The Mertens conjecture

Herman J.J. te Riele*

Title of This book
ALM ??, pp. ?–?

April 8, 2015

Abstract

The Mertens conjecture states that $|M(x)|x^{-1/2} < 1$ for $x > 1$, where $M(x) = \sum_{1 \le n \le x} \mu(n)$ and where $\mu(n)$ is the Möbius function defined by: $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is the product of k distinct primes, and $\mu(n) = 0$ if n is divisible by a square > 1. The truth of the Mertens conjecture implies the truth of the Riemann hypothesis and the simplicity of the complex zeros of the Riemann zeta function. This paper gives a concise survey of the history and state-of-affairs concerning the Mertens conjecture. Serious doubts concerning this conjecture were raised by Ingham in 1942 [12]. In 1985, the Mertens conjecture was disproved by Odlyzko and Te Riele [23] by making use of the lattice basis reduction algorithm of Lenstra, Lenstra and Lovász [19]. The best known results today are that $|M(x)|x^{-1/2} \ge 1.6383$ and there exists an $x < \exp(1.004 \times 10^{33})$ for which $|M(x)|x^{-1/2} > 1.0088$.

2000 Mathematics Subject Classification: Primary 11-04, 11A15, 11M26, 11Y11, 11Y35

Keywords and Phrases: Mertens conjecture, Riemann hypothesis, zeros of the Riemann zeta function

1 Introduction

Let $\mu(n)$ be the Möbius function with values $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is the product of k distinct primes, and $\mu(n) = 0$ if n is divisible by a square > 1. The Mertens conjecture [21] states that

$$|M(x)|/\sqrt{x} < 1 \text{ for } x > 1, \text{ where } M(x) = \sum_{1 \le n \le x} \mu(n). \tag{1.1}$$

*Centrum Wiskunde en Informatica, Science Park 123, 1098 XG Amsterdam, The Netherlands. E-mail: Herman.te.Riele@cwi.nl.

$M(x)$ counts the difference of the number of squarefree positive integers with an *even* number of prime factors and those with an *odd* number of prime factors.

In the "Comptes Rendues de l'Académie des Sciences de Paris" of July 13, 1885, Thomas Stieltjes published a two-page note under the rather vague title: "Sur une fonction uniforme". In this note he announced a proof of the Riemann hypothesis as follows: "I have succeeded to put this proposition (*HtR: the Riemann hypothesis*) beyond doubt by a rigorous proof". The only explanation Stieltjes gave for this remarkable claim was that he asserted that he was able to prove that the series

$$\frac{1}{\zeta(s)} = 1 - \frac{1}{2^s} - \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{6^s} - \cdots = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \tag{1.2}$$

"converges and defines an analytic function as long as the real part of s exceeeds $\frac{1}{2}$". Here, $\zeta(s), s = \sigma + it$, is the Riemann zeta function, i.e., it is the analytic function, defined for $\sigma > 1$, by: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, and for $\sigma \leq 1, \sigma \neq 1$ by means of analytic continuation. This assertion indeed would imply that all the complex zeros of $\zeta(s)$ have real part $\frac{1}{2}$ by the following argument: For $\sigma = \Re s > 1$ we find, by partial summation, that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} \frac{M(n) - M(n-1)}{n^s} = \sum_{n=1}^{\infty} M(n) \left\{ \frac{1}{n^s} - \frac{1}{(n+1)^s} \right\}$$

$$= \sum_{n=1}^{\infty} M(n) \int_n^{n+1} \frac{s dx}{x^{s+1}} = s \sum_{n=1}^{\infty} \int_n^{n+1} \frac{M(x) dx}{x^{s+1}} = s \int_1^{\infty} \frac{M(x) dx}{x^{s+1}},$$

because $M(x)$ is constant on every interval $[n, n+1)$. The boundedness of $M(x)/\sqrt{x}$ implies that the last integral in the above formula defines a function which is analytic in the half-plane $\sigma > \frac{1}{2}$, and this would give an analytic continuation of $1/\zeta(s)$ from $\sigma > 1$ to $\sigma > \frac{1}{2}$. In particular, this would imply that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1/2$. By the functional equation for $\zeta(s)$ this is equivalent with the Riemann hypothesis. In addition it is not difficult to derive from the formulas above that all complex zeros of $\zeta(s)$ are simple (assuming that $M(x)/\sqrt{x}$ is bounded) [23].

Stieltjes never published his "proof". From correspondence with his friend Hermite and with Mittag-Leffler [2] we know that Stieltjes believed that he could prove that the function $M(x)/\sqrt{x}$ always stays within two fixed limits, possibly $+1$ and -1. In his heritance a table was found with values of $M(n)$, for $1 \leq n \leq 1200, 2000 \leq n \leq 2100$, and $6000 \leq n \leq 6100$ and this could well be the basis of his "belief".

After Stieltjes, many others have computed tables of $M(x)$, in order to collect more numerical data about the behaviour of $M(x)/\sqrt{x}$. The first one after Stieltjes was Mertens who, in 1897, published a 50-page table of $\mu(n)$ and $M(n)$ for $n = 1, 2, \ldots, 10000$ [21]. Based on this, Mertens concluded that the inequality $|M(x)| < \sqrt{x}$ for $x > 1$ is very probable. This is known since then as the *Mertens conjecture*.

In this paper, we will give a survey of the history of the Mertens conjecture up to the current state-of-affairs. First we give a concise survey of explicit computations of $M(x)$ carried out after Stieltjes. Next we describe research which has resulted in strong evidence for the *unboundedness* of the function $M(x)/\sqrt{x}$, as opposed to what Stieltjes and Mertens hoped to be able to prove. Finally we will describe the technique and numerical computations by which it has been shown that the Mertens conjecture is false, and beyond. We conclude by giving the smallest known upper bound on x for which it is known that $|M(x)|/\sqrt{x} > 1$.

2 Explicit computations of $M(x)$

Stieltjes and Mertens started to compute and publish values of $M(x)$ in order to collect more numerical evidence for the possible boundedness of the function $M(x)/\sqrt{x}$. In order to compute $M(x)$, it seems necessary at first sight to know all the values of $\mu(m)$ for $1 \le m \le x$. However, below we will encounter fomulas where $M(n)$ is expressed in terms of $M(j)$ with $j \le n/k$, for some fixed $k \ge 2$ (these formulas become increasingly more complicated as k increases). In this way it is possible to compute $M(n)$ for large isolated values of n in ranges where it is infeasible to compute *all* the values of $M(n)$ for $1 \le n \le x$.

