JACOBI SERIES AND APPROXIMATION

H. BAVINCK

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ACADEMISCH PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE WISKUNDE EN NATUURWETENSCHAPPEN AAN DE UNIVERSITEIT VAN AMSTERDAM, OP GEZAG VAN DE RECTOR MAGNIFICUS, DR. A. DE FROE, HOOGLERAAR IN DE FACULTEIT DER GENEESKUNDE, IN HET OPENBAAR TE VERDEDIGEN IN DE AULA DER UNIVERSITEIT (TIJDELIJK IN DE LUTHERSE KERK, INGANG SINGEL 411, HOEK SPUI) OP WOENSDAG 12 APRIL 1972 DES VOORMIDDAGS TE 11.00 UUR

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CONTENTS

SUMMARY AND INTRODUCTION

This tract deals with the approximation theoretic aspects of summation methods for expansions in terms of Jacobi polynomials $R_n^{(\alpha,\beta)}(\cos\theta)$. By studying Jacobi polynomials a broad class of orthogonal polynomials is covered including Chebyshev polynomials ($\alpha=\beta=-\frac{1}{2}$), Legendre polynomials ($\alpha=\beta=0$) and Gegenbauer or ultraspherical polynomials ($\alpha=\beta$). For certain discrete values of α and β the Jacobi polynomials can be interpreted as spherical functions on compact symmetric spaces of rank 1 (Gangolli [25]). Especially, if $\alpha = \beta = (n-3)/2$ the Jacobi polynomials are zonal spherical harmonics on the unit sphere in R^n. When a function f is expanded in terms of Jacobi polynomials many summation methods for this Fourier-Jacobi series may be considered as approximation processes for the function f. The main object of this tract is to investigate the order of approximation of these processes and to characterize the functions which allow a certain order of approximation. Many of these processes exhibit the phenomenon of saturation, which is equivalent to the existence of an optimal order of approximation (the saturation order). For the summation methods treated in this tract the saturation order and the saturation class, that is the class of functions which can be approximated with the optimal order, are derived.

In recent years much progress has been made separately in both subjects combined in this tract, in Jacobi series as well as in approximation theory. The joint work of Askey and Wainger led to the discovery of convolution structures which give rise to Banach algebras for Jacobi series [5] and their duals, Jacobi coefficients [6]. Gasper [26 until 28, 3] determined the exact regions in the (α,β) plane, where these Banach algebras can be defined and where the generalized translation operator or its dual, which are used in the definition of the convolution, are positive operators. The convolution structure for Jacobi series is one of the basic tools in the present investigation. It will be used here only in the case, where the generalized translation operator is positive, that is if $\alpha \geq \beta$ and either $\beta \geq -\frac{1}{2}$ or $\alpha + \beta \geq 0$. However, most of the results dealt with in this work can easily be carried over to the case, where a weaker condition, uniform boundedness instead of positivity, is required for the generalized translation operator, that is if $\alpha \geq \beta$, $\alpha + \beta \geq -1$. Only at a few places the positivity is essential. But the author has restricted his considerations in order to keep the formulas as simple as possible, while all the interesting special cases are still covered. A survey of well-known facts on Jacobi polynomials and series is given in the first sections of chapter I. The convolution structure for Jacobi series is dealt with in section 1.4. Section 1.5 is devoted to a special class of Jacobi series. A more general class has been considered by the author in [8], but for the applications in this tract the easier class treated in section 1.5 suffices.

In approximation theory the concept of saturation has been introduced by Favard in 1947 (cf. [24]). During the following years many authors contributed to the subject and general methods were developed for determining the saturation classes of families of convolution operators on the real line, on the unit circle in the plane, on the n-dimensional unit sphere, on the Euclidean n-dimensional space, the n-dimensional torus, etc. For historical details the reader is referred to [18], section 12.6. Methods due to Peetre [39] of constructing intermediate spaces between two Banach spaces turned out to be a useful framework for characterizing saturation classes and classes of non-optimal approximation. For semi-groups of operators on Banach spaces Butzer and Berens used this setting in [16], while more general families of operators on Banach spaces were considered with these intermediate spaces in Berens [12], Butzer and Scherer [20] and in a number of papers. In concrete cases as are dealt with in this investigation, these general results on approximation processes on Banach spaces are quite useful. They indicate which particular inequalities or limit relations for the approximation processes in question are sufficient in order to draw conclusions about their saturation class and classes of non-optimal approximation. The theorems on approximation processes on Banach spaces, necessary for use in this tract are stated in chapter II. For the proofs the reader is referred to [12], [16], [20], [21].

In the first section of chapter III kernels and approximation kernels are introduced. If X is written for a member of a class of function spaces, which become Banach spaces by choosing suitable norms, the convolution of a function $f \in X$ with an approximation kernel furnishes an approximation process for f in the X norm. Next, summation methods for Jacobi series are defined and with each summation method a kernel is associated. If this kernel happens to be an approximation kernel then the Fourier-Jacobi series of a function $f \in X$, summed up by means of this method, converges to f in the X norm. This is the case for many classical summation methods.

The main part of chapter IV consists in proving theorems of the Jackson and the Bernstein type, which relate the smoothness of the function and the order of best approximation by polynomials to each other. The smoothness here is introduced by means of the modulus of continuity defined with respect to the generalized translation. If the space of continuous functions endowed with the supremum norm is considered, this modulus of continuity turns out to be equivalent with the ordinary symmetric modulus of continuity.

In chapter V the author investigates a number of more or less classical summation methods and characterizes the functions which allow a certain order of approximation by these processes. The methods of proof used in this chapter are indicated by the general theory on approximation processes on Banach spaces, stated in chapter II.

Processes generated by the convolution of a function $f \in X$ with a positive kernel are considered in chapter VI. If positivity of the kernel is assumed the conditions for norm convergence can be relieved considerably. On the other hand, the order of approximation by positive polynomial kernels is usually limited for non-constant functions. In section 6.3 a saturation theorem is given for processes generated by positive kernels, which satisfy a certain condition on the Fourier-Jacobi coefficients. By means of this theorem the saturation class and the saturation order of some kernels are determined in section 6.4.

Finally, certain classes of functions which were used in the preceding chapters are characterized in chapter VII.

Most of the results obtained in this tract were known for Fourier cosine series ($\alpha=\beta=-\frac{1}{2}$). However, at many places the way to generalize was not at all obvious, since many specific facts for cosines, used in the classical proofs, do not go over to Jacobi polynomials. Also, in [13] some results are proven for Laplace series, which include ultraspherical series ($\alpha=\beta$). But the investigations there are restricted to the saturation of the summation methods. In the present work the author has succeeded in characterizing almost all the classes of functions which arise as classes of optimal or non-optimal approximation for the summation methods under consideration in terms of the modulus of continuity of the functions.

NOTATIONS

Let $g(x)$ be defined and postive on $[a,b]$, let $f(x)$ be any function defined on $[a,b]$ and x_0 be an element of $[a,b]$. Then we mean by the notation $f(x) = O(g(x))$, $(x\to x_0)$, that there exists a positive constant M such that

$$\frac{|f(x)|}{g(x)} \leq M$$

in a neighbourhood of x_0 and by $f(x) = o(g(x))$, $(x\to x_0)$, that

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0.$$

If we write $f(x) \approx g(x)$, $(x\to x_0)$, we mean that there exist two positive constants M_1 and M_2 such that in a neighbourhood of x_0

$$M_1\, g(x) \leq f(x) \leq M_2\, g(x) \; .$$

We write $f(x) \simeq g(x)$, $(x\to x_0)$, if

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1 \; .$$

For two Banach spaces X and Y we write $X \cong Y$, if the spaces are equal and have equivalent norms. If we put $Y \subset X$ we mean a continuous embedding of Y in X. By $\mathbb{R}$ we denote the set of real numbers; for the real numbers > 0 we write $\mathbb{R}^+$. Let $\mathbb{Z}$ be the set of all integers, $\mathbb{P}$ the set of all nonnegative integers and $\mathbb{Z}^+$ the set of all natural numbers 1, 2, ... By $\mathbb{A}$ we mean either $\mathbb{R}^+$ or $\mathbb{Z}^+$.

LIST OF SYMBOLS

CHAPTER I JACOBI SERIES

The purpose of this chapter is to state the facts about Jacobi polynomials and Jacobi series which are needed in the following chapters. For the proofs of the formulas stated in the first two sections the reader is referred to Szegö's book [46]. Section 1.1 contains a number of formulas whereas in section 1.2 estimates are mentioned. The expansion of functions and finite measures in Jacobi series is treated in section 1.3. The next section deals with the generalized translation operator and the convolution structure for Jacobi series, which play a crucial rôle in this tract. In section 1.5 a special class of Jacobi series is studied. Theorem 1.5.4 is essential for the proof of the theorems concerning fractional integration and differentiation stated in section 5.5 and proved in Bavinck [8], section 5. The author has preferred to present these rather technical investigations at this stage of the tract, although some results on the Abel-Poisson summation method, treated in section 3.3, are needed.

1.1. FORMULAS

By $P_n^{(\alpha,\beta)}(x)$ we denote the Jacobi polynomial of degree n and order (α,β), defined by

$$(1.1.1)\qquad (1-x)^{\alpha}(1+x)^{\beta}\, P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\,[(1-x)^{\alpha+n}(1+x)^{\beta+n}],\quad \alpha,\beta > -1.$$

The polynomials in $\cos\theta$

$$R_n^{(\alpha,\beta)}(\cos\theta) = \frac{P_n^{(\alpha,\beta)}(\cos\theta)}{P_n^{(\alpha,\beta)}(1)} \qquad n \in \mathbf{P},$$

are orthogonal on $[0,\pi]$ with respect to the weight function

$$(1.1.2)\qquad \rho^{(\alpha,\beta)}(\theta) = (\sin\tfrac{\theta}{2})^{2\alpha+1}(\cos\tfrac{\theta}{2})^{2\beta+1}$$

and normalized to be 1 at $\theta = 0$. Hence,

$$(1.1.3)\qquad \int_0^{\pi} R_n^{(\alpha,\beta)}(\cos\theta)\, R_m^{(\alpha,\beta)}(\cos\theta)\, \rho^{(\alpha,\beta)}(\theta)d\theta = \delta_{n,m}[\omega_n^{(\alpha,\beta)}]^{-1},$$

where $\delta_{n,m} = 1$ if $n = m$ and $\delta_{n,m} = 0$ if $n \neq m$, and

$$(1.1.4)\qquad \omega_n^{(\alpha,\beta)} = \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)\Gamma(n+1)\Gamma(\alpha+1)\Gamma(\alpha+1)}\ .$$

For large n we have $\omega_n^{(\alpha,\beta)} \approx n^{2\alpha+1}$.

Also, $R_n^{(\alpha,\beta)}(\cos\theta)$ satisfies the differential equation

$$(1.1.5)\qquad -[\rho^{(\alpha,\beta)}(\theta)]^{-1}\frac{d}{d\theta}\{\rho^{(\alpha,\beta)}(\theta)\frac{d}{d\theta}R_n^{(\alpha,\beta)}(\cos\theta)\} =$$
$$= n(n+\alpha+\beta+1)\ R_n^{(\alpha,\beta)}(\cos\theta).$$

Occasionally, we shall use the notation

$$(1.1.6)\qquad P(\tfrac{d}{d\theta}) = -[\rho^{(\alpha,\beta)}(\theta)]^{-1}\frac{d}{d\theta}\{\rho^{(\alpha,\beta)}(\theta)\frac{d}{d\theta}\}$$

and

$$(1.1.7)\qquad \lambda_n = n(n+\alpha+\beta+1)\ .$$

For $R_n^{(\alpha,\beta)}(\cos\theta)$ we have the following representation as a hypergeometric function:

$$(1.1.8)\qquad R_n^{(\alpha,\beta)}(\cos\theta) = {}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;\sin^2\tfrac{\theta}{2})$$
$$= \sum_{k=0}^{n}(-1)^k\frac{\Gamma(n+k+\alpha+\beta+1)\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n-k+1)\Gamma(n+\alpha+\beta+1)\Gamma(k+\alpha+1)\Gamma(k+1)}\sin^{2k}\frac{\theta}{2}.$$

An easy consequence of (1.1.8) is the useful differentiation formula

$$(1.1.9)\qquad \frac{d}{dx}R_n^{(\alpha,\beta)}(x) = \frac{n(n+\alpha+\beta+1)}{2(\alpha+1)}R_{n-1}^{(\alpha+1,\beta+1)}(x).$$

We shall need the Christoffel-Darboux formula for Jacobi polynomials. In a special case it can be written in the form

$$(1.1.10)\qquad D_n^{(\alpha,\beta)}(x) \overset{\text{def}}{=} \sum_{k=0}^{n}\omega_k^{(\alpha,\beta)}R_k^{(\alpha,\beta)}(x)$$
$$= \frac{\alpha+1}{2n+\alpha+\beta+2}\,\omega^{(\alpha+1,\beta)}R_n^{(\alpha+1,\beta)}(x).$$

1.2. ESTIMATES

It is derived in Szegö [46], section 7.32, that for $\alpha \geq \max(\beta,-\frac{1}{2})$

(1.2.1) $$\sup_{0\le\theta\le\pi} |R_n^{(\alpha,\beta)}(\cos\theta)| = R_n^{(\alpha,\beta)}(1) = 1$$

and the following relation holds uniformly in any compact subinterval of $(0,\pi)$ ([46], (7.32.5)):

(1.2.2) $$R_n^{(\alpha,\beta)}(\cos\theta) = \theta^{-\alpha-\frac{1}{2}}(\pi-\theta)^{-\beta-\frac{1}{2}}\, O(n^{-\alpha-\frac{1}{2}}) \quad n\to\infty.$$

An important asymptotic formula is Hilb's formula ([46], (8.21.17)), which is valid uniformly for $0 \le \theta \le \pi-\varepsilon$:

(1.2.3) $$(\sin\tfrac{\theta}{2})^{\alpha}(\cos\tfrac{\theta}{2})^{\beta}\, R_n^{(\alpha,\beta)}(\cos\theta) = N^{-\alpha}\,\Gamma(\alpha+1)\left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}} J_\alpha(N\theta) + \begin{cases} \theta^{\frac{1}{2}}O(n^{-3/2-\alpha}), & n\to\infty,\ cn^{-1}\le\theta\le\pi-\varepsilon, \\ \theta^{\alpha+2}O(1), & n\to\infty,\ 0<\theta\le cn^{-1}, \end{cases}$$

where $N = n+\frac{1}{2}(\alpha+\beta+1)$ and $J_\alpha(z)$ denotes the Bessel function of order α.

We now prove the following inequalities, which we shall use in chapter VI:

(1.2.4) $$1 - R_n^{(\alpha,\beta)}(\cos\theta) \le \frac{n(n+\alpha+\beta+1)}{\alpha+1}\sin^2\frac{\theta}{2} \qquad 0\le\theta\le\pi,$$

(1.2.5) $$c_\alpha\frac{n(n+\alpha+\beta+1)}{\alpha+1}\sin^2\frac{\theta}{2} \le 1 - R_n^{(\alpha,\beta)}(\cos\theta) \qquad 0\le\theta\le\varepsilon,$$

where $c_\alpha \in \mathbb{R}^+$ is suitably chosen and $0 < \varepsilon < 4/(2n+\alpha+\beta+2)$.
By the mean-value theorem and formula (1.1.9) we obtain

(1.2.6) $$1 - R_n^{(\alpha,\beta)}(\cos\theta) = \frac{n(n+\alpha+\beta+1)}{\alpha+1}\sin^2\frac{\theta}{2}\, R_{n-1}^{(\alpha+1,\beta+1)}(\cos\bar\theta),$$
$$0\le\bar\theta\le\theta.$$

Formula (1.2.4) is a direct consequence of (1.2.6) and (1.2.1). For the proof of (1.2.5) we use (1.2.3). The power series expansion of $(\frac{z}{2})^{-\alpha} J_\alpha(z)$ has terms which have alternating signs and are monotonically decreasing for real z, $0 \le z \le 2$. Hence, for $N\to\infty$

$$R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta) \geq \Gamma(\alpha+2)\left(\frac{2}{N\theta}\right)^{\alpha+1} J_{\alpha+1}(N\theta) + \theta^2 O(1) \quad 0 \leq \theta < 2N^{-1},$$

$$\geq 1 - \frac{(N\theta)^2}{4(\alpha+2)} + \theta^2 O(1)$$

$$(1.2.7) \qquad > \frac{\alpha+1}{\alpha+2} - O(N^{-2}).$$

The inequality (1.2.5) follows from (1.2.6) and (1.2.7) for $n \geq n_0$. On the other hand, the constant c_α can be chosen such that (1.2.5) remains valid for $0 \leq n \leq n_0$.

We shall also need the following estimate, which can be derived from (1.2.1), (1.2.2) and (1.2.3) in a way similar to Szegö [46], (7.34.1):

$$(1.2.8) \qquad \int_0^\pi |R_n^{(\alpha+\lambda,\beta)}(\cos\theta)|^p \rho^{(\alpha+\mu,\beta)}(\theta)d\theta \approx$$

$$\approx \begin{cases} n^{-2\alpha-2\mu-2} & p\lambda > (2-p)\alpha+2\mu+2-p/2 \\ n^{-2\alpha-2\mu-2}\log n & p\lambda = (2-p)\alpha+2\mu+2-p/2 \\ n^{-p/2-\alpha p-\lambda p} & p\lambda < (2-p)\alpha+2\mu+2-p/2 \end{cases}$$

with $\lambda \geq 0$, $\mu \geq 0$, $1 \leq p < \infty$, $\alpha+\lambda \geq \beta$, $n \to \infty$.

1.3. EXPANSIONS IN TERMS OF JACOBI POLYNOMIALS

In this tract we shall be concerned with the summation of the Jacobi series which are associated with functions belonging to certain spaces of functions on [-1,1]. By C we denote the space of continuous functions, L^∞ is written for the essentially bounded functions and the L^p spaces are introduced with respect to the weight function (1.1.2) with $x = \cos\theta$. We call M the space of all regular finite Borel measures on [-1,1]. The spaces C, L^p $(1 \leq p \leq \infty)$ and M are Banach spaces if they are endowed with the following norms

$$||f||_C = \sup_{0\leq\theta\leq\pi} |f(\cos\theta)|,$$

$$||f||_p = [\int_0^\pi |f(\cos\theta)|^p \rho^{(\alpha,\beta)}(\theta)d\theta]^{1/p}, \qquad 1 \leq p < \infty,$$

$$||f||_\infty = \operatorname*{ess\,sup}_{0\le\theta\le\pi} |f(\cos\theta)|,$$

$$||\mu||_M = \int_0^\pi |d\mu(\cos\theta)|.$$

With an element of one of the spaces C or L^p ($1\le p\le\infty$) we associate an expansion in terms of Jacobi polynomials, the so-called Fourier-Jacobi expansion

$$(1.3.1) \qquad f(\cos\theta) \sim \sum_{n=0}^{\infty} f^\wedge(n)\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta),$$

where $\omega_n^{(\alpha,\beta)}$ is defined by (1.1.4) and

$$(1.3.2) \qquad f^\wedge(n) = \int_0^\pi f(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\theta)\, \rho^{(\alpha,\beta)}(\theta)d\theta, \qquad n\in \mathrm{P}.$$

The numbers $f^\wedge(n)$ are called the Fourier-Jacobi coefficients of f.

With a measure $\mu \in M$ we associate the Jacobi-Stieltjes expansion

$$(1.3.3) \qquad d\mu(\cos\theta) \sim \sum_{n=0}^{\infty} \mu^\vee(n)\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta),$$

where

$$(1.3.4) \qquad \mu^\vee(n) = \int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)\, d\mu(\cos\theta), \qquad n\in \mathrm{P}.$$

The numbers $\mu^\vee(n)$ are called the Jacobi-Stieltjes coefficients of μ. The expansions (1.3.1) and (1.3.3) are only formal expansions, which need not to converge. In chapter III we shall direct our attention to the norm convergence of these series and general methods will be investigated for the summation.

1.4. CONVOLUTION STRUCTURE

One of the main tools in this tract is the convolution structure for Jacobi series. In the case of ultraspherical polynomials ($\alpha=\beta$) there is an old addition formula due to Gegenbauer [29], which gives rise to a type of convolution, introduced by Gelfand [30] and Bochner [14]. Later Gangolli [25] discovered a convolution of this type for Jacobi series with these values of α and β, for which the Jacobi polynomials can be interpreted as spherical functions. For general Jacobi series ($\alpha\geq\beta\geq-\frac{1}{2}$) the convolution structure was found by Askey and Wainger [5], who used asymptotic formulas for Jacobi polynomials to prove the uniform boundedness of the generalized translation operator T_ϕ. This operator maps a function f belonging to one of the spaces C or L^p ($1\leq p\leq\infty$) and with Fourier-Jacobi expansion (1.3.1) into

$$(1.4.1) \qquad T_\phi\, f(\cos\theta) \sim \sum_{n=0}^{\infty} f^\wedge(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\phi).$$

It was pointed out by Gasper [27] that the operator T_ϕ is a positive operator, that is if $f(\cos\theta)\geq 0$ almost everywhere on $[0,\pi]$ then $T_\phi\, f(\cos\theta)\geq 0$ almost everywhere on $[0,\pi]$.By the orthogonality of the polynomials $R_n^{(\alpha,\beta)}(x)$ and the positivity of the operator it follows that T_ϕ has operator norm 1. Gasper gave an explicit formula for $R_n^{(\alpha,\beta)}(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\phi)$ as an integral with Bessel functions, from which resulted the positivity of the operator T . Recently, Koornwinder [34] has discovered the generalization to Jacobi polynomials of Gegenbauer's addition formula for ultraspherical polynomials by means of group theoretical considerations. His formula enables us to write the generalized translation T_ϕ f in the form

$$T_\phi\, f(\cos\theta) = \frac{1}{\mu_{\alpha,\beta}} \int_{r=0}^{1}\int_{\psi=0}^{\pi} f(2|\cos\tfrac{\theta}{2}\cos\tfrac{\phi}{2} + \sin\tfrac{\theta}{2}\sin\tfrac{\phi}{2}\, re^{i\psi}|^2-1)d\mu_{\alpha,\beta},$$

where

$$d\mu_{\alpha,\beta} = (1-r^2)^{\alpha-\beta-1}\, r^{2\beta+1}(\sin\psi)^{2\beta}\, dr d\psi \qquad (\alpha>\beta>-\tfrac{1}{2}).$$

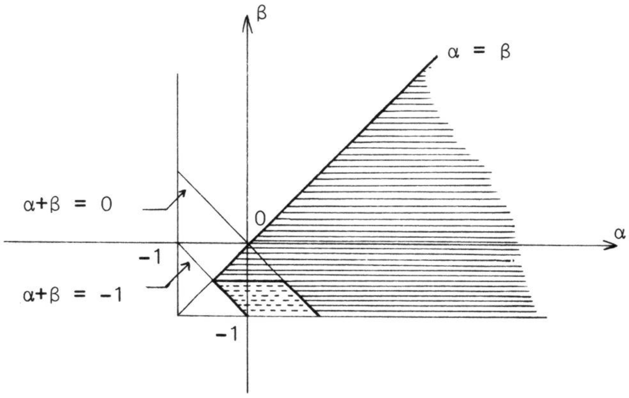
From this representation the positivity of the operator T_ϕ is obvious. Finally, Gasper showed in [28], that the operator T_ϕ is uniformly bounded if and only if $\alpha\geq\beta$, $\alpha+\beta\geq -1$. Moreover, he proved that T_ϕ is a positive operator if and only if $\alpha\geq\beta$ and either $\beta\geq -\frac{1}{2}$ or $\alpha+\beta\geq 0$. In the rest of this tract we shall always assume $\alpha\geq\beta$ and either $\beta\geq-\frac{1}{2}$ or $\alpha+\beta\geq 0$.
Hence, we have

$$(1.4.2) \qquad ||T_\phi\, f(\cdot)|| \leq ||f(\cdot)|| \qquad \text{in C and } L^p\ (1\leq p\leq\infty)$$

and by the Banach-Steinhaus theorem (Hille-Phillips [32], p. 41) we may also conclude that

$$(1.4.3) \qquad \lim_{\phi\to 0^+} ||T_\phi\, f(\cdot) - f(\cdot)|| = 0 \qquad \text{in C and } L^p\ (1\leq p<\infty),$$

since (1.4.3) holds for polynomials and the polynomials are dense in the spaces C and L^p $(1 \leq p < \infty)$.



region where T_ϕ is bounded

region where T_ϕ is uniformly bounded but not positive.

Following Askey and Wainger [5] we define for $f_1, f_2 \in L^1$ the convolution $f_1 * f_2$ by

$$(1.4.4) \qquad (f_1 * f_2)(\cos\theta) = \int_0^\pi (T_\phi f_1(\cos\theta))\, f_2(\cos\phi)\, \rho^{(\alpha,\beta)}(\phi)d\phi.$$

This convolution has the following properties (see Askey and Wainger [5]).

1.4.1. Proposition. Let $f_1, f_2, f_3 \in L^1$. Then $f_1 * f_2 \in L^1$ and

i) $\qquad f_1 * f_2 = f_2 * f_1$,

ii) $\qquad f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3$,

iii) $\qquad (f_1 * f_2)^\wedge(n) = f_1^\wedge(n)\, f_2^\wedge(n)$,

iv) $\qquad$ If $g \in L^p$ $(1 \leq p \leq \infty)$, then $f_1 * g \in L^p$ and $||f_1 * g||_p \leq ||f||_1 ||g||_p$.

By F. Riesz's representation theorem the space M is the dual space of C. We use this fact to give an implicit definition of the convolution of measures. Suppose $\mu, \nu \in M$ and $f \in C$. Then the map

$$f \to \int_0^\pi \int_0^\pi T_\phi f(\cos\theta) d\mu(\cos\theta) d\nu(\cos\phi)$$

defines a bounded linear functional on C and thus there exists a unique measure $\mu * \nu$ such that

$$\int_0^{\pi} f(\cos\theta)d(\mu*\nu)(\cos\theta) = \int_0^{\pi}\int_0^{\pi} T_{\phi} f(\cos\theta)d\mu(\cos\theta)d\nu(\cos\phi).$$

The following properties are easily verified:

1.4.2. Proposition. Let $\mu_1,\mu_2,\mu_3 \in M$. Then $\mu_1*\mu_2 \in M$ and

i) $\mu_1 * \mu_2 = \mu_2 * \mu_1$,

ii) $\mu_1 * (\mu_2*\mu_3) = (\mu_1*\mu_2) * \mu_3$,

iii) $(\mu_1*\mu_2)^{\vee}(n) = \overset{\vee}{\mu}_1(n)\overset{\vee}{\mu}_2(n)$,

iv) $||\mu_1*\mu_2||_M \leq ||\mu_1||_M\ ||\mu_2||_M$.

There is an obvious embedding of L^1 into M, namely $f \to m_f$, where $dm_f = f(\cos\theta)\ \rho^{(\alpha,\beta)}(\theta)d\theta$. The space L^1 consists of all the measures which are absolutely continuous with respect to $\rho^{(\alpha,\beta)}(\theta)d\theta$. The convolution defined for L^1 coincides with the convolution defined for M restricted to the absolutely continuous measures. In fact,

$$d(m_{f_1}*m_{f_2})(\cos\theta) = \{\int_0^{\pi} T_{\phi}\ f_1(\cos\theta)f_2(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi\}\rho^{(\alpha,\beta)}(\theta)d\theta.$$

If $f \in L^p$ $(1\leq p\leq\infty)$ and $\mu \in M$ we define $f * \mu$ by

$$(f*\mu)(\cos\theta) = \int_0^{\pi} T_{\phi}\ f(\cos\theta)d\mu(\cos\phi). \tag{1.4.5}$$

It follows that $m_{f*\mu} = m_f*\mu$ and that $f*\mu \in L^p$ satisfying

$$||f*\mu||_p \leq ||f||_p\ ||\mu||_M. \tag{1.4.6}$$

Remark. By using the well-known relation (Szegö [46], (4.1.3))

$$P_n^{(\alpha,\beta)}(x) = (-1)^n\ P_n^{(\beta,\alpha)}(-x)$$

an analogous Banach algebra can be obtained in the case $\beta \geq \alpha$ and either $\alpha \geq -\frac{1}{2}$ or $\alpha+\beta \geq 0$, if one considers the polynomials

$$S_n^{(\alpha,\beta)}(\cos\theta) = \frac{P_n^{(\alpha,\beta)}(\cos\theta)}{P_n^{(\alpha,\beta)}(-1)}$$

and defines the generalized translation and the convolution in a way similar to (1.4.2) and (1.4.4).

