INFORMATION AND COMPUTATION 75, 264–288 (1987)

Fixed Point Equations with Parameters in the Projective Model

EVANGELOS KRANAKIS*

Centre for Mathematics and Computer Science, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

Existence and uniqueness theorems are given for solving infinite and finite systems of fixed point equations with parameters in the projective model (a natural model in the calculus of communicating processes). The results obtained are derived by exploiting the special topological and combinatorial properties of the projective model and the polynomial operators defined on it. The topological methods employed in the proofs of the existence and uniqueness theorems use a combination of the following three ideas: compactness argument, density argument, and Banach's contraction principle. As a converse to the uniqueness theorem it is also shown, using combinatorial methods, that in certain signatures guarded equations are the only ones that have unique fixed points. © 1987 Academic Press, Inc.

1. INTRODUCTION

There has been a lot of effort in the current literature to understand the mathematical behavior of processes. Beginning with Milner's (1980) seminal work on "Calculus of Communicating Systems," an attempt was made to bring the provability of correctness of computer programs under a solid mathematical foundation. In fact, one of Milner's main contributions is to regard the basic concepts of communication and parallelism as algebraic in nature. Motivated from this, Bergstra and Klop gave an axiomatic-algebraic framework for studying processes (see Bergstra and Klop, 1986, for a survey introduction to their equational laws), which is more easily amenable to formal analysis and mathematical proof verification. In many respects their axiomatization constitutes a formal analog of some basic concepts of Milner's "Calculus of Communicating Systems."

* This research was carried out while the author was visiting the Computer Science Department of the University of Amsterdam and it was partially supported by Esprit under Contract 432, Meteor. I would like to thank all the participants of the P.A.M. seminar for their numerous comments. However, I am particularly indebted to J. Baeten, W. Bouma, and J. W. Klop, whose valuable criticisms have guided me throughout my research on the present paper. The comments of an anonymous referee helped in improving the presentation of the proof of Theorem 1.6.

0890-5401/87 $3.00

Starting from a given set of *atomic processes* (or *actions*) one can assemble large systems of processes. The atomic processes of such a system may interact with one another, communicate, be executed in parallel, or even lead to a deadlock (see Hoare, 1985). The experience accumulated from studying the bahavior of processes has led to a set of equational laws (see Bergstra and Klop, 1986). In the Table I the axioms of the theory of the algebras of communicating processes are given (the reader is advised to look in Bergstra and Klop (1986) for details and further discussion of the axiom system, as well as to (Milner, 1980; Hoare, 1985) for a justification of the introduction of these equational rules.) The given signature of the axiom system consists of: an **atom** a, for each $a \in A$, where A is the set of atomic processes, $+$ (**alternative composition** or **sum**), $\cdot$ (**sequential composition** or **product**), $\|$ (**parallel composition** or **merge**), $\mathbin{\|\!\lfloor}$ (**left merge**), $|$ (**communication merge**), ∂_H (**encapsulation**, where H is a subset of the set A of atoms), the atomic process δ (**deadlock** or **failure**), and the atomic process τ (**silent** or **internal action**). (Both τ, δ are atomic actions but since they behave differently from the rest of the atomic processes it will be more appropriate in the sequel not be considered members of the set A.) The axioms of process algebras are listed in Table I.

TABLE I

Axioms of Process Algebras

Basic axioms

$$x + y = y + x$$
$$x + (y + z) = (x + y) + z$$
$$x + x = x$$
$$(x + y) \cdot z = x \cdot z + y \cdot z$$
$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$
$$x + \delta = x$$
$$\delta \cdot x = \delta$$

τ-axioms

$$x \cdot \tau = x$$
$$\tau \cdot x + x = \tau \cdot x$$
$$a \cdot (\tau \cdot x + y) = a \cdot (\tau \cdot x + y) + a \cdot x$$
$$\tau \mathbin{\|\!\lfloor} x = \tau \cdot x$$
$$(\tau \cdot x) \mathbin{\|\!\lfloor} y = \tau \cdot (x \| y)$$
$$\tau | x = x | \tau = \delta$$
$$(\tau \cdot x) | y = x | (\tau \cdot y) = x | y$$

Merge axioms

$$x \| y = y \mathbin{\|\!\lfloor} x + x \mathbin{\|\!\lfloor} y + x | y$$
$$a \mathbin{\|\!\lfloor} x = a \cdot x$$
$$(a \cdot x) \mathbin{\|\!\lfloor} y = a \cdot (x \| y)$$
$$(x + y) \mathbin{\|\!\lfloor} z = x \mathbin{\|\!\lfloor} z + y \mathbin{\|\!\lfloor} z$$
$$(a \cdot x) | b = a | (b \cdot x) = (a | b) \cdot x$$
$$(a \cdot x) | (b \cdot y) = (a | b) \cdot (x \| y)$$
$$(x + y) | z = x | z + y | z$$
$$z | (x + y) = z | x + z | y$$

Communication axioms

$$a | b = b | a$$
$$(a | b) | c = a | (b | c)$$
$$\delta | a = \delta$$

Encapsulation axioms

$$\partial_H(\tau) = \tau$$
$$\partial_H(a) = a \text{ if } a \in H$$
$$\partial_H(a) = \delta \text{ if } a \in A - H$$
$$\partial_H(x + y) = \partial_H(x) + \partial_H(y)$$
$$\partial_H(x \cdot y) = \partial_H(x) \cdot \partial_H(y)$$

The communication function $|: A_\delta \times A_\delta \to A_\delta$(where A_δ consists of the atoms in A including δ) is initially defined only on atomic processes. Then it is extended to all finite terms (including τ) using the τ-, merge, and communication axioms. In the absence of communication, axiom $x \| y = y \mathbin{\|\!\_} x + x \mathbin{\|\!\_} y + x | y$ should be replaced with the new axiom $x \| y = y \mathbin{\|\!\_} x + x \mathbin{\|\!\_} y$, (i.e., the absence of communication is interpreted by the equation: $x | y = \delta$, for all x, y). The theory consisting of the first five basic axioms together with the first four merge axioms is known as (**basic**) **process algebra** and is abbreviated by PA. ACP, the **algebra of communicating processes** consists of the basic, merge, communication, and encapsulation axioms. Finally, ACP_τ consists of ACP plus the τ-axioms. As usual, the universal quantifiers, which quantify the variables x, y, z in the axioms in Table I are omitted. In addition, the letters a, b range over atoms in A_δ. From now on and for the rest of the paper the process-product sign $\cdot$ will be omitted.