As said before, Mertens was the first [21] to publish a table of $\mu(n)$ and $M(n)$. He does not explain how he computed this table. From the well-known formula

$$\sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor \mu(i) = \sum_{i=1}^{n} M\left(\frac{n}{i}\right) = 1 \tag{2.3}$$

(where $\lfloor x \rfloor$ means the greatest integer $\le x$) he derives the following relation, which may be used to express $M(n)$ in terms of $\mu(1), \mu(2), \ldots, \mu(k), M(k)$ and $M(n/2), M(n/3), \ldots, M(n/k)$, where $k = \lfloor \sqrt{n} \rfloor$:

$$\sum_{i=1}^{k} \left\lfloor \frac{n}{i} \right\rfloor \mu(i) + \sum_{i=1}^{k} M\left(\frac{n}{i}\right) - kM(k) = 1. \tag{2.4}$$

This relation served as a check, as Mertens states on p. 763 of [21], during the computation of his table. Moreover, Mertens derives a second relation, viz.,

$$M(n) = 2M(k) - \sum_{r,s=1}^{k} \left\lfloor \frac{n}{rs} \right\rfloor \mu(r)\mu(s), \quad k = \lfloor \sqrt{n} \rfloor, \tag{2.5}$$

which expresses $M(n)$ in terms of $M(k)$ and $\mu(1), \ldots, \mu(k)$. This "allows to compute $M(n)$ without knowing the decomposition of the numbers $k+1$ up to n in their prime factors" [21, p. 764].

In the year that Mertens published his table, Von Sterneck started a series of four papers presenting tables of $M(n)$, for $n = 1, 2, \ldots, 150000$ [35], for $n = 150000(50)500000$ [36] and for 16 selected values of n between 5×10^5 and 5×10^6 [37, 38]. The latter values were computed by means of a refined version of Mertens' formula (2.4), viz.,

$$\sum_{i=1}^{k} \omega_j \left(\frac{n}{i}\right) \mu(i) + \sum_{i' \le k} M\left(\frac{n}{i'}\right) - \omega_j(k)M(k) = 0, \quad k = \lfloor \sqrt{n} \rfloor, \quad j = 0, 1, \ldots, \tag{2.6}$$

where $\omega_j(n)$ denotes the number of positive integers $\leq n$ which are not divisible by any of the first j primes, and where i' runs through all such positive integers $\leq k$. For $j = 0$, (2.6) reduces to (2.4). Von Sterneck applied (2.6) for $j = 1, 2, 3$ and 4. For $j = 4$, e.g., i' runs through the integers $1, 11, 13, 17, \ldots$ so that it is possible to compute $M(n)$ from a table of M-values up to $\lfloor n/11 \rfloor$. From his results, Von Sterneck draws the conclusion [38] that the inequality $|M(n)| < \frac{1}{2}\sqrt{n}$, for $n > 200$, "represents an unproved, but extremely probable number-theoretic law".

Fifty years after Von Sterneck, Neubauer [22] published an empirical study in which *all* the values $M(n)$, $1 \leq n \leq 10^8$, were computed. Neubauer computed $\mu(m)$ for a series of 1000 values of m: $1000n < m \leq 1000(n+1)$, for $n = 0, 1, \ldots, 10^5 - 1$, by means of a sieving process which strongly resembles the well-known sieve of Eratosthenes for finding all the primes below a given limit. This is considerably cheaper than computing $\mu(m), \mu(m+1), \ldots$ by factoring $m, m+1, \ldots$. Neubauer checked the computations of Von Sterneck [36, 37] and he found several errors in [36] and errors in 9 of the 16 sample values of $M(n)$ which Von Sterneck had published in [37]. Neubauer also computed many sample values of $M(n)$ for several n between 10^8 and 10^{10}, by means of (2.6). As a result, he found four values of n for which $M(n) > \frac{1}{2}\sqrt{n}$ (but none for which $M(n) < -\frac{1}{2}\sqrt{n}$), the smallest being $n_0 = 7\,760\,000\,000$ with $M(n_0) = 47465$ and $M(n_0)/\sqrt{n_0} = 0.5388....$ The largest $M(n)/\sqrt{n}$–value he found was 0.5572..., for $n = 7\,770\,000\,000$.

Yorinaga [40] computed all the values of $M(n)$ for $n \leq 4 \times 10^8$, by factoring all $n \leq 4 \times 10^8$.

Cohen and Dress [7] have extended Yorinaga's computations with the purpose to find the smallest $n > 200$ for which $M(n)/\sqrt{n} > \frac{1}{2}$, knowing from Neubauer's computations that this n is $< 7.76 \times 10^9$. Without taking the trouble to mention their method, they state that they have carried out their computations in one week on a TI 980B mini-computer. They computed all the values of $M(n)$ for n up to 7.8×10^9 and saved a table of $M(n)$ for $n = 10^7(10^7)7.8 \times 10^9$. The smallest $n > 200$ for which $M(n)/\sqrt{(n)} > \frac{1}{2}$ turned out to be $n_0 = 7\,725\,038\,629$ with $M(n_0) = 43947$.

Dress [9] extended the computations of $M(n)$ to $n \leq 10^{12}$ with the purpose to find the smallest $n > 200$ for which $M(n)/\sqrt{n} < -\frac{1}{2}$. This turned out to be $n_0 = 330\,486\,258\,610$ with $M(n_0) = -287440$ and $M(n_0)/\sqrt{n_0} = -0.500$ (rounded to 3 decimals).

The most extensive systematic computations of $M(n)$ have been carried out by Lioen and Van de Lune [20] (up to $n \leq 10^{13}$) and by Kotnik and Van de Lune [15] (up to $n \leq 10^{14}$). They established the bounds $-0.525 < M(x)/\sqrt{x} < 0.571$ for $200 < x \leq 10^{14}$ where for $n_0 = 71\,578\,936\,427\,177$ they found: $M(n_0) = -4440015$ and $M(n_0)/\sqrt{n_0} = -0.525$ (rounded to three decimals).

