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Factorization formulas for $2D$ critical percolation, revisited

R.P. Conijn

VU University Amsterdam, De Boelelaan 1081, Amsterdam, The Netherlands

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Abstract

We consider critical site percolation on the triangular lattice in the upper half-plane. Let u_1, u_2 be two sites on the boundary and w a site in the interior. It was predicted by Simmons et al. (2007) that the ratio $\mathbb{P}(nu_1 \leftrightarrow nu_2 \leftrightarrow nw)^2 / \mathbb{P}(nu_1 \leftrightarrow nu_2) \cdot \mathbb{P}(nu_1 \leftrightarrow nw) \cdot \mathbb{P}(nu_2 \leftrightarrow nw)$ converges to K_F as $n \to \infty$, where $x \leftrightarrow y$ denotes that x and y are in the same cluster, and K_F is a constant. Beliaev and Izyurov (2012) proved an analog of this in the scaling limit. We prove, using their result and a generalized coupling argument, the earlier mentioned prediction. Furthermore we prove a factorization formula for $\mathbb{P}(nu_2 \leftrightarrow [nu_1, nu_1 + s];\ nw \leftrightarrow [nu_1, nu_1 + s])$, where $s > 0$.

MSC: 60K35; 82B43

Keywords: Critical percolation; Scaling limit

1. Introduction and main results

We consider critical site percolation on the triangular lattice. See [6] for a general introduction and [13,14] for more recent progress in two dimensional percolation. A lot of attention has been given to crossing probabilities and critical exponents, which are believed to be universal. In particular it is believed that in the continuum limit of many two dimensional critical percolation models, crossing probabilities are conformally invariant. However this has only been proved

E-mail address: R.P.Conijn@vu.nl.

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for site percolation on the triangular lattice by Smirnov [12]. Another interesting question is whether it is possible to examine the higher order correlation functions. These are the functions $\mathbb{E}[X_{v_1}X_{v_2}\cdots X_{v_n}]$, where v_i is a vertex and $X_{v_i} = \mathbf{1}\{0 \leftrightarrow v_i\}$ is the indicator function of the event that v_i is in the open cluster of the origin. A possible approach to compute these correlation functions might be via factorization formulas.

To state our main results we consider the hexagonal lattice, where every centre of a hexagon is a site of the triangular lattice $\mathbb{T}$ in the closure of the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$. In this lattice two neighbouring sites $x, y \in \mathbb{T}$ have $|x - y| = 1$. By $\mathbb{P}_\eta$ we denote the probability measure of critical percolation on $\eta\mathbb{T}$, for $\eta > 0$. Let $\eta > 0$ and let the random set $Q \subset \overline{\mathbb{H}}$ be the union of all hexagons for which the centre is open. The points $u, v \in \overline{\mathbb{H}}$ are connected if u, v are in the same connected component of Q. We denote this by $u \leftrightarrow v$. Let, for $u \in \eta\mathbb{T}$, $\mathcal{C}(u)$ denote the open cluster containing u. Let, for $A \subset \overline{\mathbb{H}}$,

$$\mathcal{C}(A) := \bigcup_{u \in A\cap\eta\mathbb{T}} \mathcal{C}(u).$$

Further we will denote the hypergeometric function by ${}_2F_1(a, b; c; d)$ (see for example [1]). We denote by $\mathbb{S} := \{z \in \mathbb{C} : \Im(z) \in (0, 1), \Re(z) > 0\}$ the semi-infinite strip.

Our first main result is a factorization formula for the probability that three given vertices are in the same cluster, where two of the vertices are on the boundary of the half-plane.

Theorem 1. *Let $u_1, u_2 \in \mathbb{R}$ and $w \in \mathbb{H}$ and $u_1 \neq u_2$, then*

$$\lim_{\eta\to 0} \frac{\mathbb{P}_\eta(u_1 \leftrightarrow u_2 \leftrightarrow w)^2}{\mathbb{P}_\eta(u_1 \leftrightarrow u_2)\mathbb{P}_\eta(u_1 \leftrightarrow w)\mathbb{P}_\eta(u_2 \leftrightarrow w)} = K_F, \tag{1.1}$$

where

$$K_F = \frac{2^7\pi^5}{3^{3/2}\Gamma(1/3)^9}.$$

This factorization formula was heuristically derived, using Conformal Field Theory arguments, by Simmons, Kleban and Ziff in [10]. Using the convergence of percolation exploration interfaces to SLE_6 (see e.g. [9,12]), a mathematical rigorous proof of an analog of this formula in the continuum scaling limit was given by Beliaev and Izyurov in [3]. See Theorem 3 for their result. That result is the starting point in the proof of Theorem 1. To obtain Theorem 1 from it we state and prove a quite general and robust form of a coupling result for one-arm like events (see Proposition 10 in Section 3.1).

Our second main result involves the limiting behaviour of $\mathbb{P}(\{u_2, w\} \subset \mathcal{C}([u_1, u_1+s]))$, where u_1, u_2 are on the boundary of the half-plane and w is in the half-plane. We have the following theorem.

Theorem 2. *Let $u_1 \in \mathbb{R}$, $w \in \mathbb{H}$, $s > 0$ and $u_2 > u_1 + s$, then*

$$\lim_{\eta\to 0} \frac{\mathbb{P}_\eta(\{u_2, w\} \subset \mathcal{C}([u_1, u_1 + s]))}{\mathbb{P}_\eta(w \in \mathcal{C}([u_1, u_1 + s]))\,\mathbb{P}_\eta(u_2 \in \mathcal{C}([u_1, u_1 + s]))} = \psi(u_1, s, u_2, w), \tag{1.2}$$

where ψ is the function

$$\psi(u_1, s, u_2, w) = e^{\pi x/3} \cdot \frac{{}_2F_1\left(-\frac{1}{2}, -\frac{1}{3}; \frac{7}{6}; e^{-2\pi x}\right)}{{}_2F_1\left(-\frac{1}{2}, -\frac{1}{3}; \frac{7}{6}; 1\right)},$$

with $x = \Re(\Psi_{u_1,s,u_2}(w))$ *where* Ψ_{u_1,s,u_2} *is the conformal map that transforms* $\{\mathbb{H}, u_1, u_1+s, u_2\}$ *to* $\{\mathbb{S}, \mathbf{i}, 0, \infty\}$.

Simmons, Ziff and Kleban studied in [11] the probability in the numerator in (1.2). They used Conformal Field Theory arguments to find several predictions for formulas of the probabilities in (1.2). Theorem 2 is a discrete analog of one of their predictions (Eq. (29) in Section III B of [11]).

Our interest in these factorization formulas came from the paper [3] by Beliaev and Izyurov. They rigorously proved an analog of the formula (1.2) above in the scaling limit, but with the probability $\mathbb{P}(w \in \mathcal{C}([u_1, u_1 + s]))$ replaced by $s_3^{5/48}$, see Theorem 4. However their theorem involves probabilities where the cluster does not necessarily touch w, but comes within a certain distance from it. More precisely, their formula is about the limits where first the mesh size, and secondly the above mentioned distance tends to zero.

Remark. We believe that our coupling argument, Proposition 10, is more generally applicable. For example Simmons, Ziff and Kleban also predicted in [11] a factorization formula for the probability $\mathbb{P}_\eta(u_2 \leftrightarrow w \leftrightarrow [u_1, u_1 + s])$. We hope that as soon as an analog of this result in the scaling limit has been proved, Proposition 10 can be used to prove this factorization formula in a discrete setting. More recently Delfino and Viti heuristically derived in [4] (see also [15]) a factorization formula for the probability $\mathbb{P}(x \leftrightarrow y \leftrightarrow w)$, where all three points are in the interior of the half-plane. We also believe that Proposition 10 might be an ingredient for a rigorous proof of a discrete analog of this factorization formula, again after the scaling limit analog has been proved.

The rest of the paper is organized as follows. In Section 2 we introduce some notation and sum up some preliminary results, which are crucial for our proofs. In Section 3.1 we state and prove a quite general and abstract ratio limit result, Proposition 10, which is based on a coupling argument. This proposition forms a key ingredient for the proofs of both main theorems. In Sections 3.2 and 3.3 we give the proofs of our main results.

