the scalar equation

$$u_t = f(u_x,x), \quad x \in \mathbb{R}, \tag{2.1}$$

where

$$f(u_x,x) = u_x + g(x). \tag{2.2}$$

Using the method of lines, we obtain a system of ordinary differential equations [3]

$$\frac{d}{dt}\mathbf{U} = \mathbf{F}(\mathbf{U},t), \tag{2.3}$$

where $\mathbf{U}$ is a grid function approximating $\mathbf{u}$, and $\mathbf{F}(\mathbf{U},t)$ a vector function approximating the right-hand side function. The function $f(u_x,x)$ in (2.2) is discretized, on a grid with mesh size h, with the usual second-order central differences

$$F_j(\mathbf{U}) = (D\mathbf{U})_j + g(x_j), \quad x_j = jh, \tag{2.4}$$

where

$$(D\mathbf{U})_j = (U_{j+1} - U_{j-1})/(2h), \tag{2.5}$$

and U_j approximates $u(x_j)$. Let us consider the Jacobian matrix of (2.3). Here, the Jacobian matrix is defined by

$$J = \left(\frac{\delta \mathbf{F}}{\delta \mathbf{U}}\right). \tag{2.6}$$

Note that for linear systems, the Jacobian matrix is independent of $\mathbf{U}$. So we simply have

$$J = D. \tag{2.7}$$

When Crank-Nicolson is applied to (2.3), with τ being the grid spacing in t-direction, we find

$$\{(I - \frac{\tau}{2} J)U^{n+1}\}_j = \{(I + \frac{\tau}{2} J)U^n\}_j + \tau g(x_j), \tag{2.8}$$

where U^n approximates the exact solution $U(t)$ of (2.3) at $t = n\tau$. Using (2.4), formula (2.8) can be rewritten to

$$\{(I - \frac{\tau}{2} J)U^{n+1}\}_j = \{(I - \frac{\tau}{2} J)U^n\}_j + \tau F_j(\mathbf{U}^n). \tag{2.9}$$

As the operator $(I - \frac{\tau}{2} J)$ is invertible, we find

$$\begin{aligned} U_j^{n+1} &= U_j^n + \tau\{(I - \frac{\tau}{2} J)^{-1} \mathbf{F}(\mathbf{U}^n)\}_j, \\ &= U_j^n + \tau\{(I - \frac{\tau^2}{4} J^2)^{-1} (I + \frac{\tau}{2} J)\mathbf{F}(\mathbf{U}^n)\}_j. \end{aligned} \tag{2.10}$$

Omitting the implicit smoothing operator $S_{\text{impl}} = (I - \frac{\tau^2}{4} J^2)^{-1}$, gives the generalized one-step Runge-Kutta method (see [2],p.44)

$$U_j^{n+1} = U_j^n + \tau\{(I + \frac{\tau}{2} J)\mathbf{F}(\mathbf{U}^n)\}_j. \tag{2.11}$$

Note that this scheme is identical to a two-stage second-order Runge-Kutta method, which is rewritten to a one-stage method. It appears that the unsmoothed scheme (2.11) is unstable for hyperbolic problems. Hence, by smoothing it is possible to stabilize the method.

Our goal is to find an explicit smoothing operator , which can replace the operator $S_{\text{impl}} = (I - \frac{\tau^2}{4} J^2)^{-1}$. Thus we have

$$U_j^{n+1} = U_j^n + \tau\{(S_{\text{expl}})(I + \frac{\tau}{2} J)\mathbf{F}(\mathbf{U}^n)\}_j, \tag{2.12}$$

where S_{expl} is an explicit smoothing operator.