In the papers [22, 20, 15] a sieving algorithm was used to compute $\mu(n)$ for all $n \in [1, N]$, which strongly resembles the well-known sieve of Eratosthenes for finding all the primes below a given limit. Van de Lune and his co-authors speeded up this algorithm by using vector computers. We use the formulation of Van de Lune to describe this algorithm:

for $n = 1$ to N: $\mu(n) = 1$
for all primes $p \le \sqrt{N}$: for all $n, p|n$: $\mu(n) = -p \cdot \mu(n)$
for all primes $p \le \sqrt{N}$: for all $n, p^2|n$: $\mu(n) = 0$
for $n = 1$ to N: if $|\mu(n)| \neq n$ then $\mu(n) = -\mu(n)$
for $n = 1$ to N: $\mu(n) = sign(\mu(n))$

J. Schröder [30, 31, 32] has derived several rather complicated formulas for computing $M(x)$. As far as we know, these formulas have never been used for the computation of extensive tables of $M(x)$.

Liouville's function $\lambda(n)$ is defined by $\lambda(n) = (-1)^r$ where r is the number of prime factors of n, multiple factors counted according to their multiplicity. Lehman [18] has published a method to compute the function $L(x) = \sum_{n \le x} \lambda(n)$ at isolated values of x in $\mathcal{O}(x^{2/3+\epsilon})$ bit operations. According to Lehman, a similar method (with the same amount of work) can be derived from (2.3) for the computation of $M(x)$. An analytic method of Lagarias and Odlyzko [17, 25, 6] for computing $\pi(x)$ (i.e., the number of primes $\le x$) can be adapted to obtain a method for computing $M(x)$ that requires on the order of $\mathcal{O}(x^{1/2+\epsilon})$ bit operations. As far as we know, neither Lehman's method, nor that of Lagarias and Odlyzko has been implemented.

3 Evidence for the unboundedness of $M(x)/\sqrt{x}$

In 1942, Ingham [12] published a paper which raised the first serious doubt concerning the conjecture of Mertens. Ingham showed that it may be possible to show that the function $|M(x)|/\sqrt{x}$ assumes *large* values without the need to compute $M(x)$ explicitly. In order to explain this, we write $x = e^y$, $-\infty < y < \infty$ and define

$$m(y) := M(x)x^{-1/2} = M(e^y)e^{-y/2} \tag{3.7}$$

and

$$\underline{m} := \liminf_{y\to\infty} m(y), \quad \overline{m} := \limsup_{y\to\infty} m(y). \tag{3.8}$$

Then we have the following

Theorem 3.1 ([12, 23]). *Let*

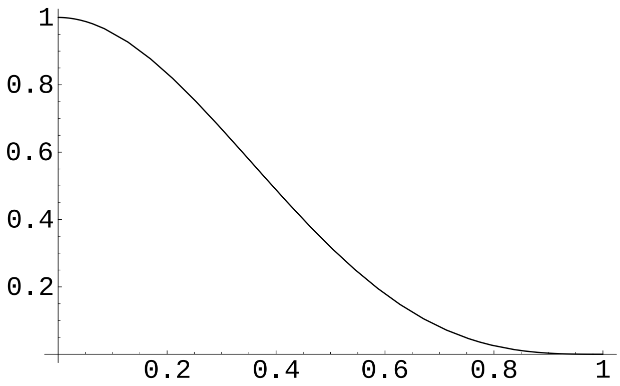
$$h(y,T) := 2 \sum_{0<\gamma<T} \Big[(1-\frac{\gamma}{T})\cos(\pi\frac{\gamma}{T}) + \pi^{-1}\sin(\pi\frac{\gamma}{T})\Big] \frac{\cos(\gamma y - \psi_\gamma)}{|\rho\zeta'(\rho)|} \tag{3.9}$$

where $\rho = \beta + i\gamma$ are the complex zeros of the Riemann zeta function with $\beta = \frac{1}{2}$ and which are simple, and where $\psi_\gamma = \arg \rho\zeta'(\rho)$. Then for any real number y_0 we have

$$\underline{m} \le h(y_0, T) \le \overline{m} \tag{3.10}$$

and any value $h(y,T)$ is approximated arbitrarily close, and infinitely often, by $M(x)/\sqrt{x}$.

```
Plot[(1-t)*Cos[Pi*t]+Sin[Pi*t]/Pi, {t, 0, 1}]
```



Notice that Theorem 3.1 assumes the truth of the Riemann hypothesis up to $|\gamma| < T$ for the complex zeros $\beta + i\gamma$ of the Riemann zeta function $\zeta(s)$ *and* that it assumes the simplicity of these zeros.

The function $h(y, T)$ plays a crucial role in the mathematics and heuristics behind the disproof of the Mertens conjecture. It is a complicated trigonometric function of the complex zeros of $\zeta(s)$. It is real-valued and although it looks far more complicated than the function $M(x)$, it turns out that in practice it is less difficult to find large values of $h(y, T)$, and hence, by (3.10), of $m(y)$, than to find large values of $M(x)/\sqrt{x}$ directly.

The following, partly heuristic, argument should clarify this. It is easy to see that the function $(1-t)\cos(\pi t) + \pi^{-1}\sin(\pi t)$, $0 \le t \le 1$ (see plot above) is nonnegative.

In addition, the sum $\sum_\rho |\rho\zeta(\rho)|^{-1}$ diverges [39, Section 14.27]. This implies that the sum of the coefficients of $\cos(\gamma y - \psi_\gamma)$ in the Theorem can be made arbitrarily large, by choosing T large enough. From this it follows that if we could find a value of y such that *all* values $\gamma y - \psi_\gamma$ would be close to integer multiples of 2π (so that all values of $\cos(\gamma y - \psi_\gamma)$ would be close to 1), we could construct arbitrarily large values of $h(y, T)$. If the values of the γ's in $h(y, T)$ would be linearly independent over the rationals then, according to a Theorem of Kronecker [11, Theorem 442], for every $\epsilon > 0$ there exist integers y and integers m_γ for each γ such that

$$|\gamma y - \psi_\gamma - 2\pi m_\gamma| < \epsilon \tag{3.11}$$

for all $\gamma \in (0, T)$. This would imply that $h(y, T)$, and hence $M(x)/\sqrt{x}$, could be made arbitrarily large. With a similar argument one could reason that $M(x)/\sqrt{x}$ could be made arbitrarily large on the negative side.