1.5. A SPECIAL CLASS OF JACOBI SERIES

In Bavinck [8] Jacobi series of the form

$$(1.5.1) \qquad \sum_{n=1}^{\infty} n^{-\gamma}\, b(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)$$

are studied, where $\gamma \in \mathbb{R}^+$ and $b(n)$ is a slowly varying function, such as a power of log n. For the applications in this tract we only need the results in the case $b(n) = 1$ for all $n \in \mathbb{Z}^+$. In this much easier case we shall treat this class of Jacobi series here. For trigonometric series of the form (1.5.1), the case $\alpha = \beta = -\frac{1}{2}$, we refer to Zygmund [51], ch. V. Askey and Wainger [4] have investigated ultraspherical series ($\alpha=\beta$) of this type.

One of the main techniques we use is summation by parts. We have the following lemma.

1.5.1. Lemma. Let $a(n)$ be a function defined on $\mathbb{Z}^+$ and let there exist an $\varepsilon > 0$ such that $a(n) = O(e^{-\varepsilon n})$ as $n \to \infty$. Let

$$H(\cos\theta) = \sum_{n=1}^{\infty} a(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

Then

$$(1.5.2) \qquad H(\cos\theta) = \sum_{n=1}^{\infty} (a(n)-a(n+1)) \frac{(\alpha+1)}{(2n+\alpha+\beta+2)}\, \omega_n^{(\alpha+1,\beta)} R_n^{(\alpha+1,\beta)}(\cos\theta).$$

Proof. We first take the sum

$$H_N(\cos\theta) = \sum_{n=1}^{N} a(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

By (1.1.10) summation by parts yields

$$H_N(\cos\theta) = \sum_{n=1}^{N-1} (a(n)-a(n+1)) \frac{(\alpha+1)}{(2n+\alpha+\beta+2)} \omega_n^{(\alpha+1,\beta)} R_n^{(\alpha+1,\beta)}(\cos\theta)$$

$$+ a(N) \frac{(\alpha+1)}{(2N+\alpha+\beta+2)} \omega_N^{(\alpha+1,\beta)} R_N^{(\alpha+1,\beta)}(\cos\theta).$$

Relation (1.5.2) follows by taking the limit as $N \to \infty$, since $\omega_N^{(\alpha+1,\beta)}$ does not grow faster than a polynomial in N and $|R_N^{(\alpha+1,\beta)}(\cos\theta)|$ is bounded by 1 (see (1.1.4) and (1.2.1)).

In the case $\gamma > \alpha + \frac{3}{2}$ and $b(n) = 1$, the behaviour of the series (1.5.1) is described in the following lemma.

1.5.2. Lemma. Let $\gamma > \alpha + \frac{3}{2}$. Then the series

$$(1.5.3) \qquad \sum_{n=1}^{\infty} n^{-\gamma} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)$$

converges absolutely and uniformly in any compact subinterval of $(0,\pi)$. If we call $F(\cos\theta)$ the sum of (1.5.3), then $F(\cos\theta)$ is continuous for $0 < \theta < \pi$ and

$$(1.5.4) \qquad F(\cos\theta) \simeq \frac{\Gamma(\alpha+1-\frac{\gamma}{2})}{\Gamma(\frac{\gamma}{2})\Gamma(\alpha+1)} (\sin\tfrac{\theta}{2})^{\gamma-2\alpha-2} \qquad (\theta\to 0^+).$$

Proof. The fact that (1.5.3) converges uniformly and absolutely in any compact subinterval of $(0,\pi)$ follows from (1.1.4) and (1.2.2). This implies the continuity of $F(\cos\theta)$ for $0 < \theta < \pi$. Moreover, the following formula is easily derived from (1.1.1) (cf. Szegö [46], (9.3.11):

$$(1.5.5) \qquad \frac{\Gamma(\alpha+1-\frac{\gamma}{2})}{\Gamma(\frac{\gamma}{2})\Gamma(\alpha+1)} (\sin\tfrac{\theta}{2})^{\gamma-2\alpha-2} =$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+1-\frac{\gamma}{2})}{\Gamma(n+\alpha+1)\Gamma(n+\beta+\frac{\gamma}{2}+1)} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta) \qquad (\gamma>\alpha+\tfrac{3}{2}).$$

For any positive j there exists a set of numbers μ_k $(k=1,2,\dots,j)$ such that

$$\frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+1-\frac{\gamma}{2})}{\Gamma(n+\alpha+1)\Gamma(n+\beta+\frac{\gamma}{2}+1)} = \frac{1}{n^{\gamma}} + \sum_{k=1}^{j} \mu_k \frac{\Gamma(n+\beta+1)\Gamma(n+\alpha-\frac{\gamma-k}{2}+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+\frac{\gamma+k}{2}+1)} + O(n^{-\gamma-j-1})$$

as $n \to \infty$. If we choose $j > 2\alpha-\gamma+1$ the Jacobi series with the coefficients $O(n^{-\gamma-j-1})$ converges absolutely and hence (1.5.4) follows from (1.5.5).

1.5.3. Lemma. Let $F_\varepsilon(\cos\theta)$ be given by

$$F_\varepsilon(\cos\theta) = \sum_{n=1}^{\infty} n^{-\gamma} e^{-\varepsilon n} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)} \qquad (\varepsilon\in R^+,\ \gamma\in R).$$

Then for $\nu \in Z^+$ we may write

$$F_\varepsilon(\cos\theta) = M_\varepsilon(\cos\theta) + E_{1,\varepsilon}(\cos\theta) + E_{2,\varepsilon}(\cos\theta),$$

where

$$(1.5.6) \qquad M_\varepsilon(\cos\theta) = \frac{\Gamma(\alpha+\nu+1)\Gamma(\frac{\gamma}{2}+\nu)}{\Gamma(\alpha+1)\Gamma(\frac{\gamma}{2})} \sum_{n=1}^{\infty} e^{-\varepsilon n} n^{-\gamma-2\nu} \cdot \omega_n^{(\alpha+\nu,\beta)} R_n^{(\alpha+\nu,\beta)}(\cos\theta),$$

$$(1.5.7) \qquad E_{1,\varepsilon}(\cos\theta) = \sum_{j=1}^{\nu} c_j^{(\nu)} \sum_{n=1}^{\infty} e^{-\varepsilon n} \varepsilon^j n^{-\gamma-2\nu+j} \cdot \omega_n^{(\alpha+\nu,\beta)} R_n^{(\alpha+\nu,\beta)}(\cos\theta),$$

$$(1.5.8) \qquad E_{2,\varepsilon}(\cos\theta) = \sum_{j=0}^{\nu} \sum_{k=0}^{\nu} \sum_{n=1}^{\infty} d_{j,k,n}^{(\nu)} e^{-\varepsilon n} \varepsilon^j n^{-\gamma-2\nu+j-1-k} \cdot \omega_n^{(\alpha+\nu,\beta)} R_n^{(\alpha+\nu,\beta)}(\cos\theta).$$

Here $c_j^{(\nu)}$ and $d_{j,k,n}^{(\nu)}$ belong to R. Moreover, $|d_{j,k,n}^{(\nu)}| < M^{(\nu)}$, where $M^{(\nu)}$ is only dependent on ν.

Proof. We apply ν times lemma 1.5.1, using the mean-value theorem to replace the differences by derivatives and noting that

$$(1.5.9) \qquad \left|\frac{1}{2n+\alpha+\beta+2} - \frac{1}{2n}\right| \leq \frac{\alpha+\beta+2}{4n^2}.$$

We then obtain

$$(1.5.10) \qquad F_\varepsilon(\cos\theta) = \frac{(-1)^\nu}{2^\nu} \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)} \sum_{n=1}^{\infty} [(\frac{1}{t}\frac{d}{dt})^\nu (t^{-\gamma} e^{-\varepsilon t})]_{t=n} \cdot \omega_n^{(\alpha+\nu,\beta)} R_n^{(\alpha+\nu,\beta)}(\cos\theta) + E_{2,\varepsilon}(\cos\theta).$$

The error term $E_{2,\varepsilon}(\cos\theta)$ can be written in the form (1.5.8). The summation over the variable k is due to the right-hand side in (1.5.9) whereas the summation over j accounts for the error made by replacing differences by derivatives, since

$$\Delta^{\nu}(n^{-\gamma}e^{-\varepsilon n}) = \frac{d^{\nu}}{dt^{\nu}}(t^{-\gamma}e^{-\varepsilon t})\Big|_{t=n+\theta\nu} \qquad 0 < \theta < 1$$

$$= \frac{d^{\nu}}{dt^{\nu}}(t^{-\gamma}e^{-\varepsilon t})\Big|_{t=n} + \theta\nu \frac{d^{\nu+1}}{dt^{\nu+1}}(t^{-\gamma}e^{-\varepsilon t})\Big|_{t=n+\theta_1\nu}, \quad 0 < \theta_1 < 1.$$

The main term $M_\varepsilon(\cos\theta)$ arises from taking derivatives only on powers of t in (1.5.10). $E_{\varepsilon,1}(\cos\theta)$ consists of the remaining terms.

1.5.4. Theorem. Let $\gamma \in R$ be such that $\gamma < 2\alpha + 2$ and $\gamma \neq 0, -2, -4, \ldots$. For $\varepsilon > 0$ we define

$$F_\varepsilon(\cos\theta) = \sum_{n=1}^{\infty} n^{-\gamma}e^{-\varepsilon n}\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta).$$

Then, for $\theta \neq 0$

$$F(\cos\theta) = \lim_{\varepsilon\to 0^+} F_\varepsilon(\cos\theta)$$

exists in the pointwise sense. Also, $F(\cos\theta)$ is continuous on $0 < \theta \leq \pi$ and

$$(1.5.11) \qquad F(\cos\theta) \simeq \frac{\Gamma(\alpha+1-\frac{\gamma}{2})}{\Gamma(\frac{\gamma}{2})\Gamma(\alpha+1)}(\sin\tfrac{\theta}{2})^{\gamma-2\alpha-2} \qquad (\theta\to 0^+).$$

Finally, if $\gamma > 0$ then $F(\cos\theta) \in L^1$ and

$$F(\cos\theta) \sim \sum_{n=1}^{\infty} n^{-\gamma}\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta).$$

Proof. We choose $\nu \in Z^+$ so large that $\gamma + \nu > \alpha + \frac{5}{2}$ and we apply lemma 1.5.3. It follows immediately from (1.2.2) that the series $M_\varepsilon(\cos\theta)$, defined in (1.5.6) converges uniformly in any closed subinterval of $(0,\pi)$. $M_\varepsilon(\cos\theta)$ with $\varepsilon = 0$ is a series of the type treated in lemma 1.5.2. Since the Abel-Poisson means (see section 3.3) define a summation process which converges uniformly to the sum of the series, whenever it exists and is continuous (this can be proved by an argument similar to theorem 3.1.4), we find that $M_0(\cos\theta) = \lim_{\varepsilon\to 0^+} M_\varepsilon(\cos\theta)$ exists and is continuous on $0 < \theta < \pi$. Moreover, by (1.5.4) we have

$$M_0(\cos\theta) \simeq \frac{\Gamma(\alpha+1-\frac{\gamma}{2})}{\Gamma(\frac{\gamma}{2})\Gamma(\alpha+1)} (\sin\frac{\theta}{2})^{\gamma-2\alpha-2} \qquad (\theta\to 0^+).$$

We now investigate $E_{1,\varepsilon}(\cos\theta)$, defined by (1.5.7). Application of (1.2.2) and the obvious estimate

$$(1.5.12) \qquad \varepsilon^j e^{-\varepsilon n} = O(n^{-j}) \qquad (n\to\infty,\ j\in R^+)$$

yields

$$E_{1,\varepsilon}(\cos\theta) = \varepsilon\theta^{-\alpha-\nu-\frac{1}{2}}(\pi-\theta)^{-\beta-\frac{1}{2}}\, O(\sum_{j=1}^{\nu}\sum_{n=1}^{\infty} n^{-\gamma+\alpha-\nu+3/2}).$$

Thus $E_{1,\varepsilon}(\cos\theta)$ converges uniformly in any closed subinterval of $(0,\pi)$. Moreover, we see that for $0 < \theta < \pi$, the function $E_{1,\varepsilon}(\cos\theta)$ tends to zero as $\varepsilon \to 0^+$.

Finally, we consider $E_{2,\varepsilon}(\cos\theta)$. Since $E_{2,\varepsilon}(\cos\theta)$ contains terms similar to those of $M_\varepsilon(\cos\theta)$ and $E_{1,\varepsilon}(\cos\theta)$, except that the exponent of n may be lower, we may apply a reasoning similar to that in the previous terms. We find, that $E_{2,\varepsilon}(\cos\theta)$ is a series which converges uniformly in any closed subinterval of $(0,\pi)$. Also, we find that

$$E_{2,0}(\cos\theta) = \lim_{\varepsilon\to 0^+} E_{2,\varepsilon}(\cos\theta)$$

exists and is continuous for $0 < \theta < \pi$. Furthermore, as $\theta \to 0^+$,

$$E_{2,0}(\cos\theta) = o(\theta^{\gamma-2\alpha-2}).$$

We now examine the behaviour of $F(\cos\theta)$ near $\theta = \pi$. It suffices to show that $F_\varepsilon(\cos\theta)$ converges uniformly to $F(\cos\theta)$ as $\varepsilon \to 0^+$ for θ sufficiently close to π. For $\theta = \pi$ the convergence follows from theorem 7 of Wainger [50], with $x = \frac{\pi}{2}$. We use the Bateman integral (see Askey and Fitch [2], formula 3.4)

$$(1+x)^\beta \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(-1)} = \frac{\Gamma(\beta+1)}{\Gamma(\frac{1}{2})\Gamma(\beta+\frac{1}{2})}\int_{-1}^{x} (1+y)^{-\frac{1}{2}} \frac{P_n^{(\alpha+\beta+\frac{1}{2},-\frac{1}{2})}(y)}{P_n^{(\alpha+\beta+\frac{1}{2},-\frac{1}{2})}(-1)} (x-y)^{\beta-\frac{1}{2}}\, dy$$

or, writing $x = 2u^2-1$, $y = 2z^2-1$,

$$\frac{u^{2\beta}}{\Gamma(n+\beta+1)} P_n^{(\alpha,\beta)}(2u^2-1) = \frac{2}{\Gamma(\beta+\frac{1}{2})\Gamma(n+\frac{1}{2})}$$

$$\cdot \int_0^u P_n^{(\alpha+\beta+\frac{1}{2},-\frac{1}{2})}(2z^2-1)(u^2-z^2)^{\beta-\frac{1}{2}}\, dz.$$

Thus, by applying Szegö [46], (4.1.5),

$$P_n^{(\alpha,\beta)}(2u^2-1) = \frac{2u^{-2\beta}\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+\frac{3}{2})\Gamma(2n+1)}{\Gamma(\beta+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(2n+\alpha+\beta+\frac{3}{2})\Gamma(n+1)}$$

$$\cdot \int_0^u P_{2n}^{(\alpha+\beta+\frac{1}{2},\alpha+\beta+\frac{1}{2})}(z)(u^2-z^2)^{\beta-\frac{1}{2}}\, dz.$$

We investigate $F_\varepsilon(\cos\theta)$ near $\theta = \pi$. If we put $\cos\theta = 2u^2-1$, we have to study $F_\varepsilon(2u^2-1)$ with u in the neighbourhood of 0. After some calculations we obtain

$$F_\varepsilon(2u^2-1) = \frac{\Gamma(\alpha+\beta+\frac{3}{2})}{2^{2\alpha+2\beta+1}\Gamma(\alpha+1)\Gamma(\beta+\frac{1}{2})} u^{-2\beta} \int_0^u (u^2-z^2)^{\beta-\frac{1}{2}}.$$

$$\Big(\sum_{n=1}^{\infty} e^{-\varepsilon n} n^{-\gamma} \omega_{2n}^{(\alpha+\beta+\frac{1}{2},\alpha+\beta+\frac{1}{2})} R_{2n}^{(\alpha+\beta+\frac{1}{2},\alpha+\beta+\frac{1}{2})}(z)\Big)dz.$$

In the first part of this theorem we have shown that the series in the integrand converges uniformly in any closed subinterval of $(-1,1)$ and that its $\lim_{\varepsilon\to 0^+}$ exists and is continuous. Indeed, if $\sum a_n R_n^{(\alpha,\alpha)}(x)$ and $\sum a_n R_n^{(\alpha,\alpha)}(-x)$ are continuous functions of x near $x = 0$, then so is their sum $\sum a_{2n} R_n^{(\alpha,\alpha)}(x)$, which is a series of the kind used in the integrand. By the dominated convergence theorem, $F_\varepsilon(2u^2-1)$ converges pointwise to a limit as $\varepsilon \to 0^+$, at least if u is sufficiently small. Moreover,

$$F(2u^2-1) = O(u^{-2\beta} \int_0^u c(z)(u^2-z^2)^{\beta-\frac{1}{2}}\, dz) \qquad (u\to 0^+),$$

where $c(z)$ is continuous near $z = 0$. And the convergence is uniform, since

$$|u^{-2\beta} \int_0^u (u^2-z^2)^{\beta-\frac{1}{2}}\, dz| = O(1) \qquad (u\to 0^+).$$

This implies that $F(\cos\theta)$ is continuous near $\theta = \pi$.

Let us now assume $\gamma > 0$. We apply lemma 1.5.3 with $\nu \in \mathbb{Z}^+$ such that $\nu > \alpha + \frac{3}{2}$. We combine (1.5.6), (1.5.7) and (1.5.8) using (1.5.12) to the following estimate

$$|F_\varepsilon(\cos\theta)| \le c_\nu \sum_{n=1}^{\infty} n^{-\gamma+2\alpha+1} \, |R_n^{(\alpha+\nu,\beta)}(\cos\theta)|.$$

Application of (1.2.8) with $p = 1$, $\lambda = \nu$ and $\mu = 0$ yields

$$\int_0^\pi |F_\varepsilon(\cos\theta)| \, \rho^{(\alpha,\beta)}(\theta)d\theta < K.$$

K does not depend on ε. By the weak* compactness of a closed sphere in M we may conclude that there is a sequence $\varepsilon_i \to 0^+$ as $i \to \infty$, and a measure $\mu \in M$ such that for each $g \in C$

$$\lim_{i\to\infty} \int_0^\pi g(\cos\theta) \, F_{\varepsilon_i}(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta = \int_0^\pi g(\cos\theta)d\mu(\cos\theta).$$

But in a preceding part we have shown that $F_{\varepsilon_i}(\cos\theta)$ converges uniformly to $F(\cos\theta)$ in any compact subinterval of $(0,\pi]$. This implies that the singular part of μ is concentrated at 0 and therefore is a Dirac δ-measure at 0. We wish to show that μ is absolutely continuous. Let $\mu = \mu_a+\mu_s$, where μ_a is absolutely continuous and μ_s is a Dirac δ-measure at 0. We have for h sufficiently small

$$\left|\int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)d\mu_a(\cos\theta)\right| \le \int_0^{h/n} |R_n^{(\alpha,\beta)}(\cos\theta)| \, |d\mu_a(\cos\theta)|$$

$$+ \int_{h/n}^\pi |R_n^{(\alpha,\beta)}(\cos\theta)| \, |d\mu_a(\cos\theta)| = o(1) \qquad (n\to\infty).$$

By (1.2.1) it follows that, if μ_s is not zero

$$\int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)d\mu_s(\cos\theta) \ne o(1) \qquad (n\to\infty).$$

On the other hand

$$\int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)d\mu(\cos\theta) = \lim_{i\to\infty} \int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)F_{\varepsilon_i}(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta$$

$$= n^{-\gamma}.$$

This is a contradiction unless μ_s is zero. The Radon-Nikodym theorem ensures us that there exists a unique function $G(\cos\theta) \in L^1$ such that

$$G(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta = d\mu(\cos\theta).$$

By corollary 3.3.2 we may conclude that

$$G(\cos\theta) = \lim_{\varepsilon\to 0^+} F_\varepsilon(\cos\theta) = F(\cos\theta) \qquad \text{almost everywhere.}$$

CHAPTER II APPROXIMATION PROCESSES ON BANACH SPACES

In this chapter we quote the main results on approximation processes on Banach spaces treated in Butzer and Berens [16], Berens [12] and Butzer and Scherer [20]. We first give a short summary of the K-method developed by Peetre [39] in the theory of interpolation spaces between two Banach spaces. We only deal with the special case when one Banach space is a normalized Banach subspace of the other. In order to keep this chapter as short as possible we avoid the J-method of interpolation, which entails that some of the theorems cannot be stated in their most general form. We use the discrete K-method introduced in Butzer-Scherer [20] instead of the original continuous K-method, since the discrete method is more appropriate for the characterization of the spaces of best approximation, to which section 2.2 is devoted. In the next section the properties of the intermediate spaces which are related to strong approximation processes on Banach spaces are given. Then, we treat the direct and inverse theorems for strong approximation processes in Banach spaces, which have been obtained by Berens [12] (see also Butzer-Nessel [18], section 13.4). Finally, results due to Hille and Butzer (see [16]) are stated for families of operators on Banach spaces which define an equi-bounded semi-group of class (C_0). For Dutch readers a convenient presentation of the K- and J-method of interpolation and of approximation processes on Banach spaces may be found in [11].

2.1. THE DISCRETE K-METHOD OF INTERPOLATION

2.1.1. Definition. Let X be a Banach space and Let Y be a Banach subspace of X, which is dense in X and such that for all $f \in Y$

$$(2.1.1) \qquad ||f||_X \leq ||f||_Y.$$

Then we call Y a normalized subspace of X.

2.1.2. Definition. By l^q_* $(1 \leq q \leq \infty)$ we denote the space of all sequences of real or complex numbers $a = \{a_n\}_{n \in Z^+}$ such that

$$(2.1.2) \qquad ||a||_{l^q_*} = \begin{cases} \{\sum_{n=1}^{\infty} |a_n|^q \, n^{-1}\}^{1/q} & (1 \leq q < \infty), \\ \sup_{n \in Z^+} |a_n| & (q=\infty), \end{cases}$$

is finite.

2.1.3. Definition. Let X be a Banach space and Y be a normalized Banach subspace. If for $0 < t < \infty$ and for every $f \in X$ we consider the function norm

$$(2.1.3) \qquad K(t;f) = K(t;f;X,Y) = \inf_{f_1 \in Y} (||f-f_1||_X + t||f_1||_Y),$$

then we denote by $(X,Y)_{\theta,q;K}$, $-\infty < \theta < \infty$, $1 \leq q \leq \infty$, the set of all elements $f \in X$ such that

$$\{n^\theta K(n^{-1};f)\}_{n\in Z^+} \in l^q_*.$$

2.1.4. Proposition. For $1 \leq q \leq \infty$, $\theta < 1$ and $q = \infty$, $\theta = 1$ the spaces $(X,Y)_{\theta,q;K}$ are Banach spaces under the norms

$$(2.1.4) \qquad ||f||_{\theta,q;K} = ||n^\theta K(n^{-1};f)||_{l^q_*} .$$

They satisfy the inclusion relation

$$Y \subset (X,Y)_{\theta,q;K} \subset X.$$

In all other cases the spaces $(X,Y)_{\theta,q;K}$ only contain the zero element. Moreover, for $1 \leq q \leq \infty$, $\theta < 0$ and $q = \infty$, $\theta = 0$

$$(X,Y)_{\theta,q;K} \cong X.$$

The spaces $(X,Y)_{\theta,q;K}$ are called spaces of K-interpolation.

2.1.5. Proposition. For $0 \leq \theta' < \theta < 1$ the following inclusions hold:

a) $\qquad (X,Y)_{\theta,q;K} \subset (X,Y)_{\theta,p;K}$, $\qquad (1\leq q\leq p\leq\infty)$,

b) $\qquad (X,Y)_{\theta,q;K} \subset (X,Y)_{\theta,p;K}$, $\qquad (1\leq q,p\leq\infty)$.

The next theorem is usually called the theorem of reiteration.

2.1.6. Theorem. Let θ_1, θ_2 be real numbers satisfying $0 \leq \theta_1 < \theta_2 < 1$ and let X_{θ_i} (i=1,2) be Banach spaces such that

(2.1.5) $$(X,Y)_{\theta_i,1;K} \subset X_{\theta_i} \subset (X,Y)_{\theta_i,\infty;K}$$

for $i = 1,2$ if $\theta_1 > 0$ and for $i = 2$ with $X_{\theta_1} \cong X$ if $\theta_1 = 0$. Then for $0 < \theta' < 1$, $1 \leq q \leq \infty$ and $\theta = (1-\theta')\theta_1+\theta'\theta_2$ it follows that

$$(X_{\theta_1},X_{\theta_2})_{\theta',q;K} \cong (X,Y)_{\theta,q;K} .$$

2.2. SPACES OF BEST APPROXIMATION

The following results on spaces of best approximation are taken from Butzer-Scherer [20], ch. 2.

2.2.1. Definition. Let X be a Banach space and let P_n $(n \in Z^+)$ be subspaces such that

$$\{0\} = P_0 \subset P_1 \subset P_2 \ldots \subset P_n \subset \ldots \subset X.$$

We define the degree of best approximation of $f \in X$ by elements of P_n by

(2.2.1) $$E(P_n;f;X) = \inf_{p_n \in P_n} ||f-p_n||_X .$$

We denote by $X^K_{\theta,q}$, $-\infty < \theta < \infty$, $1 \leq q \leq \infty$ the set of all elements $f \in X$ such that

$$\{n^\theta E(P_n;f;X)\}_{n \in Z^+} \in l^q_*.$$

2.2.2. Proposition. For $1 \leq q \leq \infty$ and all real θ the spaces $X^K_{\theta,q}$ are Banach spaces under the norms

(2.2.2) $$||f||^K_{\theta,q} = ||f||_X + ||n^\theta E(P_n;f;X)||_{l^q_*} .$$

They satisfy the inclusion relation

$$P_n \subset X^K_{\theta,q} \subset X \qquad (n \in Z^+).$$

Moreover, for $1 \leq q \leq \infty$, $\theta < 0$ and $q = \infty$, $\theta = 0$

$$X^K_{\theta,q} \cong X.$$

The spaces $X^K_{\theta,q}$ are called spaces of best approximation.

2.2.3. Proposition. The following inclusions hold:

a) $$X^K_{\theta,q} \subset X^K_{\theta,p} \,, \qquad (1\le q\le p\le\infty,\ \theta>0),$$

b) $$X^K_{\theta,q} \subset X^K_{\theta',p} \,, \qquad (1\le p,\ q\le\infty,\ \theta'<\theta<1).$$

2.2.4. Definition. A Banach space Y, satisfying the inclusion $P_n \subset Y \subset X$, $n \in Z^+$, is said to belong to

a) the class $D^K_\theta(X)$, $\theta \ge 0$, if for all $n \in Z^+$ and for $f \in Y$ the relation

(2.2.3) $$n^\theta E(P_n;f;X) \le C||f||_Y \qquad \text{(C independent of n)}$$

holds;

b) the class $D^J_\theta(X)$, $\theta \ge 0$, if for $p_n \in P_n$, $n \in Z^+$, the relation

(2.2.4) $$||p_n||_Y \le Cn^\theta ||p_n||_X \qquad \text{(C independent of n)}$$

holds;

c) the class $D_\theta(X)$ if it belongs to the classes $D^K_\theta(X)$ and $D^J_\theta(X)$.