3. Stability

In VAN DER HOUWEN [2] derivations of the stability condition are given for generalized one-step Runge-Kutta methods. We will omit the inhomogeneous term $g(x)$, because this term is not significant in the local stability analysis. Let

$$\tilde{F}_j(\mathbf{U}^n) = (D\mathbf{U})_j \quad \text{and} \tag{3.1}$$

$$\Theta_0(z) = 1 + \frac{z}{2}.$$

Then scheme (2.12) can be rewritten to

$$U_j^{n+1} = U_j^n + \tau\{(S_{\text{expl}})\,\Theta_0(\tau J)\,\tilde{\mathbf{F}}(\mathbf{U}^n)\}_j. \tag{3.2}$$

For the stability of equation (3.2), the Jacobian matrix of J is decisive, in particular tl eigenvalues of J. Let $\hat{J}(\omega)$ be the continuous spectrum of eigenvalues of the Jacobian matr J, corresponding to the eigenvectors $\exp(i\omega jh)$, where the frequencies ω are arbitrary. It easy to verify that

$$\hat{J}(\omega) = \frac{i \sin(\omega h)}{h} . \tag{3.3}$$

Note that we have purely imaginary eigenvalues. The stability polynomial $R(z)$ (s [2].p.124) is given by

$$R(z) = 1 + \lambda_{es}\, z\, \Theta_0(z) , \tag{3.4}$$

where

$$z = \tau \hat{J}(\omega) \tag{3.5}$$

and $\lambda_{es}(\omega h)$ is the real eigenvalue of the explicit smoothing operator S, which will t described in the next section. The stability region of the stability function R is defined t the set of points (in the complex plane)

$$\{ z \mid | R(z) | \leqslant 1 \} . \tag{3.6}$$

Let β_{imag} be the maximal value on the imaginary axis, which satisfies (3.6). Then the integr ion step should satisfy the stability condition

$$\tau \leqslant \frac{\beta_{\text{imag}}}{\rho(\hat{J}(\omega))} , \tag{3.7}$$

where $\rho(.)$ denotes the spectral radius (see e.g. [2],p.83).

In Wubs [5] a smoothing technique was used to reduce the eigenvalues of the Jacobia matrix J and so reducing the spectral radius. Here, we will use a smoothing technique t extend the stability region (in particular the imaginary stability boundary). It appears th the smoothing depends on the spectrum of J.

Let

$$p = \frac{\tau}{h} , \quad s = \sin(\omega h) \text{ and } \lambda_{es} = \lambda_{es}(\omega h) . \tag{3.8}$$

Then the stability condition (3.7) can be rewritten to

$$\sqrt{(1 - \frac{1}{2} p^2 \lambda_{es} s^2)^2 + p^2 \lambda_{es}^2 s^2} \leqslant 1 , \tag{3.9}$$

which leads to

$$\lambda_{es} (1 + \frac{1}{4} p^2 \sin^2(\omega h)) \leqslant 1 . \tag{3.10}$$

Notice that from (3.9) it follows immediately that

$$\lambda_{es}(\omega h) \geqslant 0 . \tag{3.11}$$

4. Explicit smoothing operators

As mentioned before, smoothing operators are often used for the stabilization of explicit methods. Here, we will review the family of explicit smoothing operators, which are proposed in WUBS [5]. At first, consider the smoothing operator S_1 defined by

$$(S_1\mathbf{F})_j := (F_{j+1} + F_{j-1})/2 . \tag{4.1}$$

The eigenvalue is

$$\lambda_{S_1} = \cos(\omega h) . \tag{4.2}$$

We now will define the smoothing operator more generally by

$$S_{k_0}^m := \prod_{k=k_0}^{m} S_k , \tag{4.3}$$

where

$$(S_k\mathbf{F})_j := \mu_k F_{j+2^{k-1}} + (1-2\mu_k)F_j + \mu_k F_{j-2^{k-1}} . \tag{4.4}$$

The corresponding eigenvalue is equal to

$$\lambda_{S_{k_0}^m} = \prod_{k=k_0}^{m} \{(1 - 2\mu_k) + 2\mu_k \cos(2^{k-1}\omega h)\} , \tag{4.5}$$

$$= \prod_{k=k_0}^{m} \{ 1 - 4\mu_k \sin^2(2^{k-2}\omega h)\} , \qquad k=k_0,\ldots,m .$$