2. Notation and preliminaries

We begin with some notation. Let $\Omega^\eta := \{0, 1\}^{\eta\mathbb{T}}$. Elements of Ω^η will typically be denoted by ω, ν and called *configurations*. We call a vertex $v \in \eta\mathbb{T}$ *open* if $\omega_v = 1$, otherwise we say that v is *closed*. For two configurations $\omega, \nu \in \Omega^\eta$ we write $\omega \le \nu$ if and only if $\omega_v \le \nu_v$ for all $v \in \eta\mathbb{T}$. Let $P \subset \mathbb{H}$, we write $\omega_P \in \{0, 1\}^{\eta\mathbb{T}\cap P}$ for the restriction of ω to the vertices which are contained in P. For two disjoint sets $P, Q \subset \mathbb{H}$, and configurations ω_P, ω_Q we define $\omega_P \times \omega_Q$ to be the configuration $\tilde{\omega}_{P\cup Q} \in \{0, 1\}^{\eta\mathbb{T}\cap(P\cup Q)}$ such that $\tilde{\omega}_P = \omega_P$ and $\tilde{\omega}_Q = \omega_Q$. Let $V \subset \Omega^\eta$ be an event and $A \subset \mathbb{H}$. We define the event

$$V_A := \{\omega \mid \exists\, \tilde{\omega}_{\mathbb{H}\setminus A} : \omega_A \times \tilde{\omega}_{\mathbb{H}\setminus A} \in V\}. \tag{2.1}$$

Further, with some abuse of notation, for $A \subset \mathbb{H}, \omega_A \in \{0, 1\}^{A\cap\eta\mathbb{T}}$ and $V \subset \Omega^\eta$ we write $\mathbb{P}_\eta(V \mid \omega_A)$ for the conditional probability of V given that the configuration on A equals ω_A. Similarly we write $\{\omega_A\}$ for the event that the configuration on A equals ω_A.

For $z = z_1 + z_2\mathbf{i} \in \mathbb{H}$ and $a > 0$, we write $B_a(z)$ for the intersection of the half-plane with the $2a \times 2a$-box centred at z. We denote annuli by $A(z; a, b) := B_b(z)\setminus B_a(z)$. A *circuit* in an annulus $A(z; a, b)$ is a sequence of neighbouring vertices in $\eta\mathbb{T}$, such that every vertex appears at most once in the sequence, the last vertex is a neighbour of the first and it surrounds $B_a(z)$. We

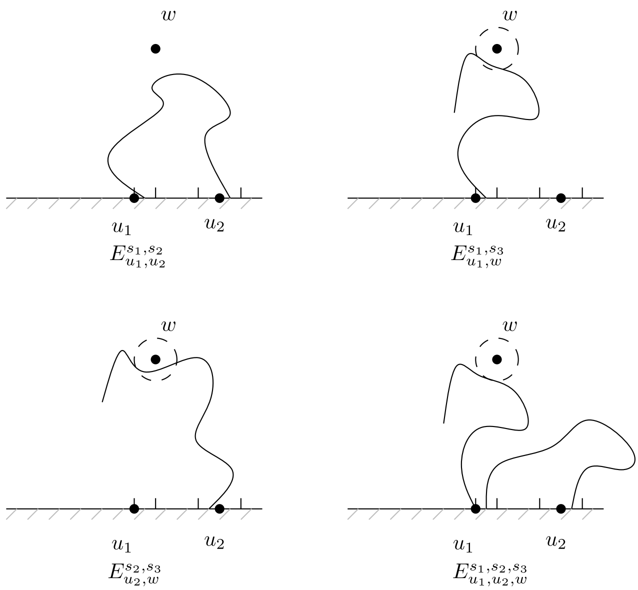

Fig. 1. The events $E^{s_1,s_2}_{u_1,u_2}$, $E^{s_1,s_3}_{u_1,w}$, $E^{s_2,s_3}_{u_2,w}$ and $E^{s_1,s_2,s_3}_{u_1,u_2,w}$. Note that the clusters in $E^{s_1,s_2,s_3}_{u_1,u_2,w}$ might be disjoint.

will often encounter annuli which intersect the boundary of $\mathbb{H}$, in that case we will also consider *semi-circuits*. A semi-circuit in an annulus $A(z; a, b)$ is a sequence of neighbouring vertices such that every vertex appears at most once in the sequence, the first and the last vertex are both on the boundary $\partial\mathbb{H}$ and the semi-circuit 'surrounds' $B_a(z)$. In other words a semi-circuit is a path in $\mathbb{H}$ from the boundary of $\mathbb{H}$ to the boundary of $\mathbb{H}$ which disconnects $B_a(z)$ from infinity. A (semi-)circuit is called open if all its vertices are open. For a (semi-)circuit γ we denote by $int(\gamma)$ the bounded connected component of $\mathbb{H}\backslash\bar{\gamma}$ containing $B_a(z)$, where $\bar{\gamma}$ is the curve in the plane described by γ. Further $ext(\gamma)$ is the unbounded connected component of $\mathbb{H}\backslash\bar{\gamma}$.

Let $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ be the open ball of radius one. For $w \in \mathbb{H}$ and a closed connected set $A \subset \mathbb{H}$ we denote by $\rho(w, A)$ the conformal radius of the component of w in $\mathbb{H}\backslash A$ seen from w. It is defined as follows. If $w \notin A$, let V be the connected component of w in $\mathbb{H}\backslash A$. Let $\phi : V \to \mathbb{U}$ be the unique conformal map with $\phi(w) = 0$ and $\phi'(w) > 0$. Then we set $\rho(w, A) := 1/\phi'(w)$. Otherwise, if $w \in A$ we set $\rho(w, A) := 0$. We can compare the conformal radius with the euclidean distance from the point to the set, namely it follows from Koebe's 1/4-Theorem and Schwarz' Lemma that

$$\frac{1}{4}\rho(w, A) \le \min_{x\in A} |w - x| \le \rho(w, A), \tag{2.2}$$

(see e.g. [2]).

We introduce the following events, which all represent the existence of clusters which come close to certain vertices. See Fig. 1. For $u_1, u_2 \in \mathbb{R}$, $w \in \mathbb{H}$ and $s_1, s_2, s_3 > 0$,

$$\begin{aligned} E^{s_1,s_2}_{u_1,u_2} &:= \{\mathcal{C}([u_1, u_1 + s_1]) \cap [u_2 - s_2, u_2 + s_2] \neq \emptyset\}; \\ E^{s_1,s_3}_{u_1,w} &:= \{\rho(w, \mathcal{C}([u_1, u_1 + s_1])) < s_3\}; \\ E^{s_2,s_3}_{u_2,w} &:= \{\rho(w, \mathcal{C}([u_2 - s_2, u_2 + s_2])) < s_3\}; \\ E^{s_1,s_2,s_3}_{u_1,u_2,w} &:= E^{s_1,s_2}_{u_1,u_2} \cap E^{s_1,s_3}_{u_1,w}. \end{aligned} \tag{2.3}$$

Although all these events depend on η, we omit this from the notation. They represent the discrete versions of the events used by Beliaev and Izyurov in [3]. Note the difference between the events $E^{s_1,s_3}_{u_1,w}$ and $E^{s_2,s_3}_{u_2,w}$. This is to stay as close as possible to the events defined in that paper. As mentioned before Beliaev and Izyurov considered the limits, as $\eta \to 0$, of the probabilities of the events above. That is

$$f^{s_1,s_2}_{u_1,u_2} := \lim_{\eta\to 0} \mathbb{P}_\eta(E^{s_1,s_2}_{u_1,u_2});$$

$$f^{s_1,s_3}_{u_1,w} := \lim_{\eta\to 0} \mathbb{P}_\eta(E^{s_1,s_3}_{u_1,w});$$

$$f^{s_2,s_3}_{u_2,w} := \lim_{\eta\to 0} \mathbb{P}_\eta(E^{s_2,s_3}_{u_2,w});$$

$$f^{s_1,s_2,s_3}_{u_1,u_2,w} := \lim_{\eta\to 0} \mathbb{P}_\eta(E^{s_1,s_2,s_3}_{u_1,u_2,w}).$$

The existence of these limits follows from the results in [8,12]. Namely the existence of the first one (which is actually given by Cardy's formula) was proved by Smirnov in [12]. The second and third are described in the article on the one-arm exponent for critical $2D$ percolation [8], using the so called exploration path, started at, respectively u_1+s_1 and u_2+s_2. The fourth one can also be described in terms of exploration path. It is the intersection of the events: (1) the exploration path starting at u_1+s_1 swallows u_2-s_2 before it swallows u_1 or u_2+s_2 and (2) the exploration path, or union of nested exploration paths, comes s_3 close to w in conformal radius. See [8] for the definition of the exploration path and more details.

