

## ON THE NULL SPACE OF A COLIN DE VERDIÈRE MATRIX

by L. LOVÁSZ and A. SCHRIJVER

---

### 1. Introduction and results.

Let  $G = (V, E)$  be an undirected graph, with vertex set  $\{1, \dots, n\}$ . Let  $\mathcal{M}(G)$  be the set of symmetric  $n \times n$  matrices  $M = (m_{i,j})$  satisfying

- (1) (i)  $M$  has exactly one negative eigenvalue, of multiplicity 1;  
(ii) for all  $i, j$  with  $i \neq j$  one has :  
if  $i$  and  $j$  are nonadjacent then  $m_{i,j} = 0$ ,  
if  $i$  and  $j$  are adjacent then  $m_{i,j} < 0$ .

Moreover,  $M$  is said to have the *Strong Arnold Property* (or to satisfy the *Strong Arnold Hypothesis*) if for each symmetric  $n \times n$  matrix  $X = (x_{i,j})$  satisfying  $x_{i,j} = 0$  if  $i = j$  or  $i$  and  $j$  are adjacent, and satisfying  $MX = 0$ , one has  $X = 0$ .

Yves Colin de Verdière [1] introduced the parameter  $\mu(G)$ , being the maximum corank of any matrix in  $\mathcal{M}(G)$  having the Strong Arnold Property. (The *corank* of a matrix is the dimension of its null space.)

Stimulated by discussions with François Jaeger (who suggested an idea to show (ii) below), Colin de Verdière showed that  $\mu(H) \leq \mu(G)$  if  $H$

is a minor of  $G$ , and that

- (2) (i)  $\mu(G) \leq 1$  if and only if  $G$  is a disjoint union of paths;  
(ii)  $\mu(G) \leq 2$  if and only if  $G$  is outerplanar;  
(iii)  $\mu(G) \leq 3$  if and only if  $G$  is planar.

In [5], we showed that

- (3) (iv)  $\mu(G) \leq 4$  if and only if  $G$  is linklessly embeddable.

(A graph  $G$  is *linklessly embeddable* if it can be embedded in  $\mathbb{R}^3$  so that any two disjoint circuits in  $G$  form unlinked closed curves in  $\mathbb{R}^3$ .)

It was shown by Hein van der Holst [3] that if  $G$  is 3-connected and planar, then any matrix in  $\mathcal{M}(G)$  has corank at most 3 — also those not having the Strong Arnold Property.

The main result in this paper is that if  $G$  is 3-connected and planar, then for any matrix  $M \in \mathcal{M}(G)$  of corank 3, the null space  $\ker M$  of  $M$  yields an embedding of  $G$  in the 2-sphere  $S^2$ . We also show the related but easier results that if  $G$  is a path, then for any matrix  $M \in \mathcal{M}(G)$  of corank 1, the null space  $\ker M$  of  $M$  yields an embedding of  $G$  in the line; furthermore, if  $G$  is 2-connected and outerplanar, then for any matrix  $M \in \mathcal{M}(G)$  of corank 2, the null space  $\ker M$  of  $M$  yields a representation of  $G$  as a convex polygon with non-crossing diagonals.

To make this more precise, assume that  $G$  is connected. In this case condition (1)(ii) implies that the eigenvector  $\pi$  belonging to the negative eigenvalue is (up to scaling) positive. We define the *null space representation* derived from the matrix  $M$  as follows. Let  $r$  be the corank of  $M$ , let  $a_1, a_2, \dots, a_r$  form a basis of  $\ker M$ , and for each vertex  $i$  of  $G$ , let  $u_i := (a_{1,i}, a_{2,i}, \dots, a_{r,i})^T \in \mathbb{R}^r$ . As  $\pi$  is orthogonal to each of  $a_1, \dots, a_r$  we have

$$\sum_i \pi_i u_i = 0.$$

This way we define a mapping  $V \rightarrow \mathbb{R}^r$ . Note that this mapping is determined only up to a linear transformation of  $\mathbb{R}^r$ . In the results below, we have to scale the vectors  $u_i$ . Note that arbitrary scaling of them can be achieved through appropriate scaling of the rows and columns of  $M$ . In particular, we consider the unit vectors

$$(4) \quad v_i = \frac{1}{\|u_i\|} u_i,$$

and the vectors

$$(5) \quad w_i = \frac{1}{\pi_i} u_i.$$

For (4) to make sense, we'll have to show that the  $u_i$  are non-null.