In this axiomatic framework one can define the so-called **term** (or **initial**) **model** A_ω (= the set of all processes built up from the atomic processes $a \in A$, including δ, τ if they belong to the signature, via the operations in the given signature), as well as the models A_n, for each $n > 0$. More formally, A_ω is the least set S of finite strings such that S contains all the constants of the given signature, and S is closed under the operations of the given signature.

(The reader should be aware of all the possible signatures arising in the present study; practically every subset of $+$, $\cdot$, $\|$, $\mathbin{\|\!\_}$, $|$, ∂_H, δ, τ, a $(a \in A)$ is a possible signature and hence it can give rise to a different term model A_ω. It would be very cumbersome, however, to keep a different notation for A_ω for each possible signature. Instead, it will be left to the reader to derive from the context what the proper signature in each case is.) In addition, it should be pointed out that if one thinks of the elements of A_ω as finite trees with edges labeled by atoms then A_n can be considered as consisting of those trees in A_ω which have height at most n (see Bergstra and Klop, 1986, for more details).

Given any term t in A_ω and any positive integer n let $(t)_n$ be the subtree of t of height at most n obtained from t by deleting all those nodes which are located at height bigger than n. In a sense. $(\cdot)_n$ can be considered as the **projection** of the term model A_ω onto the model A_n. Now the **projective** (or **standard**) **model**, denoted by A^∞ consists of all infinite sequences $(p_1, p_2, \ldots, p_n, \ldots)$ such that $p_n \in A_n$ and $(p_{n+1})_n = p_n$, for all $n > 0$.

In the study of the theory of concurrent processes one is particularly interested in solving **fixed point equations**, i.e., equations of the form $x = T(x)$, where $T(x)$ is a (**polynomial**) **operator** built up from the atomic processes, the variable x, and the operations of the given signature. Such equations or even systems of equations are often used to implicitly describe the behavior of processes that arise naturally in the description of several well-known concepts in computer science, like *stack*, *bag*, *counter*, *mutual exclusion*, etc. (see Bergstra and Klop, 1986; Hoare, 1985, for a description of such concepts via the terminology of process algebras).

However, this methods of self-referential description of a process x through the fixed point equation $x = T(x)$ is meaningful only if it can be guaranteed that $x = T(x)$ has a unique solution, e.g., any process x is a solution of $x = x$. To ensure the uniqueness of solutions of $x = T(x)$ one is led to define the notion of **guardedness**. Intuitively, x is **guarded** in $T(x)$, if every occurrence of x in $T(x)$ occurs within the scope of a subterm of $T(x)$ of the form $a(\ldots x \ldots)$ (see Section 5 for a definition of the concept of guardedness).

The following is an example of an infinite system of fixed point equations. For more examples of both infinite and finite systems the reader is advised to consult (Bergstra and Klop, 1986; Hoare, 1985; Milner, 1980).

EXAMPLE 1.1 (movements of an object on an infinite chess board). Let the possible moves of an object moving on an infinite chess board be u = up, d = down, l = left, r = right. For each integer $n = \ldots, -2, -1, 0, 1, 2, \ldots$, let x_n be the behavior of the object when it is on the nth row. Then the complete bahvior of the object can be described by the following system of fixed point equations.

$$x_n = u \cdot x_{n+1} + d \cdot x_{n-1} + (l + r) \cdot x_n,$$

where $n = \ldots, -2, -1, 0, 1, 2, \ldots$ is an integer and the set of atomic processes is $A = \{u, d, l, r\}$.

In general, one is particularly interested in establishing criteria that will guarantee both the existence as well as the uniqueness of solutions of systems of fixed point equations. For finite systems without parameters two such theorems are given in Bergstra and Klop, 1986, 1987) for the above mentioned projective model (in the signature $+, \cdot, \|, \lfloor\!\lfloor, |, \partial_H, a\ (a \in A)$).

THEOREM 1.1 (existence theorem). *Every finite system*

$$\Sigma = \{x_k = T_k(x_1, \ldots, x_n) : k = 1, \ldots, n\}$$

of fixed point equations has a solution in $(A^\infty)^n$.

Theorem 1.1 is stated in Bergstra and Klop, 1986, Theorem 1.1.9) only for the signature $+, \cdot, \|, \lfloor\!\lfloor, a\ (a \in A)$. In addition, (Bergstra and Klop, 1982) provides a proof of the theorem for the case $n = 1$ and the signature $+, \cdot, \|, \lfloor\!\lfloor, a\ (a \in A)$. The full statement of Theorem 1.1, as this is stated above, was communicated to the author by J. W. Klop and will appear in a forthcoming revised version of (Bergstra and Klóp, 1982). A similar existence theorem for systems of arbitrary size (without parameters) has recently been proved by R. J. Van Glabbeek for the case of the **countably branching graph model** (and is implicitly mentioned in (Van Glabbeek, 1987).