The linear indendency of the γ's over the rationals has never been proved but, on the other hand, no numerical evidence whatsoever of the opposite assertion is known. Given a small positive ϵ and values of γ with $0 < \gamma < T$ and corresponding ψ_γ, to find integers y and m_γ satisfying (3.11) is known as an *inhomogeneous*

i	γ_i	$\lvert\rho_i\zeta'(\rho_i)\rvert^{-1}$	i	γ_i	$\lvert\rho_i\zeta'(\rho_i)\rvert^{-1}$
1	14.135	0.0891	11	52.970	0.00778
2	21.022	0.0418	12	56.446	0.00748
3	25.011	0.0291	13	59.347	0.01211
4	30.425	0.0252	14	60.832	0.00994
5	32.935	0.0220	15	65.113	0.00671
6	37.586	0.0137	16	67.080	0.00836
7	40.919	0.0164	17	69.546	0.00658
8	43.327	0.0126	18	72.067	0.00468
9	48.005	0.0133	19	75.705	0.00742
10	49.774	0.0142	20	77.145	0.00889

Table 1: Behaviour of $|\rho\zeta'(\rho)|^{-1}$ for the first 20 zeros $\rho = \rho_i = \frac{1}{2} + \gamma_i$ of $\zeta(s)$ (all values given are rounded)

Diophantine approximation problem.

Further doubt on the validity of the Mertens conjecture, and of the weaker conjecture that $m(y)$ is bounded, was generated by the work of Bateman et al. [3]. They showed that if $m(y)$ is bounded, then there exist infinitely many relations among the γ's of the form $\sum_\gamma c_\gamma \gamma = 0$, where the $c_\gamma = 0, \pm 1$, or ± 2, and at most one of the c_γ satisfies $|c_\gamma| = 2$. This enforced skepticism about the Mertens conjecture, especially since Bateman et al. looked at linear combinations of the first few γ's with coefficients of the above form and did not find anything that might suggest evidence of linear relations of the required type. Later, Bailey and Ferguson [1] have shown that if there exists any linear relation of the form $\sum c_\gamma \gamma = 0$ where $c_\gamma \in \mathbb{Z}$ and the sum runs over the imaginary parts of the first 8 complex zeros of the Riemann zeta function, then the Euclidean norm of the vector (c_γ) exceeds 5.1×10^{24}.

4 Numerical computations leading to the disproof of the Mertens conjecture, and beyond

Ingham's paper [12] and Theorem 3.1 form the basis of the developments which led to the disproof of the Mertens conjecture.

Spira [33] was the first who tried to find large values of (a slightly simplified version of) the function $h(y, T)$ in (3.9). For $T = 100$ he computed $h(y, T)$ on a fine grid of values of $y \in [0, 1000]$ and subsequently he computed $h(y, T)$ for $T = 200, 500$ and 1000 for a selection of "promising" y-values, i.e., for which $h(y, 100)$ was locally maximal. In this way, he showed that $\overline{m} \geq 0.5355$ and $\underline{m} \leq -0.6027$.

Jurkat et al. [13] realized that the size of the sum $h(y, T)$ is determined largely by the first few terms since, numerically, the numbers $|\rho\zeta'(\rho)|^{-1}$ appear to be of order ρ^{-1}. Table 1 illustrates this. Therefore, they looked for values of y such that $\cos(\gamma_1 y - \pi\psi_1) = 1$ and $\cos(\gamma_i y - \pi\psi_i) > 0.9$ (say), for $i = 2, 3, \ldots, N+1$, where N is taken as large as possible. This gives an inhomogeneous Diophantine approximation problem for which Jurkat et al. devised an ingenious algorithm. By applying it with $N = 12$ they proved that $\overline{m} \geq 0.779$ and $\underline{m} \leq -0.638$. While Jurkat et al. used a programmable desk calculator, Te Riele [26] implemented the algorithm of Jurkat et al. (together with a few improvements, one of them realized by arranging the γ_i's in the Diophantine approximation problem in such a manner that the values of $|\rho_i \zeta'(\rho_i)|^{-1}$ are as large as possible and decreasing [1]) on a high speed computer and proved that $\overline{m} \geq 0.860$ and $\underline{m} \leq -0.843$.

In order to find a y which solves (3.11) for a subset of small γ's, say $\gamma_1, \gamma_2, \ldots, \gamma_n$ (which in general are not the first n γ's) and a small ϵ, Odlyzko and Te Riele used a (in that time) new algorithm due to A.K. Lenstra, H.W. Lenstra, Jr. and L. Lovász [19], now known as the L^3 algorithm. This was designed to find short vectors in lattices, and since then it has found widespread applications, e.g., in polynomial factorizations and in public key cryptography.
If $v_1, v_2, \ldots, v_m$ is a set of basis vectors of an m-dimensional lattice L in $\mathbb{R}^m$, then the L^3 algorithm finds another basis, $v_1^*, v_2^*, \ldots, v_m^*$, called *reduced*, which satisfies

$$||v_1^*|| \leq \left(\frac{4}{4u-1}\right)^{\frac{m-1}{2}} \min_{\underline{v} \in L, \underline{v} \neq \underline{0}} ||\underline{v}||, \quad \text{and} \quad \prod_{i=1}^{m} ||v_i^*|| \leq \left(\frac{4}{4u-1}\right)^{\frac{m(m-1)}{4}} d(L), \tag{4.12}$$

where $u \in (1/4, 1)$ is a parameter chosen beforehand, $d(L)$ is the determinant of the lattice, and $||\underline{v}||$ denotes the euclidean norm of the vector $\underline{v}$.
In order now to find a y such that each of

$$\eta_j := \gamma_j y - \psi_j - 2\pi m_j, 1 \leq j \leq n, \tag{4.13}$$

is small, where

$$\psi_j = \arg\left(\frac{1}{2} + i\gamma_j\right)\zeta'\left(\frac{1}{2} + i\gamma_j\right), \tag{4.14}$$

this problem is transformed into a problem about finding short vectors in lattices as follows. The lattice L used by Odlyzko and Te Riele is generated by the columns $v_1, v_2, \ldots, v_{n+2}$ of the following $(n+2) \times (n+2)$ matrix (here $[x]$ means the greatest integer $\leq x$):

$$\begin{pmatrix}
-[\alpha_1\psi_1 2^\nu] & [\alpha_1\gamma_1 2^{\nu-10}] & [2\pi\alpha_1 2^\nu] & 0 & & 0 \\
-[\alpha_2\psi_2 2^\nu] & [\alpha_2\gamma_2 2^{\nu-10}] & 0 & [2\pi\alpha_2 2^\nu] & & 0 \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
-[\alpha_n\psi_n 2^\nu] & [\alpha_n\gamma_n 2^{\nu-10}] & 0 & 0 & & [2\pi\alpha_n 2^\nu] \\
2^\nu n^4 & 0 & 0 & 0 & & 0 \\
0 & 1 & 0 & 0 & & 0
\end{pmatrix} \tag{4.15}$$