**THEOREM 1.1.** — *Let  $G$  be a path. Then the mapping  $i \mapsto w_i$ , together with the segments connecting  $w_i$  and  $w_j$  for  $ij \in E(G)$ , gives an embedding of  $G$  in the line.*

**THEOREM 1.2.** — *Let  $G$  be a 2-connected outerplanar graph. Then the mapping  $i \mapsto v_i$ , together with the segments connecting  $v_i$  and  $v_j$  for  $ij \in E(G)$ , gives an embedding of  $G$  in the plane as a convex polygon with non-crossing diagonals.*

**THEOREM 1.3.** — *Let  $G$  be a 3-connected planar graph. Then the mapping  $i \mapsto v_i$ , together with the geodesic curves on  $S^2$  connecting  $v_i$  and  $v_j$  for  $ij \in E(G)$ , gives an embedding of  $G$  in  $S^2$ .*

*Remarks.* — 1. It seems curious that we had to scale differently in the case of the path as in the other two cases. It may be the case that for planar and outerplanar graphs the scaling  $(w_i)$  provides an embedding with interesting properties, say in the planar case an embedding in the skeleton of a convex polyhedron; but we cannot prove this.

2. Our motivation has been to derive a similar result for one dimension higher, that is, for linklessly embeddable graphs, where we would like to obtain a linkless embedding of  $G$  in  $S^3$ . Suppose that  $G$  is a 4-connected linklessly embeddable graph and  $M$  is a matrix in  $\mathcal{M}(G)$  of corank 4. Perhaps  $\ker M$  yields a proper embedding of  $G$  in  $S^3$ : vertices of  $G$  are embedded in  $S^3$  in such a way that adding shortest arcs in  $S^3$  connecting adjacent vertices, gives a linkless embedding of  $G$  in  $S^3$ . We do not know if this is true.

## 2. Proofs.

### 2.1. Van der Holst's lemma and its extensions.

For any vector  $x \in \mathbb{R}^n$ , the *support* of  $x$ , denoted by  $\text{supp}(x)$ , is the set  $\{i \in V \mid x_i \neq 0\}$ . The *positive support*  $\text{supp}^+(x)$  of  $x$  is the set  $\{i \in V \mid x_i > 0\}$ , and the *negative support*  $\text{supp}^-(x)$  is the set  $\{i \in V \mid x_i < 0\}$ . For any  $U \subseteq V$ ,  $N(U)$  denotes the set of vertices  $i \notin U$  adjacent to at least one vertex in  $U$ .

The following observation is useful:

- (6) Let  $M \in \mathcal{M}(G)$  and let  $x \in \ker M$ . Then each vertex in  $N(\text{supp}(x))$  belongs to both  $N(\text{supp}^+(x))$  and  $N(\text{supp}^-(x))$ .

To see (6), consider any  $i \in N(\text{supp}(x))$  (so  $x_i = 0$ ), and note that

$$(7) \sum_j m_{i,j} x_j = 0,$$

implying that, if  $x_j > 0$  for some  $j$  adjacent to  $i$ , then  $x_j < 0$  for some  $j$  adjacent to  $i$ , and conversely. (Here we use that  $m_{i,j} < 0$  if  $j$  is adjacent to  $i$ , and  $m_{i,j} x_j = 0$  otherwise.)