THEOREM 1.2 (uniqueness theorem). *Every finite system*

$$\Sigma = \{x_k = T_k(x_1, ..., x_n): k = 1, ..., n\}$$

of guarded fixed point equations has a unique solution in $(A^\infty)^n$.

The present paper generalizes both of the above theorems in two directions: on the one hand it allows the systems to have parameters in the projective model and on the other hand it permits systems with a countable number of fixed point equations. In the case of finite systems without parameters the finiteness of the set A of atoms is not an issue; one can assume, without loss of generality, that A is a finite set containing all the atoms occurring in all the specifications of the given finite system of fixed point equations. However, that situation is different in the case of a system of fixed point equations with parameters. This is due to the fact that for infinite A there exist processes in A^∞ with an infinite number of occurrences of atoms, which can be parameters in a fixed point equation of the form $x = T(x)$, e.g., $p = (a_1, a_1 \cdot a_2, ..., a_1 \cdots a_n, ...)$, where $a_1, ..., a_n, ...$ is an infinite list of mutually distinct atoms in A. In particular, the following result will be proved.

THEOREM 1.3 (extended existence theorem; A is finite). *Every countable system*

$$\Sigma = \{x_k = T_k(x_1, ..., x_{n(k)}, p_1, ..., p_{m(k)}): k = 1, ..., n, ...\}$$

of fixed point equations with parameters $p_1, ..., p_m, ... \in A^\infty$ *has a solution in* $(A^\infty)^\omega$.

In the case of arbitrary A (i.e., posssibly infinite) the assertion of Theorem 1.3 is only known for finite systems Σ, as this is stated in the next theorem.

THEOREM 1.4. (extended existence theorem; A is arbitrary). *Every finite system*

$$\Sigma = \{x_k = T_k(x_1, ..., x_n, p_1, ..., p_m): k = 1, ..., n\}.$$

of fixed point equations with parameters $p_1, ..., p_m \in A^\infty$ *has a solution in* $(A^\infty)^n$.

As an immediate corollary of Theorem 1.4 one obtains, for arbitrary A, an existence theorem for countable, infinite, diagonal systems of fixed points equations with parameters. (Call an arbitrary system $\Sigma = \{x_k = T_k(x_1, ..., x_{n(k)}, p_1, ... p_{m(k)}): k = 1, ..., n, ...\}$ of fixed point equations **diagonal**, if for all k, $n(k) \leqslant k$.) Moreover, the notion of **guardedness** given in (Bergstra and Klop, 1986) is generalized to include fixed point equations with parameters. **Guarded** operators $T(x)$ do not always provide equations $x = T(x)$ which have unique solutions in every model of process algebra (pathological counterexamples are in fact easy to give). However, it is one of the many interesting properties of the projective model A^∞ that for the above mentioned full signature $+$, $\cdot$, $\|$, $\|\!\_$, $|$, ∂_H, δ, τ, a $(a \in A)$, *which includes the silent action*, one can prove the following uniqueness result (**notice the omission of the abstraction operator** τ_I, which is defined in *op. cit.*)

THEOREM 1.5 (extended uniqueness theorem; A is arbitrary). *Every countable system*

$$\Sigma = \{x_k = T_k(x_1, ..., x_{n(k)}, p_1, ..., p_{m(k)}): k = 1, ..., n, ...\}$$

of guarded fixed point equations with parameters $p_1, ..., p_m, ... \in A^\infty$ *has a unique solution in* $(A^\infty)^\omega$.

This extends the previous results of (Bergstra and Klop, 1986; Rounds, 1985) by allowing τ in the signature. In fact, the last theorem is proved for arbitrary (even uncountable) systems of fixed point equations. However, it appears that it is only the countable case which is applicable in practice.

The converse of the uniqueness theorem appears to be much more intricate. In general, one is interested to know if the notion of guardedness given in the paper fully captures all the specifications which have unique solutions in the projective model. (Experience shows, e.g., see Chang and Keisler, 1973, that results of this type are not very easy to prove.) To be more specific the following partial converse to Theorem 1.5 is proved.

THEOREM 1.6 (converse of the uniqueness theorem). *Let* $T(x)$ *be an operator in the signature* $+$, $\cdot$, $\|$, $\|\!\_$, a, $(a \in A)$ *such that the equation* $x = T(x)$ *has a unique solution in* A^∞. *If* A *has an atom which does not occur in the operator* $T(x)$ *then there exists a guarded operator* $S(x)$, *without any parameters other than the atomic processes in* A, *such that the equations* $x = T(x)$, $x = S(x)$ *have exactly the same solution in* A^∞. *In addition, if* A *has at least two distinct atoms then* $T(x)$ *itself must be guarded.*

The arrangement of the following sections is as follows: Section 2 gives all the preliminary results concerning the topological nature of the projective model which will be used throughout the paper; Section 3 outlines the three main ideas to be used for solving fixed equations with parameters; Section 4 proves the existence theorems; Section 5 introduces the notion of guarded equation, which includes the silent action and is to be used in the proof of the uniqueness theorem in Section 6; in addition, Section 6 gives a proof of the converse of the uniqueness theorem; Section 7 raises related issues and discusses some open problems.

In addition, the results of the paper are stated and proved in the formalism of (Bergstra and Klop, 1986). This, however, is not necessary and the formalism of either (Hoare, 1985 or Milner, 1980) could have been used.

Remark on Notation. Throughout the present paper $T(x_1, ..., x_n)$, $S(x_1, ..., x_n)$, etc., with or without subscripts and superscripts, will always denote (polynomial) operators, i.e., terms built up from the variables $x_1, ..., x_n$, the atoms in A, and the operations of the given signature. More formally, the set $A_\omega[x_1, ..., x_n]$ of such polynomial operators is the least set F of finite strings such that

F contains all the variables $x_1, ..., x_n$,
F contains all the constants of the given signature, and
F is closed under the operations of the given signature.