2.2.5. Lemma. For $\theta > 0$ the Banach space Y, satisfying $P_n \subset Y \subset X$, $n \in Z^+$, belongs to

a) the class $D^K_\theta(X)$ if and only if

$$Y \subset X^K_{\theta,\infty} \,;$$

b) the class $D^J_\theta(X)$ if and only if

$$X^K_{\theta,1} \subset Y \,;$$

c) the class $D_\theta(X)$ if and only if

$$X^K_{\theta,1} \subset Y \subset X^K_{\theta,\infty} \,.$$

2.2.6. Theorem. Let X_i be spaces of the classes $D_{\theta_i}(X)$, $i = 1,2$, and $\theta_2 > \theta_1 \geq 0$. Then for $0 < \theta' < 1$ and $1 \leq q \leq \infty$

$$(X_1,X_2)_{\theta',q;K} \cong X^K_{\theta,q} ,$$

where

$$\theta = (1-\theta')\theta_1+\theta'\theta_2 .$$

2.3. SPACES OF S-APPROXIMATION

2.3.1. Definition. Let the set A be R^+ or Z^+. A family $\{S_\rho\}_{\rho\in A}$ of operators mapping a Banach space X into itself and satisfying the properties

$$(2.3.1) \quad \begin{cases} \text{i)} & ||S_\rho f||_X \leq \Lambda||f||_X, \; f \in X, \; \Lambda \in R^+, \text{ uniformly for } \rho \in A, \\ \text{ii)} & \lim_{\rho\to\infty} ||S_\rho f-f||_X = 0, \qquad f \in X, \\ \text{iii)} & S_\rho(S_\tau f) = S_\tau(S_\rho f) \quad \text{for all } \rho,\tau \in A, \; f \in X, \end{cases}$$

is called a (commutative) strong approximation process on X.

2.3.2. Definition. Let $\{S_\rho\}_{\rho\in A}$ be a strong approximation process on the Banach space X and let $f \in X$. The expression

$$(2.3.2) \qquad \omega_S(\rho;f;X) = \sup_{\sigma\geq\rho} ||S_\sigma f-f||_X$$

is called the modulus of S-approximation of f. By $X_{\lambda,q;S}$, $-\infty < \lambda < \infty$, $1 \leq q \leq \infty$, we denote the collection of all elements $f \in X$ such that

$$\{n^\lambda\omega_S(n;f;X)\}_{n\in Z^+} \in l^q_*.$$

The spaces $X_{\lambda,q;S}$ are called spaces of S-approximation.

2.3.3. Proposition. For $\lambda > 0$, $1 \leq q \leq \infty$ the spaces $X_{\lambda,q;S}$ are normalized Banach subspaces of X under the norms

$$(2.3.3) \qquad ||f||_{\lambda,q;S} = ||f||_X + ||n^\lambda\omega_S(n;f;X)||_{l^q_*} .$$

2.3.4. Proposition. For $\lambda > \lambda' > 0$ the following inclusions hold:

a) $$X_{\lambda,q;S} \subset X_{\lambda,p;S} \ , \qquad (1 \leq q \leq p \leq \infty),$$

b) $$X_{\lambda,q;S} \subset X_{\lambda',p;S} \ , \qquad (1 \leq q, p \leq \infty).$$

2.4. SATURATION AND NON-OPTIMAL APPROXIMATION

The concept of saturation has been introduced by Favard [24].

2.4.1. Definition. Let $\{S_\rho, \rho \in A\}$ be a strong approximation process on the Banach space X and let $\phi(\rho)$ be a positive non-increasing function on A tending to zero as $\rho \to \infty$. Let X_0 be a subspace of X. We shall say that the process $\{S_\rho\}$ is saturated with order $\phi(\rho)$ and with trivial subspace X_0 if every $f \in X$ for which

$$\omega_S(\rho;f;X) = o(\phi(\rho)) \qquad (\rho \to \infty)$$

belongs to X_0 and if the set

$$F(X,S) = \{f \in X\colon \omega_S(\rho;f;X) = O(\phi(\rho)),\ \rho \to \infty\}$$

contains at least one element which does not belong to X_0. The set F(X,S) is called the saturation or Favard class of the process $\{S_\rho\}$.

Let B be a closed linear operator mapping the Banach space X into itself with a domain D(B) which is dense in X. Under the norm

$$(2.4.1) \qquad ||f||_{D(B)} = ||f||_X + ||Bf||_X.$$

D(B) is a normalized Banach subspace of X.

We now state the following saturation theorem.

2.4.2. Theorem. Let $f \in X$, $\{S_\rho\}_{\rho \in A}$ be a strong approximation process on X and let B be a closed linear operator. Suppose that the ranges $S_\rho[X]$ of the operators S_ρ belong to D(B) for all $\rho \in A$ and that there exists a positive number γ_0 such that for all $f \in D(B)$

$$(2.4.2) \qquad \lim_{\rho\to\infty} ||\rho^{\gamma_0}\{S_\rho f - f\} - Bf||_X = 0.$$

Then the process $\{S_\rho\}$ is saturated with order $\rho^{-\gamma_0}$ and the saturation class $F(X,S)$ is the space of K-interpolation $(X,D(B))_{1,\infty;K}$. The trivial subspace X_0 mentioned in definition 2.4.1 is the null space $N(B) = \{f \in D(B): Bf = 0\}$ of the operator B.

In the case of non-optimal approximation we need an additional condition in the form of an inequality of the Bernstein-type.

2.4.3. Theorem. Let f, $\{S_\rho\}$, γ_0 and B satisfy the conditions of theorem 2.4.2 and, in addition, suppose that

$$(2.4.3) \qquad ||BS_\rho f||_X \leq N\rho^{\gamma_0}||f||_X \quad , \qquad (\rho\in A,\ f\in X),$$

where N is a constant ≥ 1. Then the spaces of S-approximation $X_{\lambda,q;S}$, $0 < \lambda < \gamma_0$, $1 \leq q \leq \infty$ or $\lambda = \gamma_0$, $q = \infty$ coincide with the spaces of K-interpolation $(X,D(B))_{\lambda/\gamma_0,q;K}$ with equivalent norms.

2.5. SEMI-GROUPS OF OPERATORS

2.5.1. Definition. A family $\{U(t),\ t \geq 0\}$ of operators mapping a Banach space X into itself and satisfying the conditions

$$(2.5.1) \quad \begin{cases} \text{i)} & ||U(t)f||_X \leq \Lambda||f||_X\,, \qquad \Lambda \in R^+, \text{ uniformly for } t \geq 0. \\ \text{ii)} & \lim_{t\to 0^+} ||U(t)f - f||_X = 0, \\ \text{iii)} & U(t_1+t_2) = U(t_1)U(t_2), \qquad t_1,t_2 \geq 0, \\ \text{iv)} & U(0) = I \qquad , \end{cases}$$

is called an equi-bounded semi-group of operators of class (C_0).

2.5.2. Definition. The infinitesimal generator B of the semi-group $\{U(t),\ t \geq 0\}$ is defined by

$$(2.5.2) \qquad Bf = \lim_{\tau\to 0^+} B_\tau f, \qquad B_\tau = \frac{1}{\tau}[U(\tau) - I] \quad ,$$

whenever the limit exists; D(B) is the set of elements for which this limit exists.

2.5.3. Proposition.

a) D(B) is a linear manifold in X and B is a linear operator.

b) If $f \in D(B)$, then $U(t)f \in D(B)$ for each $t \geq 0$ and

$$\frac{d}{dt} U(t)f = BU(t)f = U(t)Bf \qquad (t \geq 0); \tag{2.5.3}$$

furthermore,

$$U(t)f - f = \int_0^t U(\tau)Bf d\tau \qquad (t>0). \tag{2.5.4}$$

c) D(B) is dense in X and B is a closed operator.

If we put $\rho = t^{-1}$, an equi-bounded semi-group of operators of class (C_0) defines a strong approximation process on the Banach space X satisfying a limit relation of the form (2.4.2), where $\gamma_0 = 1$ and B is written for the infinitesimal generator. Hence, if $U(t)f \in D(B)$ for all $f \in X$, then, by theorem 2.4.2, the process $\{U(\rho^{-1}), \rho \in R^+$ is saturated with order ρ^{-1} and the saturation class F(X,U) is the space of K-interpolation $(X,D(B))_{1,\infty;K}$. The 'trivial' subspace mentioned in definition 2.4.1 is the null space of the infinitesimal generator.

The characterization of the spaces of U-approximation $X_{\lambda,q;U}$, $0 < \lambda < 1$, $1 \leq q \leq \infty$ or $\lambda = 1$, $q = \infty$ can be obtained by means of the following relation for equi-bounded semi-groups of operators of class (C_0):

$$K(\rho^{-1},f;X,D(B)) \approx \min(1,\rho)||f||_X + \omega_U(\rho;f;X) \qquad (\rho\to\infty, f\in X), \tag{2.5.5}$$

(see Butzer-Berens [16], prop. 3.4.1). Here $\omega_U(\rho;f;X)$ denotes the modulus of U-approximation (see definition 2.3.2). Summarizing the results we have

2.5.4. Theorem. The process $\{U(t), t \geq 0\}$ is saturated with order t. The saturation class F(X,U) is the space $(X,D(B))_{1,\infty;K}$. Moreover, the following statements are equivalent for $0 < \theta < 1$, $1 \leq q \leq \infty$ or $\theta = 1$, $q = \infty$

i) $f \in (X, D(B))_{\theta,q;K}$,

ii) $\{n^{\theta}\omega_U(n;f;X)\}_{n\in Z^+} \in l^q_*$.

CHAPTER III SUMMABILITY OF FOURIER-JACOBI SERIES

The main part of this chapter consists in introducing summation methods for the Fourier-Jacobi expansion of a function and in showing that many summation methods can be looked upon as strong approximation processes on certain spaces of functions. In section 3.1 we define approximation kernels and we show that the convolution of a function with an approximation kernel generates a strong approximation process on the function space to which the function belongs. The next section deals with results, due to Rau [42] and Pollard [40] concerning the norm convergence of the partial sums of the Fourier-Jacobi expansion of a function. In section 3.3 we introduce summation methods for Jacobi series and we associate a kernel with each summation process. For many classical summation methods this kernel is an approximation kernel, which enables us to prove the norm convergence of the summation process. In the last section of this chapter we prove some results concerning operators of the factor sequence type, which will be used in the next chapters. For analogous theorems on summation methods for Fourier series we refer to the recent textbook by Butzer and Nessel [18], ch. 1, which we follow to a large extent in our treatment. In [13], Berens, Butzer and Pawelke deal with similar results for expansions in terms of spherical harmonics.

Notation. In the rest of this tract X will always denote one of the function spaces C or L^p ($1 \leq p < \infty$), defined in section 1.3. By the set A we mean either Z^+ or R^+.

3.1. APPROXIMATION KERNELS

We first give the definition of a kernel.

3.1.1. Definition. A set of functions $\{K_\lambda(\cos\theta)\}_{\lambda\in\mathsf{A}}$ is called a kernel if $K_\lambda \in L^1$ for each $\lambda \in \mathsf{A}$ and

$$(3.1.1) \qquad \int_0^\pi K_\lambda(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta = 1.$$

A kernel $\{K_\lambda(\cos\theta)\}$ is called real if $K_\lambda(\cos\theta)$ is a real function for each $\lambda \in \mathsf{A}$. A real kernel $\{K_\lambda(\cos\theta)\}$ is said to be positive if $K_\lambda(\cos\theta) \geq 0$ a.e. for each $\lambda \in \mathsf{A}$.

3.1.2. Definition. A kernel $\{K_\lambda(\cos\theta)\}$ is called an approximation kernel if, with some constant $\Lambda > 0$,

$$(3.1.2) \qquad ||K_\lambda||_1 \leq \Lambda \qquad \text{(uniformly for all } \lambda\in A).$$

$$(3.1.3) \qquad \lim_{\lambda\to\infty} K_\lambda^\wedge(n) = 1 \qquad (n\in P).$$

Every positive kernel satisfies condition (3.1.2) with $\Lambda = 1$ as follows from (3.1.1). The following theorem justifies the name approximation kernel. We write $K_\lambda(f;\cos\theta)$ for the operator which is defined by $(K_\lambda * f)(\cos\theta)$.

3.1.3. Theorem. Let $\{K_\lambda(\cos\theta)\}$ be an approximation kernel. Then for $f \in X$

$$(3.1.4) \qquad ||K_\lambda(f;\cdot)||_X \leq \Lambda||f(\cdot)||_X \qquad (\lambda\in A),$$

$$(3.1.5) \qquad \lim_{\lambda\to\infty} ||K_\lambda(f;\cdot)-f(\cdot)||_X = 0,$$

$$(3.1.6) \qquad K_\lambda * (K_\mu * f) = K_\mu * (K_\lambda * f) \qquad \text{for all } \lambda,\mu \in A.$$

Proof. Relation (3.1.4) is an immediate consequence of (3.1.2) and prop. 1.4.1 (iv). Relation (3.1.5) follows by application of the Banach-Steinhaus theorem (see Hille-Phillips [32], p. 41) by using (3.1.2) and the fact that (3.1.5) holds for a dense set, the polynomials in $\cos\theta$, as follows from (3.1.3). Relation (3.1.6) is a consequence of prop. 1.4.1 (i) and (ii).

Thus, we have shown that convolution of the elements of X with an approximation kernel leads to a strong approximation process on the Banach space X (see definition 2.3.1).

The condition (3.1.3) can be replaced by the 'peaking property'

$$(3.1.7) \qquad \lim_{\lambda\to\infty} \int_h^\pi |K_\lambda(\cos\theta)|\rho^{(\alpha,\beta)}(\theta)d\theta = 0, \qquad \text{for each } h,\ 0 < h \leq \pi.$$

In this case we have

3.1.4. Theorem. Let $\{K_\lambda(\cos\theta)\}$ be a kernel satisfying (3.1.2) and (3.1.7). Then (3.1.4), (3.1.5) and (3.1.6) are valid and $\{K_\lambda(\cos\theta)\}$ is an approximation kernel.

Proof. Relation (3.1.4) and (3.1.6) follow as in the preceding theorem. In order to derive (3.1.5) we consider

$$||K_\lambda(f;\cdot)-f(\cdot)||_X = ||\int_0^\pi K_\lambda(\cos\phi)(T_\phi f(\cdot)-f(\cdot))\rho^{(\alpha,\beta)}(\phi)d\phi||_X$$

$$\leq \int_0^\pi |K_\lambda(\cos\phi)|\ ||T_\phi f(\cdot)-f(\cdot)||_X\ \rho^{(\alpha,\beta)}(\phi)d\phi,$$

where we have used the Hölder-Minkowski inequality (see [18], prop. 0.1.7). We break up the range of integration into the parts $[0,h]$ and $[h,\pi]$. By (1.4.3) we can choose $h < \pi$ such that $||T_\phi f(\cdot)-f(\cdot)||_X < \varepsilon$ for $0 \leq \phi \leq h$. Hence, by (3.1.2) we have

$$\int_0^h |K_\lambda(\cos\phi)|\ ||T_\phi f(\cdot)-f(\cdot)||_X\ \rho^{(\alpha,\beta)}(\phi)d\phi < \varepsilon\Lambda.$$

On the other hand, using (1.4.2) and (3.1.7), we obtain

$$\int_h^\pi |K_\lambda(\cos\phi)|\ ||T_\phi f(\cdot)-f(\cdot)||_X\ \rho^{(\alpha,\beta)}(\phi)d\phi$$

$$\leq 2||f||_X \int_h^\pi |K_\lambda(\cos\phi)|\rho^{(\alpha,\beta)}(\phi)d\phi < \varepsilon, \quad \text{if } \lambda > \lambda_1(\varepsilon).$$

This proves relation (3.1.5). If we take for f the functions $R_n^{(\alpha,\beta)}(\cos\theta)$, $(n\in P)$ it follows from prop. 1.4.1 (iii) that (3.1.5) implies (3.1.3), which shows that $\{K_\lambda(\cos\theta)\}$ is an approximation kernel.

3.2. NORM CONVERGENCE OF THE PARTIAL SUMS

When we study the convergence of the Fourier-Jacobi series associated with a function $f \in X$, we first consider the partial sums. In view of (1.3.1) and (1.3.2) we have

$$(3.2.1)\qquad S_N(f;\cos\theta) = \sum_{n=0}^N f^\wedge(n)\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta)$$

$$= \int_0^\pi f(\cos\phi)\{\sum_{n=0}^N \omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta)R_n^{(\alpha,\beta)}(\cos\phi)\}\rho^{(\alpha,\beta)}(\phi)d\phi.$$

Recalling formula (1.1.10) and prop. 1.4.1 (i) we derive

$$S_N(f;\cos\theta) = \int_0^\pi T_\phi f(\cos\theta) D_N^{(\alpha,\beta)}(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi$$

$$= \frac{(\alpha+1)\omega_N^{(\alpha+1,\beta)}}{2N+\alpha+\beta+2} \int_0^\pi T_\phi f(\cos\theta) R_N^{(\alpha+1,\beta)}(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi.$$

The functions $\{D_N^{(\alpha,\beta)}(\cos\theta)\}_{N\in Z^+}$ define a kernel, which is called the Dirichlet kernel. But the Dirichlet kernel fails to be an approximation kernel, since by formula (1.2.8) with $\lambda = p = 1$, $\mu = 0$,

$$\int_0^\pi |D_n(\cos\theta)|\rho^{(\alpha,\beta)}(\theta)d\theta = \frac{(\alpha+1)\omega_n^{(\alpha+1,\beta)}}{2n+\alpha+\beta+2} \int_0^\pi |R_n^{(\alpha+1,\beta)}(\cos\theta)|\rho^{(\alpha,\beta)}(\theta)d\theta$$

$$\approx \begin{cases} n^{\alpha+\frac{1}{2}} & (\alpha>-\frac{1}{2},\ n\to\infty), \\ \log n & (\alpha=-\frac{1}{2},\ n\to\infty). \end{cases} \tag{3.2.2}$$

This result is due to Rau [42]. It follows that the operator norm of the partial sum operator S_N as an operator from X into X satisfies

$$||S_N|| \leq ||D_N||_1. \tag{3.2.3}$$

If we choose X = C it is not hard to show that equality holds in (3.2.3). Hence, by the uniform boundedness principle, there exists at least one element $f \in C$ for which the partial sums (3.2.1) do not converge to f in the supremum norm.

On the other hand if $X = L^2$ the partial sums converge to f in the norm by the Riesz-Fischer theorem. The norm convergence of $S_N(f;\cos\theta)$ in the L^p spaces has been treated by Pollard [40]. He showed that there is norm convergence if

$$\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1} \quad \text{and} \quad \frac{4(\beta+1)}{2\beta+3} < p < \frac{4(\beta+1)}{2\beta+1}$$

and there is no norm convergence, if p is outside one of these ranges.

3.3. SUMMATION METHODS

Since the partial sums of the Fourier-Jacobi expansion in general do not furnish a process which converges in the norm to the function, we are looking for the processes which produce norm convergence. We introduce a sequence $\{c_\lambda(n)\}_{n\in P}$, $\lambda \in A$, which for each $\lambda \in A$ satisfies the conditions

(3.3.1) $\quad c_\lambda(n)$ is real for all $n \in P$,

(3.3.2) $\quad \sum_{n=0}^{\infty} \omega_n^{(\alpha,\beta)} \, |c_\lambda(n)| < \infty$,

(3.3.3) $\quad c_\lambda(0) = 1.$

Then for $f \in X$ we may form the c-means

$$C_\lambda(f;\cos\theta) = \sum_{n=0}^{\infty} c_\lambda(n) f^\wedge(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta). \tag{3.3.4}$$

By (3.3.2) and the fact that the Fourier-Jacobi coefficients $f^\wedge(n)$ are all bounded by $||f||_1$, the series (3.3.4) converges absolutely and uniformly in $\cos\theta$. If we substitute (1.3.2) into (3.3.4) we obtain

$$C_\lambda(f;\cos\theta) = \int_0^\pi T_\phi f(\cos\theta)\,\{\sum_{n=0}^{\infty} c_\lambda(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\phi)\}\rho^{(\alpha,\beta)}(\phi)d\phi$$

$$= (C_\lambda * f)(\cos\theta), \tag{3.3.5}$$

where we have put

$$C_\lambda(\cos\theta) = \sum_{n=0}^{\infty} c_\lambda(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta). \tag{3.3.6}$$

The interchange of summation and integration is justified by the uniform convergence of the series. On account of (3.3.1), (3.3.2) and (3.3.3) the functions $\{C_\lambda(\cos\theta)\}$ define a real, continuous kernel. If the c-means $C_\lambda(f;\cos\theta)$ of $f \in X$ converge in some sense (pointwise, in the norm etc.) to a limit as $\lambda \to \infty$ and if the limit coincides with the usual sum of the series in case the Fourier-Jacobi series converges in ordinary sense, we say that the factors $\{c_\lambda(n)\}$ define a summation method or summation process and we call the Fourier-Jacobi series c-summable. By theorem 3.1.3 we have

3.3.1. Proposition. Let $\{c_\lambda(n)\}_{n\in P}$, $\lambda \in A$, satisfy (3.3.1), (3.3.2) and (3.3.3) and be such that the corresponding kernel $\{C_\lambda(\cos\theta)\}$ is an approximation kernel. Then for each $f \in X$

$$\lim_{\lambda\to\infty} ||C_\lambda(f;\cdot)-f(\cdot)||_X = 0,$$

that is, the Fourier-Jacobi series of f is c-summable to f in X.

We give two examples. First we take

$$(3.3.7)\qquad c_\lambda(n) = e^{-n/\lambda}, \qquad \lambda \in R^+,$$

which after the substitution $r = e^{-1/\lambda}$ are called the Abel-Poisson factors. The corresponding means

$$(3.3.8)\qquad A_r(f;\cos\theta) = \sum_{n=0}^{\infty} r^n f^\wedge(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)$$

are called the Abel-Poisson means of the Fourier-Jacobi series of f. In view fo (3.3.5) we may set

$$A_r(f;\cos\theta) = (A_r * f)(\cos\theta),$$

where A_r is written for the Abel-Poisson kernel

$$(3.3.9)\qquad A_r(\cos\theta) = \sum_{n=0}^{\infty} r^n \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

Bailey [7] has given the following explicit representation for the Abel-Poisson kernel

$$(3.3.10)\qquad A_r(\cos\theta) =$$

$$= \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{(1-r)}{(1+r)^{\alpha+\beta+2}} \; {}_2F_1\left(\frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}; \beta+1; \frac{4r\cos^2\frac{\theta}{2}}{(1+r)^2}\right),$$

which shows that it is a positive kernel. Since condition (3.1.3) is trivially satisfied, if $r \to 1^-$ or in (3.3.7) if $\lambda \to \infty$ we have

3.3.2. Corollary. The Fourier-Jacobi series (1.3.1) of $f \in X$ is Abel-Poisson summable to f in X, that is

$$\lim_{r\to 1^-} \|A_r(f;\cdot)-f(\cdot)\|_X = 0.$$

As a second example we treat the Cesàro-means of order μ. The factors are defined by ($N \in Z^+$, $\mu \in R^+$)

$$(3.3.11)\qquad c_N(n) = \begin{cases} a^\mu_{N-n} \,/\, a^\mu_N & n \le N, \\ 0 & n > N, \end{cases}$$

where

$$a^\mu_n = \frac{\Gamma(n+\mu+1)}{n!\,\Gamma(\mu+1)} .$$

The corresponding means, the (C,μ)means, are

$$(3.3.12)\qquad S^\mu_N(f;\cos\theta) = (a^\mu_N)^{-1} \sum_{n=0}^{N} a^\mu_{N-n}\, f^\wedge(n)\,\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

We may put

$$S^\mu_N(f;\cos\theta) = (S^\mu_N * f)(\cos\theta),$$

where the S^μ_N is written for the kernel

$$(3.3.13)\qquad S^\mu_N(\cos\theta) = (a^\mu_N)^{-1} \sum_{n=0}^{N} a^\mu_{N-n}\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

The kernel (3.3.13) clearly satisfies (3.1.3) if $N \to \infty$. It has been shown by Szegö [46], theorem 9.41, that S^μ_N satisfies

$$(3.3.14)\qquad ||S^\mu_N||_1 \le \Lambda_\mu \qquad \text{(uniformly for all } N \in Z^+;\ \mu > \alpha+\tfrac{1}{2})$$

and that (3.3.14) is not satisfied if $\mu \le \alpha+\frac{1}{2}$.
By theorem 3.1.2 we conclude

3.3.3. Corollary. The Fourier-Jacobi series (1.3.1) of $f \in X$ is (C,μ) summable to f in X if $\mu > \alpha+\frac{1}{2}$, that is

$$\lim_{N\to\infty} ||S^\mu_N(f;\cdot)-f(\cdot)||_X = 0, \qquad \mu > \alpha+\tfrac{1}{2}.$$

3.3.4. Definition. If for a summation process with factors $\{c_\lambda(n)\}_{n\in P}$ the index set $A = Z^+$ and $\lambda = N$ and there exists an increasing function $m(N)$ on Z^+ with values in Z^+ such that $c_N(n) = 0$ for $n > m(N)$, then the kernel $C_N(\cos\theta)$, which corresponds with this summation process, is called a polynomial kernel of degree $m(N)$.

In chapter V we shall investigate a number of summation methods, which have the property that their corresponding kernels are approximation kernels. In particular, we shall study the order of approximation of these processes by means of the general theory on approximation processes on Banach spaces, treated in chapter II. We have to show that relations of the form (2.4.2) and (2.4.3) hold. For the processes we investigate, the operator B that occurs in (2.4.2) is of the factor sequence type, as is defined in the next section.

3.4. OPERATORS OF THE FACTOR SEQUENCE TYPE

3.4.1. Definition. Let $\psi(n)$ be an arbitrary real or complex valued function defined on P. The operator B_ψ, which maps $f \in X$ with the Fourier-Jacobi expansion (1.3.1) into $g \in X$, where

$$(3.4.1) \qquad g(\cos\theta) = B_\psi f(\cos\theta) \sim \sum_{n=0}^{\infty} \psi(n) f^\wedge(n) \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta),$$

is called an operator of the factor sequence type with factors $\psi(n)$, $n \in$ P.

The following lemma is proved in the same way as in the case of Fourier series (see Butzer-Scherer [20], lemma 4.1.1).

3.4.2. Lemma. Let B_ψ be an operator of the factor sequence type with factors $\psi(n)$, $n \in$ P. Then B_ψ is a closed, linear operator with domain

$$(3.4.2) \qquad D(B_\psi) = \{f \in X: \exists g \in X,\ g(\cos\theta) \sim \sum_{n=0}^{\infty} \psi(n) f^\wedge(n) \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)\}$$

and range in X. The domain $D(B_\psi)$ is a normalized Banach subspace of X under the norm

$$(3.4.3) \qquad ||f||_{D(B_\psi)} = ||f||_X + ||B_\psi f||_X.$$

Proof. We assume that $\{f_i\}_{i=0}^{\infty}$ is a sequence in $D(B_\psi)$ with $\lim_{i\to\infty} f_i = f$ and $\lim_{i\to\infty} B_\psi f_i = g$ in X. It follows that $\lim_{i\to\infty} f_i^\wedge(n) = f^\wedge(n)$ $(n \in P)$ and $\lim_{i\to\infty} \psi(n) f_i^\wedge(n) = g^\wedge(n)$, which implies that $\psi(n) f^\wedge(n) = g^\wedge(n)$ $(n \in P)$. This means that $f \in D(B_\psi)$ and $B_\psi f = g$ or B_ψ is closed. The linearity of the operator B_ψ is obvious. Also, $D(B_\psi)$ is dense in X, since it contains all the polynomials in $\cos\theta$. The last assertion of the theorem is a consequence of section 2.4.