An important further tool we need is the following lemma proved by van der Holst [2]. A vector  $x$  in  $\ker M$  is said to have *minimal support* if  $x \neq 0$ , and each nonzero vector  $y \in \ker M$  with  $\text{supp}(y) \subseteq \text{supp}(x)$  satisfies  $\text{supp}(y) = \text{supp}(x)$ .

LEMMA 2.1 [van der Holst]. — *Let  $G$  be connected, let  $M \in \mathcal{M}(G)$  and let  $x \in \ker M$  have minimal support. Then both  $\text{supp}^+(x)$  and  $\text{supp}^-(x)$  are nonempty and induce connected subgraphs of  $G$ .*

Geometrically, this lemma expresses the following nice property of the null space embedding: every hyperplane in  $\mathbb{R}^r$  spanned by some vectors  $u_i$  separates the graph into two non-empty connected subgraphs.

We need the following variation on this lemma.

LEMMA 2.2. — *Let  $M \in \mathcal{M}(G)$  and let  $x \in \mathbb{R}^V$  satisfy  $Mx < 0$ . Then  $\text{supp}^+(x)$  is nonempty and induces a connected subgraph of  $G$ .*

One way to get a vector  $x$  with  $Mx < 0$  is to choose a vector in the nullspace of  $M$  and add a positive multiple of  $\pi$  to it. In this special case, Lemma 2.2 has a nice geometric formulation: *the nodes  $i$  for which*

$w_i$  belongs to an open halfspace in  $\mathbb{R}^r$  containing the origin induce a non-empty connected subgraph.

We give a proof of a more general lemma, which also includes van der Holst's lemma and its extension given in [4]. To formulate this, we need to describe a class of exceptions. Let  $G = (V, E)$  be a graph,  $M \in \mathcal{M}(G)$  and  $S \subset V$ . Let  $G_1, \dots, G_r$  ( $r \geq 2$ ) be the connected components of  $G - S$ , and let  $S_i = V(G_i)$ . We say that  $S$  is a *regular cutset* (in  $G$ , with respect to  $M$ ), if there exist non-zero, non-negative vectors  $x_1, \dots, x_r$  and  $y$  in  $\mathbb{R}^V$  such that

- (i)  $\text{supp}(x_i) = S_i$  and  $\text{supp}(y) \subseteq S$ ;
- (ii)  $Mx_1 = Mx_2 = \dots = Mx_r = -y$ .

Note that if  $M_i$  and  $x'_i$  are the restrictions of  $M$  and  $x_i$  to  $S_i$ , then  $x'_i > 0$  and  $M_i x_i = 0$ . Thus by the Perron-Frobenius Theorem, the  $x_i$  are uniquely determined up to positive scaling.

LEMMA 2.3. — *Let  $M \in \mathcal{M}(G)$  and let  $x \in \mathbb{R}^V$  satisfy  $Mx \leq 0$ . Then  $\text{supp}^+(x)$  is nonempty. Furthermore, if  $\text{supp}^+(x)$  induces a disconnected graph then  $G$  has a regular cutset  $S$  such that  $x$  is a linear combination*

$$x = \sum_i \alpha_i x_i$$

*of the corresponding vectors  $x_1, \dots, x_r$ , where  $\sum_i \alpha_i \geq 0$ , and at least two  $\alpha_i$  are positive.*

*Proof.* — We may assume that the negative eigenvalue of  $M$  is  $-1$ . Then

$$\pi^T x = -\pi^T Mx \geq 0,$$

and since  $x \neq 0$ , it follows that  $\text{supp}^+(x)$  is nonempty.