2. Topology of the Projective Model

Let A_ω be the term model defined in Section 1. It consists of all finite terms modulo the equivalence relation determined by the corresponding theory in the given signature. In addition, let the projection function $(\cdot)_n$ be defined as follows on A_ω:

$$\begin{aligned}
(a)_n &= a, \\
(at)_1 &= a, \\
(at)_n &= a(t)_{n-1}, \qquad \text{for} \quad n > 1, \\
(t+t')_n &= (t)_n + (t')_n, \\
(\tau)_n &= \tau, \\
(\tau t)_n &= \tau(t)_n.
\end{aligned}$$

Notice that unlike τ, the atomic process δ (deadlock) is treated just like any other atom $a \in A$ in the definition above. Let $A_n = \{(t)_n : t \in A_\omega\}$. The projective model A^∞ consists of all sequences $(p_1, ..., p_n, ...)$ such that each $p_n \in A_n$ and $(p_{n+1})_n = p_n$, for all $n > 0$. The operations are defined on A^∞

in a natural way; thus, following (Bergstra and Klop, 1986), if $\square$ is any binary operation on A_ω one defines a new binary operation $\square'$, which for convenience will also be denoted by $\square$, as

$$(p_1, ..., p_n, ...) \square (q_1, ..., q_n, ...) = ((p_1 \square q_1)_1, ..., (p_n \square q_n)_n, ...).$$

(The unitary operation ∂_H is treated in a similar fashion.) The term model A_ω can be embedded in a natural way in the projective model A^∞; to any finite term t associate the infinite sequence $((t)_1, ..., (t)_n, ...)$. Because of this it is identified with a subset of the projective model (this also explains the convention above to use the same symbol for the corresponding operations in A_ω, A^∞). Extend the projection functions to A^∞ by defining $(p)_n = p_n$, for all $p = (p_1, ..., p_n, ...) \in A^\infty$ and all $n > 0$. For any two distinct elements p, q of A^∞ let $k(p, q)$ = the least $n > 0$ such that $(p)_n$ is not equal to $(q)_n$. This definition makes it possible to endow A^∞ with a **metric space structure**. Indeed, define the **distance** $d(p, q)$ between p, q by

$$d(p, q) = \begin{cases} 2^{-k(p,q)} & \text{if } p \neq q \\ 0 & \text{if } p = q. \end{cases}$$

This metric, due to Hausdorff, was used by Arnold and Nivat (1980) in the context of "denotational semantics of concurrency." An essentially equivalent metric was also defined by de Bakker and Zucker (1982). For additional information and further properties of this metric the reader is advised to consult (Lloyd, 1984; Rounds, 1985).

The following important result summarizes all the basic properties of the metric space (A^∞, d) and will be used frequently in the sequel. Its proof is omitted, but the interested reader can find the essential details in (Lloyd, 1984; Arnold and Nivat, 1980).

THEOREM 2.1 (in the signature $+, \cdot, \|, \lfloor\!\lfloor, |, \partial_H, \delta, \tau, a \ (a \in A)$). (i) (A^∞, d) *is an ultrametric space, i.e., it satisfies the following three properties for all elements* $p, q, r \in A^\infty$,

(a) $d(p, q) = 0$ *if and only if* $p = q$,

(b) $d(p, q) = d(q, p)$,

(c) $d(p, q) \leqslant \max\{d(p, r), d(r, q)\}$.

(ii) $p^{(r)} \to p$ *if and only if* $\forall n \, \exists m \, \forall k \geqslant m \, [(p^{(k)})_n = (p)_n]$.

(iii) (A^∞, d) *is the metric completion of the metric space* (A_ω, d'), *where* d' *is the restriction of* d *on* A_ω.

(iv) *For all* $p \in A^\infty$ *and each* $n > 0$, $d(p, (p)_n) \leqslant 2^{-n}$. *Hence,* $\lim_{n \to \infty} (p)_n = p$.

(v) *The operations* $(\cdot)_n \colon A^\infty \to A_n$ *are continuous.*

The forthcoming results of the paper will require a finer analysis of the relationship between the algebraic and topological structure of A^∞. The signature of the next few results (unless otherwise specified) is $+$, $\cdot$, $\|$, $\lfloor\!\lfloor$, $|$, ∂_H, δ, τ, a $(a \in A)$.

LEMMA 2.2 (Bergstra and Klop). *For any process $p \in A^\infty$ and any positive integers n, m, $((p)_n)_m = (p)_{\min\{n,m\}}$.*

Proof. It is enough to show that $((u)_n)_m = (u)_{\min\{n,m\}}$ holds for all finite terms $u \in A_\omega$ (the lemma will then follow from the continuity of $(\cdot)_n$ by passing to the limit and using the fact that A_ω is dense in A^∞). The proof is by induction on the length of the given term u. The result is obviously true at the initial step of the induction. Next, write u as a finite sum

$$u = \sum_i a_i u_i + \sum_r \tau v_r + \sum_j b_j + \tau,$$

where $a_i, b_j \in A_\delta$ are atomic processes and u_i, v_r are finite terms. In the representaton of u above some sums may be empty, in which case they are set equal to δ, and the single summand τ may or may not be missing (in the presence of the single summand τ, however, the next to the last summand is not necessary since $b_j \tau = b_j$ and sums of this type are included in the first summand). Then it is true that

$$\begin{aligned}
((u)_n)_m &= \left(\left(\sum_i a_i u_i + \sum_r \tau v_r + \sum_j b_j + \tau\right)_n\right)_m \\
&= \left(\sum_i a_i (u_i)_{n-1} + \sum_r \tau (v_r)_n\right)_m + \sum_j b_j + \tau \\
&= \sum_i a_i ((u_i)_{n-1})_{m-1} + \sum_r \tau ((v_r)_n)_m + \sum_j b_j + \tau \\
&= \sum_i a_i (u_i)_{\min(n-1,m-1)} + \sum_r \tau (v_r)_{\min(n,m)} + \sum_j b_j + \tau \\
&= (u)_{\min\{n,m\}}.
\end{aligned}$$