As Beliaev and Izyurov already mentioned in [3, Remark 4], the factorization formula they proved, Proposition 4.1 in their paper, implies the following theorem.

Theorem 3 (*Remark 4 in [3]*)**.** *Let u_1, u_2, w and K_F be as in Theorem 1. For every $\varepsilon, s_0 > 0$ there exist $s_1, s_2, s_3 < s_0$ such that*

$$\left| \frac{(f^{s_1,s_2,s_3}_{u_1,u_2,w})^2}{f^{s_1,s_2}_{u_1,u_2} \cdot f^{s_1,s_3}_{u_1,w} \cdot f^{s_2,s_3}_{u_2,w}} - K_F \right| < \varepsilon. \tag{2.4}$$

The following theorem is the main result in [3], and will be used in the proof of Theorem 2.

Theorem 4 (*Theorem 1.1 in [3]*)**.** *Let u_1, u_2, w, s be as in Theorem 2. One has*

$$\lim_{s_3\to 0}\lim_{s_2\to 0} s_3^{-5/48} \cdot \frac{f^{s,s_2,s_3}_{u_1,u_2,w}}{f^{s,s_2}_{u_1,u_2}} = K_1 |\Psi'_{u_1,s,u_2}(w)|^{5/48} G\left(\Re(\Psi_{u_1,s,u_2}(w)), \Im(\Psi_{u_1,s,u_2}(w))\right), \tag{2.5}$$

where Ψ_{u_1,s,u_2} is the conformal map that transforms $\{\mathbb{H}, u_1, u_1+s, u_2\}$ to $\{\mathbb{S}, \mathbf{i}, 0, \infty\}$ and

$$K_1 = \frac{18\pi^{5/48}}{5\pi \cdot 2^{5/48}} H(0)^{-1}$$

$$G(x, y) = e^{\pi x/3} H(x) \sinh(\pi x)^{-1/3} \left(\frac{\sinh(\pi x)^2 \sin(\pi y)^2}{\sinh(\pi x)^2 + \sin(\pi y)^2}\right)^{11/96}, \tag{2.6}$$

with

$$H(x) = {}_2F_1\left(-\frac{1}{2}, -\frac{1}{3}; \frac{7}{6}; e^{-2\pi x}\right). \tag{2.7}$$

The lemma below, proved by Beliaev and Izyurov, is an improvement of a result by Lawler, Schramm and Werner in [8].

Lemma 5 (*Lemma 2.2 in [3]*)**.** *Let u_1, w be as in Theorem 1 and let $s > 0$. One has*

$$\lim_{s_3 \to 0} s_3^{-5/48} \cdot f_{u_1,w}^{s,s_3} = K_2 |\phi'(w)|^{5/48} (\sin(\pi\omega/2))^{1/3}, \tag{2.8}$$

where ω is the harmonic measure of $(u_1, u_1 + s)$ seen from w; ϕ is a conformal map from $\mathbb{H}$ to the unit disc such that $\phi(w) = 0$, and

$$K_2 = \frac{18}{5\pi}. \tag{2.9}$$

We end this section with a lemma which is a simple generalization of the FKG inequality.

Lemma 6. *Let $A \subset \mathbb{H}$ and let B, E be increasing events. Let $\nu_A \in \{0, 1\}^{\eta\mathbb{T}\cap A}$. If B is completely determined by the vertices in $\mathbb{H}\backslash A$, that is $B = B_{\mathbb{H}\backslash A}$, then*

$$\mathbb{P}_\eta(B \cap E \cap \{\nu_A\}) \geq \mathbb{P}_\eta(B)\mathbb{P}_\eta(E \cap \{\nu_A\}).$$

Proof of Lemma 6. The proof of this lemma is straightforward and we omit it. □

3. Proofs of the main results

3.1. Coupling of one-arm like events

The proof of our first main result, Theorem 1, has two ingredients. The first is Theorem 3. The second ingredient for our proof is a coupling argument for one-arm like events which appeared in somewhat different forms in [7] and more recently in [5]. However our coupling result is developed in a more general framework of one-arm like events; see Definitions 7–9.

Our second main result, Theorem 2, also has this coupling argument as one of the main ingredients. The other main ingredients for the proof of Theorem 2 are Theorem 4 and Lemma 5.

The proof of our coupling argument is along the lines of the sketch in [5]. In that paper, among other very interesting results, a ratio limit theorem was proved. They proved that, for every $a > 0$

$$\lim_{\eta \to 0} \frac{\mathbb{P}_\eta(0 \leftrightarrow \mathbb{C}\backslash[-a, a]^2)}{\mathbb{P}_\eta(0 \leftrightarrow \mathbb{C}\backslash[-1, 1]^2)} = a^{-5/48},$$

see Section 5.1 in that paper. Here we show that their arguments can be modified, which makes them more generally applicable. In the arguments of [5], when a cluster comes s close to a point z it means that the cluster touches the boundary of $B_s(z)$. Hence the configuration in $B_s(z)$ is independent of the event that the cluster comes close. However, in our situation, when a cluster comes close to a vertex z it means in some occasions that the conformal radius is small and in other occasions it means that the cluster touches the interval $[z - s, z + s]$, as we saw in Section 2. Hence in our situation the configuration in $B_s(z)$ is not independent from the event that the cluster comes s close to z. This difference in measuring the distance of a cluster to a point makes the arguments more complicated. Our way to solve these complications is to grasp the essence which makes things work. This led us to the following formal definition of a class of events which intuitively describe the occurrence of a cluster coming within a distance s from z.

Definition 7. Let $s, C > 0$. Let $z \in \mathbb{H}$ and $V \subset \Omega^\eta$ be an increasing event. We say that V is an *(s, C)-one-arm like event around z* if, for every (semi-)circuit γ in $A(z; s, C)$,

$$V \begin{cases} \subset & \{B_s(z) \leftrightarrow \mathbb{H}\backslash B_C(z)\} \\ \supset & \{\gamma \textit{ open}\} \cap V_{ext(\gamma)} \cap V_{int(\gamma)} \end{cases} \tag{3.1}$$

and

$$\{I(z, s) \leftrightarrow \gamma\} \subset V_{int(\gamma)},$$

where $I(z, s)$ is the horizontal line segment $[z, z + s/8] \subset \overline{\mathbb{H}}$ and $V_{int(\gamma)}, V_{ext(\gamma)}$ as in (2.1).

For example, for every $x, s, C \in \mathbb{R}$ and $a \in [1/8, 1]$, the events $\{B_{as}(x\mathbf{i}) \leftrightarrow (x\mathbf{i} + 2C(1 + \mathbf{i}))\}$ and $\{I(x, s) \leftrightarrow \mathbb{H}\backslash B_{2C}(x)\}$ are (s, C)-one-arm like events around $x\mathbf{i}$, respectively x. In the proof of Theorem 1 we will see that also certain events concerning a small conformal radius from z to a certain cluster are (s, C)-one-arm like events.

Observe that the definition above implies that for every (semi-)circuit γ in $A(z; s, C)$,

$$V \cap \{\gamma \text{ open}\} = V_{ext(\gamma)} \cap V_{int(\gamma)} \cap \{\gamma \text{ open}\}, \tag{3.2}$$

where V is an (s, C)-one-arm like event around z.

If V is an (s, C)-one-arm like event around z, there is a certain open cluster which comes within a distance s from z. For any such event V we will also consider a related event where this cluster hits z. Intuitively a good candidate for such an event would be $V \cap \{z \leftrightarrow \mathbb{H}\backslash B_C(z)\}$, but this is not appropriate: under this event the cluster $\mathcal{C}(z)$ and the earlier mentioned cluster, could be disjoint. In other words, this event is too large. It turns out that the following definition is suitable for our purposes.