Suppose next that  $\text{supp}^+ x$  can be decomposed into two disjoint nonempty sets  $A$  and  $B$  such that no edge connects  $A$  and  $B$ . Let  $C := \text{supp}^-(x)$  (this may be empty), and  $S = V \setminus \text{supp}(X)$ . Let  $a, b$  and  $c$  arise from  $x$  by setting to 0 the entries with index out of  $A, B$  and  $C$ , respectively. Define

$$w := (\pi^T b)a - (\pi^T a)b.$$

Then  $\pi^T w = 0$ , and hence by our assumption that  $M$  has exactly one negative eigenvalue with eigenvector  $\pi$ ,

$$w^T M w \geq 0.$$

On the other hand, we have  $a^T Mb = 0$  (as there are no edges between  $A$  and  $B$ ) and  $a^T Mc \geq 0$  (as  $a \geq 0$ ,  $c \leq 0$  and  $M$  is non-positive outside the diagonal). Furthermore,  $Mx \leq 0$  implies that

$$a^T Ma \leq -a^T Mb - a^T Mc \leq 0.$$

Thus

$$w^T Mw = (\pi^T b)^2 a^T Ma - 2(\pi^T b)(\pi^T a) a^T Mb + (\pi^T a)^2 b^T Mb \leq 0.$$

This implies that  $w^T Mw = 0$ . Moreover, we must have equality in  $a^T Mc \geq 0$ , which implies that there are no edges between  $A$  and  $C$ , and similarly, between  $B$  and  $C$ .

Thus  $(Ma)_i = (Mx)_i \leq 0$  for  $i \in A$ , and so  $a^T Ma = 0$  implies that  $(Ma)_i = (Mx)_i = 0$  for  $i \in A$ . Similarly,  $(Mb)_i = (Mx)_i = 0$  for  $i \in B$ . Thus  $\text{supp}(Ma), \text{supp}(Mb), \text{supp}(Mc) \subseteq S$ .

Since  $M$  is positive semidefinite on vectors orthogonal to  $\pi$ ,  $w^T Mw = 0$  implies  $Mw = 0$ . This says that

$$(8) \quad \frac{1}{(\pi^T a)} Ma = \frac{1}{(\pi^T b)} Mb.$$

To conclude, let  $S_1, S_2, \dots, S_r$  be the connected components of  $G - S$ . The fact that  $\text{supp}(Mc) \subseteq S$  implies that if  $z \in \mathbb{R}^V$  is defined by  $z_i = |x_i|$ , then  $Mz \leq 0$ . Thus we can apply the above argument choosing any two components  $S_i$  as  $A$  and  $B$ . If  $z_i$  denotes the restriction of  $z$  to  $S_i$ , and  $x_i = (1/\pi^T z_i)z_i$ , then (8) above implies that  $Mx_1 = \dots = Mx_r$ . We have also seen that this vector has support in  $S$ , and since its support is disjoint from the support of the  $x_i$ , it is non-positive. Clearly  $x$  is a linear combination of the  $x_i$ , and the conditions on the coefficients are trivially verified. □

### 2.2. Proof of Theorem 1.1.

Look at the null space representation  $(u_i)$  of  $M \in \mathcal{M}(P)$  with corank 1, where  $P$  is a path with  $n$  nodes labelled  $1, 2, \dots, n$  (in this order on the path), and the scaled version  $(w_i)$ . Note that the  $w_i$  are numbers now. We claim that either  $w_1 < w_2 < \dots < w_n$  or  $w_1 > w_2 > \dots > w_n$ . It suffices to argue that  $w_{i-1} < w_i \geq w_{i+1}$  cannot occur. There are 3 cases to consider:

(a)  $w_i > 0$ . Then the vector  $x \in \mathbb{R}^n$  defined by  $x_j = (w_i - w_j)\pi_j$  satisfies  $Mx < 0$ , and has  $x_i = 0$ ,  $x_{i-1} > 0$ , and  $x_{i+1} \geq 0$ . Let

$$y_j = \begin{cases} x_j - \varepsilon, & \text{if } j = i + 1, \\ x_j, & \text{otherwise,} \end{cases}$$

for a small positive  $\varepsilon$ . Then  $My < 0$ , and  $y_i = 0$ ,  $y_{i-1}, y_{i+1} > 0$ , which means that  $\text{supp}^+(y)$  is disconnected, contradicting Lemma 2.3.