This completes the proof of the lemma. ∎

LEMMA 2.3. *Let $\square$ (resp. ∂_H) denote any of the binary (resp. unary) operations in the signature $+$, $\cdot$, $\|$, $\lfloor\!\lfloor$, $|$, ∂_H, a $(a \in A)$. Then for any p, $q \in A^\infty$ and any integer n the following equalities hold:*

$$(p \square q)_n = ((p)_n \square (q)_n)_n, \qquad (\partial_H(p))_n = (\partial_H((p)_n))_n.$$

Proof. As before it is enough to prove the assertion of the lemma for finite terms $u, v \in A_\omega$ (the lemma will then follow by passing to the limit using the fact that A_ω is dense in A^∞). The proof is tedious but straightforward and can be given by induction on the construction of the terms u, v simultaneously for all the operations in the given signature; it is similar to the proof of Lemma 2.2 and is left as an exercise to the reader. ∎

As an immediate corollary it can be shown that

LEMMA 2.4. *Let* $\square$ *(resp.* ∂_H*) denote any of the binary (resp. unary) operations in the signature* $+, \cdot, \|, \lfloor\!\lfloor, |, \partial_H, a$ $(a \in A)$. *Then for any* $p, p_1, q, q_1 \in A^\infty$ *the following inequalities hold*:

$$d(p \square p_1, q \square q_1) \leqslant \max\{d(p, q), d(p_1, q_1)\},$$

$$d(\partial_H(p), \partial_H(q)) \leqslant d(p, q).$$

Consequently, for any operator $T(x_1, ..., x_n)$ *and any* $p_1, ..., p_n, q_1, ..., q_n \in A^\infty$,

$$d(T(p_1, ..., p_n), T(q_1, ..., q_n)) \leqslant \max\{d(p_1, q_1), ..., d(p_n, q_n)\}.$$

Proof. The second part of the lemma concerning operators is an immediate consequence of the first part using induction on the construction of the operator T. To prove the first part put $k = k(p, q)$, $k_1 = k(p_1, q_1)$, $s = \min\{k, k_1\}$. Then it is immediate that for all $i < s$, $(p)_i = (q)_i$ and $(p_1)_i = (q_1)_i$. Now it follows from Lemma 2.3 that $(p \square p_1)_{s+1} = (q \square q_1)_{s+1}$ and hence, $s \leqslant k(p \square p_1, q \square q_1)$, which completes the proof of the lemma. ∎

In the sequel it will be useful to know exactly when the metric space (A^∞, d) is compact. In the full signature $+, \cdot, \|, \lfloor\!\lfloor, |, \partial_H, \delta, \tau, a$ $(a \in A)$ it is not compact as this is shown by the following example.

EXAMPLE 2.5 (J. W. Klop, unpublished). In the presence of τ the metric space (A^∞, d) is not compact. To see this, construct a sequence $\{t_n\}$ of finite processes such that for all $n > m$, t_n is different from t_m and $(t_n)_1 = t_n$. (Such a sequence can not have any convergent subsequence since $d(t_n, t_m) = \frac{1}{2}$, for $n > m$.) The first five members of the sequence are given by $t_0 = a$, $t_1 = \tau a$, $t_2 = \tau$, $t_3 = \tau(a + \tau)$, $t_4 = a + \tau a$. For higher indices one defines by induction

$$t_{4k+i} = \begin{cases} \tau t_{4k+i-1} & \text{if } i = 1, 3, \\ t_{4k+i-3} + t_{4k+i-5} & \text{if } i = 0, 2. \end{cases}$$

On the contrary, if τ is not present in the signature then the space A^∞ can be compact as the theorem below shows.

THEOREM 2.6 (In the full signature without τ). (i) *A is finite if and only if (A^∞, d) is compact.*

(ii) *In fact, if A is finite then (A^∞, d) must be topologically homeomorphic to the Cantor set.*

Proof. (i) ($\Leftarrow$) Assume on the contrary that A is infinite and let $a_1, ..., a_n, ..$ be an infinite sequence of pairwise distinct atoms in A. Then the sequence $\{a_n\}$ cannot have any convergent subsequence since $d(a_n, a_m) = \frac{1}{2}$, for $n > m$. Clearly, this is a contradiction.

($\Rightarrow$) Since τ is not in the signature and A is finite each A_n is finite and hence compact. It follows that A^∞ is compact, as the projective limit of compact spaces (see Dugundji, 1966, p. 429).