Definition 8. Let V be an (s, C)-one-arm like event around z. Let $V^\bullet$ be an increasing event. We call $V^\bullet$ a *point version of* V if, for every (semi-)circuit γ in $A(z; s, C)$,

$$V^\bullet \begin{cases} \subset & V \cap \{z \leftrightarrow \mathbb{H}\backslash B_C(z)\} \\ \supset & \{\gamma \text{ open}\} \cap V_{ext(\gamma)} \cap \{z \leftrightarrow \gamma\}. \end{cases} \tag{3.3}$$

For example, for every $x, s, C \in \mathbb{R}$ and $a \in [1/8, 1]$, the event $\{x\mathbf{i} \leftrightarrow (x\mathbf{i} + 2C(1 + \mathbf{i}))\}$ is a point version of $\{B_{as}(x\mathbf{i}) \leftrightarrow (x\mathbf{i} + 2C(1 + \mathbf{i}))\}$ and $\{x \leftrightarrow \mathbb{H}\backslash B_{2C}(x)\}$ is a point version of $\{I(x, s) \leftrightarrow \mathbb{H}\backslash B_{2C}(x)\}$. To state the coupling proposition we need one more definition.

Definition 9. Let $z \in \mathbb{H}$ and $s, C > 0$. Let V and W be (s, C)-one-arm like events around z. We say that V, W are *(s, C)-comparable around z* if the events $V_{B_C(z)}$ and $W_{B_C(z)}$ are equal.

It follows easily from this definition, that equality also holds for any subset of $B_C(z)$. In other words, let V, W be (s, C)-comparable around z, then $V_A = W_A$ for every $A \subset B_C(z)$.

Our coupling argument is contained in the following proposition.

Proposition 10. *Let $C > 0$ and $z \in \mathbb{H}$. There exist increasing functions $\varepsilon(s), m(s) : \mathbb{R}_+ \to (0, 1)$, with $\varepsilon(s) \to 0$ and $m(s) \to 0$ as $s \to 0$ such that the following holds. For all $s > 0$, for all $\eta < m(s)$ and for every pair $V, W \subset \Omega^\eta$ of (s, C)-comparable events around z and point versions $V^\bullet$ of V and $W^\bullet$ of W we have*

$$\left| \frac{\mathbb{P}_\eta(V^\bullet \mid V)}{\mathbb{P}_\eta(W^\bullet \mid W)} - 1 \right| < \varepsilon(s). \tag{3.4}$$

Before we give a proof of this proposition, we introduce some notation and state a lemma which is crucial in the proof of Proposition 10.

Let $C, s > 0$ and $z \in \mathbb{H}$. Let $l(i) := 4^{-i}C$. Let $N(s, C) = \lfloor \log_4(C/s) \rfloor - 2$ and let $P_i := \mathbb{H} \backslash B_{l(i)}(z)$. We define for every $i \in \{0, 1, 2, \dots, N(s, C)\}$ the annuli $AI_i := A(z; \frac{1}{4}l(i), \frac{1}{2}l(i))$, $AO_i := A(z; \frac{1}{2}l(i), l(i))$ and $A_i := AI_i \cup AO_i$. We denote by ΓI_i the outermost open (semi-)circuit in AI_i and by ΓO_i the innermost open (semi-)circuit in AO_i, if they exist. Otherwise, if there is no (semi-)circuit in AI_i (resp. AO_i) we set $\Gamma I_i = \emptyset$ (resp. $\Gamma O_i = \emptyset$). Let γ_I be a fixed (semi-)circuit in AI_i and γ_O be a fixed (semi-)circuit in AO_i. The following observation is quite standard. Conditioned on $\{\Gamma I_i = \gamma_I; \Gamma O_i = \gamma_O\}$, the configuration in $int(\gamma_I) \cup ext(\gamma_O)$ is a fresh independent copy of a percolation configuration.

Lemma 11. *There exists a universal constant $C_1 \in (0, 1)$ such that the following holds. Let $z \in \mathbb{H}, s, C > 0$, $i \leq N(s, C)$ and let γ_I be a deterministic (semi-)circuit. Let V be an (s, C)-one-arm like event around z. Then, for every $\nu \in V_{P_i}$ we have*

$$\mathbb{P}_\eta(\Gamma I_i = \gamma_I \mid V \cap \{\nu_{P_i}\}) \geq C_1 \, \mathbb{P}_\eta(\{\Gamma I_i = \gamma_I\} \cap \{\Gamma O_i \text{ exists}\} \cap \{\gamma_I \leftrightarrow \Gamma O_i\}). \tag{3.5}$$

Proof of Lemma 11. It is sufficient to prove that, for every (semi-)circuit γ_O,

$$\begin{aligned} &\mathbb{P}_\eta(\{\Gamma I_i = \gamma_I\} \cap \{\Gamma O_i = \gamma_O\} \cap \{\gamma_I \leftrightarrow \gamma_O\} \mid V \cap \{\nu_{P_i}\}) \\ &\quad \geq C_1 \, \mathbb{P}_\eta(\{\Gamma I_i = \gamma_I\} \cap \{\Gamma O_i = \gamma_O\} \cap \{\gamma_I \leftrightarrow \gamma_O\}). \end{aligned} \tag{3.6}$$

Namely (3.5) immediately follows from (3.6) after summing over the possible (semi-)circuits γ_O.

Let γ_O be an arbitrary (semi-)circuit and

$$D = \{\Gamma I_i = \gamma_I\} \cap \{\Gamma O_i = \gamma_O\} \cap \{\gamma_I \leftrightarrow \gamma_O\}.$$

Then the left hand side of (3.6) is equal to

$$\frac{\mathbb{P}_\eta(D \cap V \cap \{\nu_{P_i}\})}{\mathbb{P}_\eta(V \cap \{\nu_{P_i}\})}. \tag{3.7}$$

It follows from (3.2) and Definition 7 that

$$\begin{aligned} \mathbb{P}_\eta(D \cap V \cap \{\nu_{P_i}\}) &= \mathbb{P}_\eta(D \cap V_{ext(\gamma_O)} \cap V_{int(\gamma_O)} \cap \{\nu_{P_i}\}) \\ &\geq \mathbb{P}_\eta(D \cap V_{ext(\gamma_O)} \cap \{I(z, s) \leftrightarrow \gamma_I\} \cap \{\nu_{P_i}\}). \end{aligned}$$

The last probability is, by the observation about inner- and outermost (semi-)circuits, equal to

$$\mathbb{P}_\eta(D)\mathbb{P}_\eta(I(z, s) \leftrightarrow \gamma_I)\mathbb{P}_\eta(V_{ext(\gamma_O)} \cap \{\nu_{P_i}\}). \tag{3.8}$$

On the other hand the denominator in (3.7) is, again by Definition 7, less than or equal to

$$\begin{aligned} \mathbb{P}_\eta(V_{ext(\gamma_O)} \cap \{\nu_{P_i}\} \cap \{B_s(z) \leftrightarrow \gamma_I\}) &= \mathbb{P}_\eta(V_{ext(\gamma_O)} \cap \{\nu_{P_i}\})\mathbb{P}_\eta(B_s(z) \leftrightarrow \gamma_I) \\ &\leq \mathbb{P}_\eta(V_{ext(\gamma_O)} \cap \{\nu_{P_i}\}) \cdot \frac{1}{C_1}\mathbb{P}_\eta(I(z, s) \leftrightarrow \gamma_I), \end{aligned} \tag{3.9}$$

where the constant C_1 comes from standard RSW and FKG arguments. A combination of (3.7)–(3.9) gives (3.6). This finishes the proof of Lemma 11. □

Proof of Proposition 10. We will describe a coupling of the conditional distributions given V and given W, denoted by $\tilde{\mathbb{P}}$. More precisely we construct $\tilde{\mathbb{P}}$ such that, for $\nu, \omega \in \Omega^\eta$,

$$\tilde{\mathbb{P}}(\nu \times \Omega^\eta) = \mathbb{P}_\eta(\nu \mid V), \qquad \tilde{\mathbb{P}}(\Omega^\eta \times \omega) = \mathbb{P}_\eta(\omega \mid W). \tag{3.10}$$

Furthermore $\tilde{\mathbb{P}}$ will be such that the probability that the two distributions are successfully coupled (in a sense defined precisely below) goes to 1 as s tends to zero, uniformly in η. We will finish the proof by showing how this coupling can be used to prove the proposition.