For operators of the factor sequence type B_ψ it is possible to give a characterization of the space of K-interpolation $(X,D(B_\psi))_{1,\infty;K}$ in terms of the Fourier-Jacobi coefficients (see Butzer-Nessel [18], theorem 10.4.6).

3.4.3. Theorem. Let B_ψ be an operator of the factor sequence type with factors $\psi(n)$, $n \in \mathbf{P}$, and let $D(B_\psi)$ be the Banach space (3.4.2) with norm (3.4.3). Then the following statements are equivalent:

i) $$f \in (X,D(B_\psi))_{1,\infty;K} ,$$

ii) $$f \in H(X,\psi(n)) = \begin{cases} f \in C & : \exists g \in L^\infty,\ \psi(n)f^\wedge(n) = g^\wedge(n) \\ f \in L^1 & : \exists \mu \in M,\ \psi(n)f^\wedge(n) = \mu^\vee(n) \\ f \in L^p\ (1<p<\infty) & : \exists g \in L^p,\ \psi(n)f^\wedge(n) = g^\wedge(n) \end{cases} .$$

Proof. We prove the theorem in the case $X = L^1$. The other cases are similar.

i → ii. If $f \in (X,D(B_\psi))_{1,\infty;K}$ there exists a sequence $\{f_k\}_{k\in Z^+} \in D(B_\psi)$ and a constant $\Lambda > 0$ such that

$$||f_k||_{D(B_\psi)} \leq \Lambda \quad \text{for all } k \in Z^+ \quad \text{and} \quad ||f-f_k||_1 \leq \frac{\Lambda}{k} .$$

Hence the sequence $\{B_\psi f_k\}_{k\in Z^+}$ is uniformly bounded in L^1. By the weak* compactness of a closed sphere in the space M and the embedding of L^1 into M (see section 1.4) we may conclude that there exists a subsequence k_i of Z^+ and a measure $\mu \in M$ such that for each function $g \in C$

$$\lim_{i\to\infty} \int_0^\pi g(\cos\theta)B_\psi f_{k_i}(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta = \int_0^\pi g(\cos\theta)d\mu(\cos\theta).$$

If we take for g the Jacobi polynomials $R_n^{(\alpha,\beta)}(\cos\theta)$ we obtain

$$\lim_{i\to\infty} \psi(n)f_{k_i}^\wedge(n) = \mu^\vee(n),$$

or by the fact that $\lim_{i\to\infty} f_{k_i} = f$ we have

$$\psi(n)f^{\wedge}(n) = \mu^{\vee}(n).$$

ii → i. If we assume that there exists a measure $\mu \in M$ with $\psi(n)f^{\wedge}(n) = \mu^{\vee}(n)$, then we consider the sequence

$$f_N = S_N^{\lambda}*f, \qquad N \in \mathbb{Z}^+,\ \lambda > \alpha+\tfrac{1}{2}.$$

Here S_N^{λ} denotes the kernel (3.3.12). By prop. 1.4.1. iii it follows that f_N is a polynomial of degree $\leq N$ and thus $\{f_N\}_{n\in P} \in D(B_\psi)$. Moreover, by Szegö's result on (C,λ) means (section 3.3),

$$||B_\psi f_N||_1 = ||S_N^{\lambda}*\mu||_1 \leq \Lambda||\mu||_M \ , \qquad \text{uniformly in } N.$$

On the other hand $f_N \to f$ as $N \to \infty$. Therefore, for $n \in \mathbb{Z}^+$,

$$nK(n^{-1},f) = \lim_{N\to\infty} nK(n^{-1},f_N) \leq \lim_{N\to\infty} ||f_N||_{D(B_\psi)} \leq \Lambda||\mu||_M \ ,$$

which shows that $f \in (X,D(B_\psi))_{1,\infty;K}$.

In order to prove that the domains of some operators of the factor sequence type coincide we need the following lemma, which generalizes a well-known result for Fourier series (Zygmund [51], I, p. 149).

3.4.4. Lemma. The numbers a_n, $n \in P$, are the Jacobi-Stieltjes coefficients of a measure $\mu \in M$ if and only if the Abel-Poisson means

$$A_r^{(a)}(\cos\theta) = \sum_{n=0}^{\infty} r^n\, a_n\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)$$

satisfy $||A_r^{(a)}(\cos\theta)||_1 \leq \Lambda$, uniformly in r $(0\leq r<1)$, $\Lambda \in \mathbb{R}^+$.

Proof. Suppose $\sum_{n=0}^{\infty} a_n\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)$ is the Jacobi-Stieltjes series of the measure μ. Then

$$A_r^{(a)}(\cos\theta) = (A_r*\mu)(\cos\theta),$$

which implies by the positivity of the kernel A_r (see 3.3) and (1.4.6)

$$||A_r^{(a)}(\cos\theta)||_1 \leq ||\mu||_M, \qquad 0 \leq r < 1.$$

This proves one part. We now suppose $||A_r^{(a)}(\cos\theta)||_1 \leq \Lambda$, $0 \leq r < 1$, $\Lambda \in \mathbb{R}^+$ and we take

$$F_r^{(a)}(\cos\theta) = \int_0^\theta A_r^{(a)}(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi.$$

The family $\{F_r^{(a)}(\cos\theta)\}_{r\in[0,1)}$ is a family of absolutely continuous measures, which have the property that

$$||F_r^{(a)}(\cos\theta)||_M \leq \Lambda.$$

Therefore, for $r \in [0,1)$ the integral

$$G_r^{(a)}(h) = \int_0^\pi h(\cos\theta)dF_r^{(a)}(\cos\theta)$$

defines for each $h \in C$ a linear continuous functional on the space C. The norm of the functional is given by $||G_r^{(a)}|| = ||F_r^{(a)}||_M$. The set of the linear functionals $G_r^{(a)}(h)$ is uniformly bounded with respect to r and by the weak* compactness there is a sequence r_j, with $r_j \to 1$ as $j \to \infty$, which determines a measure $\mu \in M$, such that for all $h \in C$

$$\lim_{j\to\infty} \int_0^\pi h(\cos\theta)dF_{r_j}^{(a)}(\cos\theta) = \int_0^\pi h(\cos\theta)d\mu(\cos\theta).$$

Since the measures $F_r^{(a)}$ are absolutely continuous we have

$$\int_0^\pi h(\cos\theta)dF_{r_j}^{(a)}(\cos\theta) = \int_0^\pi h(\cos\theta)A_{r_j}^{(a)}(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta.$$

If we choose $h(\cos\theta) = R_n^{(\alpha,\beta)}(\cos\theta)$ $(n\in\mathbb{P})$, we obtain

$$\int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)d\mu(\cos\theta) = \lim_{j\to\infty} r_j\, a_n = a_n,$$

which proves the lemma.

The method used in the proof of the following lemma is taken from Taibleson [47], p. 465 (see also [13], p. 256).

3.4.5. Lemma. There exists a measure $\mu \in M$ with

$$\mu^\vee(n) = \left(\frac{n}{n+\alpha+\beta+1}\right)^\lambda \qquad (n\in\mathbb{Z}^+),$$

where λ is an arbitrary real number.

Proof. For $n \in Z^+$ we have

$$(3.4.4) \qquad \left(\frac{n}{n+\alpha+\beta+1}\right)^{\lambda} = \left(1 + \frac{\alpha+\beta+1}{n}\right)^{-\lambda}$$

$$= 1 + \frac{t_1(\lambda)}{n} + \frac{t_2(\lambda)}{n^2} + \ldots + f_\lambda^{(p)}(\xi_n) \frac{(\alpha+\beta+1)^p}{p!n^p},$$

$$\left(0 < \xi_n < \frac{\alpha+\beta+1}{n}\right),$$

because the function $f_\lambda(t) = 1+t)^{-\lambda}$ has an arbitrary number of derivatives for $t > -1$. At the right-hand side, the first term, 1 for each n, furnishes the coefficients of a finite measure, the Dirac measure, as follows from (3.3.10) and lemma 3.4.4. In theorem 1.5.4 we have shown that the terms n^{-k} $(k>0)$ are the Fourier-Jacobi coefficients of an L^1 function. If we choose $p > 2\alpha+2$, the last term leads to the coefficients of an absolutely convergent series in view of

$$\sum_{n=1}^{\infty} |f_\lambda^{(p)}(\xi_n)| \frac{(\alpha+\beta+1)^p}{p!n^p} \omega_n^{(\alpha,\beta)} |R_n^{(\alpha,\beta)}(\cos\theta)| \le \sum_{n=1}^{\infty} K(\alpha,\beta,\lambda,p)n^{2\alpha+1-p}.$$

We have used (1.1.4), (1.2.1) and $|f_\lambda^{(p)}(t)| \le K'(\lambda,p)$, $0 \le t \le \frac{\alpha+\beta+1}{n}$. Thus, together the right-hand side supplies the coefficients of a measure $\mu \in M$.

3.4.6. Corollary. For the operators of the factor sequence type $B_{[n(n+\alpha+\beta+1)]^\lambda}$ with factors $[n(n+\alpha+\beta+1)]^\lambda$ $(n \in Z^+)$ and $B_{n^{2\lambda}}$ with factors $n^{2\lambda}$, $(n \in Z^+$, λ real) the domain coincides. Also, for the corresponding Banach spaces

$$D(B_{n^{2\lambda}}) \simeq D(B_{[n(n+\alpha+\beta+1)]^\lambda}).$$

Proof. This is an immediate consequence of lemma 3.4.5 and (1.4.6).

CHAPTER IV BEST APPROXIMATION BY POLYNOMIALS

In this chapter we deal with the connection between the modulus of continuity of a function f, defined with respect to the generalized translation, and the degree of approximation of f by polynomials in $\cos\theta$. We obtain theorems of the Jackson and the Bernstein type, which in the case X = C enables us to show the equivalence of the modulus of continuity (4.1.1) with the usual symmetric modulus of continuity

$$\omega^*(\phi;f;C) = \sup_{0\le\psi\le\phi} ||f(\cdot+\psi) + f(\cdot-\psi) - 2f(\cdot)||_C ,$$

to which (4.1.1) reduces in the case $X = C$, $\alpha = \beta = -\frac{1}{2}$.

4.1. PROPERTIES OF THE MODULUS OF CONTINUITY

4.1.1. Definition. In the space X we define the modulus of continuity of $f \in X$ by

$$(4.1.1) \qquad \omega(\phi;f;X) = \sup_{0\le\psi\le\phi} ||T_\psi f(\cdot) - f(\cdot)||_X , \qquad \phi \ge 0.$$

Here T_ψ is written for the generalized translation operator, which is introduced in section 1.4. We list the main properties of $\omega(\phi;f;X)$ in the following

4.1.2. Proposition. Let $f \in X$.

i) $\omega(\phi;f;X)$ is a monotonely increasing function of ϕ, $\phi \ge 0$.

ii) $\lim_{\phi\to 0^+} \omega(\phi;f;X) = 0$.

iii) There exists a constant c (independent of f) such that

$$\omega(\lambda\phi;f;X) \le c \max(1,\lambda^2)\, [\phi^2||f||_X+\omega(\phi;f;X)].$$

iv) If $\omega(\phi;f;X) = o(\phi^2)$ as $\phi \to 0^+$, then f is constant (a.e.).

v) Let $f_1,f_2 \in X$. Then

$$\omega(\phi;f_1+f_2;X) \le \omega(\phi;f_1;X) + \omega(\phi;f_2;X).$$

Proof. Properties i) and v) are obvious from definition 4.1.1. Property ii) follows by (1.4.3). For the proof of iii) and iv) we need more knowledge about the generalized translation. We shall postpone the proof of iii) and iv) until the end of this section.

In section 1.1 we have already stated, that the Jacobi polynomials satisfy the differential equation (1.1.5), which we write in the form

$$(4.1.2)\qquad P\left(\frac{d}{d\theta}\right) R_n^{(\alpha,\beta)}(\cos\theta) = \lambda_n R_n^{(\alpha,\beta)}(\cos\theta) \qquad (n\in P),$$

where $P(\frac{d}{d\theta})$ and λ_n are given by (1.1.6) and (1.1.7).
We shall denote by A the operator of the factor sequence type with factor λ_n, that is, A maps $f \in X$ with the expansion (1.3.1) onto

$$(4.1.3)\qquad Af \sim \sum_{n=0}^{\infty} \lambda_n f^{\wedge}(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

In view of (4.1.2) the operator A is the realization of $P(\frac{d}{d\theta})$ in X. The domain of A is denoted by D(A). For each $f \in D(A)$ the following equation is satisfied:

$$(4.1.4)\qquad P\left(\frac{d}{d\phi}\right) T_\phi f = T_\phi Af.$$

Generalized translations connected with an equation of the form (4.1.4) have been investigated by Löfström and Peetre [36]. Following them we prove

4.1.3. Lemma. Let $f \in D(A)$. Then there exists a constant C, independent of f, such that

$$(4.1.5)\qquad \omega(\phi;f;X) \le C\phi^2 ||Af||_X \qquad 0 \le \phi \le \pi.$$

Proof. We introduce the function

$$(4.1.6)\qquad h(\phi,\tau) = \begin{cases} -\int_\tau^\phi \{\rho^{(\alpha,\beta)}(\sigma)\}^{-1}\, d\sigma, & 0 < \tau < \phi,\\ 0, & \text{otherwise.}\end{cases}$$

Since $\rho^{(\alpha,\beta)}(\tau) \approx \tau^{2\alpha+1}$ as $\tau \to 0^+$, it follows that $\rho^{(\alpha,\beta)}(\tau)h(\phi,\tau) \to 0$ as $\tau \to 0^+$. Thus, by (4.1.4), we have for $f \in D(A)$:

$$\int_0^\phi h(\phi,\tau)T_\tau Af\ \rho^{(\alpha,\beta)}(\tau)d\tau = \int_0^\phi h(\phi,\tau)P(\frac{d}{d\tau})[T_\tau f]\rho^{(\alpha,\beta)}(\tau)d\tau$$

$$= [-\rho^{(\alpha,\beta)}(\tau)h(\phi,\tau)\ \frac{d}{d\tau}\ T_\tau f]_0^\phi + \int_0^\phi \frac{d}{d\tau}\ T_\tau f d\tau$$

$$(4.1.7) \qquad = T_\phi f - f.$$

The integration is meant in the sense of Bochner (see Hille-Phillips [32]). It is easy to estimate the function

$$C_1(\phi) = \int_0^\phi h(\phi,\tau)\rho^{(\alpha,\beta)}(\tau)d\tau$$

$$= \int_0^\phi [\rho^{(\alpha,\beta)}(\tau)]^{-1}\ (\int_0^\tau \rho^{(\alpha,\beta)}(\sigma)d\sigma)d\tau$$

on the interval $0 \leq \phi \leq \frac{\pi}{2}$ by means of the inequality $\frac{\sqrt{2}}{2} \leq \cos\frac{\phi}{2} \leq 1$. We obtain

$$(4.1.8) \qquad \frac{1}{2^\beta(\alpha+1)}\sin^2\frac{\phi}{2} \leq C_1(\phi) \leq \frac{2^{\beta+1}}{\alpha+1}\sin^2\frac{\phi}{2}, \qquad 0 \leq \phi \leq \frac{\pi}{2}.$$

Recalling definition 4.1.1 and formula (1.4.2), we obtain for $f \in D(A)$

$$\omega(\phi;f;X) \leq C_1(\phi)\ ||Af||_X \qquad 0 < \phi \leq \frac{\pi}{2}$$

$$\leq C\phi^2\ ||Af||_X.$$

In the case $\frac{\pi}{2} \leq \phi \leq \pi$ we use a computation similar to that in the paper of Butzer and Johnen [17] and attributed there to Chernoff and Ragozin:

$$T_\phi f - f = \int_0^\phi h(\phi,\tau)\ T_\tau Af\ \rho^{(\alpha,\beta)}(\tau)d\tau$$

$$= \int_0^\phi \{\rho^{(\alpha,\beta)}(\tau)\}^{-1}\ (\int_0^\tau T_\sigma Af\ \rho^{(\alpha,\beta)}(\sigma)d\sigma)d\tau$$

$$= \int_0^{\frac{\pi}{2}} \{\rho^{(\alpha,\beta)}(\tau)\}^{-1}\ (\int_0^\tau T_\sigma Af\ \rho^{(\alpha,\beta)}(\sigma)d\sigma)d\tau +$$

$$\int_{\frac{\pi}{2}}^\phi \{\rho^{(\alpha,\beta)}(\tau)\}^{-1}\ (\int_0^\tau T_\sigma Af\ \rho^{(\alpha,\beta)}(\sigma)d\sigma)d\tau = I_1+I_2.$$

The Fourier-Jacobi expansion of Af shows that

$$\int_0^\pi T_\sigma Af\ \rho^{(\alpha,\beta)}(\sigma)d\sigma = 0.$$

Hence,

$$I_2 = -\int_{\frac{\pi}{2}}^{\phi} \{\rho^{(\alpha,\beta)}(\tau)\}^{-1} \left(\int_\tau^\pi T_\sigma Af\ \rho^{(\alpha,\beta)}(\sigma)d\sigma\right)d\tau,$$

which leads to

$$||T_\phi f-f||_X \leq \phi^2 \sup_{\frac{\pi}{2}\leq\psi\leq\pi} \Big[\int_0^{\frac{\pi}{2}} \{\rho^{(\alpha,\beta)}(\tau)\}^{-1} \left(\int_0^\tau \rho^{(\alpha,\beta)}(\sigma)d\sigma\right)d\tau$$

$$+ \int_{\frac{\pi}{2}}^{\psi} \{\rho^{(\alpha,\beta)}(\tau)\}^{-1} \left(\int_\tau^\pi \rho^{(\alpha,\beta)}(\sigma)d\sigma\right)d\tau\Big]\ ||Af||_X ,$$

for $\frac{\pi}{2} \leq \phi \leq \pi$, and thus (4.1.5) is valid for $0 \leq \phi \leq \pi$.

We now want to express the modulus of continuity $\omega(\phi;f;X)$ in terms of the K function norm (2.1.3) with the spaces X and D(A).

4.1.4. Lemma. For $f \in X$ and $0 \leq \phi \leq \pi$

(4.1.9) $$K(\phi^2;f;X,D(A)) \approx \min(1,\phi^2)\ ||f||_X + \omega(\phi;f;X).$$

Proof. Let $f_1 \in D(A)$ and $f_0 = f-f_1$. By (1.4.2), (4.1.5) and prop. 4.1.2. v) we have, for some constant $C > 0$,

$$\omega(\phi;f;X) \leq C(||f_0||_X + \phi^2||Af_1||_X).$$

Since

$$\min(1,\phi^2)\ ||f||_X \leq ||f_0||_X + \phi^2||f_1||_X ,$$

we obtain

$$\min(1,\phi^2)\ ||f||_X + \omega(\phi;f;X) \leq C(||f_0||_X + \phi^2||f_1||_{D(A)}).$$

Taking the minimum of the right-hand side over all $f_1 \in D(A)$, we deduce

$$\min(1,\phi^2)\ ||f||_X + \omega(\phi;f;X) \leq CK(\phi^2;f;X,D(A)).$$

For the proof of the converse inequality we take

$$(4.1.10) \qquad f_{1,\phi} = [C_1(\phi)]^{-1} \int_0^\phi h(\phi,\tau)(T_\tau f)\rho^{(\alpha,\beta)}(\tau)d\tau,$$

$$(4.1.11) \qquad f_{0,\phi} = f - f_{1,\phi} .$$

Then, by (4.1.7) and the closeness of the operator A (lemma 3.4.2), we may conclude that $f_{1,\phi} \in D(A)$ and that the following relations hold:

$$(4.1.12) \qquad T_\phi f - f = C_1(\phi)\ Af_{1,\phi} ,$$

$$(4.1.13) \qquad ||Af_{1,\phi}||_X \leq [C_1(\phi)]^{-1}\ \omega(\phi;f;X).$$

Furthermore, by (1.4.2) and (4.1.10),

$$(4.1.14) \qquad ||f_{1,\phi}||_X \leq ||f||_X .$$

On the other hand we have

$$f_{0,\phi} = -\ [C_1(\phi)]^{-1} \int_0^\phi h(\phi,\tau)(T_\tau f - f)\rho^{(\alpha,\beta)}(\tau)d\tau,$$

which leads to

$$(4.1.15) \qquad ||f_{0,\phi}||_X \leq \omega(\phi,f;X).$$

Combining the estimates (4.1.8), (4.1.13), (4.1.14) and (4.1.15) we conclude

$$K(\phi^2;f;X,D(A)) \leq ||f_{0,\phi}||_X + \phi^2 ||f_{1,\phi}||_{D(A)}$$

$$\leq C(\phi^2||f|| + \omega(\phi;f;X)), \quad 0 \leq \phi \leq \frac{\pi}{2} .$$

Noticing that $K(\phi^2;f;X,D(A)) \leq ||f||_X$, we derive

$$(4.1.16) \qquad K(\phi^2;f;X,D(A)) \leq C(\min(1,\phi^2)||f||_X + \omega(\phi;f;x)), \qquad 0 \leq \phi \leq \frac{\pi}{2}.$$

If $\frac{\pi}{2} < \phi \leq \pi$, we observe that

$$K(\phi^2;f;X,D(A)) \leq 4K((\frac{\phi}{2})^2;f;X,D(A))$$

and we apply (4.1.16) to the right-hand side of this inequality, noticing the monotonicity of $\omega(\phi;f;X)$. Hence (4.1.16) holds for $0 \leq \phi \leq \pi$. This completes the proof of lemma 4.1.4.

We are now in a position to prove proposition 4.1.2. iii) and iv). Part iii) is a direct consequence of lemma 4.1.4 and the corresponding property for the K function norm:

$$K(\lambda t;f) \leq \max(1,\lambda)K(t;f),$$

which follows immediately from definition 2.1.3.
In order to prove part iv) let us assume $\omega(\phi;f;X) = o(\phi^2)$, $\phi \to 0^+$. If we define $f_{1,\phi}$ and $f_{0,\phi}$ by (4.1.10) and (4.1.11), then, in view of (4.1.5) and (1.4.3), we know that $f_{0,\phi} \to 0$ and $f_{1,\phi} \to f$ as $\phi \to 0^+$. Moreover, we may conclude that $Af_{1,\phi} \to 0$ as $\phi \to 0^+$. Since A is a closed operator, $Af = 0$ or, on account of (4.1.7), we have $T_\phi f = f$. This proves part iv).

4.2. DIRECT AND INVERSE THEOREMS

4.2.1. Definition. We say that $f \in X$ belongs to the space $\mathrm{Lip}(\gamma,X)$, $0 < \gamma \leq 2$, if there exists a $c \in \mathbb{R}^+$ such that

$$\omega(\phi;f;X) \leq c\phi^\gamma.$$

The spaces $\mathrm{Lip}(\gamma,X)$ can be characterized in terms of spaces of K-interpolation between X and D(A).

4.2.2. Theorem. The subspace $\mathrm{Lip}(\gamma,X)$ of X is a Banach space with respect to the norm

$$||f||_{\mathrm{Lip}(\gamma,X)} = ||f||_X + \sup_{n\in Z^+} (n^\gamma \, \omega(n^{-1};f;X)).$$

Moreover,

$$\mathrm{Lip}(\gamma,X) \cong (X,D(A))_{\gamma/2,\infty;K} .$$

Proof. The theorem is an immediate consequence of lemma 4.1.4 and propositions 2.1.4 and 2.3.3.

4.2.3. Definition. Let $f \in X$ and let P_n be the $(n+1)$ dimensional subspace of polynomials in $\cos\theta$ of degree $\leq n$. Then we call

$$(4.2.1) \qquad E(P_n;f;X) = \inf_{p_n \in P_n} ||f-p_n||_X$$

the best approximation of f by polynomials of degree n in X.

It is not hard to show that the infinum (4.2.1) is attained (Lorentz [37], ch. 2). A polynomial p_n with this property is called a polynomial of best approximation to f. It is also possible to establish the unicity of the polynomial of best approximation. For the spaces C and L^1 this follows from the fact that the Jacobi polynomials $R_k^{(\alpha,\beta)}(\cos\theta)$, $k = 0, \ldots, n$, which are a base in P_n, form a Chebyshev system. For the spaces L^p, $1 < p < \infty$, it is a consequence of the strict convexity of the spaces. For the details we refer to Cheney [22].

We now prove a direct theorem of the Jackson type.

4.2.4. Theorem. There exists a constant C_0 such that for each $f \in X$ and each $n \in Z^+$

$$(4.2.2) \qquad E(P_n;f;X) \leq C_0[n^{-2}||f||_X + \omega(n^{-1};f;X)].$$

Proof. We use the kernel

$$(4.2.3) \qquad J_{n,2}^{(\alpha+2,\beta)}(\cos\theta) = c_n[R_n^{(\alpha+2,\beta)}(\cos\theta)]^2 \quad (n \in Z^+),$$

where

$$(4.2.4) \qquad c_n^{-1} = \int_0^\pi [R_n^{(\alpha+2,\beta)}(\cos\theta)]^2 \rho^{(\alpha,\beta)} d\theta.$$

Application of the estimate (1.2.8) with $\lambda = 2$, $p = 2$, $\mu = 0$, shows that

$c_n \approx n^{2\alpha+2}$.

If we put

$$(4.2.5) \qquad K_n^{(\alpha+2,\beta)}(\cos\theta) = J_{n',2}^{(\alpha+2,\beta)}(\cos\theta), \qquad n' = [\tfrac{n}{2}],$$

then $K_n^{(\alpha+2,\beta)}(\cos\theta)$ is a positive polynomial kernel of degree $\le n$. Moreover, by (1.2.8) with $\lambda = 2$, $p = 2$, $\mu = \gamma/2$, the kernel $K_n^{(\alpha+2,\beta)}(\cos\theta)$ has the useful property

$$(4.2.6) \qquad \int_0^\pi \theta^\gamma \, K_n^{(\alpha+2,\beta)}(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta \approx n^{-\gamma}, \quad 0 < \gamma \le 2.$$

By the Hölder-Minkowski inequality we have

$$E(P_n;f;X) \le ||(K_n^{(\alpha+2,\beta)}(f;\cdot) - f(\cdot)||_X$$

$$= ||\int_0^\pi K_n^{(\alpha+2,\beta)}(\cos\phi)(T_\phi f(\cdot) - f(\cdot))\rho^{(\alpha,\beta)}(\phi)d\phi||_X$$

$$\le \int_0^\pi K_n^{(\alpha+2,\beta)}(\cos\phi)\omega(\phi;f;X)\rho^{(\alpha,\beta)}(\phi)d\phi.$$

Using proposition 4.1.2. iii), we obtain

$$E(P_n;f;X) \le C \max[1, n^2 \int_0^\pi K_n^{(\alpha+2,\beta)}(\cos\phi)\phi^2 \, \rho^{(\alpha,\beta)}(\phi)d\phi]$$

$$[n^{-2}||f||_X + \omega(n^{-1};f;X)],$$

which, by (4.2.6), leads to

$$E(P_n;f;X) \le C_0[n^{-2}||f||_X + \omega(n^{-1};f;X)].$$

This proves theorem 4.2.4.