(b)  $w_i = 0$ . It is trivial from  $Mu = 0$  that if a node  $i$  with  $u_i = 0$  has a neighbor  $j$  with  $u_j < 0$ , then it must also have a neighbor  $k$  with  $u_k > 0$  (by (6)), which is not satisfied here.

(c)  $w_i < 0$ . We know by Lemma 2.1 that there is a node  $k$  with  $w_k > 0$ . If  $i < k$ , then let  $j$  be the index with  $i < j < k$  and  $w_j$  minimum; this  $j$  then violates (a) (with signs flipped). If  $k > i$ , the argument is similar.  $\square$

### 2.3. Proof of Theorem 1.2.

As  $G$  is outerplanar, we may fix an embedding of  $G$  in  $\mathbb{R}^2$ . When speaking of faces below, we mean faces with respect to this embedding. Similarly for the outer face. We can assume that the vertices  $1, \dots, n$  occur in this order along the outer face.

We first show:

- (9) if  $a$  and  $b$  are consecutive vertices along the outer face, then  $u_a$  and  $u_b$  are linearly independent.

Suppose not. Then there exists a nonzero vector  $x \in \ker M$  such that  $x_a = x_b = 0$ . Take such an  $x$  with  $\text{supp}(x)$  minimal. Lemma 2.1 implies that both  $\text{supp}^+(x)$  and  $\text{supp}^-(x)$  are nonempty and induce connected subgraphs of  $G$ .

As  $G$  is 2-connected,  $G$  has vertex-disjoint paths  $P_1$  and  $P_2$ , each starting in  $N(\text{supp}(x))$ , and ending in  $a$  and  $b$  respectively. By (6),  $P_i$  starts in a vertex that is both in  $N(\text{supp}^+(x))$  and  $N(\text{supp}^-(x))$ . However, since  $\text{supp}^+(x)$  and  $\text{supp}^-(x)$  induce connected subgraphs of  $G$ , this is topologically not possible, showing (9).

(9) implies that each  $u_i$  is nonzero, and hence  $v_i$  is defined. To show that  $v_1, \dots, v_k$  occur in this order along  $S^1$ , it suffices to show that (using the symmetry of  $1, \dots, k$ ):

- (10) let  $l$  be the line through  $v_1$  and  $0$ . Then there is a  $k$  such that  $v_2, \dots, v_{k-1}$  all are at one side of  $l$  and  $v_{k+1}, \dots, v_n$  are at the other side of  $l$ .

Suppose this is not true. Let  $x$  be a nonzero vector  $x \in \ker M$  with  $x_1 = 0$ . Then there are  $h, i, j$  with  $2 \leq h < i < j \leq n$  such that  $x_h > 0$ ,  $x_i \leq 0$ , and  $x_j > 0$ . Since  $\text{supp}^+(x)$  induces a connected subgraph of  $G$ , there is a path  $P$  connecting  $h$  and  $j$  and traversing vertices in  $\text{supp}^+(x)$  only. Then  $1$  and  $i$  are in different components of  $G - P$ . Both components contain vertices  $m$  with  $x_m \leq 0$  (namely  $1$  and  $i$ ), and hence vertices  $m \in N(\text{supp}^+(x))$  (using the 2-connectivity of  $G$ ). Any such vertex belongs to  $\text{supp}^-(x)$  or is adjacent to a vertex in  $\text{supp}^-(x)$ . So both regions intersect  $\text{supp}^-(x)$ . This contradicts the fact that  $\text{supp}^-(x)$  induces a connected subgraph of  $G$ , which proves (10).  $\square$

#### 2.4. Proof of Theorem 1.3.

As  $G$  is planar, we may fix an embedding of  $G$  in  $S^2$ . When speaking of faces below, we mean faces with respect to this embedding.

We first show:

- (11) if  $a, b, c$  are distinct vertices on a face of  $G$ , then  $u_a, u_b$  and  $u_c$  are linearly independent.