Let us first describe the coupling procedure. First we draw, independently of each other, ν_{P_0} and ω_{P_0} according to, respectively $\mathbb{P}_\eta(\cdot \mid V)$ and $\mathbb{P}_\eta(\cdot \mid W)$. Next we draw, step by step, the random elements ν_{A_i}, ω_{A_i}, starting from $i = 0$.

Every step goes as follows. The outermost (semi-)circuits $\Gamma I_i(\nu)$, $\Gamma I_i(\omega)$ are drawn from the optimal coupling of $\mathbb{P}_\eta(\Gamma I_i(\nu) = \cdot \mid V; \nu_{P_i})$ and $\mathbb{P}_\eta(\Gamma I_i(\omega) = \cdot \mid W; \omega_{P_i})$. That is, the coupling is such that $\tilde{\mathbb{P}}(\Gamma I_i(\nu) = \Gamma I_i(\omega) \neq \emptyset \mid \nu_{P_i};\ \omega_{P_i})$ is as large as possible.

We say that this step of the coupling is successful if $\Gamma I_i(\nu) \neq \emptyset$ and $\Gamma I_i(\nu) = \Gamma I_i(\omega) =: \gamma$. In that case we can finish the coupling procedure as follows. First we draw $\nu_{ext(\Gamma I_i(\nu))\cap A_i}$ and $\omega_{ext(\Gamma I_i(\omega))\cap A_i}$ from the appropriate conditional probability measures, independently of each other. So $\nu_{ext(\Gamma I_i(\nu))\cap A_i}$ is drawn from the probability measure $\mathbb{P}_\eta(\cdot \mid \Gamma I_i(\nu) = \gamma; V; \nu_{P_i})$. Since V is an (s, C)-one-arm like event we have for every $\nu_{int(\gamma)} \in \{0, 1\}^{\eta\mathbb{T}\cap int(\gamma)}$

$$\begin{aligned}\mathbb{P}_\eta(\nu_{int(\gamma)} \mid \Gamma I_i(\nu) = \gamma; V; \nu_{ext(\gamma)}) &= \mathbb{P}_\eta(\nu_{int(\gamma)} \mid V_{int(\gamma)};\ V_{ext(\gamma)};\ \Gamma I_i(\nu) = \gamma;\ \nu_{ext(\gamma)}) \\ &= \mathbb{P}_\eta(\nu_{int(\gamma)} \mid V_{int(\gamma)}),\end{aligned}$$

where we used (3.2) in the first equality and independence of $\nu_{int(\gamma)}$ and $V_{int(\gamma)}$ from the rest in the second. The same holds for W. Now we use that V and W are (s, C)-comparable around z. As we saw immediately after Definition 9 this implies that $V_{int(\gamma)} = W_{int(\gamma)}$, hence the two conditional distributions of the interior of γ are equal. Thus we can draw $\nu_{int(\gamma)}$ according to $\mathbb{P}_\eta(\cdot \mid V_{int(\gamma)})$ and take $\omega_{int(\gamma)} := \nu_{int(\gamma)}$.

If this step of the coupling was not successful, let γ_ν and γ_ω be the outcomes of $\Gamma I_i(\nu)$ and $\Gamma I_i(\omega)$ respectively, we draw the random elements ν_{A_i}, ω_{A_i} according to $\mathbb{P}_\eta(\cdot \mid \Gamma I_i(\nu) = \gamma_\nu; V; \nu_{P_i})$ and $\mathbb{P}_\eta(\cdot \mid \Gamma I_i(\omega) = \gamma_\omega; W; \omega_{P_i})$ independently of each other and continue to the next step with $i + 1$.

If all steps, $i = 0, \ldots, N(s, C)$, of the coupling were not successful, we draw ν_{RM} and ω_{RM} according to the appropriate conditional probabilities, independently of each other, where

$$RM := B_{l(N(s,C)+1)}(z) \supset B_{2s}(z). \tag{3.11}$$

That this procedure defines a coupling for the measures in (3.10) follows from standard arguments.

Let S denote the event that the coupling is successful (i.e. that some step in the above described procedure is successful). The crucial property of this coupling is that

$$(\Omega^\eta \times W^\bullet) \cap S = (V^\bullet \times \Omega^\eta) \cap S, \tag{3.12}$$

which follows easily from Definition 8. To see that $\tilde{\mathbb{P}}(S) \to 1$ as $s \to 0$, note that it follows easily from Lemma 11 together with RSW, FKG arguments that there exists a constant $C_2 > 0$ such that for every i

$$\sum_{\gamma_I} \min_{\substack{E\in\{V,W\} \\ \omega_{P_i}\in\{0,1\}^{P_i}}} \left(\mathbb{P}_\eta(\Gamma I_i = \gamma_I \mid E;\ \omega_{P_i})\right) \geq C_2.$$

Hence, for every step in the procedure described above, the probability that the coupling is successful is at least C_2. Thus

$$\tilde{\mathbb{P}}(S) \geq 1 - (1 - C_2)^{N(s,C)+1} \tag{3.13}$$

if η is small enough.

Now we show how this coupling can be used to prove the proposition. First rewrite the quotient in (3.4)

$$\frac{\mathbb{P}_\eta(V^\bullet \mid V)}{\mathbb{P}_\eta(W^\bullet \mid W)} = \frac{\tilde{\mathbb{P}}((V^\bullet \times \Omega^\eta) \cap S) + \tilde{\mathbb{P}}(V^\bullet \times \Omega^\eta \mid S^c)\tilde{\mathbb{P}}(S^c)}{\tilde{\mathbb{P}}((\Omega^\eta \times W^\bullet) \cap S) + \tilde{\mathbb{P}}(\Omega^\eta \times W^\bullet \mid S^c)\tilde{\mathbb{P}}(S^c)}. \tag{3.14}$$

We claim that

$$\tilde{\mathbb{P}}(V^\bullet \times \Omega^\eta \mid S^c) \asymp \mathbb{P}_\eta(z \leftrightarrow \mathbb{H}\backslash B_{2s}(z)); \tag{3.15}$$

$$\tilde{\mathbb{P}}(V^\bullet \times \Omega^\eta \mid S) \asymp \mathbb{P}_\eta(z \leftrightarrow \mathbb{H}\backslash B_{2s}(z)); \tag{3.16}$$

for η small enough. Similarly for $\Omega^\eta \times W^\bullet$. Applying these claims together with (3.12) and the fact that $\tilde{\mathbb{P}}(S^c)$ converges to zero as s tends to zero, uniformly in η as follows from (3.13), proves the proposition.

It remains to prove the claims (3.15) and (3.16). At first sight one might think that these bounds are easy consequences of RSW, FKG arguments. This is not completely true since we have to deal with the condition that the coupling was not successful, respectively successful, which are neither increasing nor decreasing events. Recall the definition of RM in (3.11). Let $PN := \mathbb{H}\backslash RM$. It is sufficient to show that, for all suitable $\nu_{PN} \times \omega_{PN}$,

$$\tilde{\mathbb{P}}(V^\bullet \times \Omega^\eta \mid \nu_{PN} \times \omega_{PN}) \asymp \mathbb{P}_\eta(z \leftrightarrow \mathbb{H}\backslash B_{2s}(z)). \tag{3.17}$$

First note that it follows from the coupling procedure that

$$\tilde{\mathbb{P}}(V^\bullet \times \Omega^\eta \mid \nu_{PN} \times \omega_{PN}) = \mathbb{P}_\eta(V^\bullet \mid V \cap \{\nu_{PN}\}).$$

First we prove that in (3.17), the left hand side is less than or equal to a constant times the right hand side. To do this we introduce the event B, that there is an open (semi-)circuit in $A(z; s, 2s)$. We will prove this upper bound by showing that there exist universal constants $C_3, C_4 > 0$ such that, for all suitable ν_{PN}

$$\mathbb{P}_\eta(V^\bullet \cap B \mid V \cap \{\nu_{PN}\}) \geq C_3\, \mathbb{P}_\eta(V^\bullet \mid V \cap \{\nu_{PN}\}); \tag{3.18}$$

$$\mathbb{P}_\eta(V^\bullet \cap B \mid V \cap \{\nu_{PN}\}) \leq C_4\, \mathbb{P}_\eta(z \leftrightarrow \mathbb{H}\backslash B_{2s}(z)). \tag{3.19}$$

First we consider the lower bound (3.18). Let ν_{PN} be arbitrary. Using Lemma 6 and standard RSW, FKG arguments we get that

$$\begin{aligned}\mathbb{P}_\eta(V^\bullet \cap B \mid V \cap \{\nu_{PN}\}) &\geq \mathbb{P}_\eta(B)\mathbb{P}_\eta(V^\bullet \mid V \cap \{\nu_{PN}\}) \\ &\geq C_3\, \mathbb{P}_\eta(V^\bullet \mid V \cap \{\nu_{PN}\}).\end{aligned}$$

This proves (3.18).