The operator A^k is defined recursively by $A^k f = A(A^{k-1}f)$, $A^0 = I$ (the identity operator) $k \in Z^+$. For functions with "higher order smoothness" we have

4.2.5. Theorem. For every $r \in Z^+$ there exists a constant C_r, such that for each $f \in D(A^r)$ and each $n \in Z^+$

(4.2.7) $$E(P_n;f;X) \le C_r\, n^{-2r}[n^{-2}(||f||_X + ||A^r f||_X) + \omega(n^{-1};A^r f;X)].$$

Proof. Let $K_n^{(\alpha+2,\beta)}(\cos\theta)$ be the kernel in theorem 4.2.4 and K_n the bounded linear operator of X into P_n defined by $(K_n f)(\cos\theta) = K_n^{(\alpha+2,\beta)}(f;\cos\theta)$. We construct the powers of K_n iteratively by $K_n^j = K_n(K_n^{j-1})$, $K_n^0 = I$, $j \in Z^+$. Then

$$(K_n-I)^r f = \sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j}K_n^j f.$$

We define the operator T_n^r by

$$T_n^r = -(I-K_n)^{r+1} + I.$$

Clearly T_n^r maps X into P_n and, by the proof of theorem 4.2.4,

$$||f-T_n^r f||_X = ||(I-K_n)^r f - K_n(I-K_n)^r f||_X$$

$$\le C_0[n^{-2}||(I-K_n)^r f||_X + \omega(n^{-1};(I-K_n)^r f;X)].$$

From the Fourier-Jacobi expansion it follows that

$$A(I-K_n)^r f = (I-K_n)^r Af, \qquad f \in D(A),$$

so that by (4.1.5) we obtain

$$||f-T_n^r f||_X \le C'[n^{-2}||(I-K_n)^r f||_X + n^{-2}||(I-K_n)^r Af||_X].$$

By continuing this process, repeated application of theorem 4.2.4 and (4.1.5) leads to

$$E(P_n;f;X) \le C(r)n^{-2r}[n^{-2}\{||f||_X + \sum_{i=1}^{r}||A^i f||_X\} + \omega(n^{-1};A^r f;X)].$$

Since for $f \in D(A^2)$ we have the relation $Af = g_2 * A^2 f$, where

$$g_2(\cos\theta) = \sum_{n=1}^{\infty} [n(n+\alpha+\beta+1)]^{-1}\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta),$$

and $g_2 \in L^1$ (by theorem 1.5.4), it follows that

$$||Af||_X \leq c||A^2f||_X .$$

Hence,

$$\sum_{i=1}^{r} ||A^if||_X \leq \bar{c}||A^rf||_X$$

and theorem 4.2.5 is proved.

An inequality of the Bernstein type for the operator A has been derived by E.M. Stein [45]. There exists a constant C such that for $p_n \in P_n$

(4.2.8) $$||Ap_n||_X \leq Cn^2||p_n||_X ,$$

and therefore for each $r \in Z^+$

(4.2.9) $$||A^rp_n||_X \leq C^rn^{2r}||p_n||_X .$$

We are now in a position to apply the general theory on spaces of best approximation, dealt with in section 2.2. From definition 2.2.4 and the formulas (4.2.7) and (4.2.9) we conclude that the Banach space $D(A^r)$, $r \in Z^+$, belongs to the class $D_{2r}(X)$. Application of theorem 2.2.6 yields

4.2.6. Theorem. For $0 < \gamma < 1$ and $1 \leq q \leq \infty$

$$(D(A^r),D(A^{r+1}))_{\gamma,q;K} \cong X^K_{2r+2\gamma,q} , \qquad r \in P.$$

We have written $D(A^0) = X$.

Theorem 4.2.6 states the equivalence of the spaces of best approximation $X^K_{2r+2\gamma,q}$, which consists of all the functions $f \in X$ such that

$$\{n^{2r+2\gamma} E(P_n;f;X)\}_{n\in Z^+} \in l^q_* ,$$

with the spaces of K-interpolation $(D(A^r),D(A^{r+1}))_{\gamma,q;K}$, which, by lemma 4.1.4, are composed of all $f \in X$ such that $f \in D(A^r)$ with

$$\{n^{2\gamma}\ \omega(n^{-1};f;D(A^r))\}_{n\in Z^+} \in l^q_* ,$$

($r\in P$, $0<\gamma<1$, $1\leq q\leq\infty$). In particular, in the case $q = \infty$ we have

4.2.7. Corollary. Let $f \in X$. Then a necessary and sufficient condition for f to belong to $D(A^r)$ with $A^r f \in Lip(\gamma,X)$, $r \in P$, $0 < \gamma < 2$, is

$$\sup_{n\in Z^+} (n^{2r+\gamma}\ E(P_n;f;X)) < \infty.$$

An important consequence of theorem 4.2.6 is the equivalence of the modulus of continuity $\omega(\phi;f;C)$ with the ordinary symmetric modulus of continuity $\omega^*(\phi;f;C)$, defined in the introduction to this chapter. Since the degree of best approximation by polynomials in $\cos\theta$ is the same in both spaces, it follows from corollary 4.2.7 and the Jackson-Bernstein theorems that

$$Lip(\gamma,C) \cong Lip^*(\gamma,C), \qquad 0 < \gamma < 2,$$

where $Lip^*(\gamma,C)$ denotes the space of functions $f \in C$ with $\omega^*(\phi;f;C) \leq C\phi^\gamma$ endowed with the norm

$$||f||_{Lip^*(\gamma,C)} = ||f||_C + \sup_{n\in Z^+} (n^\gamma\ \omega^*(n^{-1};f;C)).$$

The results obtained in this chapter are generalizations of theorems of Jackson and Bernstein, for which we refer to Butzer and Nessel [18] and the literature quoted there. Similar theorems for spherical harmonic expansions are obtained by Butzer and Johnen [17]. The direct part (theorem 4.2.5) in the case $X = C$ has been established by Ragozin [41] who considers approximation of continuous functions by polynomials on projective spaces.

CHAPTER V SATURATION AND NON-OPTIMAL APPROXIMATION OF SOME SUMMATION METHODS

The main object of this chapter is to investigate the order of approximation of a number of summation processes for Jacobi series and to characterize the functions for which the Fourier-Jacobi series, summed up by this particular summation method, approximates the function with a certain order. The summation methods we consider are all saturated. This means that, except for constant functions (the 'trivial' class), there exists an optimal order of approximation (the saturation order) for these processes. The saturation problem consists in determining this optimal order and in finding the class of functions (saturation class) which can be approximated with this optimal order. By the general theory on approximation processes on Banach spaces, which is dealt with in chapter II, the problem of characterizing the classes of optimal and non-optimal approximation can be solved, if one can find a closed linear operator B, satisfying relations of the form (2.4.2) and (2.4.3). If, on the other hand, the summation process defines a semi-group of contraction operators, it is also possible to give a characterization of the classes of optimal and non-optimal approximation by applying the results mentioned in section 2.5.

5.1. THE WEIERSTRASS APPROXIMATION PROCESS

The first summation process we investigate is the Weierstrass approximation process (W-summation) which is generated by the factor sequence

$$(5.1.1)\qquad \{c_\lambda(n)\}_{n\in P} = \{e^{-n(n+\alpha+\beta+1)\lambda^{-1}}\}_{n\in P} \qquad \lambda \in R^+.$$

The sequence (5.1.1) obviously satisfies the conditions (3.3.1), (3.3.2) and (3.3.3). Hence, putting $t = 1/\lambda$, we may form the Weierstrass means for $f \in X$:

$$(5.1.2)\qquad W_t(f;\cos\theta) = \sum_{n=0}^{\infty} e^{-n(n+\alpha+\beta+1)t}\, f^\wedge(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

The corresponding kernel is defined by

$$(5.1.3)\qquad W_t(\cos\theta) = \sum_{n=0}^{\infty} e^{-n(n+\alpha+\beta+1)t}\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta),\ t \in R^+.$$

It is not hard to show that the kernel (5.1.3) is an approximation kernel (see definition 3.1.2) with $\lambda = t^{-1}$. Condition (3.1.3) is trivially satisfied. The positivity of (5.1.3), which implies (3.1.2), is a consequence of the positivity of the generalized translation operator by the following argument due to Bochner [15], (see also Gasper [28]). Let the operator of the factor sequence type B_ψ with factors $\psi(n)$, $(n \in P)$, be a positive operator, i.e. if $f \geq 0$ $(f \in X)$ then $B_\psi f \geq 0$. Then it follows that the operator of the factor sequence type with factors $e^{t\psi(n)}$, $(t \geq 0,\ n \in P)$, is also positive. Taking $\psi(n) = R_n^{(\alpha,\beta)}(\cos\phi)$, $(0<\phi\leq\pi,\ n\in P)$, and multiplying by e^{-t} we obtain that the operator with factors

$$\mu(n) = e^{-t(1-R_n^{(\alpha,\beta)}(\cos\phi))} \qquad (t\geq 0,\ n\in P)$$

is positive. If we now replace t by $t(2\alpha+2)(1-\cos\phi)^{-1}$ and we let $\phi \to 0^+$, we conclude by (1.1.9) that the operator of the factor sequence type with factor

$$e^{-n(n+\alpha+\beta+1)t} \qquad (t\geq 0,\ n\in P)$$

is positive, which is equivalent to the positivity of (5.1.3).

The function $W_t(f;\cos\theta)$ is a solution of the generalized heat equation

$$\frac{\partial}{\partial t} U(\theta,t) = - A\, U(\theta,t),$$

which, by theorem 3.1.3, satisfies the initial condition

$$\lim_{t\to 0^+} U(\theta,t) = f(\cos\theta) \qquad (f\in X).$$

Here A is written for the operator defined by (4.1.3).
The family of convolution operators $\{W_t,\ t\geq 0\}$ define an equi-bounded semi-group of operators of class (C_0). The conditions (2.5.1. i) and ii)) follow from theorem 3.1.3; the other conditions are obviously satisfied. It is not hard to show that -A is the infinitesimal generator of this semi-group. Let B be the infinitesimal generator of the semi-group $\{W_t,\ t\geq 0\}$ and suppose $f \in D(B)$. Then by definition

$$(Bf)^\wedge(n) = \lim_{\tau\to 0^+} \int_0^\pi \frac{[W_\tau(f;\cos\theta)-f(\cos\theta)]}{\tau} R_n^{(\alpha,\beta)}(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta$$

$$= \lim_{\tau\to 0^+} \frac{e^{-\lambda_n \tau}-1}{\tau} f^\wedge(n) = -\lambda_n f^\wedge(n) \qquad (n\in P),$$

which proves that B = -A and

$$D(B) \subset D(A).$$

On the other hand, if $f \in D(A)$, then we obtain for the N-th Cesàro mean of order μ ($\mu>\alpha+\frac{1}{2}$) of the element Af that

$$BS_N^\mu(f;\cos\theta) = -S_N^\mu(Af;\cos\theta) \qquad (N\in P).$$

Since, by proposition 2.5.3, B is a closed operator and $S_N^\mu(f;\cos\theta)$ and $BS_N^\mu(f;\cos\theta)$ converge in norm to f and -Af, respectively, as $N \to \infty$, this proves that $f \in D(B)$ and Bf = -Af which completes the proof. Application of theorems 2.5.4 and 4.2.2 yields

5.1.1. Theorem. The Fourier-Jacobi series (1.3.1) of $f \in X$ is W-summable to f in X. The process $\{W_t;\ t\geq 0\}$ is saturated with order t. The saturation class F(X,W) is the class Lip(2,X). Moreover, the following statements are equivalent for $0 < \theta < 1$, $1 \leq q \leq \infty$ or $\theta = 1$, $q = \infty$:

i) $\qquad f \in (X,D(A))_{\theta,q;K}$,

ii) $\qquad \{n^\theta\ \omega_W(n;f;X)\}_{n\in Z^+} \in l_*^q$.

5.2. THE GENERALIZED WEIERSTRASS APPROXIMATION PROCESS

A generalization of the Weierstrass summation method has been introduced by Bochner [15] for expansions in terms of ultraspherical polynomials. In order to define W_σ-summability we consider the following sequence

$$(5.2.1) \qquad \{c_\lambda(n)\}_{n\in P} = \{e^{-\lambda^{-1}(n(n+\alpha+\beta+1))^{\sigma/2}}\}_{n\in P}, \quad \lambda \in R^+,\ 0 < \sigma \leq 2.$$

The sequence (5.2.1) defines a summation process as is easily checked. By the substitution $t = \lambda^{-1}$ the generalized Weierstrass means of $f \in X$ can be written in the form

$$(5.2.2) \qquad W_t^\sigma(f;\cos\theta) = \sum_{n=0}^{\infty} e^{-t(n(n+\alpha+\beta+1))^{\sigma/2}} f^\wedge(n)\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta).$$

In order to show that (5.2.2) defines a strong approximation process on X (with $\lambda = t^{-1}$), we only need to prove the positivity of the kernel

$$W_t^\sigma(\cos\theta) = \sum_{n=0}^{\infty} e^{-t[n(n+\alpha+\beta+1)]^{\sigma/2}} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta). \tag{5.2.3}$$

This can be deduced from the positivity of (5.1.3) by Bochner's method of subordinators (see Bochner [15], p. 46).

The function $W_t^\sigma(f;\cos\theta)$ satisfies the equation

$$\frac{\partial}{\partial t} U(\theta,t) = -D_\sigma\, U(\theta,t)$$

with the initial condition

$$\lim_{t\to 0^+} U(\theta,t) = f(\cos\theta) \qquad (f\in X).$$

Here D_σ denotes the fractional differential operator of order σ, defined as the operator of the factor sequence type with factors $[n(n+\alpha+\beta+1)]^{\sigma/2}$. By the same argument as in section 5.1 we may conclude that the family of convolution operators $\{W_t^\sigma, t\geq 0\}$ defines an equi-bounded semi-group of operators of class (C_0) with infinitesimal generator $-D_\sigma$. Hence we have by theorems 2.5.4 and 3.4.3

5.2.1. Theorem. The Fourier-Jacobi series (1.3.1) of $f \in X$ is W^σ-summable to f in X $(0<\sigma\leq 2)$. The process $\{W_t^\sigma, t\geq 0\}$ is saturated with order t. The saturation class $F(X,W^\sigma)$ is the class $H(X,[n(n+\alpha+\beta+1)]^{\sigma/2})$ (see theorem 3.4.3). Moreover, the following statements are equivalent for $0 < \theta < 1$, $1 \leq q \leq \infty$ or $\theta = 1$, $q = \infty$:

i) $\quad f \in (X,D(D_\sigma))_{\theta,q;K}$,

ii) $\quad \{n^\theta\, \omega_{W^\sigma}(n;f;X)\}_{n\in Z+} \in l_*^q$.

Later on we shall be able to characterize the spaces $(X,D(D_\sigma))_{\theta,q;K}$, $0 < \theta < 1$, $1 \leq q \leq \infty$, $0 < \sigma < 2$ as spaces of K-interpolation between X and D(A) (see theorem 7.4.1).

5.3. THE ABEL-POISSON SUMMATION METHOD

The Abel-Poisson summation method has already been introduced in section 3.3. Here we wish to determine the saturation order of this process and to characterize the functions which allow a certain order of A-approximation. After the substitution $r = e^{-t}$ the Abel-Poisson means (3.3.8) define a equi-bounded semi-group of class (C_0) with infinitesimal generator B_{-n}, the operator of the factor sequence type with factors $-n$, $n \in P$. By theorems 2.5.4, 3.4.3 and corollary 3.4.6 we conclude:

5.3.1. Theorem. The process $\{A_r, 0 \le r < 1\}$ is saturated with order $(1-r)$. The saturation class $F(X,A)$ is the class $H(X,[n(n+\alpha+\beta+1)]^{\frac{1}{2}})$. Moreover, the following statements are equivalent for $0 < \theta < 1$, $1 \le q \le \infty$ or $\theta = 1$, $q = \infty$:

i) $\qquad f \in (X,D(D_1))_{\theta,q;K}$,

ii) $\qquad \{n^{\theta}\, \omega_A(n;f;X)\}_{n\in Z+} \in l^q_*$.

5.4. THE GENERALIZED ABEL-POISSON SUMMATION METHOD

The generalized Abel-Poisson summation process is defined by the sequence

$$(5.4.1) \qquad \{c^{\sigma}_{\lambda}(n)\}_{n\in P} = \{e^{-\frac{1}{\lambda}n^{\sigma}}\}_{n\in P} , \qquad \lambda \in R^+,\ 0 < \sigma \le 1.$$

In the case $\sigma = 1$, this process reduces to the Abel-Poisson summation method after the substitution $r = e^{-1/\lambda}$. The generalized Abel-Poisson means of $f \in X$ have the form $(t=\lambda^{-1})$

$$(5.4.2) \qquad A^{\sigma}_t(f;\cos\theta) = \sum_{n=0}^{\infty} e^{-tn^{\sigma}}\, f^{\wedge}(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

The positivity of the corresponding kernel

$$(5.4.3) \qquad A^{\sigma}_t(\cos\theta) = \sum_{n=0}^{\infty} e^{-tn^{\sigma}}\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)$$

can be deduced from the positivity of (3.3.9) by Bochner's method of subordinators (see Bochner [15], p. 46). This implies that (5.4.2) defines a strong approximation process on X (with $\lambda = t^{-1}$). Furthermore, it is easily derived that the family of operators $\{A_t^\sigma,\ t\geq 0,\ 0<\sigma\leq 1\}$ defines an equi-bounded semi-group of operators of class (C_0) with infinitesimal generator $-B_{-n^\sigma}$, where B_{-n^σ} is the operator of the factor sequence type with factors $-n^\sigma$. Application of theorems 2.5.4, 3.4.3 and corollary 3.4.6 yields

5.4.1. Theorem. The Fourier-Jacobi series (1.3.1)of $f \in X$ is A^σ-summable to f in X $(0<\sigma\leq 1)$. The process $\{A_t^\sigma,\ t\geq 0\}$ is saturated with order t. The saturation class $F(X,A^\sigma)$ is the class $H(X,[n(n+\alpha+\beta+1)]^{\sigma/2})$. Moreover, the following statements are equivalent for $0 < \theta < 1$, $1 \leq q \leq \infty$ or $\theta = 1$, $q = \infty$:

i) $\qquad f \in (X,D(X_\sigma))_{\theta,q;K}$,

ii) $\qquad \{n^\theta\ \omega_{A^\sigma}(n;f;X)\}_{n\in Z+} \in l_*^q$.

For a characterization of the spaces $(X,D(D_\sigma))_{\theta,q;K}$ in terms of the spaces of K-interpolation between X and D(A) we refer to chapter VII.

5.5. SOME RESULTS ON FRACTIONAL INTEGRALS AND DERIVATIVES

So far we have only treated summation methods which define an equi-bounded semi-group of operators of class (C_0). The next sections will be devoted to summation methods which are generated by convolution with a polynomial kernel. In order to treat the non-optimal approximation of these processes we need a generalization of the inequality (4.2.8) to fractional powers of the operator A. We shall use the function

$$(5.5.1) \qquad g_\sigma(\cos\theta) \sim \sum_{n=1}^{\infty} [n(n+\alpha+\beta+1)]^{-\sigma/2}\ \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta), \quad \sigma > 0,$$

and the following properties of g_σ.

5.5.1. Lemma. For $\sigma > 0$ the function $g_\sigma \in L^1$. Furthermore,

i) $\qquad g_{\sigma_1} * g_{\sigma_2} = g_{\sigma_1+\sigma_2} \qquad\qquad \sigma_1,\sigma_2 > 0$

ii) $\qquad |g_\sigma(\cos\theta)| = O(\theta^{\sigma-2\alpha-2}) \qquad\qquad (\theta\to 0^+,\ 0<\sigma<2\alpha+2).$

iii) If $\theta \neq 0$, then $P(\frac{d}{d\theta})\ g_\sigma(\cos\theta)$ exists, is continuous and

$$\sup_{0<\theta\leq\pi} |P(\tfrac{d}{d\theta})g_\sigma(\cos\theta)| = O(\theta^{\sigma-2\alpha-4}) \qquad (\theta\to 0^+,\ 0<\sigma<2\alpha+4).$$

The operator $P(\frac{d}{d\theta})$ is defined by (1.1.6).

Proof. Property i) is obvious. Property ii) and iii) follow from theorem 1.5.4 and formula (3.4.4).

5.5.2. Lemma. There exists a constant $C(\sigma)$, such that for each polynomial p_n of degree $\leq n$ ($n\in Z^+$) in $\cos\theta$

$$(5.5.2) \qquad ||D_\sigma p_n||_X \leq C(\sigma)n^\sigma ||p_n||_X \qquad (0<\sigma\leq 2).$$

Proof. We shall give the proof in the case $X = L^p$ ($1\leq p<\infty$).

$$||D_\sigma p_n||_p = ||\int_0^\pi T_\phi Ap_n(\cdot)g_{2-\sigma}(\phi)\rho^{(\alpha,\beta)}(\phi)d\phi||_p$$

$$\leq ||\int_0^{\frac{1}{n}}||_p + ||\int_{\frac{1}{n}}^\pi||_p = I_1+I_2.$$

If we put $p' = p/(p-1)$ and notice that $\rho^{(\alpha,\beta)}(\phi) = O(\phi^{2\alpha+1})$, $\phi\to 0^+$, then by Hölder's inequality

$$I_1^p = \int_0^\pi |\int_0^{\frac{1}{n}} T_\phi Ap_n(\cos\theta)g_{2-\sigma}(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi|^p \rho^{(\alpha,\beta)}(\theta)d\theta$$

$$\leq C(\int_0^{\frac{1}{n}} (\phi^{-1+\frac{\lambda(p-1)+1}{p}})^{p'} d\phi)^{p/p'}$$

$$\int_0^\pi (\int_0^{\frac{1}{n}} |T_\phi Ap_n(\cos\theta)|^p |g_{2-\sigma}(\cos\phi)|^p \phi^{(2\alpha+2)p-\lambda(p-1)-1}d\phi)\rho^{(\alpha,\beta)}(\theta)d\theta$$

$$\leq Cn^{-\lambda(p-1)} \int_0^{\frac{1}{n}} ||T_\phi Ap_n||_p^p \; |g_{2-\sigma}(\cos\phi)|^p \phi^{(2\alpha+2)p-\lambda(p-1)-1}d\phi.$$

Now, using (1.4.2), (4.2.8) and lemma 5.5.1. ii), we have

$$I_1^p \leq Cn^{-\lambda(p-1)}n^{2p}, ||p_n||_p^p \int_0^{\frac{1}{n}} \phi^{(2-\sigma)p-\lambda(p-1)-1}d\phi$$

$$\leq Cn^{\sigma p} ||p_n||_p^p \,,$$

if we choose $0 < \lambda < (2-\sigma)p'$.

On the other hand, using (1.4.2) and lemma 5.5.1. iii), we have

$$I_2^p = \int_0^\pi \left| \int_{\frac{1}{n}}^\pi T_\phi P_n(\cos\theta)\, P(\tfrac{d}{d\phi}) g_{2-\sigma}(\cos\phi) \rho^{(\alpha,\beta)}(\phi) d\phi \right|^p \rho^{(\alpha,\beta)}(\theta) d\theta$$

$$\leq C\Big(\int_{\frac{1}{n}}^\pi \phi^{-1-\frac{\mu(p-1)-1}{p}} d\phi\Big)^{p/p'} \Big(\int_0^\pi \Big(\int_{\frac{1}{n}}^\pi |T_\phi p_n(\cos\theta)|^p$$

$$\cdot\, |P(\tfrac{d}{d\phi}) g_{2-\sigma}(\cos\phi)|^p\, \phi^{(2\alpha+2)p+\mu(p-1)-1} d\phi\Big) \rho^{(\alpha,\beta)}(\theta) d\theta\Big)$$

$$\leq Cn^{\mu(p-1)} \int_{\frac{1}{n}}^\pi ||T_\phi p_n||_p^p\, \phi^{-\sigma p+\mu(p-1)-1} d\phi$$

$$\leq Cn^{\mu(p-1)}\, ||p_n||_p^p \int_{\frac{1}{n}}^\infty \phi^{-\sigma p+\mu(p-1)-1} d\phi$$

$$\leq Cn^{\sigma p}\, ||p_n||_p^p ,$$

if we choose $0 < \mu < \sigma p'$. Combination of the estimates proves (5.5.2).

We define the fractional integral of order σ of $f \in X$, $I_\sigma f$ by the convolution of f with g_σ, defined by (5.5.1). In [8] we have proved the following results, mainly by the method used in the proof of lemma 5.5.2:

5.5.3. Theorem.

a) If $f \in X$ and $0 < \sigma < 2$, then $I_\sigma f \in \mathrm{Lip}(\sigma,X)$.

b) If $f \in \mathrm{Lip}(\tau,X)$, $0 < \sigma$, $\tau < 2$ and $\sigma + \tau < 2$, then $I_\sigma f \in \mathrm{Lip}(\sigma+\tau,X)$.

c) If $f \in L^p$ $(1 \leq p < \infty)$ and $\frac{2\alpha+2}{p} < \sigma < 2 + \frac{2\alpha+2}{p}$, then $I_\sigma f \in \mathrm{Lip}(\sigma - \frac{2\alpha+2}{p}, C)$.

d) If $f \in L^p$ $(1<p<\infty)$ and $0 < \sigma < \frac{2\alpha+2}{p}$, then $I_\sigma f \in L^r$, where $\frac{1}{r} = \frac{1}{p} - \frac{\sigma}{2\alpha+2}$.

For the fractional derivative of order σ of $f \in X$, $D_\sigma f$, defined as the operator of the factor sequence type with factors $[n(n+\alpha+\beta+1)]^{\sigma/2}$, we have shown in [8]:

5.5.4. Theorem. Let $0 < \sigma < \tau < 2$ and suppose $f \in \mathrm{Lip}(\tau,X)$. Then $D_\sigma f \in \mathrm{Lip}(\tau-\sigma,X)$.

5.6. THE CESÀRO SUMMATION METHOD

This summation method has already been introduced in section 3.3. We now derive a limit relation of the form (2.4.2) for this process by a well-known method (cf. [13], p. 249).

5.6.1. Lemma. Let $B = B_{-\mu n}$ be the operator of the factor sequence type with factors $-\mu n$, $n \in \mathbb{P}$, and let $S_N^\mu(f;\ \cos\theta)$, $N \in \mathbb{Z}^+$, denote the Cesàro means of order μ ($\mu>\alpha+\frac{1}{2}$) of the expansion (1.3.1) of $f \in D(B)$, then

$$(5.6.1)\qquad \lim_{N\to\infty} ||N\{S_N^\mu(f;\cdot)-f(\cdot)\} - Bf(\cdot)||_X = 0.$$

Proof. We apply the following identities (see Zygmund [51], p. 269)

$$(5.6.2)\qquad S_N^{\mu+1}(f;\cos\theta) - S_{N-1}^{\mu+1}(f;\cos\theta) = -\frac{\mu+1}{\mu}\,\frac{1}{N(N+\mu+1)}\,S_N^\mu(Bf;\cos\theta),$$

$$(5.6.3)\qquad S_N^\mu(f;\cos\theta) = S_N^{\mu+1}(f;\cos\theta) - \frac{1}{\mu(N+\mu+1)}\,S_N^\mu(Bf;\cos\theta),$$

$$(N\in\mathbb{Z}^+,\ f\in D(B)).$$

Repeated application of (5.6.2) in the case $\mu > \alpha+\frac{1}{2}$ yields by corollary 3.3.3

$$||f(\cdot)-S_N^{\mu+1}(f;\cdot)||_X \le \sum_{l=N+1}^{\infty} ||S_l^{\mu+1}(f;\cdot)-S_{l-1}^{\mu+1}(f;\cdot)||_X$$

$$\le \frac{\mu+1}{\mu}\sum_{l=N+1}^{\infty}\frac{1}{l(l+\mu+1)}\,||S_l^\mu(Bf;\cdot)||_X.$$

If we put $C_N = \sum_{l=N+1}^{\infty}\frac{1}{l(l+\mu+1)}$, then by (5.6.3) we deduce for $f \in D(B)$

$$||C_N^{-1}\{S_N^\mu(f;\cdot)-f(\cdot)\} - Bf(\cdot)||_X \le \frac{\mu+1}{\mu} \sup_{l\ge N+1} ||S_l^\mu(Bf;\cdot)-Bf(\cdot)||_X$$

$$+ \frac{C_N^{-1}}{\mu(N+\mu+1)} ||S_N^\mu(Bf;\cdot) - Bf(\cdot)||_X + \frac{1}{\mu} |1- \frac{1}{C_N(N+\mu+1)}| \; ||Bf(\cdot)||_X$$

$$= o(1) \qquad\qquad (N\to\infty),$$

using corollary 3.3.3 and the fact that

$$1 - \frac{1}{C_\mu(N+\mu+1)} \simeq 1 - \frac{1}{N+\mu+1} \left(\int_{N+1}^\infty \frac{dx}{x(x+\mu+1)}\right)^{-1}$$

$$= o(1) \qquad\qquad (N\to\infty).$$

Since $\lim_{N\to\infty} C_N N = 1$ relation (5.6.1) follows.