Using (4.3), the number of smoothing factors is equal to $(m-k_0+1)$. Notice that from (3.11) and (4.5) it follows that

$$\forall\, k :(k_0 \leqslant k \leqslant m) : 0 \leqslant \mu_k \leqslant \tfrac{1}{4} . \tag{4.6}$$

4.1. The order of accuracy of the smoothing operator

The smoothing operator (4.3) should be sufficiently close to the identity operator I. In order to define the order of this smoothing operator we apply $S_{k_0}^m$ to the test vector $\mathbf{w} := (w(jh))$, where $w(x)$ is a sufficiently differentiable function of x. From equation (4.4) we find

$$S_k\mathbf{w} = [1 + \mu_k 2^{2k-2} h^2 \frac{d^2}{dx^2} + O(h^4)] w(jh) . \tag{4.7}$$

Let us assume that

$$\mu = \mu_k , \qquad \text{for } k_0 \leqslant k \leqslant m . \tag{4.8}$$

Then we find

$$S_{k_0}^m \mathbf{w} = [\prod_{k=k_0}^{m} \{1 + \mu 2^{2k-2} h^2 \frac{d^2}{dx^2} + O(h^4)\}] w(jh) \tag{4.9}$$

$$= [1 + \mu h^2 \frac{2^{2m} - 2^{2k_0-2}}{3} \frac{d^2}{dx^2} + O(h^4)] w(jh) .$$

Thus, the smoothing operator is second-order accurate. The error constant depends on m and k_0. In the following sections we will derive values for m and k_0.

5. Construction of a stabilized generalized one-step Runge-Kutta scheme

In this section it will be shown that, using the explicit smoothing operator (4.3), it is possible to construct a generalized one-step Runge-Kutta scheme, which is stable for any given integration step.

Consider the scheme (2.12) with stability condition (see (3.10))

$$\lambda_{RK} = \lambda_{es}(\omega h).(1 + \frac{1}{4}(\frac{\tau}{h})^2 \sin^2(\omega h)) \leqslant 1 . \tag{5.1}$$

Without smoothing operator (i.e $\lambda_{es}(\omega h) = 1$), it is obvious that this scheme is unstable. Hence, given an integration step τ, it is our goal to satisfy (5.1) by choosing m, k_0 and the coefficients μ_k in an appropriate way. Let

$$\alpha = \frac{1}{4}(\frac{\tau}{h})^2 , \tag{5.2}$$

then (5.1) leads to

$$\lambda_{RK} = \lambda_{es}(\omega h).(1 + \alpha \sin^2(\omega h)) \leqslant 1 . \tag{5.3}$$

The first appropriate smoothing factor seems S_2, with eigenvalues

$$\lambda_{S_2} = (1 - 4\mu_2 \sin^2(\omega h)) . \tag{5.4}$$

Substitution in (5.3) leads to

$$\lambda_{RK} = (1 - 4\mu_2 \sin^2(\omega h))(1 + \alpha \sin^2(\omega h)) \leqslant 1 . \tag{5.5}$$

This can be rewritten to

$$\lambda_{RK} = 1 + \sin^2(\omega h)\{\alpha - 4\mu_2\} + \sin^4(\omega h)\{-4\alpha\mu_2\} \leqslant 1 , \tag{5.6}$$

$$= 1 + \sin^2(2\omega h)\{\frac{\alpha - 4\mu_2}{4}\} + \sin^4(\omega h)\{-4\alpha\mu_2 + \alpha - 4\mu_2\} \leqslant 1 .$$

Let

$$\beta_{k_0} = \beta_2 = \frac{\alpha - 4\mu_2}{4} \text{ and } \gamma_2 = -4\alpha\mu_2 + \alpha - 4\mu_2 . \tag{5.7}$$