Next we prove the upper bound (3.19). Therefore let Γ denote the outermost open (semi-)circuit in $A(z; s, 2s)$. Since V is an (s, C)-one-arm like event, we have by Definition 7,

$$\bigcup_\gamma V_{ext(\gamma)} \cap \{\Gamma = \gamma\} \cap \{I(z, s) \leftrightarrow \gamma\} \subset V. \tag{3.20}$$

This, together with standard RSW, FKG arguments, implies that there exists a constant $C_5 > 0$ such that

$$\begin{aligned}\mathbb{P}_\eta(B \cap V \mid \nu_{PN}) &\geq \mathbb{P}_\eta(B \cap V_{ext(\Gamma)} \cap \{I(z,s) \leftrightarrow \Gamma\} \mid \nu_{PN}) \\ &\geq C_5\, \mathbb{P}_\eta(B \cap V_{ext(\Gamma)} \mid \nu_{PN}),\end{aligned} \tag{3.21}$$

since $\mathbb{P}_\eta(I(z,s) \leftrightarrow \Gamma \mid B;\ V_{ext(\Gamma)};\ \nu_{PN}) \geq C_5$. Hence

$$\begin{aligned}\mathbb{P}_\eta(V^\bullet \cap B \mid V \cap \{\nu_{PN}\}) &\leq \mathbb{P}_\eta(\{z \leftrightarrow \Gamma\} \cap B \mid V \cap \{\nu_{PN}\}) \\ &\leq \mathbb{P}_\eta(z \leftrightarrow \mathbb{H}\backslash B_s(z)) \cdot \frac{\mathbb{P}_\eta(B \cap V_{ext(\Gamma)} \mid \nu_{PN})}{\mathbb{P}_\eta(V \mid \nu_{PN})} \\ &\leq \frac{1}{C_5 C_6} \mathbb{P}_\eta(z \leftrightarrow \mathbb{H}\backslash B_{2s}(z)) \cdot \frac{\mathbb{P}_\eta(B \cap V \mid \nu_{PN})}{\mathbb{P}_\eta(V \mid \nu_{PN})} \\ &\leq \frac{1}{C_5 C_6} \mathbb{P}_\eta(z \leftrightarrow \mathbb{H}\backslash B_{2s}(z)),\end{aligned} \tag{3.22}$$

where we used in the first inequality Definition 8. In the second inequality we used the fact that $V \subset V_{ext(\Gamma)}$ together with the fact that $\{z \leftrightarrow \Gamma\}$ is independent of everything outside Γ (which exists because of B). The third inequality follows from (3.21) and the existence of a universal constant $C_6 > 0$ such that $\mathbb{P}_\eta(z \leftrightarrow \mathbb{H}\backslash B_{2s}(z)) \geq C_6\, \mathbb{P}_\eta(z \leftrightarrow \mathbb{H}\backslash B_s(z))$. This gives the desired inequality (3.19) and completes the proof of the upper bound in (3.17).

Next we consider the lower bound in (3.17). We prove that

$$\mathbb{P}_\eta(V^\bullet \mid V \cap \{\nu_{PN}\}) \geq C_3\, \mathbb{P}_\eta(z \leftrightarrow \mathbb{H}\backslash B_{2s}(z)). \tag{3.23}$$

To prove this, we again use the event B. The inequality (3.23) follows immediately from the following inequality

$$\mathbb{P}_\eta(V^\bullet \cap B \mid V \cap \{\nu_{PN}\}) \geq C_3\, \mathbb{P}_\eta(z \leftrightarrow \mathbb{H}\backslash B_{2s}(z)), \tag{3.24}$$

where $C_3 > 0$ is the same as in (3.18). Similarly to (3.20), but now using Definition 8, we have

$$\bigcup_\gamma \{\Gamma = \gamma\} \cap V_{ext(\gamma)} \cap \{z \leftrightarrow \gamma\} \subset V^\bullet, \tag{3.25}$$

where Γ is the outermost circuit in $A(z; s, 2s)$. Hence

$$\begin{aligned}\mathbb{P}_\eta(V^\bullet \cap B \mid V \cap \{\nu_{PN}\}) &\overset{(3.25)}{\geq} \sum_\gamma \frac{\mathbb{P}_\eta(\{\Gamma = \gamma\} \cap V_{ext(\gamma)} \cap \{z \leftrightarrow \gamma\} \cap \{\nu_{PN}\})}{\mathbb{P}_\eta(V \cap \{\nu_{PN}\})}, \\ &\geq \mathbb{P}_\eta(z \leftrightarrow \mathbb{H}\backslash B_{2s}(z)) \sum_\gamma \frac{\mathbb{P}_\eta(\{\Gamma = \gamma\} \cap V_{ext(\gamma)} \cap \{\nu_{PN}\})}{\mathbb{P}_\eta(V \cap \{\nu_{PN}\})}, \\ &\geq \mathbb{P}_\eta(z \leftrightarrow \mathbb{H}\backslash B_{2s}(z)) \frac{\mathbb{P}_\eta(B \cap V \cap \{\nu_{PN}\})}{\mathbb{P}_\eta(V \cap \{\nu_{PN}\})}.\end{aligned} \tag{3.26}$$

It follows from Lemma 6 together with the fact that $\mathbb{P}_\eta(B) \geq C_3$ that

$$\mathbb{P}_\eta(B \cap V \cap \{\nu_{PN}\}) \geq C_3 \cdot \mathbb{P}_\eta(V \cap \{\nu_{PN}\}). \tag{3.27}$$

This completes the proof of (3.24) and finishes the proof of Proposition 10. □

3.2. Proof of Theorem 1

Let u_1, u_2, w be fixed. Because of Theorem 3 it is sufficient to show that for every $\varepsilon > 0$, there exists $s > 0$, such that $\forall s_1, s_2, s_3 < s : \exists \eta_0 > 0$ with the property that

$$\left| \frac{\mathbb{P}_\eta(u_1 \leftrightarrow u_2 \leftrightarrow w \mid E^{s_1,s_2,s_3}_{u_1,u_2,w})^2}{\mathbb{P}_\eta(u_1 \leftrightarrow u_2 \mid E^{s_1,s_2}_{u_1,u_2})\, \mathbb{P}_\eta(u_1 \leftrightarrow w \mid E^{s_1,s_3}_{u_1,w})\, \mathbb{P}_\eta(u_2 \leftrightarrow w \mid E^{s_2,s_3}_{u_2,w})} - 1 \right| < \varepsilon, \tag{3.28}$$

for all $\eta < \eta_0$.

In order to prove (3.28) we define the following events:

$$\begin{aligned} E^{s_1,\bullet}_{u_1,u_2} &:= \{[u_1, u_1 + s_1] \cap \mathcal{C}(u_2) \neq \emptyset\}; \\ E^{\bullet,s_3}_{u_1,w} &:= \{\rho(w, \mathcal{C}(u_1)) < s_3\}; \\ E^{\bullet,s_3}_{u_2,w} &:= \{\rho(w, \mathcal{C}(u_2)) < s_3\}; \\ E^{s_1,\bullet,s_3}_{u_1,u_2,w} &:= \{[u_1, u_1 + s_1] \cap \mathcal{C}(u_2) \neq \emptyset\} \cap \{\rho(w, \mathcal{C}([u_1, u_1 + s_1])) < s_3\}; \\ E^{\bullet,\bullet,s_3}_{u_1,u_2,w} &:= \{u_1 \leftrightarrow u_2\} \cap \{\rho(w, \mathcal{C}(u_1)) < s_3\}. \end{aligned} \tag{3.29}$$

Let $C := (\min\{|u_1 - u_2|, |u_1 - w|, |u_2 - w|\})/(2\sqrt{2})$. We claim the following about the events defined in (2.3) and (3.29).