[1]For the first 20 zeros in Table 1 this means ordering the γ_i's according to the indices 1, 2, 3, 4, 5, 7, 10, 6, 9, 8, 13, 14, 20, 16, 11, 12, 19, 15, 17, 18.

where ν is some integer (usually $2n \le \nu \le 4n$) and

$$\alpha_j = \left| \left(\frac{1}{2} + i\gamma_j \right) \zeta' \left(\frac{1}{2} + i\gamma_j \right) \right|^{-\frac{1}{2}}.$$

The L^3 algorithm generates a reduced basis $\underline{v}_1^*, \underline{v}_2^*, \ldots, \underline{v}_{n+2}^*$ for the lattice L. This reduced basis usually contains some very short vectors. However, one is actually interested in the *longest* vector in the reduced basis. Since the reduced basis is a basis for L, it has to contain at least one vector $\underline{w}$ which has a nonzero coordinate in the $(n+1)$-st position. Since that coordinate is a multiple of $2^\nu n^4$, it is very large compared to all the other entries in the original basis, and this makes $\underline{w}$ quite long. Therefore, in order to obtain a set of short basis vectors, a good basis transformation algorithm ought to contain exactly one vector $\underline{w}$ with a nonzero $(n+1)$-st coordinate, and that coordinate then has to be $\pm 2^\nu n^4$. In all the tests run by Odlyzko and Te Riele, the L^3 algorithm did indeed behave in this desirable fashion.

Given that there is a single vector $\underline{w}$ in the reduced basis with nonzero $(n+1)$-st coefficient, which one may take to be $2^\nu n^4$ without loss of generality, its j-th coordinate for $1 \le j \le n$ equals

$$z[\alpha_j \gamma_j 2^{\nu-10}] - [\alpha_j \psi_j 2^\nu] - m_j [2\pi \alpha_j 2^\nu]$$

and the $(n+2)$-nd coordinate is z, for some integers $z, m_1, m_2, \ldots, m_n$. To minimize the length of $\underline{w}$, these all have to be small which means that all of the

$$z\alpha_j \gamma_j 2^{\nu-10} - \alpha_j \psi_j 2^\nu - m_j 2\pi \alpha_j 2^\nu$$

have to be small, so all of the

$$\beta_j := \alpha_j(\gamma_j y - \psi_j - 2\pi m_j)$$

have to be very small, where $y = z/1024$. In practice, the vectors $\underline{w}$ produced by the L^3 algorithm did indeed have this desired property.

The reason for the presence of the α_j's in the basis is that one wishes to make the sum

$$\sum_{j=1}^{n} \alpha_j^2 \cos(\gamma_j y - \psi_j - 2\pi m_j)$$

large. Now, if all of the $\gamma_j y - \psi_j - 2\pi m_j$ are small, this sum approximately equals

$$\sum_{j=1}^{n} \alpha_j^2 - \frac{1}{2} \sum_{j=1}^{n} \alpha_j^2 (\gamma_j y - \psi_j - 2\pi m_j)^2,$$

and one wishes to have the second sum above small. This, however, corresponds to minimizing the euclidean norm of the vector $(\beta_1, \beta_2, \ldots, \beta_n)$, which is what the L^3 algorithm attempts to do.

In order to obtain values of y for which the chosen zeros contribute *negative* amounts, so that $h(y, T)$ will hopefully be negative, Odlyzko and Te Riele used similar lattices, but with ψ_j replaced by $\psi_j + \pi$.

The above discussion explains why Odlyzko and Te Riele chose the lattice L as they did. This choice was made on heuristic grounds and did not guarantee that they would disprove the Mertens conjecture, but in the end they did.

If the first 400, say, γ_i's are ordered and numbered $\gamma_1^*, \gamma_2^*, \dots$ such that the quantities $|\rho_j^* \zeta'(\rho_j^*)|^{-1}$ for $\rho_j^* = \frac{1}{2} + i\gamma_j^*$ are monotonically decreasing and maximal, then the sum $2\sum_{j=1}^{n} |\rho_j^* \zeta'(\rho_j^*)|^{-1}$ exceeds 1 for $n \geq 54$ and equals $1.0787\dots$ for $n = 70$. This suggests that a disproof of the Mertens conjecture could be obtained if the L^3 algorithm could find a value of y that made each of the quantities η_j in (4.13) quite small for $n = 70$. Moreover, the number T which determines the length of the finite sum in (3.9), should be so large that the cosine-values in that sum which come from the chosen 70 zeros (and so are close to 1) should have a weight factor which is also close to 1. Odlyzko and Te Riele chose $T = \gamma_{2,000} = 2515.286482\dots$ and the accuracy of the first $2,000$ γ's to be at least 100 decimal digits. Details of these computations can be found in Section 4.2 of [23].

The L^3 algorithm as decribed above was programmed by Odlyzko on a Cray-1 super computer at AT&T Bell Laboratories in Murray Hill using Brent's MP Package [5]. This was applied next with various subsets V_n, for $n = 20(5)70$, of the first 400 zeros $\rho = \frac{1}{2} + i\gamma$ of the zeta function, ordered such that the corresponding n values of $|\rho\zeta'(\rho)|^{-1}$ are largest. For $n = 70$ this yielded two values of y for which $|h(y, \gamma_{2000})| > 1$ (h defined in (3.9)), one on the positive side $(1.061\dots)$ and one on the negative side $(-1.009\dots)$, disproving the Mertens conjecture: see Table 2. The number of decimal digits of y is given in parentheses after the value of y.

In 2006, Kotnik and Te Riele [14] have improved these results by working with $10,000$ zeros (with $T = \gamma_{10,000} = 9877.782654\dots$), computed with an accuray of 250 decimal digits. Their best results are given in Table 2. Figure 2 compares the typical behaviour of $M(e^y)/e^{y/2}$ (top) with the behaviour of $h(y, \gamma_{10,000})$ around the 1.218-spike (middle) and around the -1.229-spike (bottom). Notice the four large negative spikes to the left and to the right of the champion *positive* spike, and the four large positive spikes to the left and to the right of the champion *negative* spike. This suggests that a very large spike in one direction may be accompanied by several large spikes in the opposite direction. Notice also that the bottom graph, when inverted with respect to the horizontal axis, very much resembles the middle graph. This is explained by the fact that the two functions plotted there are sums of cosines of which the first 98 main terms are aligned very well in $\Delta y = 0$.