The method is stable if

$$\beta_2 \leqslant 0 \text{ and } \gamma_2 \leqslant 0 . \tag{5.8}$$

Furthermore (4.6) has to be satisfied. (5.8) can be rewritten to

$$\alpha \leqslant 4\mu_2 \tag{5.9a}$$

and

$$\mu_2 \geqslant \frac{1}{4}(1 - \frac{1}{1+\alpha}) . \tag{5.9b}$$

Condition (5.9b) can always be satisfied, but (5.9a) only for $\alpha \leqslant 1$. If (5.9a) cannot be satisfied the second smoothing factor of operator (4.3) is applied. The corresponding eigenvalues are

$$\lambda_{S_3} = (1-4\mu_3 \sin^2(2\omega h)) . \tag{5.10}$$

By adding this factor, we obtain the stability condition

$$\lambda_{RK} = (1 - 4\mu_3 \sin^2(2\omega h)).(1 + \beta_2 \sin^2(2\omega h) + \gamma_2 \sin^4(\omega h)) \leqslant 1 . \tag{5.11}$$

If (5.9b) is satisfied, then (5.11) can be simplified to

$$(1 - 4\mu_3 \sin^2(2\omega h)).(1 + \beta_2 \sin^2(2\omega h)) \leqslant 1 . \tag{5.12}$$

because

$$\lambda_{S_3} \geqslant 0 \text{ and } \gamma_2 \sin^4(\omega h) \leqslant 0 . \tag{5.13}$$

Notice that (5.12) is of the same form as (5.5). Hence, the method is stable if

$$\beta_3 \leqslant 0 \text{ and } \gamma_3 \leqslant 0 , \qquad (5.14$$

where

$$\beta_3 = \frac{\beta_2 - 4\mu_3}{4} \text{ and } \gamma_3 = -4\beta_2\mu_3 + \beta_2 - 4\mu_3 . \tag{5.15}$$

This can be rewritten to

$$\alpha \leqslant \sum_{k=k_0}^{3} \mu_k 4^{(k-k_0+1)} \text{ and } \mu_3 \geqslant \frac{1}{4}(1 - \frac{1}{1+\beta_2}) , \tag{5.16}$$

with $k_0 = 2$.

In general, the above-mentioned process has to be continued. By applying the smoothing operator (4.3), (i.e. $(m-k_0+1)$ smoothing factors), we obtain the stability condition

$$\alpha \leqslant \sum_{k=k_0}^{m} \mu_k 4^{(k-k_0+1)} . \tag{5.17}$$

The index m has to be chosen as small as possible, because m determines the computational costs of the smoothing operator. Furthermore, the coefficients μ_k have to satisfy

$$\mu_k \geqslant \frac{1}{4}(1 - \frac{1}{1+\beta_{k-1}}) , \quad k=k_0,...,m , \tag{5.18}$$

with

$$\beta_k = \frac{\beta_{k-1}}{4} - \mu_k \text{ and } \beta_{k_0-1} = \alpha . \tag{5.19}$$

6. Choice of the coefficients μ_k

In this section we will determine a suitable choice for the coefficients μ_k. According to (4.6) and (5.18), we already have the conditions

$$0 \leqslant \mu_k \leqslant \frac{1}{4} \tag{6.1}$$

and

$$\mu_k \geqslant \frac{1}{4}(1 - \frac{1}{1+\beta_{k-1}}) , \quad k = k_0,...,m , \tag{6.2}$$

where

$$\beta_k = \frac{\beta_{k-1}}{4} - \mu_k \text{ and } \beta_{k_0-1} = \alpha . \tag{6.3}$$

By choosing

$$\mu_k = \frac{1}{4}(1 - \frac{1}{1+\beta_{k-1}}) , \tag{6.4}$$

we obtain

$$\lim_{k\to\infty} (\beta_k) \downarrow 0 . \tag{6.5}$$

However, we have to satisfy ($\beta_k \leqslant 0$) for a certain index k. Thus, we have to impose an extra condition in order to guarantee stability after a finite number of smoothing factors. Rewriting stability condition (5.17) leads to

$$p^2 \leqslant 4 \sum_{k=k_0}^{m} \mu_k 4^{(k-k_0+1)} . \tag{6.6}$$

Hence, for every applied smoothing factor, about a factor two can be gained. Thus, we can estimate the necessary number of smoothing factors, by

$$m = [\,{}^2\log p\,] + 1 , \tag{6.7}$$

where [ ] denotes the entier function. Let

$$q^{(m-k_0+1)} = p . \tag{6.8}$$

We require that for every applied smoothing factor the right-hand side of (6.6) is multiplied by at least a factor q. From (6.6) and (6.7) it follows that $q \leqslant 2$. Now our extra condition for the coefficients μ_k is given by

$$q^{2(k-k_0+1)} \leqslant 4 \sum_{i=k_0}^{k} \mu_k 4^{(i-k_0+1)}, \quad k=k_0,\dots,m . \tag{6.9}$$