1. Every event of the form $E^{s_1,s_2,s_3}_{a_1,a_2,a_3}$ or $E^{s_1,s_2}_{a_1,a_2}$ where the a_i's are in $\{u_1, u_2, w\}$ and each s_i is in $\mathbb{R}_+$ or $s_i = \bullet$, defined in (2.3) and (3.29), is, for each $s_j \neq \bullet$ an (s_j, C)-one-arm like event around a_j. For example $E^{s_1,\bullet,s_3}_{u_1,u_2,w}$ is an (s_1, C)-one-arm like event around u_1, and an (s_3, C)-one-arm like event around w.
2. The events $\{u_1 \leftrightarrow u_2\}$, $\{u_1 \leftrightarrow w\}$, $\{u_2 \leftrightarrow w\}$, $\{u_1 \leftrightarrow u_2 \leftrightarrow w\}$ are point versions of respectively $E^{s_1,\bullet}_{u_1,u_2}$, $E^{\bullet,s_3}_{u_1,w}$, $E^{\bullet,s_3}_{u_2,w}$ and $E^{\bullet,\bullet,s_3}_{u_1,u_2,w}$.
3. Each event in (3.29) is a point version of the corresponding event $E^{s_1,s_2,s_3}_{a_1,a_2,a_3}$ or $E^{s_1,s_2}_{a_1,a_2}$, where the "$\bullet$" is replaced by a positive number s_j. E.g. $E^{\bullet,\bullet,s_3}_{u_1,u_2,w}$ is a point version of $E^{s_1,\bullet,s_3}_{u_1,u_2,w}$ and $E^{s_1,\bullet}_{u_1,u_2}$ is a point version of $E^{s_1,s_2}_{u_1,u_2}$.
4. Each pair of events of the form $E^{s_1,s_2,s_3}_{a_1,a_2,a_3}$ and $E^{s_1,s_2}_{a_1,a_2}$ where the a_i's are in $\{u_1, u_2, w\}$ and each s_i is in $\mathbb{R}_+$ or $s_i = \bullet$, defined in (2.3) and (3.29), are, for each j where both events have $s_j \neq \bullet$, (s_j, C)-comparable around a_j. For example the events $E^{s_1,s_2}_{u_1,u_2}$, $E^{s_1,s_3}_{u_1,w}$, $E^{s_1,s_2,s_3}_{u_1,u_2,w}$, $E^{s_1,\bullet}_{u_1,u_2}$, $E^{s_1,\bullet,s_3}_{u_1,u_2,w}$ are pairwise (s_1, C)-comparable around u_1.

Before we give proofs of these claims we show how Theorem 1 follows from them. We factorize the numerator in (3.28) as follows

$$\begin{aligned} &\mathbb{P}_\eta(u_1 \leftrightarrow u_2 \leftrightarrow w \mid E^{s_1,s_2,s_3}_{u_1,u_2,w})^2 \\ &\quad = \mathbb{P}_\eta(u_1 \leftrightarrow u_2 \leftrightarrow w \mid E^{\bullet,\bullet,s_3}_{u_1,u_2,w})^2 \cdot \mathbb{P}_\eta(E^{\bullet,\bullet,s_3}_{u_1,u_2,w} \mid E^{s_1,\bullet,s_3}_{u_1,u_2,w})^2 \\ &\qquad \cdot \mathbb{P}_\eta(E^{s_1,\bullet,s_3}_{u_1,u_2,w} \mid E^{s_1,s_2,s_3}_{u_1,u_2,w})^2. \end{aligned} \tag{3.30}$$

The probabilities in the denominator in (3.28) can be factorized as follows

$$\mathbb{P}_\eta(u_1 \leftrightarrow u_2 \mid E^{s_1,s_2}_{u_1,u_2}) = \mathbb{P}_\eta(u_1 \leftrightarrow u_2 \mid E^{s_1,\bullet}_{u_1,u_2})\mathbb{P}_\eta(E^{s_1,\bullet}_{u_1,u_2} \mid E^{s_1,s_2}_{u_1,u_2}) \tag{3.31}$$

$$\mathbb{P}_\eta(u_1 \leftrightarrow w \mid E^{s_1,s_3}_{u_1,w}) = \mathbb{P}_\eta(u_1 \leftrightarrow w \mid E^{\bullet,s_3}_{u_1,w})\mathbb{P}_\eta(E^{\bullet,s_3}_{u_1,w} \mid E^{s_1,s_3}_{u_1,w}) \tag{3.32}$$

$$\mathbb{P}_\eta(u_2 \leftrightarrow w \mid E^{s_2,s_3}_{u_2,w}) = \mathbb{P}_\eta(u_2 \leftrightarrow w \mid E^{\bullet,s_3}_{u_2,w})\mathbb{P}_\eta(E^{\bullet,s_3}_{u_2,w} \mid E^{s_2,s_3}_{u_2,w}). \tag{3.33}$$

Plugging this into the quotient in (3.28) and applying Proposition 10 to the 6 pairs of (s_i, C)-comparable events completes the proof.

It remains to prove claims 1–4 above. Some of these claims follow immediately, for the others we use two standard properties of conformal radius. The first is (2.2). The second property is *monotonicity*: the conformal radius is non-decreasing as the domain A decreases, (as is well known and follows easily from Schwarz' Lemma. See for example [2]).

We prove claim 1 for a particular event, namely $E^{\bullet,s_3}_{u_1,w}$.

(a) It is increasing: Let $\omega \in E^{\bullet,s_3}_{u_1,w}$ and $\nu \geq \omega$, then $\mathcal{C}(u_1)(\omega) \subset \mathcal{C}(u_1)(\nu)$. Here $\mathcal{C}(u_1)(\omega)$ means the cluster of u_1 under the configuration ω. Thus by monotonicity of the conformal radius $\rho(w, \mathcal{C}(u_1)(\nu)) \leq \rho(w, \mathcal{C}(u_1)(\omega)) < s_3$ and $\nu \in E^{\bullet,s_3}_{u_1,w}$.

(b) $E^{\bullet,s_3}_{u_1,w} \subset \{B_{s_3}(w) \leftrightarrow \mathbb{H}\backslash B_C(w)\}$: Suppose that $\omega \in E^{\bullet,s_3}_{u_1,w}$. It follows from (2.2) that $\min_{x\in\mathcal{C}(u_1)} |w - x| < s_3$. Further $\sqrt{2}C \leq |u_1 - w|/2$, which implies that $\omega \in \{B_{s_3}(w) \leftrightarrow \mathbb{H}\backslash B_C(w)\}$.

Let γ be an arbitrary (semi-)circuit in $A(w; s_3, C)$. Let $D := E^{\bullet,s_3}_{u_1,w}$

(c) $\{\gamma \text{ open}\} \cap D_{ext(\gamma)} \cap D_{int(\gamma)} \subset D$: Let $\omega \in D_{int(\gamma)}$ and $\nu \in D_{ext(\gamma)}$. By definition there exists $\tilde{\nu}$ such that $\nu_{ext(\gamma)} \times \tilde{\nu} \in D$. With the second inequality in (2.2) this implies that $u_1 \leftrightarrow \gamma$ in $ext(\gamma)$. Next let $\tilde{\omega}$ be such that $\omega_{int(\gamma)} \times \tilde{\omega} \in D$. Then it is easy to see that $\mathcal{C}(u_1)(\omega_{int(\gamma)} \times \tilde{\omega}) \cap int(\gamma) \subset \mathcal{C}(\gamma)(\omega) \cap int(\gamma)$. Monotonicity of the conformal radius implies now that

$$\rho\,(w, \mathcal{C}(\gamma)(\omega)) \leq \rho\left(w, \mathcal{C}(u_1)(\omega_{int(\gamma)} \times \tilde{\omega})\right) < s_3.$$

Let $\upsilon := \omega_{int(\gamma)} \times \{1\}^\gamma \times \nu_{ext(\gamma)}$. Note that $\mathcal{C}(u_1)(\upsilon) \cap int(\gamma) = \mathcal{C}(\gamma)(\omega) \cap int(\gamma)$. Thus $\rho(w, \mathcal{C}(u_1)(\upsilon)) = \rho(w, \mathcal{C}(\gamma)(\omega))$, and hence $\upsilon \in D$.