(ii) This is immediate from (Rinow, 1975, p. 223). A more direct proof can be given along the following lines. For each $u \in A_\omega$ let $C(u) = \{p \in A^\infty : (p)_n = u$, for some integer $n > 0\}$ and let $n(u) =$ the least n such that $(u)_n = u$. It can be shown that $\{C(u) : u \in A_\omega\}$ is a family of non-empty subsets of A^∞ such that for all $u, v \in A_\omega$ exactly one of the following three conditions holds:

$$C(u) \subseteq C(v), \qquad C(v) \subseteq C(u), \qquad C(u) \cap C(v) = \varnothing.$$

Moreover, each $C(u)$ is the (finite) disjoint intersection of those sets $C(v)$ such that $n(v) = n(u) + 1$ and $(v)_{n(u)} = u$. Finally, the homeomorphism between A^∞ and the Cantor set can be constructed as in (Dieudonné, 1968, p. 84). Details are left to the reader. ∎

3. SOLVING EQUATIONS WITH PARAMETERS

Suppose that it is desired to find a solution to an equation of the form $x = T(x)$. If the operator $T(x)$ is contractive (see Idea 3.3 below) then for any element $q \in A^\infty$, $\lim_{n \to \infty} T^n(q)$ (where T^n is the nth iterate of T) is the unique solution of the equation $x = T(x)$. However, if $T(x)$ is not contractive then Banach's contraction principle does not apply. Thus, one is faced with the problem of finding solutions to $x = T(x)$ for an arbitrary (not necessarily contractive) operator $T(x)$. Motivated by Banach's contraction principle one is tempted to prove that for any $q \in A_\omega$, $\lim_{n \to \infty} T^n(q)$ is a solution of the above mentioned equation. In fact, this idea works. An outline of the idea of the combinatorial proof, due to (Bergstra and Klop, 1982), is as follows. Let q be an arbitrary element of A_ω. One shows by induction on m that the sequence $\{T^n(q))_m : n \geqslant 0\}$ is constant, for all but a finite number of n's. This is done by induction on the construction of the operator T; to handle the operation $+$, which is also the most complex case, one needs to use König's infinity lemma (i.e., any infinite, finite branching tree has an infinite branch). For more details the reader should consult (Bergstra and Klop, 1982).

Now suppose that it is required to solve fixed point equations of the form $x = T(x, p)$, where $p \in A^\infty$ is a parameter and $T(x, y)$ is an operator in $A_\omega[x, y]$. Depending on the topological properties of the space (A^∞, d) and the topological structure of the operator T one of the following ideas can be used.

Idea 3.1: *Compactness Argument*

For each positive integer n consider the fixed point equation $x = T(x, (p)_n)$. Each such equation has a solution in A^∞ (by the existence theorem 1.1), say x_n, such that $x_n = T(x_n, (p)_n)$. However, if A^∞ is compact then the sequence $\{x_n\}$ must have a convergent subsequence, say $\{x_{n(k)}\}$, such that $\lim_{k\to\infty} x_{n(k)} = x \in A^\infty$. But it is clear from the continuity of the operator that

$$\begin{aligned} x &= \lim_{k\to\infty} x_{n(k)} \\ &= \lim_{k\to\infty} T(x_{n(k)}, (p)_{n(k)}) \\ &= T(\lim_{k\to\infty} x_{n(k)}, \lim_{k\to\infty} (p)_{n(k)}) \\ &= T(x, p). \end{aligned}$$

Thus, the limit point x is the desired solution of the given fixed point equation.

The main limitation of this method is that it works only in the case where the metric space (A^∞, d) is compact (this makes it impossible to use this idea in the case of an infinite set A of atoms or even include τ to the signature).

Idea 3.2: *Density Argument*

For each $t \in A_\omega$ let T_t be the operator obtained from T by substituting each occurrence of the variable y in $T(x, y)$ by t. The solution of the equation $x = T(x, t)$ is obtained as the limit of the sequence $\{T_t^n(a)\}$, where a is any atom and T_t^n is the nth iteration of the operator T_t (the existence of this limit is guaranteed by the main result of (Bergstra and Klop, 1982), as this was mentioned at the beginning of this section). Let $\sigma_T : A_\omega \to A^\infty$ be the function defined by

$$t \to \sigma_T(t) = \lim_{n\to\infty} T_t^n(a).$$

It can be shown that the function σ_T is uniformly continuous; in fact, the claim below states that it is nonexpansive.

CLAIM. $d(\sigma_T(u), \sigma_T(v)) \leqslant d(u, v)$, for all $u, v \in A_\omega$.

Proof of the Claim. Using the continuity of the distance function d one obtains that

$$d(\sigma_T(u), \sigma_T(v)) = \lim_{n \to \infty} d(T_u^n(a), T_v^n(a)) \leqslant d(u, v),$$

which proves the claim.

Thus, σ_T is a uniformly continuous mapping from the dense subset A_ω of A^∞ into the complete metric space A^∞. It follows that σ_T can be extended **by continuity** (see Dieudonné, 1968) to a continuous mapping ω_T: $A^\infty \to A^\infty$. Moreover, for any $p \in A^\infty$ it is true that $\omega_T(p) = \lim_{n \to \infty} \sigma_T((p)_n)$. Now it is possible to find a fixed point of the original equation. Indeed, $\sigma_T((p)_n) = T(\sigma_T((p)_n), (p)_n)$, for all $n > 0$. Using the continuity of the operator T and passing to the limit as $n \to \infty$ it follows that

$$\omega_T(p) = T(\omega_T(p), p),$$

as desired.

The main advantage of this method is that it works for any arbitrary set of atomic actions (i.e., finite or infinite). In fact, one uses only the density of A_ω (in A^∞) as well as the completeness of the metric space A^∞. Its main disadvantage, compared to the compactness argument, is that one must have a priori a **uniform** way of obtaining solutions of the equations $x = T(x, t)$ (i.e., uniform in t) as indeed was the case described above.

Idea 3.3: *Banach's Contraction Principle*

Any operator $T(x, p)$ (with parameter $p \in A^\infty$) determines a continuous (in fact, nonexpansive) mapping $x \to T(x, p)$ from A^∞ into A^∞ (see Lemma 2.4). In case it is a contraction, i.e., there exists a constant $1 > c \geqslant 0$ such that $d(T(x, p), T(y, p)) \leqslant c \cdot d(x, y)$, for all $x, y \in A^\infty$, one can find fixed points by simply iterating the operator. (An operator satisfying the above property is called **contractive**.) It follows from Banach's contraction principle (see Dugundji and Granas, 1982) that for any $q \in A^\infty$, $\lim_{n \to \infty} T^n(q, p)$ is the unique fixed point of the equation $x = T(x, p)$. For a more extensive discussion of the use of Banach's contraction principle in the theory of concurrent processes the reader is advised to consult (Lloyd, 1984; Rounds, 1985).