This can be rewritten to

$$\mu_k \geqslant \frac{q^{2(k-k_0+1)} - \sum_{i=k_0}^{k-1} \mu_i 4^{(i-k_0+2)}}{4^{(k-k_0+2)}}, \quad k=k_0,\dots,m . \tag{6.10}$$

Summarising, for a given integration step τ let

$$p = \frac{\tau}{h}, \quad \alpha = \frac{1}{4}p^2, \quad k_0 = 2, \tag{6.11}$$

$$\beta_{k_0-1} = \beta_1 = \alpha, \quad m = [\,^2\log p] + 1 \text{ and } q = p^{\frac{1}{(m-k_0+1)}} .$$

Then we will use the generalized one-step Runge-Kutta scheme (2.12)

$$U_j^{n+1} = U_j^n + \tau\{(S_{\text{expl}})(I + \frac{\tau}{2} J)\mathbf{F}(\mathbf{U}^n)\}_j , \tag{6.12}$$

with the smoothing operator $S_{\text{expl}} = S_{k_0}^m \ (= \prod_{k=k_0}^{m} S_k)$, where

$$0 \leqslant \mu_k \leqslant \frac{1}{4}, \tag{6.13a}$$

$$\mu_k \geqslant \frac{1}{4}(1 - \frac{1}{1+\beta_{k-1}}), \tag{6.13b}$$

$$\mu_k \geqslant \frac{q^{2(k-k_0+1)} - \sum_{i=k_0}^{k-1} \mu_i 4^{(i-k_0+2)}}{4^{(k-k_0+2)}}, \quad k=k_0,\dots,m \tag{6.13c}$$

and

$$\beta_k = \frac{\beta_{k-1}}{4} - \mu_k . \tag{6.14}$$

Moreover, the coefficients μ_k will be chosen as small as possible.

6.1. The smoothing error

Here, we will give an approximation of the error due to the smoothing operator (4.3) with coefficients given by (6.11) and (6.13). Since $0 \leqslant \mu_k \leqslant \frac{1}{4}$, we assume that

$$\mu = \mu_k = \frac{1}{4}, \quad \text{for } k_0 \leqslant k \leqslant m . \tag{6.15}$$

Using formula (4.9) we obtain that the smoothing error is approximated by

$$(S_{k_0}^m - I)\mathbf{w} = [\mu h^2 \frac{2^{2m} - 4}{3} \frac{d^2}{dx^2} + O(h^4)]w(jh). \tag{6.16}$$

Now this error can be expressed in terms of the time step. From (6.11) we have that the stabilized Runge-Kutta scheme (2.12) is stable if

$$\frac{\tau}{h} \leqslant 2^{m-k_0+1}, \quad m \geqslant k_0 = 2. \tag{6.17}$$

Let us assume that the maximal stable time step is used. Then equation (6.16) yields

$$(S_{k_0}^m - I)\mathbf{w} = [\frac{1}{4} \frac{\tau^2}{2^{2(m-1)}} \frac{2^{2m} - 4}{3} \frac{d^2}{dx^2} + O(h^4)]w(jh) \tag{6.18}$$

$$\approx [\frac{\tau^2}{3} \frac{d^2}{dx^2} + O(h^4)]w(jh).$$

Thus, the error due to the smoothing operator decreases quadratically with the time step. This second-order behaviour is reflected in the numerical experiments. Moreover, increasing the number of smoothing factors hardly affects the accuracy.