As $S_N^\mu(f;\cos\theta)$ is a polynomial of degree $\le N$ in $\cos\theta$, it is obvious that $S_N^\mu(f;\cos\theta) \in D(B)$ for all $N \in Z^+$ and all $f \in X$. Also, by lemma 3.4.5., we know that there exists a measure $\nu \in M$ such that $Bf = D_1(\nu * f)$, where D_1 is the operator of the factor sequence type with factors $[n(n+\alpha+\beta+1)]^{\frac{1}{2}}$. Application of (5.5.2) and (3.3.14) leads to

$$||BS_N^\mu(f;\cdot)||_X = ||D_1 S_N^\mu(\nu * f;\cdot)||_X \le C\Lambda_\mu N||\nu||_M \; ||f||_X$$

$$\le C_1 N||f||_X.$$

Now all the conditions of theorems 2.4.2 and 2.4.3 are satisfied. Hence we have

5.6.2. Theorem. The process $\{S_N^\mu,\ N\in Z^+,\ \mu>\alpha+\frac{1}{2}\}$ is saturated with order N^{-1}. The saturation class $F(X,S^\mu)$ is the class $H(X,[n(n+\alpha+\beta+1)]^{\frac{1}{2}})$. Moreover, the following statements are equivalent for $0 < \theta < 1$, $1 \le q \le \infty$ or $\theta = 1$, $q = \infty$:

i) $\quad f \in (X,D(D_1))_{\theta,q;K}$,

ii) $\quad \{n^\theta\ \omega_{S^\mu}(n;f;X)\}_{n\in Z^+} \in l_*^q$.

5.7. THE DE LA VALLÉE-POUSSIN SUMMATION METHOD

This summation method has been introduced by De la Vallée-Poussin [49] for Fourier series and it has been generalized for ultraspherical series by Kogbetliantz [33]. The N-th De la Vallée-Poussin mean $V_N(f;\cos\theta)$ of the Fourier-Jacobi series of a function $f \in X$ is defined by

$$(5.7.1)\qquad V_N(f;\cos\theta) = \sum_{n=0}^{N} \frac{\Gamma(N+1)\Gamma(N+\alpha+\beta+2)}{\Gamma(N-n+1)\Gamma(N+n+\alpha+\beta+2)} f^\wedge(n)\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta) \qquad (N\in Z^+).$$

All the conditions for a summation method (section 3.3) are trivially satisfied, since the corresponding kernel $V_N(\cos\theta)$ has the explicit representation

$$(5.7.2)\qquad V_N(\cos\theta) = \frac{\Gamma(N+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(N+\beta+1)} (\cos\tfrac{\theta}{2})^{2N} \qquad (N\in Z^+).$$

This can be verified by computing the Fourier-Jacobi coefficients of $(\cos\frac{\theta}{2})^{2N}$ by means of formula (1.1.1). The representation (5.7.2) allows us to conclude that the kernel $V_N(\cos\theta)$ is a positive approximation kernel, since the 'peaking property' (3.1.7) is satisfied. We prove

5.7.1. Lemma. Let A be the operator of the factor sequence type with factors $n(n+\alpha+\beta+1)$, $n \in P$, and let $V_N(f;\cos\theta)$, $N \in Z^+$ denote the De la Vallée-Poussin means of the expansion (1.3.1) of $f \in D(A)$, then

$$(5.7.3)\qquad \lim_{N\to\infty} ||N\{V_N(f;\cdot)-f(\cdot)\} - Af(\cdot)||_X = 0.$$

Proof. A direct calculation, based on comparison of the Fourier-Jacobi coefficients, leads to the following identity, which generalizes an identity due to Butzer and Pawelke [19]:

$$(5.7.4)\qquad N(N+\alpha+\beta+1)\,[V_N(\cos\theta)-V_{N-1}(\cos\theta)] = -AV_N(\cos\theta) \qquad (N\in Z^+).$$

Repeated application of (5.7.4) yields by proposition 3.3.1

$$||f(\cdot)-V_N(f;\cdot)||_X \le \sum_{l=N+1}^{\infty} \frac{1}{l(l+\alpha+\beta+1)} ||V_l(Af;\cdot)||_X \;.$$

If we write $C_N = \sum_{l=N+1}^{\infty} \frac{1}{l(l+\alpha+\beta+1)}$, we obtain for $f \in D(A)$

$$||C_N^{-1}\{V_N(f;\cdot)-f(\cdot)\} - Af(\cdot)||_X \leq \sup_{l \geq N+1} ||V_l(Af;\cdot)-Af(\cdot)||_X$$

$$= o(1) \qquad (N\to\infty).$$

Since $\lim_{N\to\infty} NC_N = 1$, (5.7.3) is proved.

The De la Vallée-Poussin means $V_N(f;\cos\theta)$ are polynomials of degree $\leq N$ and therefore belong to $D(A)$ for all $N \in Z^+$. For $V_N(f;\cos\theta)$ we now derive a Bernstein type inequality, which is much stronger than (4.2.8) (cf. Butzer-Scherer [20], p. 137).

5.7.2. Lemma. Let $f \in X$ and let A and $V_N(f;\cos\theta)$ be defined as in lemma 5.7.1. Then,

$$||AV_N(f;\cdot)||_X \leq 2(\alpha+1)N||f||_X \,. \tag{5.7.5}$$

Proof. By (5.7.4) we have

$$||AV_N(f;\cdot)||_X = ||\int_0^\pi T_\phi f(\cdot)AV_N(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi||_X$$

$$= N(N+\alpha+\beta+1)||\int_0^\pi \{V_{N-1}(\cos\phi)-V_N(\cos\phi)\}T_\phi f(\cdot)\rho^{(\alpha,\beta)}(\phi)d\phi||_X$$

$$\leq N(N+\alpha+\beta+1)||f||_X \int_0^\pi |V_{N-1}(\cos\phi)-V_N(\cos\phi)|\rho^{(\alpha,\beta)}(\phi)d\phi$$

$$= N(N+\alpha+\beta+1)||f||_X \frac{\Gamma(N+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(N+\beta+1)}$$

$$\cdot \int_0^\pi (\cos\tfrac{\phi}{2})^{2N-2}|(N+\beta)-(N+\alpha+\beta+1)\cos^2\tfrac{\phi}{2}|\rho^{(\alpha,\beta)}(\phi)d\phi$$

$$\leq \frac{N\Gamma(N+\alpha+\beta+2)}{\Gamma(N+\beta)\Gamma(\alpha+1)} ||f||_X \{\int_0^\pi (\sin\tfrac{\phi}{2})^{2\alpha+3}(\cos\tfrac{\phi}{2})^{2N+2\beta+1}d\phi$$

$$+ \int_0^\pi \frac{(\alpha+1)}{(N+\beta)} (\sin\tfrac{\phi}{2})^{2\alpha+1} (\cos\tfrac{\phi}{2})^{2N+2\beta+1}d\phi\}$$

$$= 2(\alpha+1)N||f||_X \,.$$

By (5.7.3) and (5.7.5) all the conditions of theorems 2.4.2 and 2.4.3 are satisfied. We summarize the results as follows:

5.7.3. Theorem. The Fourier-Jacobi series (1.3.1) of $f \in X$ is V-summable to f in X. The process $\{V_N, N \in Z^+\}$ is saturated with order N^{-1}. The saturation class F(X,V) is the class Lip(2,X). Moreover, the following statements are equivalent for $0 < \theta < 1$, $1 \le q \le \infty$ or $\theta = 1$, $q = \infty$:

i) $\quad f \in (X,D(A))_{\theta,q;K}$,

ii) $\quad \{n^{\theta}\, \omega_V(n;f;X)\}_{n \in Z^+} \in l^q_*$.

5.8. THE JACOBI POLYNOMIAL KERNEL

The formula

$$(5.8.1)\qquad R_N^{(\alpha+k+1,\beta)}(\cos\theta) = \frac{\Gamma(N+\beta+1)\Gamma(N+1)\Gamma(\alpha+k+2)\Gamma(\alpha+1)}{\Gamma(N+\alpha+\beta+k+2)\Gamma(N+\alpha+k+2)\Gamma(k+1)}$$

$$\sum_{n=0}^{N} \frac{\Gamma(N+n+\alpha+\beta+k+2)\Gamma(N-n+k+1)}{\Gamma(N+n+\alpha+\beta+2)\Gamma(N-n+1)}\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta) \qquad (N \in Z^+,\ k \in R^+)$$

is due to Szegö [46], section 9.4. The proof uses the formulas (1.1.1) and (1.1.9). We shall study the polynomial kernel $\{J_{N,1}^{(\alpha+k+1,\beta)}(\cos\theta)\}_{N \in Z^+}$, defined by (see Bavinck [10])

$$(5.8.2)\qquad J_{N,1}^{(\alpha+k+1,\beta)}(\cos\theta)$$

$$= \frac{\Gamma(N+\alpha+\beta+2)\Gamma(N+\alpha+k+2)\Gamma(k+1)}{\Gamma(N+\beta+1)\Gamma(N+k+1)\Gamma(\alpha+k+2)\Gamma(\alpha+1)}\, R_N^{(\alpha+k+1,\beta)}(\cos\theta).$$

The convolution of $f \in X$ with the kernel (5.8.2) generates a summation method for the Fourier-Jacobi expansion (1.3.1) of f, which we will call the Jacobi means of order k. The factors are

$$(5.8.3)\qquad c_N(n) = \begin{cases} \dfrac{\Gamma(N+1)\Gamma(N+\alpha+\beta+2)\Gamma(N+n+\alpha+\beta+k+2)\Gamma(N-n+k+1)}{\Gamma(N+\alpha+\beta+k+2)\Gamma(N+k+1)\Gamma(N+n+\alpha+\beta+2)\Gamma(N-n+1)} & (n \le N), \\ 0 & (n > N). \end{cases}$$

In the case $k \in Z^+$ the factors can be written in the form

$$(5.8.4)\qquad c_N(n) = \begin{cases} \prod_{j=1}^{k} \left(1 - \frac{\lambda_n}{\lambda_{N+j}}\right) & (n \le N) \\ 0 & (n > N) \end{cases}$$

where $\lambda_n = n(n+\alpha+\beta+1)$. The representation (5.8.4) shows that for $k \in Z^+$ this summation method is a possible generalization of the typical means (see Butzer and Nessel [18], p. 262).

From the estimate (1.2.8) with $\lambda = k+1$, $p = 1$, $\mu = 0$ we may conclude that there exists a $\Lambda \in R^+$ such that

$$(5.8.5)\qquad \int_0^\pi |J_{N,1}^{(\alpha+k+1,\beta)}(\cos\theta)|\rho^{(\alpha,\beta)}(\theta)d\theta \le \Lambda \quad \text{uniformly for all } N \in Z^+,$$

if $k > \alpha+\frac{1}{2}$. Since the factors $c_N(n)$ tend to 1 as N tends to ∞, it follows that if $k > \alpha+\frac{1}{2}$ the kernel $J_{N,1}^{(\alpha+k+1,\beta)}(\cos\theta)$ is an approximation kernel, which implies that

$$(5.8.6)\qquad \lim_{N\to\infty} ||J_{N,1}^{(\alpha+k+1,\beta)}(f;\cdot) - f(\cdot)||_X = 0 \qquad (k>\alpha+\tfrac{1}{2}).$$

We now derive a limit relation of the form (2.4.2).

5.8.1. Lemma. Let A be the operator of the factor sequence type with factors $n(n+\alpha+\beta+1)$, $n \in P$, and let $J_{N,1}^{(\alpha+k+1,\beta)}(f;\cos\theta)$, $N \in Z^+$ denote the Jacobi means of order k ($k>\alpha+\frac{1}{2}$) of the expansion (1.3.1) of $f \in D(A)$, then

$$(5.8.7)\qquad \lim_{N\to\infty} ||N^2\{J_{N,1}^{(\alpha+k+1,\beta)}(f;\cdot)-f(\cdot)\} - (-k)Af(\cdot)||_X = 0.$$

Proof. The following identities are valid for $f \in D(A)$:

$$(5.8.8)\qquad J_{N,1}^{(\alpha+k+2,\beta)}(f;\cos\theta) - J_{N-1,1}^{(\alpha+k+2,\beta)}(f;\cos\theta)$$

$$= \frac{(k+1)(2N+k+\alpha+\beta+2)}{N(N+\alpha+\beta+1)(N+k+1)(N+\alpha+\beta+k+2)} J_{N,1}^{(\alpha+k+1,\beta)}(Af;\cos\theta),$$

$$(5.8.9)\qquad J_{N,1}^{(\alpha+k+2,\beta)}(f;\cos\theta) - J_{N,1}^{(\alpha+k+1,\beta)}(f;\cos\theta)$$

$$= -\frac{1}{(N+k+1)(N+\alpha+\beta+k+2)} J_{N,1}^{(\alpha+k+1,\beta)}(Af;\cos\theta).$$

By (5.8.6) and by repeated application of (5.8.8) we obtain ($k>\alpha+\frac{1}{2}$)

$$(5.8.10) \qquad ||f(\cdot) - J_{N,1}^{(\alpha+k+2,\beta)}(f;\cdot)||_X$$

$$\leq (k+1) \sum_{l=N+1}^{\infty} \frac{(2l+k+\alpha+\beta+2)}{l(l+\alpha+\beta+1)(l+k+1)(l+\alpha+\beta+k+2)} ||J_{l,1}^{(\alpha+k+1,\beta)}(Af;\cdot)||_X.$$

If we put

$$(5.8.11) \qquad d_N = k \sum_{l=N+1}^{\infty} \frac{(2l+k+\alpha+\beta+2)}{l(l+\alpha+\beta+1)(l+k+1)(l+\alpha+\beta+k+2)} \approx kN^{-2} \qquad (N\to\infty)$$

then, applying (5.8.9) and (5.8.6) we find

$$||d_N^{-1}\{f(\cdot)-J_{N,1}^{(\alpha+k+1,\beta)}(f;\cdot)\} - Af(\cdot)||_X$$

$$\leq \frac{(k+1)}{k} \sup_{l\geq N+1} ||J_{l,1}^{(\alpha+k+1,\beta)}(Af;\cdot) - Af(\cdot)||_X$$

$$+ \frac{d_N^{-1}}{(N+k+1)(N+\alpha+\beta+k+2)} ||J_{N,1}^{(\alpha+k+1,\beta)}(Af;\cdot) - Af(\cdot)||_X$$

$$+ ||Af(\cdot)||_X \left|\frac{1}{k} - \frac{d_N^{-1}}{(N+k+1)(N+\alpha+\beta+k+2)}\right| = o(1) \qquad (N\to\infty).$$

By using (5.8.11), relation (5.8.7) follows.

Since $J_{N,1}^{(\alpha+k+1,\beta)}(f;\cos\theta)$ is a polynomial in $\cos\theta$ of degree $\leq N$, it clealry belongs to $D(A)$ and by (4.2.8) and (5.8.5) we may conclude that

$$||A(J_{N,1}^{(\alpha+k+1,\beta)}(f;\cdot)||_X \leq C\Lambda N^2||f||_X \qquad (k>\alpha+\tfrac{1}{2}).$$

All the conditions of theorems 2.4.2 and 2.4.3 are satisfied. We finally obtain

5.8.2. Theorem. The Fourier-Jacobi series (1.3.1) of $f \in X$ is $J^{(\alpha+k+1,\beta)}$-summable to f in X ($k>\alpha+\frac{1}{2}$). The process $\{J_{N,1}^{(\alpha+k+1,\beta)}, N\in Z^+\}$ is saturated with order N^{-2}. The saturation class $F(X,J^{(\alpha+k+1,\beta)})$ is the class $\mathrm{Lip}(2,X)$. Moreover, the following statements are equivalent for $0 < \theta < 1$, $1 \leq q \leq \infty$ or $\theta = 1$, $q = \infty$:

i) $\quad f \in (X, D(A))_{\theta,q;K}$,

ii) $\quad \{n^{2\theta} \omega_J(n;f;X)\}_{n \in Z+} \in l^q_*$.

CHAPTER VI SOME GENERAL RESULTS ON POSITIVE KERNELS

This chapter deals with the approximation processes which are generated by the convolution of $f \in X$ with a positive kernel. It turns out that for norm convergence of such a process for an arbitrary $f \in X$ it is necessary and sufficient that the process converges in the norm for the function $R_1^{(\alpha,\beta)}(\cos\theta)$. Furthermore, we show that all strong approximation processes, which are generated by the convolution of $f \in X$ with a positive polynomial kernel of degree N, approximate the function $R_1^{(\alpha,\beta)}(\cos\theta)$ with an order which does not exceed CN^{-2}. This is proved by constructing the optimal positive polynomial kernel of degree N. In the trigonometric case ($\alpha=\beta=-\frac{1}{2}$) these results are due to Korovkin [35]. In the remainder of this chapter we generalize a theorem of De Vore [23] which determines the saturation order and the saturation class of a family of positive convolution operators, satisfying a certain condition on the Fourier-Jacobi coefficients of the kernel. Finally, some applications of this theorem are given (see also Bavinck [9]).

6.1. RELATIONS BETWEEN TRIGONOMETRIC MOMENTS AND JACOBI COEFFICIENTS

The following expansion is a simple consequence of (1.1.1):

(6.1.1)
$$(\sin\tfrac{\theta}{2})^{2m} = \frac{\Gamma(m+1)\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)} \sum_{n=0}^{m} (-1)^n \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(m-n+1)\Gamma(m+n+\alpha+\beta+2)\Gamma(n+1)} R_n^{(\alpha,\beta)}(\cos\theta)$$
$$(m\in Z^+).$$

For a kernel $\{K_\lambda(\cos\theta)\}_{\lambda\in A}$ the trigonometric moment of order $2m$ ($m\in Z^+$) is defined by

(6.1.2)
$$T(K_\lambda;2m) = \int_0^\pi (\sin\tfrac{\theta}{2})^{2m} K_\lambda(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta.$$

From (6.1.1) we derive for $m \in Z^+$

(6.1.3)
$$T(K_\lambda;2m) = \frac{\Gamma(m+1)\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)} \sum_{k=1}^{m} (-1)^{k+1} \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(m-k+1)\Gamma(m+k+\alpha+\beta+2)\Gamma(k+1)} (1-K_\lambda^\wedge(k)).$$

Here the term $k = 0$ is eliminated by putting $\theta = 0$ in (6.1.1). Conversely, formula (1.1.8) leads to

$$(6.1.4) \qquad 1 - \hat{K}_\lambda(k) =$$

$$= \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+\beta+1)} \sum_{m=1}^{k} (-1)^{m+1} \frac{\Gamma(k+\alpha+\beta+m+1)}{\Gamma(k-m+1)\Gamma(m+\alpha+1)\Gamma(m+1)} T(K_\lambda;2m).$$

Hence, we easily obtain from (6.1.3)

$$(6.1.5) \qquad T(K_\lambda;2) = \frac{\alpha+1}{\alpha+\beta+2}(1-\hat{K}_\lambda(1))$$

and

$$(6.1.6) \qquad \frac{T(K_\lambda;4)}{T(K_\lambda;2)} = \frac{(\alpha+2)(\alpha+\beta+2)}{(\alpha+\beta+3)(\alpha+\beta+4)} \left[\frac{2(\alpha+\beta+3)}{\alpha+\beta+2} - \frac{1-\hat{K}_\lambda(2)}{1-\hat{K}_\lambda(1)}\right] .$$

Also, from (6.1.4) and (6.1.5) we conclude

$$(6.1.7) \qquad \frac{1-\hat{K}_\lambda(k)}{1-\hat{K}_\lambda(1)} = \frac{k(k+\alpha+\beta+1)}{\alpha+\beta+2}$$

$$- \frac{\Gamma(k+1)\Gamma(\alpha+2)}{(\alpha+\beta+2)\Gamma(k+\alpha+\beta+1)} \sum_{m=2}^{k} (-1)^m \frac{\Gamma(k+\alpha+\beta+m+1)}{\Gamma(k-m+1)\Gamma(m+\alpha+1)\Gamma(m+1)} \frac{T(K_\lambda;2m)}{T(K_\lambda;2)} .$$

Similar relations between trigonometric moments and Fourier coefficients have been established by Stark [44]. We also have the following theorem, which generalizes a result of Görlich and Stark [31] (see also Stark [44]).

6.1.1. Theorem. Let $\{K_\lambda(\cos\theta)\}_{\lambda\in A}$ be a positive kernel. Then the following assertions are equivalent:

i) $$\lim_{\lambda\to\infty} \frac{1-\hat{K}_\lambda(k)}{1-\hat{K}_\lambda(1)} = \frac{k(k+\alpha+\beta+1)}{\alpha+\beta+2} \qquad (k\in Z^+),$$

ii) $$\lim_{\lambda\to\infty} \frac{1-\hat{K}_\lambda(2)}{1-\hat{K}_\lambda(1)} = \frac{2(\alpha+\beta+3)}{\alpha+\beta+2} ,$$

iii) $$\lim_{\lambda\to\infty} \frac{T(K_\lambda;4)}{T(K_\lambda;2)} = 0.$$

Proof. Relation ii) is a trivial consequence of i). Relation iii) follows from ii) by (6.1.6). Since $0 \leq \sin^2 \frac{\theta}{2} \leq 1$ and the kernel $\{K_\lambda(\cos\theta)\}_{\lambda\in A}$ is positive, it is obvious that

$$T(K_\lambda;2m) \leq T(K_\lambda;4) \qquad (m\in Z^+,\ m\geq 2).$$

Therefore, relation iii) implies $\lim_{\lambda\to\infty} \frac{T(K_\lambda;2m)}{T(K_\lambda;2)} = 0$, $m \geq 2$. Thus, by formula (6.1.7) relation i) follows.

6.2. APPROXIMATION PROPERTIES OF POSITIVE KERNELS

If $\{K_\lambda(\cos\theta)\}_{\lambda\in A}$ is a positive kernel, then the following theorem shows that the process $K_\lambda(f;\cos\theta)$ is a strong approximation process on X if and only of the test function $R_1^{(\alpha,\beta)}(\cos\theta)$ is approximated in the X norm. For a similar theorem for Fourier series we refer to Butzer-Nessel [18], section 1.3.3.

6.2.1. Theorem. If $\{K_\lambda(\cos\theta)\}_{\lambda\in A}$ is a positive kernel, the following assertions are equivalent:

i) $$\lim_{\lambda\to\infty} ||K_\lambda(f;\cdot) - f(\cdot)||_X = 0 \qquad (f\in X),$$

ii) $$\lim_{\lambda\to\infty} ||K_\lambda(R_1^{(\alpha,\beta)};\cdot) - R_1^{(\alpha,\beta)}(\cdot)||_X = 0,$$

iii) $$\lim_{\lambda\to\infty} T(K_\lambda;2) = 0,$$

iv) $$\lim_{\lambda\to\infty} \int_h^\pi K_\lambda(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta = 0, \qquad \text{for each } h,\ 0 < h \leq \pi.$$

Proof. It is obvious that i) implies ii) since $R_1^{(\alpha,\beta)}(\cos\theta) \in X$. If we assume ii) we have

$$\begin{aligned}
&\lim_{\lambda\to\infty} ||K_\lambda(R_1^{(\alpha,\beta)}(\cos\theta);\cdot) - R_1^{(\alpha,\beta)}(\cdot)||_X = \\
&= \lim_{\lambda\to\infty} ||\int_0^\pi R_1^{(\alpha,\beta)}(\cdot)K_\lambda(\cos\phi)[R_1^{(\alpha,\beta)}(\cos\phi)-1]\rho^{(\alpha,\beta)}(\phi)d\phi||_X \\
&= \lim_{\lambda\to\infty} ||R_1^{(\alpha,\beta)}(\cdot)[K_\lambda^\wedge(1) - 1]||_X \\
&= 0.
\end{aligned}$$

Hence, $\lim_{\lambda\to\infty} [1-K_\lambda^\wedge(1)] = 0$, which by (6.1.5) implies iii). If iii) is assumed, then by the inequality

$$\int_0^\pi (\sin \tfrac{\theta}{2})^2 K_\lambda(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta \geq \int_h^\pi (\sin\tfrac{\theta}{2})^2 K_\lambda(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta$$

$$\geq (\sin\tfrac{h}{2})^2 \int_h^\pi K_\lambda(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta,$$

relation iv) follows. Relation iv) implies i) by theorem 3.1.4.

Since the norm convergence of $K_\lambda(f;\cos\theta\ )$ to $f(\cos\theta)$ holds if and only if

$$(6.2.1)\qquad \lim_{\lambda\to\infty} (1-K_\lambda^\wedge(1)) = 0,$$

we may expect that the rate of convergence will also be determined by the rate of convergence of (6.2.1). The following direct theorem which is a generalization of a theorem of Korovkin [35], p. 72, confirms this.

6.2.2. Theorem. Let $f \in X$ and the kernel $\{K_\lambda(\cos\theta)\}_{\lambda\in A}$ be positive. Then

$$(6.2.2)\qquad ||K_\lambda(f;\cdot) - f(\cdot)||_X \leq C((1-K_\lambda^\wedge(1))||f||_X + \omega(\sqrt{1-K_\lambda^\wedge(1)};f;X)).$$

Proof. By the Hölder-Minkowski inequality we have

$$||K_\lambda(f;\cdot)-f(\cdot)||_X = ||\int_0^\pi (T_\phi f(\cdot)-f(\cdot))K_\lambda(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi||_X$$

$$\leq \int_0^\pi ||T_\phi f(\cdot)-f(\cdot)||_X K_\lambda(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi||_X.$$

Thus, it follows from proposition 4.1.2. iii) and formula (6.1.5) that for each $\mu > 0$

$$||K_\lambda(f;\cdot)-f(\cdot)||_X \leq \int_0^\pi \omega(\phi;f;X)K_\lambda(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi$$

$$\leq \int_0^\pi K_\lambda(\cos\phi)\rho^{(\alpha,\beta)}(\phi)[C\max(1,\phi^2\mu^2)\{\tfrac{1}{\mu^2}||f||_X+\omega(\tfrac{1}{\mu};f;X)\}]d\phi$$

$$\leq C\{\tfrac{1}{\mu^2}||f||_X+\omega(\tfrac{1}{\mu};f;X)\}\ \max(1,C_1\mu^2(1-K_\lambda^\wedge(1))),$$

where C and C_1 are constants. The theorem follows by putting $\mu = (1-\hat{K}_\lambda(1))^{-\frac{1}{2}}$.

When we restrict ourselves to positive polynomial kernels, it is possible to construct the kernel for which the rate of convergence is optimal, that is $1 - \hat{K}_\lambda(1)$ or $T(K_\lambda;2)$ is minimal.

6.2.3. Lemma. Let Π_N ($N\in Z^+$) be the class of all positive polynomial kernels of degree $\leq N$. Among all kernels which belong to Π_N, the kernel

$$(6.2.3)\qquad F_N(\cos\theta) = \frac{\omega_n^{(\alpha,\beta)}\sin^2\theta_{1,n}}{(2n+\alpha+\beta+1)}\left(\frac{R_n^{(\alpha,\beta)}(\cos\theta)}{\cos\theta-\cos\theta_{1,n}}\right)^2$$

has the minimal trigonometric moment of order 2. Here $n = [\frac{N}{2}] + 1$ and $\theta_{1,n}$ is written for the smallest value of θ for which $R_n^{(\alpha,\beta)}(\cos\theta)$ vanishes.