An extension of the three ideas considered above will be used extensively in the sequel in order to solve arbitrary systems of fixed point equations with parameters in A^∞.

4. Existence Theorems in the Projective Model

This section includes the complete proofs of Theorems 1.3 and 1.4. In the case of Theorem 1.3, it will be convenient to handle first countable systems without parameters; the general theorem will then follow by applying a compactness argument as in Idea 3.1. It is well known that the *Tychonoff product* $(A^{\infty})^{\omega}$ of countably many copies of A^{∞} is compact, if A^{∞} is compact. This can be seen by defining a new metric d_1 on $(A^{\infty})^{\omega}$ as

$$d_1(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d(x_n, y_n),$$

where $x = (x_n : n \geqslant 1)$, $y = (y_n : n \geqslant 1) \in (A^{\infty})^{\omega}$ (see Dieudonné, 1968). Returning to the proof of Theorem 1.3 the following lemma, which is a nonparametric version of Theorem 1.3, can be proved.

Lemma 4.1 (A is finite). *Every countable system (without parameters)*

$$\Sigma = \{x_k = T_k(x_1, ..., x_{n(k)}) : k > 0\}$$

has a solution in $(A^{\infty})^{\omega}$.

Proof. Without loss of generality it can be assumed that each $n(k) \geqslant k$. Let $a \in A$ be an arbitrary but fixed atomic process. For each positive integer m consider the following finite system Σ_m of m fixed point equations:

$$x_1 = T_1(x_1, ..., x_{r_m(1)}, a, ..., a)$$
$$x_2 = T_2(x_1, ..., x_{r_m(2)}, a, ..., a)$$
$$\cdots$$
$$x_m = T_m(x_1, ..., x_{r_m(m)}, a, ..., a),$$

where for each $k \geqslant m$,

$$r_m(k) = \begin{cases} n(k) & \text{if } n(k) \leqslant m \\ m & \text{if } n(k) > m \end{cases}$$

and the number of a's occurring in the kth equation is $m - r_m(k)$ (in other words, one replaces each occurrence of the variables $x_{r_m(k)+1}, ..., x_{n(k)}$ by a). Thus, $r_m(m) = m$, for all m. Theorem 1.1 implies that each system Σ_m has a solution, say $s_{1,m}, ..., s_{m,m}$ such that for all $k = 1, ..., m$,

$$s_{k,m} = T_k(s_{1,m}, ..., s_{r_m(k),m}, a, ..., a).$$

For each m let s_m denote the infinite sequence $(s_{1,m}, ..., s_{m,m}, a, ..., a, ...)$. Since A^∞ is compact so is $(A^\infty)^\omega$. It follows that the sequence $\{s_m\}$ has a convergent subsequence, say

$$s_{m(i)} \to u = (u_1, ..., u_m, ...), \qquad \text{as} \quad i \to \infty.$$

By the definition of s_m it is true that for all integers i and all $k = 1, ..., m(i)$,

$$s_{k,m(i)} = T_k(s_{1,m(i)}, ..., s_{r_{m(i)}(k),m(i)}, a, ..., a).$$

Now fix the integer k. Then there exists an integer i_0 such that for all $i \geqslant i_0$, $m(i) \geqslant n(k)$, and hence $r_{m(i)}(k) = n(k)$. Thus, the above equation becomes

$$s_{k,m(i)} = T_k(s_{1,m(i)}, ..., s_{n(k),m(i)}, a, ..., a).$$

Using the continuity of T_k and passing to the limit as $i \to \infty$ one easily obtains that

$$u_k = T_k(u_1, ..., u_{n(k)}).$$

This completes the proof of the lemma. ∎

Proof of Theorem 1.3. Let Σ be the system of Theorem 1.3. For each integer r consider the countably infinite system Σ_r given by the equations

$$x_1 = T_1(x_1, ..., x_{n(1)}, (p_1)_r, ..., (p_{m(1)})_r),$$
$$x_2 = T_2(x_1, ..., x_{n(2)}, (p_1)_r, ..., (p_{m(2)})_r),$$
$$\cdots$$
$$x_k = T_k(x_1, ..., x_{n(k)}, (p_1)_r, ..., (p_{m(k)})_r),$$
$$\cdots.$$

Each Σ_r is a countable system of fixed point equations **without parameters**. Hence, Lemma 4.1 applies to each Σ_r. For each r, let $s_r = (s_{1,r}, ..., s_{k,r}, ...)$ be a solution of the system Σ_r, i.e., for all integers k, r, it is true that

$$s_{k,r} = T_k(s_{1,r}, ..., s_{n(k),r}, (p_1)_r, ..., (p_{m(k)})_r).$$

Using the compactness of the Tychonoff product space $(A^\infty)^\omega$, it follows that the sequence $\{s_r\}$ has a convergent subsequence,

$$s_{r(i)} \to u = (u_1, ..., u_k, ...),$$

as $i \to \infty$. Let k be fixed. It follows from the definition of s_r that for all integers i,

$$s_{k,r(i)} = T_k(s_{1,r(i)}, ..., x_{n(k),r(i)}, (p_1)_{r(i)}, ..., (p_{m(k)})\, r(i)).$$