In 2006, Damien Stehle [34] communicated to the author a new record, namely a y-value for which $h(y, \gamma_{10,000}) = 1.485\dots$ (Table 2).

In 2010, Andreas Decker published his bachelor thesis, written under supervision of Jens Franke [8]. Decker used $20,000$ zeros of $\zeta(s)$ (with $\gamma_{20,000} = 18046.464296\dots$) and found new records on both sides: a value of y for which $h(y, \gamma_{20,000}) = 1.568\dots$ and a value of y for which $h(y, \gamma_{20,000}) = -1.562\dots$ (Table 2).

Very recently, Best and Trudgian [4] have given a remarkable alternative disproof of the Mertens conjecture, based on the work of Bateman et al. [3] and Grosswald [10] on linear relations of zeros of the Rieman zeta function. Moreover, with this approach, they were able to beat the best result concerning the disproof of the

T	$y \quad (\log_{10}\lvert y\rvert)$	$h(y,T)$
$\gamma_{2,000}$	−1404528968 0592998046 7903616303 9978112740 0591999789 7380399659\ 60762.521505 (64.1)	1.061545[23]
$\gamma_{2,000}$	3209702577 2922655869 7400001862 1130709979 7144540349 0626828053\ 21651.697419 (64.5)	−1.009749[23]
$\gamma_{10,000}$	−2330292715 1345312150 1401819967 7234010204 4567850916 6815575186\ 7434340369 2402308908 9332617069 0292339582 730162362.807965 (108.3)	1.218429[14]
$\gamma_{10,000}$	−1608734975 4400091981 7483964016 5505468521 2472228477 8177553930\ 3027535069 0810795719 4829643360 2695144210 2295321275 4000.679958 (114.2)	−1.229385[14]
$\gamma_{10,000}$	2931339489 7319888309 9543329140 9361767361 8491594610 0664427517\ 9801933427 0711381134 8106612811 1768567779 8144778689 2682777627\ 2928928347 2370395900 2979033056 0382283806 8270937917 5098799198\ 3948460287 6306897440 9595135948 9807747029 0296785610 2481000966\ 6150263302 2376233781 4346582246 3794426753 4170493490 3463251318\ 0871363608 0573025076 6808650216 8366510503 2906325007 1604913393\ 9804383123 3331689638 1399844414 3002376959 6010312472 1051207456\ 8338383040 3640040744 8715816999 0686530085 0089551031 9531190724\ 0602927838 5444586484 0678226792 8376289863 7980444020 7288912145\ 8240220448 0614745062 9114470721 0124424515 6762328790 5471286374\ 2762402195 9139461238 1426938229 4987022063 4933770987 6945335222\ 1719286404 1506834912 1658.864934 (683.4)	1.485852[34]
$\gamma_{20,000}$	6347293713 6347496565 5368805841 8132885340 9037359019 7157964968\ 9931914597 5323363442 9142637819 6779117628 8789830571 7927690834\ 2212685058 0347378498 3930901055 8601949542 3756802549 0914939321\ 4052137175 3043592857 6436799853 1062471458 1195424989 0292441997\ 4901321745 4840828916 9068242478 1204644487 4333910170 1557984066\ 0595081435 7757960661 1981722324 8210040788 2292832465 8786023935\ 8106692592 7576281309 6227605527 1130465015 5741242337 3504693485\ 0360016262 7557977265 7809306424 8998683542 0656882310 7598056766\ 7134854271 7951193462 1302516582 8333273288 5379326487 9278506440\ 9086107115 9042507554 4529739014 2397858772 7697329654 4606344529\ 2009770381 9434871956 9199208707 9845359628 8865491592 9442166344\ 5575308903 5108362268 6479397863 0968887026 1304497995 4268460862\ 3733282368 0280358133 5472375399 9372354687 0305456153 6337914775\ 9718436385 8681034040 4422246122 4925099723 5227716089 7566331253\ 8688715523 4666994240 5837733041 6671736030 8254014591 4733357658\ 8036402750 3024078433 144.87079375 (922.8)	1.568901[8]
$\gamma_{20,000}$	2858942972 7801091032 7392053975 2167905852 6820804801 7494149919\ 5872390703 6844037914 8505670723 9320510605 4368012430 2467297475\ 6128012644 7849711726 6169466025 7222896793 0242677463 7132216524\ 1519275043 2238659718 7862061737 2558288571 9839057394 2685436112\ 8521099684 8202248858 7850209744 8444386259 4464633581 1513518168\ 0880628650 2934940525 4133647088 4011592884 3604618012 2802401214\ 2660445822 1185142176 9487216526 8710653067 3298985825 2203071132\ 8083314717 5521500146 5251574801 6171926835 9592640879 0124457208\ 9422942167 3597448686 9276693632 8417245334 7087064100 2615760286\ 7121841232 0087094510 1338074673 8161490744 2186027391 3773269832\ 9790759745 7126319385 7301485519 9185571701 0476086213 4837813819\ 5744335608 6576766815 0299080890 4021957935 1967748583 3963437683\ 3866880984 4489974665 2447772528 4681808171 7538482822 5591326801\ 3699693556 9026995542 3495017107 5637836701 1883726226 0077489124\ 5603770055 6427816823 5018761006 5373200075 6958855664 0785605710\ 2208796675 1791993714 972.6826171875 (922.5)	−1.562908[8]