Suppose not. Then there exists a nonzero vector  $x \in \ker M$  such that  $x_a = x_b = x_c = 0$ . Take such an  $x$  with  $\text{supp}(x)$  minimal. Lemma 2.1 implies that both  $\text{supp}^+(x)$  and  $\text{supp}^-(x)$  are nonempty and induce connected subgraphs of  $G$ .

As  $G$  is 3-connected,  $G$  has pairwise vertex-disjoint paths  $P_1, P_2, P_3$ , each starting in  $N(\text{supp}(x))$  and ending in  $a, b$  and  $c$  respectively. By (6), the  $P_i$  start in a vertex that are both in  $N(\text{supp}^+(x))$  and  $N(\text{supp}^-(x))$ . As  $\text{supp}^+(x)$  and  $\text{supp}^-(x)$  induce connected subgraphs of  $G$ , we can contract each of  $\text{supp}^+(x)$ ,  $\text{supp}^-(x)$ ,  $P_1$ ,  $P_2$ , and  $P_3$  to one vertex, so as to obtain a  $K_{2,3}$  with the three vertices of degree 2 on one common face, which is not possible. This shows (11).

This implies that each  $u_i$  is nonzero, and hence  $v_i$  is defined. It also implies that  $v_i$  and  $v_j$  are linearly independent if  $i$  and  $j$  are adjacent. So for adjacent vertices  $i$  and  $j$  there exists a unique shortest geodesic on  $S^2$  connecting  $v_i$  and  $v_j$ . This gives an extension of  $i \rightarrow u_i$  to a mapping  $\psi : G \rightarrow S^2$ . We will show that this is in fact an embedding.

To this end we show:



- (12) Let (say)  $1, \dots, k$  be the vertices around a face  $F$  of  $G$ , in this cyclic order. Then  $u_1, \dots, u_k$  are the extremal rays of the convex cone  $C$  they generate in  $\mathbb{R}^3$ , in this cyclic order.

Consider two consecutive vertices along  $F$ , say 1 and 2. Let  $H$  be the plane spanned by  $u_1, u_2$  and 0. It suffices to prove that  $u_3, \dots, u_k$  are all at the same side of  $H$ . Let  $x$  be a nonzero vector in  $\ker M$  satisfying  $x_1 = x_2 = 0$  ( $x$  is unique up to scalar multiplication, as  $u_1$  and  $u_2$  are linearly independent.) By (11),  $3, \dots, k$  belong to  $\text{supp}(x)$ , and hence none of  $u_3, \dots, u_k$  are on  $H$ . Suppose that they are not all at the same side of  $H$ . That is, both  $\text{supp}^+(x)$  and  $\text{supp}^-(x)$  intersect  $\{3, \dots, k\}$ . As  $G$  is 3-connected, there exist vertex-disjoint paths  $P_1$  and  $P_2$  starting in  $N(\text{supp}(x))$  and ending in 1 and 2 respectively. Contracting each of  $\text{supp}^+(x)$ ,  $\text{supp}^-(x)$ ,  $P_1$ , and  $P_2$  to one vertex, we obtain a  $K_4$  embedded in  $S^2$  with all four vertices on one face, which is not possible. This shows (12).

Next:

- (13) Let  $a \in V$ , and let (say)  $1, \dots, k$  be the vertices adjacent to  $a$ , in this cyclic order. Then the geodesics on  $S^2$  connecting  $v_a$  to  $v_1, \dots, v_k$  issue from  $v_a$  in this cyclic order.

Consider a nonzero vector  $x \in \ker M$  with  $x_a = x_1 = 0$ . We claim

- (14) there are no  $h, i, j$  with  $2 \leq h < i < j \leq k$  such that  $x_h > 0$ ,  $x_i \leq 0$ , and  $x_j > 0$ .