(d) $\{I(w, s_3) \leftrightarrow \gamma\} \subset D_{int(\gamma)}$: Let $\omega \in \{I(w, s_3) \leftrightarrow \gamma\}$ and $\nu \in \{u_1 \leftrightarrow \gamma\}$. Then the first inequality in (2.2) implies that $\omega_{int(\gamma)} \times \{1\}^\gamma \times \nu_{ext(\gamma)} \in D$, hence $\omega \in D_{int(\gamma)}$.

This completes the proof of claim 1 for this particular event. The proofs for the other events and claims are very similar and we omit them. □

3.3. *Proof of Theorem 2*

We will use the notation

$$E^{s_1,\bullet,\bullet}_{u_1,u_2,w} := \{\{u_2, w\} \subset \mathcal{C}([u_1, u_1 + s])\}. \tag{3.34}$$

With this notation we can write the quotient in (1.2) as

$$\frac{\mathbb{P}(\{u_2, w\} \subset \mathcal{C}([u_1, u_1 + s]))}{\mathbb{P}(w \in \mathcal{C}([u_1, u_1 + s]))\,\mathbb{P}(u_2 \in \mathcal{C}([u_1, u_1 + s]))} = \frac{\mathbb{P}_\eta(E^{s,\bullet,\bullet}_{u_1,u_2,w})}{\mathbb{P}_\eta(E^{s,\bullet}_{u_1,w})\mathbb{P}_\eta(E^{s,\bullet}_{u_1,u_2})}. \tag{3.35}$$

Similarly to the proof of Theorem 1 we factorize this as follows

$$\begin{aligned}\frac{\mathbb{P}_\eta(E^{s,\bullet,\bullet}_{u_1,u_2,w})}{\mathbb{P}_\eta(E^{s,\bullet}_{u_1,w})\mathbb{P}_\eta(E^{s,\bullet}_{u_1,u_2})} &= \frac{\mathbb{P}_\eta(E^{s,\bullet,\bullet}_{u_1,u_2,w} \mid E^{s,\bullet,s_3}_{u_1,u_2,w})}{\mathbb{P}_\eta(E^{s,\bullet}_{u_1,w} \mid E^{s,s_3}_{u_1,w})} \cdot \frac{\mathbb{P}_\eta(E^{s,\bullet,s_3}_{u_1,u_2,w} \mid E^{s,s_2,s_3}_{u_1,u_2,w})}{\mathbb{P}_\eta(E^{s,\bullet}_{u_1,u_2} \mid E^{s,s_2}_{u_1,u_2})} \\ &\quad\cdot \frac{\mathbb{P}_\eta(E^{s,s_2,s_3}_{u_1,u_2,w})}{\mathbb{P}_\eta(E^{s,s_3}_{u_1,w})\mathbb{P}_\eta(E^{s,s_2}_{u_1,u_2})}.\end{aligned} \tag{3.36}$$

The first two ratio's converge to 1 by Proposition 10, uniformly in η. Namely the involved events are point versions and (s, C)-comparable, by similar arguments as in the proof of Theorem 1.

We claim that the ratio

$$\frac{\mathbb{P}_\eta(E^{s,s_2,s_3}_{u_1,u_2,w})}{\mathbb{P}_\eta(E^{s,s_3}_{u_1,w})\mathbb{P}_\eta(E^{s,s_2}_{u_1,u_2})} \tag{3.37}$$

converges to the function $\psi(u_1, s, u_2, w)$, as η, s_2, s_3 tend to zero. To prove this claim we note that

$$\frac{\mathbb{P}_\eta(E^{s,s_2,s_3}_{u_1,u_2,w})}{\mathbb{P}_\eta(E^{s,s_3}_{u_1,w})\mathbb{P}_\eta(E^{s,s_2}_{u_1,u_2})} = \frac{s_3^{-5/48} \cdot \mathbb{P}_\eta(E^{s,s_2,s_3}_{u_1,u_2,w} \mid E^{s,s_2}_{u_1,u_2})}{s_3^{-5/48} \cdot \mathbb{P}_\eta(E^{s,s_3}_{u_1,w})}. \tag{3.38}$$

Theorem 4 and Lemma 5 imply that the following limit of (3.38) exists: First send η to zero, after that send s_2 to zero and finally let s_3 go to zero. This, together with the uniform convergence in η of the first two ratio's in (3.36), implies that the limit in (1.2) exists and is equal to

$$\frac{\pi^{5/48}|\Psi'_{u_1,s,u_2}(w)|^{5/48} G\left(\Re(\Psi_{u_1,s,u_2}(w)), \Im(\Psi_{u_1,s,u_2}(w))\right)}{2^{5/48} H(0) \cdot |\phi'(w)|^{5/48} (\sin(\pi\omega/2))^{1/3}}, \tag{3.39}$$

where Ψ, G, ϕ, H, ω are as in Theorem 4 and Lemma 5.

To finish the proof of Theorem 2 we have to simplify (3.39) and show that it is equal to the function $\psi(u_1, s, u_2, w)$ given in that theorem. Hereto let $\Pi : \mathbb{H} \to \mathbb{H}$ be a conformal map such that the points $u_1, u_1 + s, u_2$ are mapped to $-1, 1, \infty$ respectively. Let $\tilde{w} = \Pi(w)$. Let $\tilde{\Psi} : \mathbb{H} \to \mathbb{S}$ be the conformal map, such that $\Psi = \tilde{\Psi} \circ \Pi$, thus

$$\tilde{\Psi}(z) = \frac{-\mathbf{i}}{\pi} \arcsin(z) + \frac{1}{2}\mathbf{i}.$$

Further let $\tilde{\phi}$ be the conformal map such that $\phi = \tilde{\phi} \circ \Pi$. We have that

$$|\phi'(w)| = \frac{|\Pi'(w)|}{2\Im(\tilde{w})}, \qquad |\Psi'(w)| = \frac{|\Pi'(w)|}{\pi\sqrt{|1 - \tilde{w}^2|}}. \tag{3.40}$$

Recall that $x = \Re(\Psi_{u_1,s,u_2}(w)),\ y = \Im(\Psi_{u_1,s,u_2}(w))$ and $\Psi_{u_1,s,u_2}(w) = \tilde{\Psi}(\tilde{w})$, thus

$$\begin{aligned} \sinh(\pi x) &= \sinh(\Im(\arcsin(\tilde{w}))), \\ \sin(\pi y) &= \cos(\Re(\arcsin(\tilde{w}))). \end{aligned}$$

It follows from standard formulas for hyperbolic functions that

$$\sinh(\pi x)^2 \sin(\pi y)^2 = \Im(\tilde{w})^2, \tag{3.41}$$

$$\sinh(\pi x)^2 + \sin(\pi y)^2 = |1 - \tilde{w}^2|. \tag{3.42}$$

Further note that

$$\begin{aligned} &\left(\frac{1}{\sinh(\pi x)}\right)^{1/3} \left(\frac{\sinh(\pi x)^2 \sin(\pi y)^2}{\sinh(\pi x)^2 + \sin(\pi y)^2}\right)^{11/96} \\ &= \left(\frac{\sin(\pi y)^2}{\sinh(\pi x)^2 + \sin(\pi y)^2}\right)^{1/6} \left(\frac{\sinh(\pi x)^2 + \sin(\pi y)^2}{\sinh(\pi x)^2 \sin(\pi y)^2}\right)^{5/96}. \end{aligned} \tag{3.43}$$

Putting together the definition of G in (2.6) and equations (3.40)–(3.43) gives that (3.39) is equal to

$$\frac{e^{\pi x/3} H(x)}{H(0)} \cdot \left(\frac{\cos(\Re(\arcsin(\tilde{w})))}{\sqrt{|1-\tilde{w}^2|}\sin(\pi\omega/2)} \right)^{1/3}. \tag{3.44}$$

Recall that $\omega\pi$ is equal to the angle at $\tilde{w}$ in the triangle with corners -1, 1, $\tilde{w}$. It follows easily that

$$\sin(\pi\omega/2) = \sqrt{\frac{1}{2} - \frac{|\tilde{w}|^2 - 1}{2|1-\tilde{w}^2|}},$$

and from formulas for hyperbolic functions, including (3.42), that

$$2\cos(\Re(\arcsin(\tilde{w})))^2 = |1-\tilde{w}^2| + 1 - |\tilde{w}|^2,$$

which together imply that the last factor in (3.44) equals 1. This completes the proof of Theorem 2. □

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