Table 2: Record values of y for which $\lvert h(y,T)\rvert > 1$

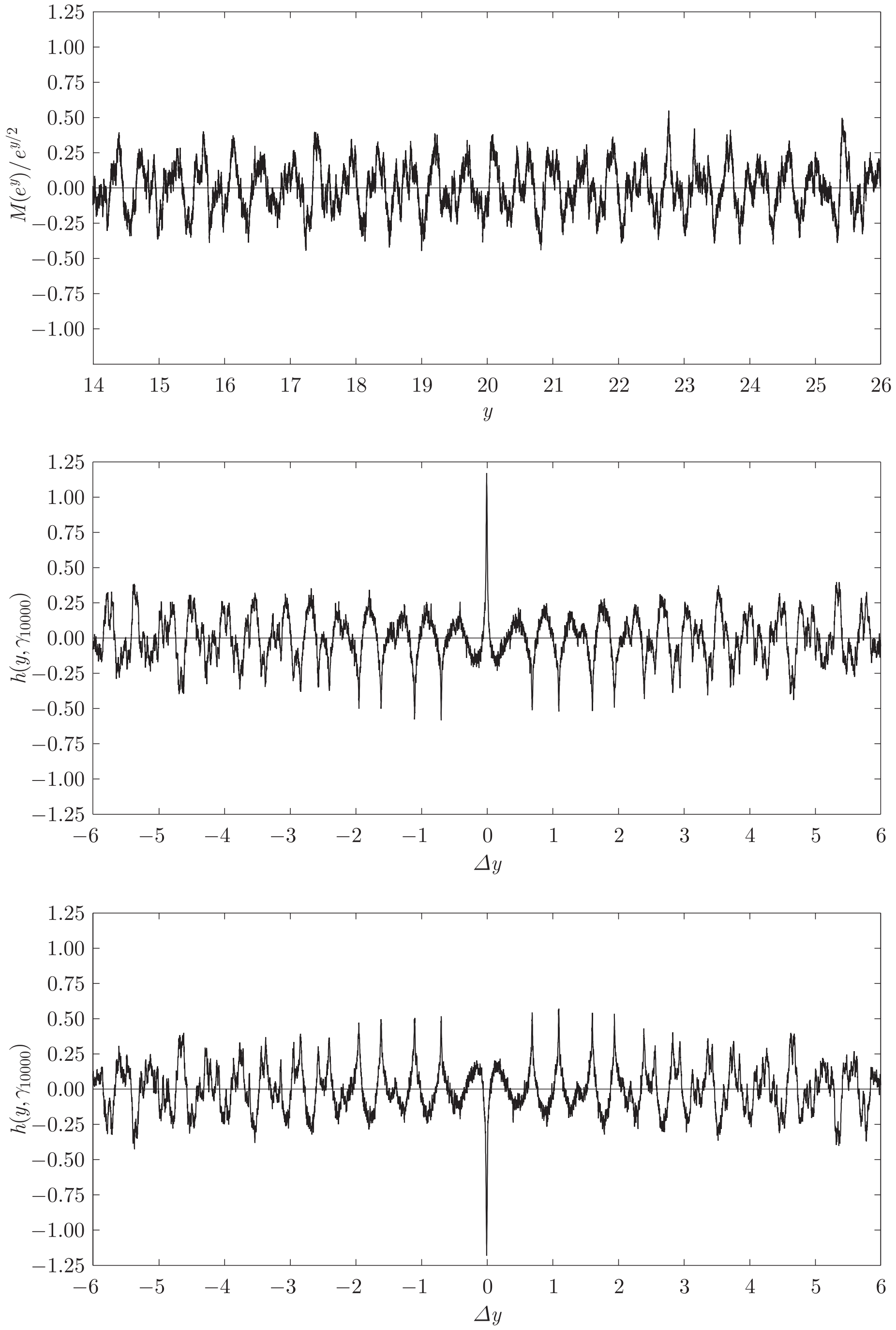


FIGURE 2

Figure 1: Typical behaviour of $M(e^y)/e^{y/2}$ compared with atypical behaviour of $h(y,T)$ near the local extremal value

Mertens conjecture, namely, they showed that $\limsup_{x\to\infty} M(x)x^{-1/2} \geq 1.6383$, and $\liminf_{x\to\infty} M(x)x^{-1/2} \leq -1.6383$.

5 Upper bounds on the smallest number for which the Mertens conjecture is false

The computations concerning the Mertens conjecture described above are not effective in the sense that they do not give an upper bound on x for which $|M(x)x^{-1/2}| > 1$. In 1987, Pintz [24] gave an effective disproof of the Mertens conjecture with the following [2]

Theorem 5.1. *Let*

$$h_P(y,T) := 2 \sum_{0<\gamma<T} e^{-1.5\times 10^{-6}\gamma^2} \frac{\cos(\gamma y - \psi_\gamma)}{|\rho\zeta'(\rho)|}. \tag{5.16}$$

If there exists a $y \in [e^7, e^{5\times 10^4}]$ *with* $|h_P(y,T)| > 1 + e^{-40}$ *for* $T = 1.4 \times 10^4$*, then* $|M(x)|x^{-1/2} > 1$ *for some* $x < e^{y+\sqrt{y}}$*.*

For the number $y = y_0 = 3.2097\cdots \times 10^{64}$ given as second entry in Table 2, Te Riele computed $h_P(y_0, \gamma_{2,000}) = -1.00223\ldots$, which implies, by Theorem 5.1, that the Mertens conjecture is false for some $x < \exp(3.21 \times 10^{64})$.

In 2006, Kotnik and Te Riele [14] have improved this result by finding the value

$$y = y_0 = 1\,5853191167\,3595000428\,9014722171\,6268116204.984802$$

for which $h_P(y_0, \gamma_{10,000}) = -1.00819\ldots$. This implies, by Theorem 5.1, that the Mertens conjecture is false for some $x < \exp(1.59 \times 10^{40})$.

In 2013, Saouter and Te Riele [29] further lowered the upper bound for the minimal counterexample to the Mertens conjecture by improving the original constructive disproof of Pintz. This showed that the Mertens conjecture is false for some $x < \exp(1.004 \times 10^{33})$.

It is known that, if the Riemann hypothesis is true and if all non-trivial zeros of the Riemann zeta-function are simple, the function $q(x) := M(x)/\sqrt{x}$ can be approximated by a series of trigonometric functions of $\log x$. Kotnik and Van de Lune [16] have carried out numerical experiments with this series and, based on extrema found of this series, they conjecture that $q(x) = \Omega_\pm(\sqrt{\log\log\log x})$. In addition, on heuristic grounds they suggest that the Mertens conjecture could be false "not too far from" $x \approx \exp(5.3 \times 10^{23})$.

Acknowledgments

The author is grateful to Jan van de Lune and Andrew Odlyzko for remarks improving the presentation of this paper.

[2]The "P" in $h_P(y,T)$ refers to Pintz.

References

[1] D.H. Bailey and H.R.P. Ferguson, Numerical results on relations between fundamental constants using a new algorithm, Math. Comp. 53 (1989), 649–656.

[2] B. Bailloud and H. Bourget, Correspondance d'Hermite et de Stieltjes (with an appendix entitled: Lettres de Stieltjes à M. Mittag-Leffler sur la fonction $\zeta(s)$ de Riemann), Gauthiers-Villars, Paris, 1905.

[3] P.T. Bateman, J.W. Brown, R.S. Hall, K.E. Kloss, and R.M. Stemler, Linear relations connecting the imaginary parts of the zeros of the zeta function, 11–19 in: A.O.L. Atkin and B.J. Birch (eds.), Computers in Number Theory, New York 1971.