Suppose that such  $i, j$  and  $h$  do exist. Since  $\text{supp}^+(x)$  induces a connected subgraph of  $G$ , there is a path  $P$  connecting  $h$  and  $j$  and traversing vertices in  $\text{supp}^+(x)$  only. Together with  $a$  this gives a circuit  $C$ , dividing  $S^2$  into two (open) regions. Both regions contain vertices  $m$  with  $x_m \leq 0$  (namely 1 and  $i$ ), and hence vertices  $m \in N(\text{supp}^+(x))$  (using the 3-connectivity of  $G$ ). Any such vertex belongs to  $\text{supp}^-(x)$  or is adjacent to a vertex in  $\text{supp}^-(x)$ . So both regions intersect  $\text{supp}^-(x)$ . This contradicts the fact that  $\text{supp}^-(x)$  induces a connected subgraph of  $G$ . This proves (14).

This implies that  $v_2$  and  $v_k$  are on different sides of the plane  $H$  through  $v_a, v_1$ , and 0. Otherwise we can assume that  $x_2 > 0$  and  $x_k > 0$ . As  $a \in N(\text{supp}(x))$ , we know that  $a \in N(\text{supp}^-(x))$ . So  $x_i < 0$  for some  $i$  with  $2 < i < k$ . This contradicts (14). Moreover, (14) implies that the nodes  $i$  with  $x_i > 0$  occur contiguously, and also the  $i$  with  $x_i < 0$  occur contiguously. This implies (13) (using the symmetry of the  $1, \dots, k$ ).

Now it is easy to complete the proof. (12) and (13) imply that we can extend the mapping  $\psi : G \rightarrow S^2$  to a mapping  $\psi : S^2 \rightarrow S^2$  such that  $\psi$  is locally one-to-one. We show that  $\psi$  is one-to-one. To see this, note that there is a number  $k$  such that  $|\psi^{-1}(p)| = k$  for each  $p \in S^2$ . Now let  $H$  be any 3-connected planar graph embedded in  $S^2$ , with  $v$  vertices,  $e$  edges, and  $f$  faces. Then  $\psi^{-1}(H)$  is a graph embedded in  $S^2$ , with  $kv$  vertices,  $ke$  edges, and  $kf$  faces. Hence by Euler's formula,  $2k = kv - ke + kf = 2$ , and so  $k = 1$ . Therefore,  $\psi$  is one-to-one, and  $\psi$  embeds  $|G|$  in  $S^2$ .  $\square$

### BIBLIOGRAPHY

- [1] Y. COLIN de VERDIÈRE, Sur un nouvel invariant des graphes et un critère de planarité, *Journal of Combinatorial Theory, Series B*, 50 (1990), 11–21 [English translation: On a new graph invariant and a criterion for planarity, in: *Graph Structure Theory* (N. Robertson, P. Seymour, eds.), Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island, 1993, 137–147].
- [2] H. van der HOLST, A short proof of the planarity characterization of Colin de Verdière, *Journal of Combinatorial Theory, Series B*, 65 (1995), 269–272.
- [3] H. van der HOLST, *Topological and Spectral Graph Characterizations*, Ph.D. Thesis, University of Amsterdam, Amsterdam, 1996.
- [4] H. van der HOLST, L. LOVÁSZ, A. SCHRIJVER, On the invariance of Colin de Verdière's graph parameter under clique sums, *Linear Algebra and its Applications*, 226 (1995), 509–517.
- [5] L. LOVÁSZ, A. SCHRIJVER, A Borsuk theorem for antipodal links and a spectral characterization of linklessly embeddable graphs, *Proceedings of the American Mathematical Society*, 126 (1998), 1275–1285.

L. LOVÁSZ,  
 Department of Computer Science  
 Yale University  
 New Haven CT 06520 (USA).  
 A. SCHRIJVER,  
 CWI  
 Kruislaan 413  
 1098 SJ Amsterdam (The Netherlands).  
 &  
 University of Amsterdam  
 Department of Mathematics  
 Plantage Muidergracht 24  
 1018 TV Amsterdam (The Netherlands).