7. Construction of a stabilized Lax-Wendroff method

In this section we will develop an analogue stabilization theory for the Lax-Wendroff method. At first, we will demonstrate that the 'box integration' method (see [4],p.191) can be rewritten to the Lax-Wendroff scheme in which an implicit smoothing operator occurs.

Consider the scalar equation (2.1) and (2.2). By applying the 'box integration' method, we obtain

$$(M^+ - \frac{\tau}{2}D^+)U_j^{n+1} = (M^+ + \frac{\tau}{2}D^+)U_j^n + \tau\frac{k(x_j,n) + k(x_j,n+1)}{2}, \tag{7.1}$$

where

$$(D^+ \mathbf{U})_j = (U_{j+1} - U_j)/h, \tag{7.2}$$

$$(M^+ \mathbf{U})_j = (U_{j+1} + U_j)/2$$

and

$$k(x_j,n) = \frac{g(jh,n\tau) + g((j+1)h,n\tau)}{2}. \tag{7.3}$$

Let

$$F_j(\mathbf{U}) = (D^+ \mathbf{U})_j + \frac{k(x_j,n) + k(x_j,n+1)}{2}. \tag{7.4}$$

Substitution into (7.1) leads to

$$(M^+ - \frac{\tau}{2} D^+)U_j^{n+1} = (M^+ - \frac{\tau}{2} D^+)U_j^n + \tau F_j(\mathbf{U}^n). \tag{7.5}$$

Let

$$(D^- \mathbf{U})_j = (U_j - U_{j-1})/h , \qquad (7.6)$$
$$(M^- \mathbf{U})_j = (U_{j-1} + U_j)/2 .$$

As the operator $(M^+ - \frac{\tau}{2} D^+)$ is invertible, we find

$$U_j^{n+1} = U_j^n + \tau\{(M^+ - \frac{\tau}{2} D^+)^{-1}\mathbf{F}(\mathbf{U}^n)\}_j , \qquad (7.7)$$
$$= U_j^n + \tau\{(M^- M^+ - \frac{\tau^2}{4} D^- D^+)^{-1} (M^- + \frac{\tau}{2} D^-)\mathbf{F}(\mathbf{U}^n)\}_j .$$

Omitting the implicit smoothing operator $(M^- M^+ - \frac{\tau^2}{4} D^- D^+)^{-1}$, there remains the Lax-Wendroff scheme, which is stable for $\tau/h \leqslant 1$. By adding an explicit smoothing operator , we obtain

$$U_j^{n+1} = U_j^n + \tau\{(S_{\text{expl}})(M^- + \frac{\tau}{2} D^-)\mathbf{F}(\mathbf{U}^n)\}_j . \qquad (7.8)$$

In order to determine the stability condition of scheme (7.8), we will use the von Neumann method (see [4],p.167). Omitting the inhomogeneous term, there remains

$$U_j^{n+1} = U_j^n + \tau\{(S_{\text{expl}})(M^- + \frac{\tau}{2} D^-)D^+ (\mathbf{U}^n)\}_j . \qquad (7.9)$$

Using the fourier components $\exp(i\omega jh)$,the amplification factor λ is easily shown to be

$$\lambda(\omega h) = 1 + ip\, \lambda_{es} \sin(\omega h) - 2p^2 \lambda_{es} \sin^2(\frac{\omega h}{2}) . \qquad (7.10)$$

The Von Neumann condition $|\lambda(\omega h)| \leqslant 1$ gives rise to

$$\lambda_{es} (1 + (p^2 - 1) \sin^2(\frac{\omega h}{2})) \leqslant 1 , \qquad (7.11)$$

where

$$\lambda_{es}(\omega h) \geqslant 0 . \qquad (7.12)$$

This stability condition is nearly identical to the stability condition for the generalized one-step Runge-Kutta scheme (2.12). By choosing $\alpha = p^2 - 1$ and $k_0 = 1$, we can apply the smoothing process, which is described in the Section 5. Hence, we obtain the stability condition (cf. 5.17)