[4] D.G. Best and T.S. Trudgian, Linear relations of zeroes of the zeta-function, Math. Comp., to appear (arXiv:1209.3843v3 [math.NT], 10 Jul. 2014).

[5] R.P. Brent, A Fortran muliple-precision arithmetic package, ACM Trans. Math. Software 4 (1978), 57–70.

[6] Jan Büthe, A practical analytic method for calculating $\pi(x)$, arXiv:1410.7008v1 [math.NT], 26 Oct. 2014.

[7] H. Cohen and F. Dress, Calcul numérique de $M(x)$, 1–13 in: Rapport de lATP A12311 Informatique 1975, CNRS, 1979.

[8] A. Decker, Die Widerlegung der Mertens-Vermutung, Math. Institut, Math. Naturw. Fakultät der Rheinischen Friedrich-Willems-Universität Bonn, Oct. 15, 2010.

[9] F. Dress, Fonction sommatoire de la fonction de Möbius, 1. Majorations expérimentales, Exp. Math. 2 (1993), 89–98.

[10] E. Grosswald, Oscillation theorems, 141–168 in Lecture Notes in Math. 251.

[11] G.H. Hardy and E.M. Wright, An Introduction to the Theory of Numbers, Oxford at the Clarendon Press, 4th Edition, 1975.

[12] A.E. Ingham, On two conjectures in the theory of numbers, Amer. J. Math. 64 (1942), 313–319.

[13] W. Jurkat and A. Peyerimhoff, A constructive approach to Kronecker approximation and its application to the Mertens conjecture, J. reine angew. Math. 286/287 (1976), 332–340.

[14] Tadej Kotnik and Herman te Riele, The Mertens conjecture revisited, 156–167 in: F. Hess, S. Pauli, and M. Pohst (eds.), Algorithmic Number Theory Symposium 2006, Lecture Notes in Computer Science 4076, Springer, 2006.

[15] T. Kotnik and J. van de Lune, Further systematic computations on the summatory function of the Möbius function, Report MAS-R0313, CWI Amsterdam, November 2003.

[16] Tadej Kotnik and Jan van de Lune, On the order of the Mertens function, Exp. Math. 13 (2004), 473–481.

[17] J.C. Lagarias and A.M. Odlyzko, Computing $\pi(x)$: an analytic method, J. Algorithms 8 (1987), 173–191.

[18] R.S. Lehman, On Liouville's function, Math. Comp. 14 (1960), 311–320.

[19] A.K. Lenstra, H.W. Lenstra, Jr. and L. Lovász, Factoring polynomials with rational coefficients, Math. Ann. 261 (1982), 515–534.

[20] W.M. Lioen and J. van de Lune, Systematic computations on Mertens' conjecture and Dirichlet's divisor problem by vectorized sieving, 421–432 in: K. Apt, A. Schrijver and N.M. Temme (eds.), From universal morphisms to megabytes, A Baayen Space Odyssey, CWI Amsterdam, 1994.

[21] F. Mertens, Über eine zahlentheoretische Funktion, Sitzungsberichte Akad. Wiss. Wien IIa 106 (1897), 761–830.

[22] G. Neubauer, Eine empirische Untersuchung zur Mertens'schen Funktion, Numer. Math. 5 (1963), 1–13.

[23] A.M. Odlyzko and H.J.J. te Riele, Disproof of the Mertens conjecture, J. reine angew. Math. 357 (1985), 138–160.

[24] J. Pintz, An effective disproof of the Mertens conjecture, Astérisque, 147-148 (1987), 325–333.

[25] David Platt, Computing $\pi(x)$ analytically, Math. Comp. 84 (2015), 1521–1535.

[26] H.J.J. te Riele, Computations concerning the conjecture of Mertens, J. reine angew. Math. 311/312 (1979), 356–360.

[27] Herman J.J. te Riele, On the history of the function $M(x)\sqrt{x}$ since Stieltjes, 69–79 in: Thomas J. Stieltjes, Œvres Complètes (Collected Papers), Vol. 1, Paper # XLIV, Berlin, Springer, 1993.

[28] Herman J.J. te Riele, Some historical and other notes about the Mertens conjecture and its recent disproof, Nieuw Archief voor Wiskunde (4), 3 (1985), 237–243.

[29] Yannick Saouter and Herman te Riele, Improved results on the Mertens conjecture, Math. Comp. 83 (2014), 421–433.

[30] J. Schröder, Zur Berechnung von Teilsummen der summatorischen Funktion der Möbius'schen Funktion $\mu(x)$, Norsk Mat. Tidsskrift 14 (1932), 45–53.

[31] J. Schröder, Beiträge zur Darstellung der Möbius'schen Funktion, Jahresberichte der Deutschen Math. Ver. 42 (1932), 223–237.

[32] J. Schröder, Zur Auswertung der zur Möbius'schen Funktion gehörenden summatorischen Funktion, Mitt. Math. Ges. Hamburg 7, no. 3 (1933), 148–163.

[33] R. Spira, Zeros of sections of the zeta function. II, Math. Comp. 22 (1966), 163–173.

[34] Damien Stehle, Private communication, August, 2006.

[35] R.D. von Sterneck, Empirische Untersuchung über den Verlauf der zahlentheoretischen Funktion $\sigma(n) = \sum_{x=1}^{n} \mu(x)$ im Intervalle von 0 bis 150000, Sitzungsberichte Akad. Wiss. Wien IIa 106(1897), 835–1024.

[36] R.D. von Sterneck, Empirische Untersuchung über den Verlauf der zahlentheoretischen Funktion $\sigma(n) = \sum_{x=1}^{n} \mu(x)$ im Intervalle von 150000 bis 500000, Sitzungsberichte Akad. Wiss. Wien IIa 110(1901), 1053–1102.

[37] R.D. von Sterneck, Die zahlentheoretischen Funktion $\sigma(n)$ bis zur Grenze 500000, Sitzungsberichte Akad. Wiss. Wien IIa 121(1912), 1083–1096.

[38] R.D. von Sterneck, Neue empirischen Daten über die zahlentheoretischen Funktion $\sigma(n)$, in: Proc 5th International Congress of Mathematicians, vol. 1, Cambridge University Press, 1913, 341–343.

[39] E.C. Titchmarsh, The theory of the Riemann zeta-function, Oxford, 1951.

[40] M. Yorinaga, Numerical investigation of sums of the Möbius function, Math. J. Okayama Univ., 21 (1979), 41–47.