Proof. We are looking for

$$(6.2.4)\qquad \min_{p_N\in\Pi_N}\int_0^\pi (\sin\tfrac{\phi}{2})^2\, p_N(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi$$

$$= \tfrac{1}{2}(1-2^{-\alpha-\beta-1}\max_{p_N\in\Pi_N}\int_{-1}^1 x\, p_N(x)(1-x)^\alpha(1+x)^\beta dx).$$

When we write $n = [\frac{N}{2}] + 1$, then $x\,p_N(x)$ is a polynomial of degree $\leq 2n-1$. It is a well-known fact (Szegö [46], section 3.4), that the integral of any polynomial of degree $\leq 2n-1$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ can be computed exactly by means of the Gauss-Jacobi quadrature formula

$$\int_{-1}^1 p_{2n-1}(x)(1-x)^\alpha(1+x)^\beta\, dx = \sum_{i=1}^n \lambda_{i,n}\, p_{2n-1}(x_{i,n}),$$

where $1 > x_{1,n} > x_{2,n} > \ldots > x_{n,n} > -1$ denote the zeros of $R_n^{(\alpha,\beta)}(x)$ and $\lambda_{i,n}$ are the Christoffel numbers, which are all ≥ 0 and have the following representation (Szegö [46], (15.3.1)):

$$(6.2.5)\quad \lambda_{i,n} = 2^{\alpha+\beta+1}(2n+\alpha+\beta+1)[\omega_n^{(\alpha,\beta)}\sin^2\theta_{i,n}(\tfrac{d}{dx}R_n^{(\alpha,\beta)}(x))_{x=\cos\theta_{i,n}}]^{-1},$$

$i = 1, 2, \ldots, n.$

Hence, the problem reduces to that of finding

$$(6.2.6)\qquad \max_{p_N\in\Pi_N} \sum_{i=1}^{n} \lambda_{i,n}\, x_{i,n}\, p_N(x_{i,n}).$$

Since by (3.1.1) we have

$$(6.2.7)\qquad \int_{-1}^{1} p_N(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx = \sum_{i=1}^{n} \lambda_{i,n}\, p_N(x_{i,n}) = 2^{\alpha+\beta+1}$$

and $1 > x_{1,n} > x_{2,n} > \dots > x_{n,n} > -1$ it follows that the maximum in (6.2.6) will be reached by the polynomial $F_N(x)$ such that $F_N(x_{i,n}) = 0$, $i = 2, \dots, n$. This implies that

$$(6.2.8)\qquad \lambda_{1,n}\, F_N(x_{1,n}) = 2^{\alpha+\beta+1}$$

and

$$(6.2.9)\qquad \int_{-1}^{1} x\, F_N(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx = x_{1,n}\, 2^{\alpha+\beta+1} .$$

It is easy to see that the polynomial kernel $\{F_N(\cos\theta)\}_{N\in Z^+}$ has the form

$$(6.2.10)\qquad F_N(x) = \frac{1}{C_N}\left(\frac{R_n^{(\alpha,\beta)}(x)}{x-x_{1,n}}\right)^2 \qquad (n=[\tfrac{N}{2}]+1)$$

where

$$C_N = \int_0^{\pi} \left(\frac{R_n^{(\alpha,\beta)}(\cos\theta)}{\cos\theta-\cos\theta_{1,n}}\right)^2 \rho^{(\alpha,\beta)}(\theta)\,d\theta .$$

Using (6.2.5) and (6.2.7) we obtain

$$C_N = \frac{(2n+\alpha+\beta+1)}{\omega_n^{(\alpha,\beta)}\sin^2\theta_{1,n}} .$$

Also, formulas (6.2.9) can be proved directly from the representation (6.2.10), since by the orthogonality of the polynomials $R_n^{(\alpha,\beta)}(x)$

$$\frac{1}{C_N}\int_{-1}^{1} (x-x_{1,n})\left(\frac{R_n^{(\alpha,\beta)}(x)}{x-x_{1,n}}\right)^2 (1-x)^{\alpha}(1+x)^{\beta}\,dx$$

$$= \frac{1}{C_N}\int_{-1}^{1} R_n^{(\alpha,\beta)}(x)\,\frac{R_n^{(\alpha,\beta)}(x)}{x-x_{1,n}}\,(1-x)^{\alpha}(1+x)^{\beta}\,dx = 0.$$

Hence we conclude by (6.2.4), (6.2.9) and (6.1.5)

$$(6.2.11) \qquad 1 - F_N^\wedge(1) = \frac{\alpha+\beta+2}{2(\alpha+1)} (1-\cos\theta_{1,n}) ,$$

or, by (1.2.3),

$$\begin{aligned} 1 - F_N^\wedge(1) &\simeq \frac{\alpha+\beta+2}{4(\alpha+1)} \theta_{1,n}^2 \\ &\simeq \frac{(\alpha+\beta+2)(j_1^{(\alpha)})^2}{(\alpha+1)(2n+\alpha+\beta+1)^2} && (n\to\infty) \\ &\approx N^{-2} && (N\to\infty). \end{aligned}$$

Here, $j_1^{(\alpha)}$ is written for the first zero of the Bessel function $J_\alpha(x)$.

6.2.4. Corollary. Let $\{p_N(\cos\theta)\}_{N\in Z^+}$ be a positive polynomial kernel of degree $\leq N$. Then the sequence $\{N^2||p_N(R_1^{(\alpha,\beta)}(\cos\theta);\cdot) - R_1^{(\alpha,\beta)}(\cdot)||_X\}$ does not tend to zero as $N \to \infty$.

Hence, even for an infinitely differentiable function such as $R_1^{(\alpha,\beta)}(\cos\theta)$, the degree of approximation by processes with a positive kernel cannot be better than $O(N^{-2})$, $N \to \infty$. The corresponding result in the case of approximation by positive trigonometric polynomial operators is due to Korovkin [35], p. 127. The method followed in the proof was communicated orally to the author by R.A. De Vore.

6.3. SATURATION OF A CLASS OF POSITIVE OPERATORS

For approximation processes which are generated by the convolution of $f \in X$ with a positive approximation kernel $\{K_\lambda(\cos\theta)\}_{\lambda\in A}$, satisfying one of the conditions stated in lemma 6.3.1, we determine the saturation class. The corresponding result in the trigonometric case is due to De Vore [23], who generalizes a theorem of Tureckii [48]. We first prove the equivalence of two different conditions on the kernel in the following lemma.

6.3.1. Lemma. Let $\{K_\lambda(\cos\theta)\}_{\lambda\in A}$ be a positive kernel. Then the following conditions are equivalent:

i) There exists a number $C_A \in R^+$ such that for each $k \in Z^+$ there is a

$\Lambda(k) \in R^+$ with

$$\frac{1-K_\lambda^\wedge(k)}{1-K_\lambda^\wedge(1)} \geq C_A\, k(k+\alpha+\beta+1) \qquad (\lambda>\Lambda(k));$$

ii) There exists a number $C_B \in R^+$ such that for each $0 < \varepsilon \leq \pi$ there is a $\Lambda(\varepsilon)$ with

$$\int_0^\varepsilon (\sin\frac{\theta}{2})^2\, K_\lambda(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta$$

$$\geq C_B \int_0^\pi (\sin\frac{\theta}{2})^2\, K_\lambda(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta \qquad (\lambda>\Lambda(\varepsilon)).$$

Proof. In order to simplify the notation we write

$$(6.3.1) \qquad d\mu_\lambda(\cos\theta) = K_\lambda(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta\ .$$

We first show that ii) implies i). If we take $\varepsilon < 4/(2k+\alpha+\beta+2)$ and $\lambda > \Lambda(\varepsilon)$, where $\Lambda(\varepsilon)$ is given by ii), we derive by using (1.2.5) and (6.1.5)

$$1 - K_\lambda^\wedge(k) = \int_0^\pi (1-R_k^{(\alpha,\beta)}(\cos\theta))\, d\mu_\lambda(\cos\theta)$$

$$\geq \int_0^\varepsilon (1-R_k^{(\alpha,\beta)}(\cos\theta)d\mu_\lambda(\cos\theta)$$

$$\geq c_\alpha \frac{k(k+\alpha+\beta+1)}{\alpha+1} \int_0^\varepsilon (\sin\frac{\theta}{2})^2\, d\mu_\lambda(\cos\theta)$$

$$\geq c_\alpha \frac{k(k+\alpha+\beta+1)}{\alpha+1}\, C_B \int_0^\pi (\sin\frac{\theta}{2})^2\, d\mu_\lambda(\cos\theta)$$

$$= \frac{c_\alpha C_B}{\alpha+\beta+2}\, k(k+\alpha+\beta+1)(1-K_\lambda^\wedge(1)).$$

Therefore, i) holds with $\Lambda(k) = \Lambda(\varepsilon)$ and $C_A = \frac{c_\alpha C_B}{\alpha+\beta+2}$.

We will now show that i) implies ii) with $C_B = C_A \frac{(\alpha+\beta+2)}{2}$, We choose $\varepsilon = \varepsilon_0$ and we consider the measures defined by $(\lambda\epsilon A)$

$$d\nu_\lambda(\cos\theta) = \begin{cases} 0 & (0\le\theta<\varepsilon_0) \\ \dfrac{1}{T(K_\lambda;2)} K_\lambda(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta & (\varepsilon_0\le\theta\le\pi). \end{cases}$$

Then for all $\lambda \in A$

$$\int_0^\pi d\nu_\lambda(\cos\theta) \le \frac{1}{(\sin\frac{\varepsilon_0}{2})^2\, T(K_\lambda;2)} \int_0^\pi (\sin\tfrac{\theta}{2})^2\, d\mu_\lambda(\cos\theta)$$

$$= (\sin\frac{\varepsilon_0}{2})^{-2}.$$

Thus,

$$\int_0^\pi \{1 - R_k^{(\alpha,\beta)}(\cos\theta)\}d\nu_\lambda(\cos\theta) \le 2(\sin\frac{\varepsilon_0}{2})^{-2}.$$

We choose k_0 so large that

$$\frac{C_A\, k_0(k_0+\alpha+\beta+1)(\alpha+\beta+2)}{4(\alpha+1)} \ge (\sin\frac{\varepsilon_0}{2})^{-2}.$$

Hence,

$$\frac{1}{T(K_\lambda;2)} \int_0^{\varepsilon_0} \{1 - R_{k_0}^{(\alpha,\beta)}(\cos\theta)\}d\mu_\lambda(\cos\theta)$$

$$= \frac{1}{T(K_\lambda;2)} \int_0^\pi \{1 - R_{k_0}^{(\alpha,\beta)}(\cos\theta)\}d\mu_\lambda(\cos\theta)$$

$$- \int_0^\pi \{1 - R_{k_0}^{(\alpha,\beta)}(\cos\theta)\}d\nu_\lambda(\cos\theta)$$

$$\ge \frac{1}{T(K_\lambda;2)} \int_0^\pi \{1 - R_{k_0}^{(\alpha,\beta)}(\cos\theta)\}d\mu_\lambda(\cos\theta)$$

$$- \frac{C_A\, k_0(k_0+\alpha+\beta+1)(\alpha+\beta+2)}{2(\alpha+1)} .$$

By virtue of condition i) and (6.1.5) we have for $\lambda \ge \Lambda(k_0)$

$$\int_0^{\varepsilon_0} \{1 - R_{k_0}^{(\alpha,\beta)}(\cos\theta)\} d\mu_\lambda(\cos\theta)$$

$$\geq C_A\, k_0(k_0+\alpha+\beta+1)\, \frac{(\alpha+\beta+2)}{2(\alpha+1)} \int_0^\pi (\sin\tfrac{\theta}{2})^2\, d\mu_\lambda(\cos\theta).$$

Finally, by (1.2.4) it follows that

$$\int_0^{\varepsilon_0} (\sin\tfrac{\theta}{2})^2\, d\mu_\lambda(\cos\theta) \geq \frac{(\alpha+1)}{k_0(k_0+\alpha+\beta+1)} \int_0^{\varepsilon_0} \{1-R_{k_0}^{(\alpha,\beta)}(\cos\theta)\} d\mu_\lambda(\cos\theta)$$

$$\geq C_A\, \frac{(\alpha+\beta+2)}{2} \int_0^\pi (\sin\tfrac{\theta}{2})^2\, d\mu_\lambda(\cos\theta),$$

which proves lemma 6.3.1.

6.3.2. Theorem. Let $\{K_\lambda(\cos\theta)\}_{\lambda\in A}$ be a positive approximation kernel satisfying either condition i) or ii) of lemma 6.3.1 and let $f \in X$. Then the process $\{K_\lambda, \lambda\in A\}$ is saturated with order $(1-K_\lambda^\wedge(1))$ and the saturation class $F(X,K)$ is the class $\mathrm{Lip}(2,X)$.

Proof. On account of lemma 6.3.1 we may assume that the kernel $\{K_\lambda(\cos\theta)\}$ satisfies both conditions i) and ii) and we will interchange them appropriately.

We first show that the process $\{K_\lambda, \lambda\in A\}$ is saturated with order $(1-K_\lambda^\wedge(1))$. If $f \in X$ and

$$||K_\lambda(f;\cdot) - f(\cdot)||_X = o(1-K_\lambda^\wedge(1)) \qquad (\lambda\to\infty),$$

then for all $k \in Z^+$

$$f^\wedge(k) - f^\wedge(k)K_\lambda^\wedge(k) = o(1-K_\lambda^\wedge(1)) \qquad (\lambda\to\infty).$$

In view of condition i) this implies $f^\wedge(k) = 0$, $k \in Z^+$ and therefore f is a constant. On the other hand the function $f_0(\cos\theta) = R_1^{(\alpha,\beta)}(\cos\theta)$ is an example of a non-constant function which satisfies

$$||K_\lambda(f_0;\cdot) - f_0(\cdot)||_X = ||R_1^{(\alpha,\beta)}(\cdot)||_X\, (1-K_\lambda^\wedge(1)).$$

Hence, the process $\{K_\lambda, \lambda\in A\}$ is saturated with order $\{1 - K_\lambda^\wedge(1)\}$. The 'trivial' subspace in definition 2.4.1 is the space of constant functions here.

We now wish to characterize the saturation class F(X,K). An element $f \in X$ belongs to F(X,K) if and only if

$$\left\|\int_0^\pi (T_\phi f(\cdot)-f(\cdot))\ K_\lambda(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi\right\|_X = O(1-K_\lambda^\wedge(1)) \quad (\lambda\to\infty),$$

or equivalently

$$\left\|\int_0^\pi \frac{(T_\phi f(\cdot)-f(\cdot))}{(\sin\frac{\phi}{2})^2}\ d\psi_\lambda(\cos\phi)\right\|_X = O(1) \qquad (\lambda\to\infty),$$

where

$$d\psi_\lambda(\cos\phi) = \frac{(\alpha+\beta+2)(\sin\frac{\phi}{2})^2\ K_\lambda(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi}{(\alpha+1)(1-K_\lambda^\wedge(1))}.$$

By (6.1.5) we have $||\psi_\lambda||_M = 1$ ($\lambda\in A$), and consequently it is clear that $f \in F(X,K)$, if $f \in Lip(2,X)$.

We still have to prove that $f \in F(X,K)$ implies $f \in Lip(2,X)$. If A denotes the operator defined by (4.1.3), then we will first show that for $f \in D(A)$, satisfying

(6.3.3) $$||f(\cdot) - K_\lambda(f;\cdot)||_X \leq \Lambda(1-K_\lambda^\wedge(1)) \qquad (\lambda\to\infty),$$

where Λ is a positive constant, the following inequality is valid:

(6.3.4) $$||Af||_X \leq C(\Lambda+||f||_X).$$

Here C is a positive constant independent of f.

Since the measures $\{\psi_\lambda\}_{\lambda\in A}$ all have norm 1, there exists a sequence $\{\lambda_j\}_{j\in Z^+}$ and a measure ψ such that $\{\psi_{\lambda_j}\}$ converges weakly* to ψ. By condition ii) and the weak* convergence it follows that for each $\varepsilon > 0$

(6.3.5) $$\int_0^\varepsilon d\psi = \lim_{j\to\infty}\int_0^\varepsilon d\psi_{\lambda_j} \geq C_B .$$

We choose ε_0 so small that $\varepsilon_0 \leq \frac{\pi}{2}$ and

(6.3.6) $$\int_{(0,\varepsilon_0)} d\psi \leq \frac{C_B}{S} \qquad \text{with } S > 2+2^{\beta+2}.$$

For $f \in D(A)$ satisfying (6.3.3) we have

$$\left|\left|\int_0^{\pi} \frac{T_\phi f(\cdot)-f(\cdot)}{(\sin \frac{\phi}{2})^2} \, d\psi(\cos \phi)\right|\right|_X$$

$$\leq \lim_{j\to\infty} \left|\left|\int_0^{\pi} \frac{T_\phi f(\cdot)-f(\cdot)}{(\sin \frac{\phi}{2})^2} \, d\psi_{\lambda_j}(\cos \phi)\right|\right| \leq \frac{\alpha+\beta+2}{\alpha+1} \Lambda \quad .$$

Hence,

$$(6.3.7) \qquad \left|\left|\int_0^{\varepsilon_0} \frac{T_\phi f(\cdot)-f(\cdot)}{(\sin \frac{\phi}{2})^2} \, d\psi(\cos \phi)\right|\right|_X$$

$$\leq \frac{\alpha+\beta+2}{\alpha+1} \Lambda + \left|\left|\int_{\varepsilon_0}^{\pi} \frac{T_\phi f(\cdot)-f(\cdot)}{(\sin \frac{\phi}{2})^2} \, d\psi(\cos \phi)\right|\right|_X$$

$$\leq \frac{\alpha+\beta+2}{\alpha+1} \Lambda + \frac{2||f||_X}{(\sin \frac{\varepsilon_0}{2})^2} \quad .$$

From (4.1.7) it follows that

$$\lim_{\phi\to 0^+} \left|\left|\frac{T_\phi f(\cdot)-f(\cdot)}{(\sin \frac{\phi}{2})^2} - (-\frac{1}{\alpha+1} Af(\cdot))\right|\right|_X = 0 \ .$$

In virtue of (6.3.5) and (6.3.6) we have

$$(6.3.8) \qquad \left|\left|\int_0^{\varepsilon_0} \frac{T_\phi f(\cdot)-f(\cdot)}{(\sin \frac{\phi}{2})^2} \, d\psi(\cos \phi)\right|\right|_X$$

$$\geq (1 - \frac{1}{S}) \, C_B \frac{1}{\alpha+1} ||Af||_X - \left|\left|\int_{(0,\varepsilon_0)} \frac{T_\phi f(\cdot)-f(\cdot)}{(\sin \frac{\phi}{2})^2} \, d\psi(\cos \phi)\right|\right|_X \ .$$

Since by (4.1.7) and (4.1.8)

$$\left|\left|\frac{T_\phi f(\cdot)-f(\cdot)}{(\sin \frac{\phi}{2})^2}\right|\right|_X \leq \frac{2^{\beta+1}}{\alpha+1} ||Af||_X \ , \qquad 0 < \phi \leq \frac{\pi}{2} \ ,$$

we derive from (6.3.8) and (6.3.6)

$$(6.3.9) \qquad \left|\left|\int_0^{\varepsilon_0} \frac{T_\phi f(\cdot)-f(\cdot)}{(\sin \frac{\phi}{2})^2} \, d\psi(\cos \phi)\right|\right|_X$$

$$\geq (1 - \frac{1}{S}) \, C_B \frac{1}{\alpha+1} ||Af||_X - \frac{C_B}{S} \frac{2^{\beta+1}}{\alpha+1} ||Af||_X \geq \frac{1}{2(\alpha+1)} C_B ||Af||_X \ ,$$

since we have chosen $S > 2+2^{\beta+2}$.
Hence (6.3.9) and (6.3.7) yield

$$||Af||_X \leq \frac{2(\alpha+\beta+2)}{C_B} \Lambda + \frac{4(\alpha+1)}{C_B(\sin \frac{\varepsilon_0}{2})^2} ||f||_X ,$$

which establishes (6.3.4), if we choose $C = \max(\frac{2(\alpha+\beta+2)}{C_B}, \frac{4(\alpha+1)}{C_B(\sin \frac{\varepsilon_0}{2})^2})$.

If we take an arbitrary element of F(X,K) such that

$$||f(\cdot) - K(f;\cdot)||_X \leq \Lambda(1-K_\lambda^\wedge(1)) \qquad (\lambda\epsilon A),$$

then we study the convolution of f with a positive polynomial kernel of degree $\leq N$ (we may take the De la Vallée-Poussin kernel $V_N(\cos\theta)$ defined by (5.7.2)). Then the function $f_N(\cos\theta) = V_N(f;\cos\theta)$ clearly belongs to D(A) and furthermore we have for each $N \epsilon Z^+$

$$\begin{aligned} ||f_N(\cdot) - K_\lambda(f_N;\cdot)||_X &= ||(f*V_N)(\cdot) - ((f*V_N)*K_\lambda)(\cdot)||_X \\ &= ||(V_N*(f-K_\lambda*f))(\cdot)||_X \\ &\leq ||V_N||_1 \, ||f(\cdot) - K_\lambda(f;\cdot)||_X \\ &\leq \Lambda(1-K_\lambda^\wedge(1)) \qquad (\lambda\epsilon A). \end{aligned}$$

Since $||f_N||_X \leq ||f||_X$, it follows from (6.3.4) that

$$||Af_N||_X \leq C(\Lambda+||f_N||_X) \leq C(\Lambda+||f||_X).$$

Hence for $\phi > 0$ we derive from (4.1.7) and (4.1.8)

$$||\frac{T_\phi f_N(\cdot)-f_N(\cdot)}{\phi^2}||_X \leq \frac{2^{\beta-1}}{\alpha+1} ||Af_N||_X \leq C'(\Lambda+||f||_X), \; N \epsilon Z^+.$$

If we take the limit as $N \to \infty$ we obtain

$$||\frac{T_\phi f(\cdot)-f(\cdot)}{\phi^2}||_X \leq C'(\Lambda+||f||_X) ,$$

which is equivalent with $f \in \mathrm{Lip}(2,X)$. This completes the proof of theorem 6.3.2.

6.4. APPLICATIONS

We shall utilize the results of the preceding sections in order to derive the saturation class of some approximation processes with a positive kernel, for which a limit relation of the form (2.4.2) is not known, so that the method used in chapter V fails.

We first consider the process, which is generated by convolution of $f \in X$ with the kernel

$$J_{N,2}^{(\alpha+k+1,\beta)}(\cos\theta) = C_N[R_N^{(\alpha+k+1,\beta)}(\cos\theta)]^2 \qquad (k\in\mathbb{R},\ N\in\mathbb{Z}^+), \tag{6.4.1}$$

where

$$C_N^{-1} = \int_0^\pi [R_N^{(\alpha+k+1,\beta)}(\cos\theta)]^2\,\rho^{(\alpha,\beta)}(\theta)d\theta . \tag{6.4.2}$$

From (1.2.8), where we choose $\lambda = k+1$, $p = 2$ and successively $\mu = 0$ and $\mu = 1$, we conclude that the trigonometric moment of order 2

$$T(J_{N,2}^{(\alpha+k+1,\beta)};2) = C_N\int_0^\pi [R_N^{(\alpha+k+1,\beta)}(\cos\theta)]^2\,\rho^{(\alpha+1,\beta)}(\theta)d\theta$$

$$= o(1) \qquad (N\to\infty),$$

if $k \geq -\frac{1}{2}$. Hence, by theorem 6.2.1, the process $J_{N,2}^{(\alpha+k+1,\beta)}(f;\cos\theta)$ is a strong approximation process on X if $k \geq -\frac{1}{2}$. Furthermore, if we take $k \geq \frac{1}{2}$, formula (1.2.8) with $\lambda = k+1$, $p = 2$ and for μ the values 0, 1 and 2 leads to

$$T(J_{N,2}^{(\alpha+k+1,\beta)};2) \approx \begin{cases} N^{-2} & (k>\frac{1}{2},\ N\to\infty), \\ N^{-2}\log N & (k=\frac{1}{2},\ N\to\infty), \end{cases} \tag{6.4.3}$$

$$T(J_{N,2}^{(\alpha+k+1,\beta)};4) = o(T(J_{N,2}^{(\alpha+k+1,\beta)};2)) \qquad (N\to\infty). \tag{6.4.4}$$

Theorem 6.1.1 shows the equivalence of (6.4.4) with the relation

$$\lim_{N\to\infty} \frac{1-(J_{N,2}^{(\alpha+k+1,\beta)})^\wedge(n)}{1-(J_{N,2}^{(\alpha+k+1,\beta)})^\wedge(1)} = \frac{n(n+\alpha+\beta+1)}{\alpha+\beta+2} \qquad (n\in Z^+),$$

which implies that condition i) of lemma 6.3.1 is satisfied. Applying theorem 6.3.2 and formula (6.1.5) we obtain

6.4.1. Theorem. The process $\{J_{N,2}^{(\alpha+k+1,\beta)}(\cos\theta),\ N\in Z^+\}$ is saturated with order N^{-2} if $k > \frac{1}{2}$ and with order $N^{-2}\log N$ if $k = \frac{1}{2}$. The saturation class is Lip(2,X).

We have already used the kernel (6.4.1) with k = 1 in chapter IV in the proof of theorem 4.2.4. We now investigate another approximation process generated by convolution with a positive approximation kernel, the generalized Jackson kernel. In the case of Fourier series this kernel is often used to prove direct theorems (see Lorentz [37]). The kernel is defined by

$$(6.4.5) \qquad L_{N,r}(\cos\theta) = c_{N,r}\left(\frac{\sin N\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)^{2r} \qquad (N\in Z^+,\ r\in Z^+),$$

where

$$c_{N,r}^{-1} = \int_0^\pi \left(\frac{\sin N\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)^{2r} \rho^{(\alpha,\beta)}(\theta)d\theta.$$

From the well-known estimates $\frac{\theta}{\pi} \le \sin\frac{\theta}{2} \le \frac{\theta}{2}$ for $0 \le \theta \le \pi$ and $\frac{\sqrt{2}}{2} \le \cos\frac{\theta}{2} \le 1$ for $0 \le \theta \le \frac{\pi}{2}$ we easily derive the asymptotic relation for $N \to \infty$

$$(6.4.6) \qquad \int_0^\pi \left(\frac{\sin N\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)^{2r} \rho^{(\alpha+\mu,\beta)}(\theta)d\theta \approx \begin{cases} N^{2r-2\alpha-2\mu-2} & (r>\alpha+\mu+1), \\ \log N & (r=\alpha+\mu+1), \\ 1 & (r<\alpha+\mu+1). \end{cases}$$

Relation (6.4.6) implies that the trigonometric moment of order 2

$$T(L_{N,r};2) = c_{N,r}\int_0^\pi \left(\frac{\sin N\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)^{2r} \rho^{(\alpha+1,\beta)}(\theta)d\theta$$

$$= o(1) \qquad (N\to\infty),$$

if $r \ge \alpha+1$. Hence, by theorem 6.2.1, we conclude, that for $f \in X$ the process $L_{N,r}(f;\cos\theta)$ is a strong approximation process on X if $r \ge \alpha+1$. Furthermore, if we take $r \geq \alpha+2$, formula (6.4.6) leads to

(6.4.7) $$T(L_{N,r};2) \approx \begin{cases} N^{-2} & (r>\alpha+2,\ N\to\infty), \\ N^{-2}\log N & (r=\alpha+2,\ N\to\infty), \end{cases}$$

(6.4.8) $$T(L_{N,r};4) = o(T(L_{N,r};2)) \qquad (N\to\infty).$$

By theorem 6.1.1, relation (6.4.8) is equivalent to

$$\lim_{N\to\infty} \frac{1-\hat{L}_{N,r}(n)}{1-\hat{L}_{N,r}(1)} = \frac{n(n+\alpha+\beta+1)}{\alpha+\beta+2} \qquad (n\in A^+),$$

which implies condition i) of lemma 6.3.1. Consequently, application of theorem 6.3.2 and formula (6.1.5) yields

6.4.2. Theorem. The process $\{L_{N,r}(\cos\theta),\ N\in Z^+\}$ is saturated with order N^{-2} if $r > \alpha+2$ and with order $N^{-2}\log N$ if $r = \alpha+2$ $(r\in Z^+)$. The saturation class is Lip(2,X).