$$p^2 \leqslant 1 + \sum_{k=k_0}^{m} \mu_k 4^{(k-k_0+1)} . \qquad (7.13)$$

However, the coefficients μ_k have to be chosen in a different way. The conditions (6.13a) and (6.13b) remain the same, but the condition (6.13c) has to be adapted. Due to the fact that the first smoothing factor ($= S_{k_0} = S_1$) has only a small influence on the stability condition, we will not impose an extra condition. However, starting with the second smoothing factor we require that

$$q^{2(k-k_0)} \leqslant 1 + \sum_{i=k_0}^{k} \mu_k 4^{(i-k_0+1)} , \quad k = k_0+1,...,m , \tag{7.14}$$

where

$$q = p^{\frac{1}{m-k_0}} \quad \text{and} \quad m = [\,^2\log p\,] + 1 . \tag{7.15}$$

This can be rewritten to

$$\mu_k \geqslant \frac{q^{2(k-k_0)} - 1 - \sum_{i=k_0}^{k-1} \mu_i 4^{(i-k_0+1)}}{4^{(k-k_0+1)}} , \quad k = k_0+1,...,m . \tag{6.13c'}$$

where

$$\alpha = p^2 - 1 \ , \ k_0 = 1 \text{ and } \beta_{k_0-1} = \beta_0 = \alpha \ . \tag{7.16}$$

8. Numerical illustration

8.1. A linear problem

Consider the linear test problem

$$u_t = u_x - (16\pi / L)\cos(32\pi x / L) , \quad 0<t<T, \quad 0<x<L , \tag{8.1}$$

and the initial condition

$$u(x,0) = \frac{1}{2}\sin(2\pi x / L) + \frac{1}{2}\sin(32\pi x / L) . \tag{8.2}$$

The exact solution is given by

$$u(x,t) = \frac{1}{2}\sin(2\pi(x+t)/L) + \frac{1}{2}\sin(32\pi x / L) , \tag{8.3}$$

where $L = 100$.

For the space mesh we have chosen $\Delta x = L / 384$. We assume that the solution $u(x,t)$ is periodic on the interval $[0,L]$. The solution consists of a non-stationary part, which is slowly varying both in the time and in the space variable, and a stationary part which varies rapidly in the space variable only. Therefore, the numerical approximation of the stationary part needs a finer space mesh than the non-stationary part. This fine space mesh severely restricts the integration step.

Here, we will give the results for the following methods. At first, we use the stabilized generalized one-step Runge-Kutta method, which is described in (6.12) and (6.13). Next we use the stabilized Lax-Wendroff scheme of the previous section. Finally we compare the methods with the Crank-Nicolson method. To measure the obtained accuracy we define

$$\text{cd} = -\,^{10}\log(\,|\text{ maximal global error at the endpoint } t = T\,|\,) , \tag{8.4}$$

denoting the number of correct digits in the numerical approximation at the endpoint. In Table 8.1 we give the cd-values of the two methods, obtained at the endpoint $T = 2.8 \times 128$, the central processing time (in sec.) and in brackets [ ] the number of smoothing factors.

Δt	Runge-Kutta		Lax-Wendroff		Crank-Nicolson	
	cd	c.p.time	cd	c.p.time	cd	c.p.time
0.7	2.0[2]	12.0	2.2[3]	17.6	1.9	27.3
1.4	1.9[3]	7.0	2.0[4]	9.8	1.7	13.7
2.8	1.5[4]	4.1	1.5[5]	5.5	1.4	6.9
5.6	0.9[5]	2.3	0.9[6]	3.1	0.9	3.4
11.2	0.3[6]	1.4	0.4[7]	1.8	0.4	1.8

Table 8.1: Numerical results for a linear test problem using an explicit smoothing operator with $T = 2.8 \times 128$, $h = 100/384$.

Concerning accuracy, the results clearly show that the three methods are comparable. When the integration step increases, the results for all methods develop in the same way. At first the number of correct digits decreases slightly. These errors are mainly due to the stationary part of the solution, which is independent of the integration step. However, when the integration step becomes larger than about 1.4, the errors due to the non-stationary part become larger. Hence, the number of correct digits decreases rapidly. The c.p.time reduction factor is roughly 1.7 for this problem. So the computational costs of the explicit smoothing operator are relatively low with respect to the right-hand side evaluations.

8.2. A non-linear problem

In this section, we will use the stabilization technique for a non-linear equation. The problem is given by

$$u_t = uu_x + g(x,t), \quad 0<t<T, \quad 0<x<L, \tag{8.5}$$

where $L = 100$. The function g is chosen such that we have a solution consisting of a part, which is slowly varying both in the time and in the space variable, and a part which varies relatively rapidly in the space variable only. The solution is given by

$$u(x,t) = \frac{1}{2}\sin(2\pi(x+t)/L) + \frac{1}{2}\sin(8\pi x/L). \tag{8.6}$$

Hence, the function g follows to be

$$g(x,t) = (2\pi/L)\{\frac{1}{2}\cos(2\pi(x+t)/L) - [\frac{1}{2}\sin(2\pi(x+t)/L) + \frac{1}{2}\sin(8\pi x/L)] * [\frac{1}{2}\cos(2\pi(x+t)/L) + 2\cos(8\pi x/L)]\}. \tag{8.7}$$

The initial condition is taken from the exact solution (8.6). We use the following method: At first, we use the improved Euler method ([2]), which is a second-order two-step Runge Kutta method. The non-linear term uu_x is discretized by

$$\frac{(U_{j+1} + 2U_j + U_{j-1})}{4} * \frac{(U_{j+1} - U_{j-1})}{2h} = \frac{(U_{j+1} + U_j)^2 - (U_j + U_{j-1})^2}{8h} . \quad (8.8)$$

Next we use the MacCormack scheme ([4],p.179), which is a Lax-Wendroff type method written as a two-step process. This scheme is given by

$$\begin{aligned} \tilde{U}_j &= U_j^n + \tau F_j(\mathbf{U},t) , \\ U_j^{n+1} &= \tfrac{1}{2}(U_j^n + \tilde{U}_j) + \tfrac{1}{2}\tau F_{j-1}(\tilde{\mathbf{U}},t+\tau) , \end{aligned} \quad (8.9)$$

where

$$F_j(\mathbf{U},t) = \frac{(U_{j+1}^2 - U_j^2)}{2h} + \frac{g(jh,t) + g((j+1)h,t)}{2} . \quad (8.10)$$

We now give the results in the same form as in Table 8.1.

Δt	Runge-Kutta		Mac-Cormack	
	cd	c.p.time	cd	c.p.time
0.8	2.0[2]	137.7	2.2[3]	243.4
1.6	1.5[3]	71.2	1.6[4]	124.6
3.2	1.2[4]	37.3	1.3[5]	63.2
6.4	1.4[5]	19.4	1.4[6]	32.5
12.8	0.8[6]	10.2	0.8[7]	16.7

Table 8.2: Numerical results for a non-linear test problem using an explicit smoothing operator with $T = 128 \times 8$, $h = 100/384$.

Globally, we observe the same effect for the non-linear problem as for the linear test problem. At first, the error due to the non-stationary part is negligible with respect to the stationary part. When the integration step increases, the error due to the non-stationary part becomes significant.

Concerning the computational costs, for every applied smoothing the reduction factor is roughly two. So the costs of the explicit smoothing factors are negligible with respect to the right-hand side evaluations. This is due to the expensive cosine and sine evaluations in (8.5). But also in the case of a cheap right-hand side function (e.g. the linear test problem (8.1)), the application of a smoothing operator is worth-while.

9. Conclusion

We have set up a theory for the stabilization of the Lax-Wendroff method and a generalized one-step Runge-Kutta method for initial-value problems. Using the smoothing technique, the integration step is not limited by stability considerations. Therefore the integration step may be freely chosen, because the number of smoothing factors and the coefficients μ_k are automatically adapted to ensure stability. Moreover, it is quite easy to implement the smoothing operator. By its simplicity, it can be easily added to existing programs.

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