The last approximation process with a positive kernel we consider is the process generated by the kernel $F_N(\cos\theta)$ defined by (6.2.3). This kernel is the generalization to Jacobi polynomials of the Fejér-Korovkin kernel (see Butzer-Nessel [18], section 1.6.1). As an immediate consequence of (6.2.4) and (6.2.9) we have

$$T(F_N;2) = \tfrac{1}{2}(1-\cos\theta_{1,n}).$$

For the notation see lemma 6.2.3. We now compute $T(F_N;4)$:

$$\begin{aligned} T(F_N;4) &= \tfrac{1}{4}\int_0^\pi (1-2\cos\theta+\cos^2\theta)F_N(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta \\ &= \frac{1}{2^{\alpha+\beta+3}}\int_{-1}^1 (1-2x+x^2)F_N(x)(1-x)^\alpha(1+x)^\beta dx \\ &= \frac{1}{2^{\alpha+\beta+3}}\int_{-1}^1 [(x-x_{1,n})^2-2(1-x_{1,n})x+(1-x_{1,n}^2)]F_N(x)(1-x)^\alpha(1+x)^\beta dx \\ &= \frac{1-x_{1,n}^2}{4(2n+\alpha+\beta+1)} + \tfrac{1}{4}(1-x_{1,n})^2 \quad . \end{aligned}$$

Hence

$$\lim_{N\to\infty} \frac{T(F_N;4)}{T(F_N;2)} = \lim_{N\to\infty} \left[\frac{(1+x_{1,n})}{2(2n+\alpha+\beta+1)} + \tfrac{1}{2}(1-x_{1,n})\right] = 0.$$

Thus, by theorem 6.1.1, condition i) of lemma 6.3.1 follows and we conclude:

6.4.3. Theorem. The process $\{F_N(\cos\theta),\ N\in Z^+\}$ is saturated with order N^{-2}. The saturation class is Lip(2,X).

CHAPTER VII CHARACTERIZATION OF CERTAIN CLASSES OF FUNCTIONS

This last chapter is devoted to the characterization of certain classes of functions which occur in the preceding chapters. In section 7.1 we give necessary and sufficient conditions for a function f on [-1,1] to belong to the domain of the operator A, defined in (4.1.3). Next, we characterize the domain of the operator D_1, the fractional differentiation operator of order 1, by means of the conjugate function $\tilde{f}$, which can be introduced in a way similar to the work of Muckenhoupt and Stein [38] on ultraspherical expansions. The method to obtain the characterizations for D(A) and $D(D_1)$ is taken from Berens, Butzer and Pawelke [13]. Finally, we direct our attention to the spaces of K-interpolation between the space X and the domain of the fractional differentiation operator of order γ, in terms of which we have characterized the spaces of non-optimal approximation of certain summation methods in chapter V. We show that the spaces $(X,D(D_\gamma))_{\theta,q;K}$, $0 < \gamma < 2$, $0 < \theta < 1$, $1 \leq q \leq \infty$ coincide with the spaces of K-interpolation $(X,D(A))_{\theta\gamma/2,q;K}$ which in the case $q = \infty$ are the Lipschitz spaces $\mathrm{Lip}(\frac{\theta\gamma}{2},X)$.

7.1. CHARACTERIZATION OF D(A)

In this section the following theorem is proved.

7.1.1. Theorem. For $f,g \in X$ the relation

$$(7.1.1) \qquad n(n+\alpha+\beta+1)f^\wedge(n) = g^\wedge(n) \qquad (n \in P)$$

is valid, if and only if f(cos θ) is locally absolutely continuous almost everywhere on (-1,1), the function $\rho^{(\alpha,\beta)}(\theta)\frac{df}{d\theta}$ is absolutely continuous almost everywhere on [-1,1] and vanishes in the points -1 and 1, and

$$(7.1.2) \qquad -\frac{d}{d\theta}\,[\rho^{(\alpha,\beta)}(\theta)\,\frac{d}{d\theta}\,f(\cos\theta)] = \rho^{(\alpha,\beta)}(\theta)\;g(\cos\theta)$$

almost everywhere on [-1,1].

Proof. If we presuppose (7.1.1), then the differential operator $P(\frac{d}{d\theta})$, defined in (1.1.6), works on the Weierstrass approximation process (section 5.1) in the following way:

$$P(\frac{d}{d\theta})\ W_t f(\cos\theta) = W_t g(\cos\theta).$$

If we integrate twice, we obtain

$$W_t f(\cos\theta) = W_t f(\cos\varepsilon) - \int_\varepsilon^\theta \{\rho^{(\alpha,\beta)}(\eta)\}^{-1}\, d\eta \int_0^\eta \rho^{(\alpha,\beta)}(\tau) W_t g(\cos\tau)d\tau.$$

From the norm convergence of $W_t f$ to f for $t \to 0^+$ we conclude, that there exists a sequence $\{t_i\}_{i\in Z^+}$ such that $W_{t_i} f(\cos\theta)$ converges to $f(\cos\theta)$ almost everywhere, for $t_i \to 0^+$ (Rudin [43], theorem 3.12). Hence

$$f(\cos\theta) = f(\cos\varepsilon) - \int_\varepsilon^\theta \{\rho^{(\alpha,\beta)}(\eta)\}^{-1}\, d\eta \int_0^\eta g(\cos\tau)\rho^{(\alpha,\beta)}(\tau)d\tau \quad \text{a.e.}$$

where the right-hand side converges for every $\varepsilon > 0$ and for all $\theta \in (0,\pi)$. Thus f is differentiable almost everywhere on $(0,\pi)$ and

$$\frac{df(\cos\theta)}{d\theta} = \{\rho^{(\alpha,\beta)}(\theta)\}^{-1} \int_0^\theta g(\cos\tau)\rho^{(\alpha,\beta)}(\tau)d\tau \qquad \text{a.e.}$$

It follows that the function $\rho^{(\alpha,\beta)}(\theta)\ \frac{df(\cos\theta)}{d\theta}$ is absolutely continuous a.e. on $[0,\pi]$, vanishes at $\theta = 0$ and, by the hypothesis $g^\wedge(0) = 0$, also at $\theta = \pi$. Moreover,

$$P(\frac{d}{d\theta})\ f(\cos\theta) = g(\cos\theta) \qquad \text{a.e.}$$

(the assertions are everywhere if $X = C$).

For the converse part we presuppose (7.1.2). Then

$$g^\wedge(n) = -\int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)\ \frac{d}{d\theta}\ \{\rho^{(\alpha,\beta)}(\theta)\ \frac{d}{d\theta}\ f(\cos\theta)\}d\theta,\ n \in \mathbf{P}.$$

If we integrate by parts twice, we get

$$g^\wedge(n) = -\int_0^\pi \frac{d}{d\theta}\ [\rho^{(\alpha,\beta)}(\theta)\ \frac{d}{d\theta}\ R_n^{(\alpha,\beta)}(\cos\theta)]\ f(\cos\theta)d\theta$$

$$= \int_0^\pi n(n+\alpha+\beta+1)\ R_n^{(\alpha,\beta)}(\cos\theta)f(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta$$

$$= n(n+\alpha+\beta+1)\ f^\wedge(n)\ , \qquad n \in \mathbf{P}.$$

We have used the fact that $\rho^{(\alpha,\beta)}(\theta)\ \frac{df(\cos\ \theta)}{d\theta}$ vanishes at $\theta = 0$ and $\theta = \pi$. The proof is complete.

7.2. THE CONJUGATE FUNCTION

It has been suggested by Askey [1], that some parts of the work of Muckenhoupt and Stein [38] on the conjugate series of ultraspherical expansions can be generalized to Jacobi series. If we take $f \in X$ with the expansion

$$f(\cos\ \theta) \sim \sum_{n=0}^{\infty} f^{\wedge}(n)\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\ \theta)$$

and the Abel-Poisson means (section 3.3)

$$A_r(f;\cos\ \theta) = \sum_{n=0}^{\infty} r^n\ f^{\wedge}(n)\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\ \theta), \tag{7.2.1}$$

then the conjugate Abel-Poisson sum $\tilde{A}_r(f;\cos\ \theta)$ can be defined by

$$\tilde{A}_r(f;\cos\ \theta) = \frac{1}{\alpha+1}\sum_{n=1}^{\infty} r^n\ n\ f^{\wedge}(n)\omega_n^{(\alpha,\beta)}R_{n-1}^{(\alpha+1,\beta+1)}(\cos\ \theta)\ \sin\frac{\theta}{2}\ \cos\frac{\theta}{2}\ . \tag{7.2.2}$$

If we put $u = A_r(f;\cos\ \theta)$ and $v = r^{\alpha+\beta+1}\ \rho^{(\alpha,\beta)}(\theta)\ \tilde{A}_r(f;\cos\ \theta)$, then u and v satisfy the equations

$$rv_r = -r^{\alpha+\beta+1}\ \rho^{(\alpha,\beta)}(\theta)u_\theta, \tag{7.2.3.a}$$

$$v_\theta = r^{\alpha+\beta+1}\ \rho^{(\alpha,\beta)}(\theta)ru_r\ . \tag{7.2.3.b}$$

By the method developed by Askey [1], theorem 1, the generalization of M. Riesz' theorem can be proved.

7.2.1. Theorem. If $f \in L^p$, $1 < p < \infty$, then we have

a) $$||\tilde{A}_r(f;\cdot)||_p \leq M_p||f(\cdot)||_p\ .$$

b) There exists a function $\tilde{f} \in L^p$ such that

$$\lim_{r\to 1^-} ||\tilde{A}_r(f;\cdot) - \tilde{f}(\cdot)||_p = 0$$

and

$$||\tilde{f}||_p \le M_p ||f||_p \quad .$$

c) $$\lim_{r\to 1^-} \tilde{A}_r(f;\cos\theta) = \tilde{f}(\cos\theta), \quad \text{for almost all } \theta, \quad 0 \le \theta \le \pi.$$

The function $\tilde{f}(\cos\theta)$ is called the conjugate function of $f(\cos\theta)$.

7.3. CHARACTERIZATION OF $D(D_1)$

The concept of the conjugate function is used in the following theorem, which by corollary 3.4.6 gives a characterization of the domain of the fractional derivative of order 1.

7.3.1. Theorem. For $f,g \in X$ the relation

$$nf^{\wedge}(n) = g^{\wedge}(n) \qquad (n \in P) \tag{7.3.1}$$

is valid if and only if $\tilde{f} \in X$, the function $\rho^{(\alpha,\beta)}(\theta)\,\tilde{f}(\cos\theta)$ is absolutely continuous almost everywhere on $[-1,1]$ and vanishes at -1 and 1, and the relation

$$\frac{d}{d\theta}\,[\rho^{(\alpha,\beta)}(\theta)\tilde{f}(\cos\theta)] = \rho^{(\alpha,\beta)}(\theta)g(\cos\theta) \tag{7.3.2}$$

holds almost everywhere on $[-1,1]$.

Proof. We first assume that (7.3.1) is satisfied for f and $g \in X$. As we have mentioned in section 7.2, the functions $u = \tilde{A}_r(f;\cos\theta)$ and $v = r^{\alpha+\beta+1}\,\rho^{(\alpha,\beta)}(\theta)\,\tilde{A}_r(f;\cos\theta)$ satisfy the equations (7.2.3). From (7.2.3) we deduce, using (7.2.1) and (7.3.1)

$$\frac{d}{d\theta}\{r^{\alpha+\beta+1}\,\rho^{(\alpha,\beta)}(\theta)\,\tilde{A}_r(f;\cos\theta)\} = r^{\alpha+\beta+1}\,\rho^{(\alpha,\beta)}(\theta)\,A_r(g;\cos\theta). \tag{7.3.3}$$

It follows from (7.3.1) that f has a representation of the form

$$f(\cos\theta) = f^{\wedge}(0)\omega_0^{(\alpha,\beta)} + (g*h)(\cos\theta)\ , \tag{7.3.4.}$$

where

$$h(\cos\theta) \sim \sum_{n=1}^{\infty} n^{-1}\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta) \;.$$

In section 1.5 we have shown that h is a continuous function in each compact subinterval of $(0,\pi]$ and that

$$h(\cos\theta) = O(\theta^{-2\alpha-1}) \qquad\qquad \theta \to 0^+.$$

Thus, for $1 < p < 1/(2\alpha+1)$ the function h belongs to L^p. By lemma 1.4.1. iv) we may conclude from (7.3.4) that f belongs to L^p. Hence, by theorem 7.2.1 it follows that $\tilde{f} \in L^p$ and consequently $\tilde{f} \in L^1$. Integration of (7.3.3) yields

$$\rho^{(\alpha,\beta)}(\theta)\, \tilde{A}_r(f;\cos\theta) = \int_0^\theta A_r(g;\cos\tau)\rho^{(\alpha,\beta)}(\tau)d\tau \;. \tag{7.3.5}$$

For $r \to 1^-$, the left-hand side converges almost everywhere to $\rho^{(\alpha,\beta)}(\theta)\, \tilde{f}(\cos\theta)$ by theorem 7.2.1. c). Since

$$\left|\int_0^\theta [A_r(g;\cos\tau)-g(\cos\tau)]\rho^{(\alpha,\beta)}(\tau)d\tau\right| \leq ||A_r(g;\cdot)-g(\cdot)||_1$$

$$= o(1) \qquad (r\to 1^-),$$

the right-hand side of (7.3.5) converges almost everywhere to $\int_0^\theta g(\cos\tau)\rho^{(\alpha,\beta)}(\tau)d\tau$, which implies that the following relation holds almost everywhere:

$$\rho^{(\alpha,\beta)}(\theta)\, \tilde{f}(\cos\theta) = \int_0^\theta g(\cos\tau)\, \rho^{(\alpha,\beta)}(\tau)d\tau \;. \tag{7.3.6}$$

At $\theta = 0$ the right-hand side vanishes and also at $\theta = \pi$, since $g^\wedge(0) = 0$. Moreover, differentiation of (7.3.6) leads to (7.3.2), which establishes the first part of the theorem.

For the proof of the converse we deduce from (7.3.2)

$$g^\wedge(n) = \int_0^\pi g(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\theta)\, \rho^{(\alpha,\beta)}(\theta)d\theta$$

$$= \int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)\, \frac{d}{d\theta}[\rho^{(\alpha,\beta)}(\theta)\, \tilde{f}(\cos\theta)]d\theta,\ n \in P.$$

If we take into account that $\rho^{(\alpha,\beta)}(\theta)\ \tilde{f}(\cos\theta)$ vanishes at $\theta = 0$ and $\theta = \pi$ and if we use formula (1.1.9), we obtain after integration by parts

$$g^{\wedge}(0) = 0\ ,$$

$$g^{\wedge}(n) = \frac{n(n+\alpha+\beta+1)}{\alpha+1} \int_0^{\pi} \tilde{f}(\cos\theta)\ R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\rho^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(\theta)d\theta,$$

$$n \in Z^{+}.$$

Since $\rho^{(\alpha,\beta)}(\theta)\ \tilde{f}(\cos\theta)$ is absolutely continuous, relation (7.3.6) holds. Thus we may conclude

$$|\tilde{f}(\cos\theta)| \leq ||g||_1\ \{\rho^{(\alpha,\beta)}(\theta)\}^{-1}\quad ,$$

which implies that $\tilde{f} \in L^p$ for $1 < p < 1 + \frac{1}{2\alpha+1}$. Hence,

$$\int_0^{\pi} [\tilde{f}(\cos\theta) - \tilde{A}_r(f;\cos\theta)]\ R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\rho^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(\theta)d\theta$$

$$\leq ||\tilde{A}_r(f;\cdot) - f(\cdot)||_p = o(1) \qquad (r\to 1^-).$$

We now investigate

$$\frac{n(n+\alpha+\beta+1)}{\alpha+1} \int_0 \tilde{A}_r(f;\cos\theta)R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\rho^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(\theta)d\theta,$$

$$n \in Z^{+}.$$

If we substitute for $\tilde{A}_r(f;\cos\theta)$ the expansion (7.2.2) and integrate term by term, noticing that the polynomials $R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)$ are orthogonal with respect to $\rho^{(\alpha+1,\beta+1)}(\theta)$, we obtain

$$g^{\wedge}(n) = \lim_{r\to 1^-} \frac{n(n+\alpha+\beta+1)}{\alpha+1} \int_0^{\pi} \tilde{A}_r(f;\cos\theta)R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\rho^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(\theta)d\theta$$

$$= \lim_{r\to 1^-} \frac{n(n+\alpha+\beta+1)}{(\alpha+1)^2} \sum_{k=1}^{\infty} kf^{\wedge}(k)r^k\ \omega_k^{(\alpha,\beta)}$$

$$\int_0^{\pi} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta)R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\rho^{(\alpha+1,\beta+1)}(\theta)d\theta$$

$$= \lim_{r\to 1^-} \frac{n^2(n+\alpha+\beta+1)}{(\alpha+1)^2}\, r^n f^\wedge(n)\omega_n^{(\alpha,\beta)}\, [\omega_{n-1}^{(\alpha+1,\beta+1)}]^{-1}$$

$$= nf^\wedge(n) \,, \qquad n \in \mathbb{Z}^+,$$

which completes the proof of theorem 7.3.1.

7.4. CHARACTERIZATION OF THE SPACES $(X,D(D_\gamma))_{\gamma,q;K}$

This last section is devoted to the following theorem:

7.4.1. Theorem. For $0 < \gamma < 2$ and $0 < \theta < 1$, $1 \leq q \leq \infty$ the following statements are equivalent for $f \in X$:

i) $\qquad f \in (X,D(D_\gamma))_{\theta,q;K}$,

ii) $\qquad f \in (X,D(A))_{\gamma\theta/2,q;K}$.

Proof. This theorem is a direct consequence of theorem 2.1.6, if we able to show that

(7.4.1) $\qquad (X,D(A))_{\gamma/2,1;K} \subset D(D_\gamma) \subset (X,D(A))_{\gamma/2,\infty;K}$.

We first prove the second inclusion. If $f \in D(D_\gamma)$, there exists a function $g \in X$, such that $f = I_\gamma g$, where I_γ denotes the fractional integration operator, introduced in section 5.5. We have shown that $f = I_\gamma g \in \mathrm{Lip}(\gamma,X)$. Hence, by theorem 4.2.2 the second inclusion follows.

For the proof of the first inclusion in (7.4.1) we need some theorems on spaces of best approximation quoted in section 2.2. For the subspaces P_n $(n\in\mathbb{Z}^+)$ are chosen the spaces of the polynomials in cos θ of degree $\leq n$. On account of the inequality (5.5.2) and definition 2.2.4 we know that $D(D_\gamma)$ is a space of the class $D_\gamma^J(X)$ and thus, by lemma 2.2.5 we have for the space of best approximation $X_{\gamma,1}^K$ the inclusion

(7.4.2) $\qquad X_{\gamma,1}^K \subset D(D_\gamma)$.

Furthermore, we show that $D(A)$ is a space of the class $D_2(X)$. The fact that $D(A)$ is a space of the class $D_2^K(X)$ follows from definition 2.2.4 and the formulas 4.1.5 and 4.2.2. The space $D(A)$ belongs to the class $D_2^J(X)$ by the inequality (4.2.8). We are now in a position to apply theorem 2.2.6 to conclude that

$$(7.4.3) \qquad X^K_{\gamma,1} \cong (X,D(A))_{\gamma/2,1;K} \,.$$

Combination of (7.4.2) and (7.4.3) leads to the first inclusion of (7.4.1). This proves the theorem.

Theorem 7.4.1 enables us to characterize in terms of the spaces of K interpolation between X and D(A) all the spaces of non-optimal approximation, that occur in chapter V.

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SAMENVATTING

In dit proefschrift worden sommatiemethoden voor reeksontwikkelingen in termen van Jacobi polynomen bestudeerd. De Jacobi polynomen $R_n^{(\alpha,\beta)}(x)$ vormen een ruime klasse van orthogonale polynomen, met als belangrijke speciale gevallen Chebyshev polynomen ($\alpha=\beta=-\frac{1}{2}$), Legendre polynomen ($\alpha=\beta=0$) en Gegenbauer of ultraspherische polynomen ($\alpha=\beta$), die in het geval $\alpha = \beta = \frac{n-3}{2}$ spherische harmonischen zijn in $\mathbb{R}^n$. Verder kunnen voor een aantal discrete waarden van α en β de Jacobi polynomen geïnterpreteerd worden als spherische functies op symmetrische ruimten van rang 1 (Gangolli [25]). Wanneer een functie f formeel wordt ontwikkeld in een Fourier-Jacobi reeks, dan hoeft deze reeks niet te convergeren, zodat het zinvol is om sommatiemethoden erop toe te passen. Sommatiemethoden voor de Fourier-Jacobi reeks van een functie f kunnen worden opgevat als approximatieprocessen voor de functie f. In dit proefschrift wordt de snelheid onderzocht, waarmee een dergelijk proces de functie benadert en tevens worden die functies gekarakteriseerd, die met een voorgeschreven snelheid worden geapproximeerd door het betreffende proces. Bij vele van deze processen treedt het verschijnsel van saturatie op. Er bestaat dan een optimale snelheid van benadering, die alleen bij functies behorende tot een triviale klasse kan worden overtroffen, terwijl deze snelheid bij minstens één niet-triviale functie wordt bereikt. Voor de sommatie methoden die in dit proefschrift voorkomen wordt deze optimale snelheid (saturatieorde) opgespoord en verder wordt de klasse van functies bepaald, die met deze optimale snelheid kunnen worden benaderd (saturatieklasse).

De laatste tijd hebben zich belangrijke ontwikkelingen voorgedaan, zowel in de theorie van de Jacobi reeksen als in de approximatietheorie. Het werk van Askey en Wainger leidde o.a. tot de ontwikkeling van convolutiealgebras voor Jacobi reeksen [5] en hun dualen, Jacobi coefficienten [6]. Gasper [27, 28] bakende in het (α,β) vlak het gebied af, waar de convolutiealgebra voor Jacobi reeksen kan worden gedefinieerd en ook bepaalde hij het gebied, waar de gegeneraliseerde translatieoperator, die gebruikt wordt bij de definitie van de convolutie, een positieve operator is. Het analoge probleem voor de duale convolutie werd ook door Gasper opgelost [26, 3]. De convolutiealgebra voor Jacobi reeksen is een van de belangrijkste hulpmiddelen in dit proefschrift. Er zal steeds van worden uitgegaan, dat de gegeneraliseerde translatie een positieve operator is, dat wil zeggen dat $\alpha \geq \beta$ en dat tevens aan een van de volgende voorwaarden moet zijn voldaan: $\beta \geq -\frac{1}{2}$ of $\alpha + \beta \geq 0$, $\beta > -1$. Dit levert de restricties, die aan α en β in dit proefschrift zullen worden opgelegd.

Na in de eerste paragrafen van hoofdstuk I enkele feiten over Jacobi polynomen en Jacobi reeksen te hebben vermeld, wordt in paragraaf 1.4 de convolutiestructuur behandeld. In paragraaf 1.5 wordt aandacht geschonken aan een speciale Jacobi klasse Jacobi reeksen. De hier verkregen resultaten blijken in de volgende hoofdstukken van groot nut te zijn.

Het begrip saturatie in de approximatietheorie is ingevoerd door Favard in 1947. Gedurende de volgende jaren is door vele auteurs aan dit onderwerp gewerkt en werden er algemene methoden ontwikkeld om de saturatieklasse te bepalen van families van convolutieoperatoren op de reële as, op de R^n, op de eenheidscirkel in R^2, op de n-dimensionale torus, op de eenheidsbol in R^n, enz. Voor historische details zij verwezen naar [18], paragraaf 12.6. De van Peetre [39] afkomstige methoden, om intermediaire ruimten tussen twee Banach ruimten te construeren, vormen een zeer geschikt kader om saturatieklassen en klassen van niet-optimale approximatie te karakteriseren. Butzer en Berens [16] maakten hiervan gebruik bij het bestuderen van halfgroepen van operatoren op Banach ruimten. Algemene families van operatoren op Banach ruimten werden met behulp van deze intermediaire ruimten behandeld in Berens [12], Butzer en Scherer [20, 21]. In concrete gevallen zijn deze algemene resultaten over approximatieprocessen op Banach ruimten bijzonder nuttig. Zij geven namelijk aan, welke ongelijkheden of limietrelaties voor de betreffende approximatieprocessen voldoende zijn om conclusies te kunnen trekken over hun saturatieorde, saturatieklasse en klassen van niet-optimale approximatie. De benodigde stellingen over approximatieprocessen op Banach ruimten staan vermeld in hoofdstuk II. Voor de bewijzen wordt de lezer verwezen naar [12], [16], [20], [21] en [11].

In de eerste paragraaf van hoofdstuk III worden kernen en approximatiekernen ingevoerd. Wanneer de ruimte X behoort tot een bepaalde klasse functieruimten, die door geschikte normkeuze Banach ruimten gemaakt kunnen worden, dan levert de convolutie van $f \in X$ met een approximatiekern een approximatieproces voor f in de X norm. Vervolgens worden sommatiemethoden voor Jacobi reeksen gedefinieerd en met iedere sommatiemethode wordt een kern geassocieerd. Als deze kern een approximatiekern blijkt te zijn, dan levert de betreffende sommatiemethode, toegepast op de Jacobi reeks van $f \in X$, een proces dat in de X norm naar f convergeert. Dit is voor veel klassieke sommatiemethoden het geval.

Het belangrijkste deel van hoofdstuk IV wordt gevormd door stellingen van het Jackson en het Bernstein type, waarin de gladheid van de functie in verband wordt gebracht met de snelheid waarmee een functie met polynomen van de graad N kan worden benaderd. De gladheid van de functie wordt hier gedefinieerd door middel van de continuïteitsmodulus ten opzichte van de gegeneraliseerde translatie. Als de ruimte van continue functies met de supremumnorm wordt beschouwd, dan blijkt deze continuïteitsmodulus equivalent te zijn met de gewone symmetrische continuïteitsmodulus.

De auteur behandelt in hoofdstuk V een aantal min of meer klassieke sommatiemethoden en karakteriseert de functies die een zekere approximatiesnelheid toelaten door deze processen. De bewijsmethoden zijn ontleend aan de algemene theorie betreffende approximatieprocessen op Banach ruimten.

Processen die ontstaan door convolutie van $f \in X$ met een positieve kern, vormen het onderwerp van hoofdstuk VI. Wanneer de kern positief wordt verondersteld kunnen de voorwaarden voor normconvergentie van het proces aanzienlijk verzwakt worden. Daar staat tegenover, dat de approximatiesnelheid van processes met een positieve polynoomkern meestal beperkt is voor niet-constante functies. In paragraaf 6.3 wordt een saturatiestelling gegeven voor processen met positieve kernen, die aan een speciale voorwaarde voor de Fourier-Jacobi coëfficiënten voldoen. Met behulp van deze stelling wordt in paragraaf 6.4 de saturatieorde en de saturatieklasse van een aantal processen met een positieve kern bepaald.

Tenslotte wordt in hoofdstuk VII een nadere karakterisering gegeven van zekere functieklassen, die in de voorafgaande hoofdstukken zijn opgetreden.