Passing to the limit as $i \to \infty$ and using the continuity of the operator T_k one easily obtains that

$$u_k = T_k(u_1, ..., u_{n(k)}, p_1, ..., p_{m(k)}).$$

This completes the proof of Theorem 1.3. ∎

Proof of Theorem 1.4. Let Σ be the system of Theorem 1.4. For each $t_1, ..., t_m \in A_\omega$ and each positive integer $i = 1, ..., n$ define the functions

$$\sigma_i: (A_\omega)^m \to A^\infty: (t_1, ..., t_m) \to \sigma_i(t_1, ..., t_m) = \lim_{k \to \infty} \sigma_{i,k}(t_1, ..., t_m),$$

where the sequence $\sigma_{i,k}(t_1, ..., t_m)$ is defined by induction on k as follows (for simplicity put $t = (t_1, ..., t_m)$):

$$\sigma_{i,0}(t) = a, \qquad (k = 1, ..., n),$$
$$\sigma_{1,k+1}(t) = T_{k+1}(\sigma_{1,k}(t), ..., \sigma_{n,k}(t), t),$$
$$\sigma_{i+1,k+1}(t) = T_{k+1}(\sigma_{1,k+1}(t), ..., \sigma_{i,k+1}(t), \sigma_{i+1,k}(t), ..., \sigma_{n,k}(t), t) \quad (k \geqslant 0),$$

where $0 \leqslant i < n$ and $a \in A$ is a fixed atomic process. (The proof of the existence of the limit $\lim_{k \to \infty} \sigma_{i,k}(t)$ will appear in a revised version of (Bergstra and Klop, 1982)). It is clear that for all i, k there exists an operator $S_{i,k}(u_1, ..., u_n, v_1, ..., v_m)$ such that for all t,

$$\sigma_{i,k+1}(t) = S_{1,k}(\sigma_{1,1}(t), ..., \sigma_{i-1,k}(t), \sigma_{i,k+1}(t), ..., \sigma_{i,n}(t), t).$$

The proof is by induction on i; for each i one proves the assertion above in succession for $i = 1$, $i = 2$, ..., $i = n$. Just as in Section 3 it can be shown that the functions σ_i are uniformly continuous (in fact they are nonexpansive). Hence, each σ_i can be extended to a uniformly continuous mapping $\omega_i: (A^\infty)^m \to A^\infty$. The rest of the details are as in Idea 3.2 and are left to the reader. This completes the proof of Theorem 1.4. ∎

Recall from the introduction that a system $\Sigma = \{x_k = T_k(x_1, ..., x_{n(k)}, p_1, ..., p_{m(k)}: k > 0\}$ is **diagonal**, if for all k, $n(k) \leqslant k$. As an immediate corollary of Theorem 1.4 one can also show that

COROLLARY 4.2 (existence theorem for diagonal systems; A is arbitrary). *Every countable diagonal system*

$$\Sigma = \{x_k = T_k(x_1, ..., x_k, p_1, ..., p_{m(k)}: k > 0\},$$

of fixed point equations with parameters $p_1, ..., p_m, ... \in A^\infty$ *has a solution in* $(A^\infty)^\omega$.

Proof. By Theorem 1.4, $x_1 = T_1(x_1, p_1, ..., p_{m(1)})$ has a solution, say s_1. Apply Theorem 1.4 once more to find a solution of $x_2 = T_1(s_1, x_2, p_1, ..., p_{m(2)})$, say s_2. Proceed in this fashion to obtain a solution $(s_1, ..., s_k, ...)$ of the system Σ. This completes the proof of the corollary. ∎

5. Guarded Equations

It will be proved in the sequel that a sufficient condition for a fixed point equation to have a unique solution is to be **guarded**. This last concept will be made precise in the sequel. However, in order to obtain the most general definition of **guardedness** it will be necessary to define first the notion of **guard** (see also Hoare, 1985, p. 28).

Definition 5.1 (in the signature $+, \cdot, \|, \lfloor\!\lfloor, |, \partial_H, \delta, \tau, a\ (a \in A)$). Call $g \in A^\infty$ a **guard** if and only if every finite branch of (the tree corresponding to) g has an edge which is labeled with an atomic process other than τ.

The definition above arises from the following informal observation. To obtain a uniqueness theorem for fixed point equations one is tempted to consider fixed point equations of the form $x = T(x)$ such that $T(x)$ is contractive. Clearly, by Lemma 2.4, $T(x)$ is not distance increasing (at least in the signature $+, \cdot, \|, \lfloor\!\lfloor, |, \partial_H, \delta, \tau, a\ (a \in A)$). Since an operator T is built up from the signature $+, \cdot, \|, \lfloor\!\lfloor, |, \partial_H, \delta, \tau, a\ (a \in A)$ and the variable x, it is apparent that one must first search for a distance contraction principle for the *nontrivial* operators of minimal length. Such terms are of the form gv, where g is a parameter and v is a variable (the operators τv, $g \| v$, $g \lfloor\!\lfloor v$, $g | v$, etc. are also of minimal length but cannot be considered guarded since they lead to fixed point equations which do not necessarily have unique solutions in the projective model). Hence, one is lead to define g to be a guard if and only if for all $x, y \in A^\infty$, $d(gx, gy) \leqslant \frac{1}{2} \cdot d(x, y)$ (see Definition 5.1 and Lemma 5.2(i)). It follows from Banach's contraction principle and the completeness of the metric space (A^∞, d) that the fixed point equation $v = gv$ has a unique solution, if g is a guard. Now it is an immediate consequence of the definition that the following result holds.

Lemma 5.2 (in the signature $+, \cdot, \|, \lfloor\!\lfloor, |, \partial_H, \delta, \tau, a\ (a \in A)$). (i) *$g$ is a guard if and only if $d(gx, gy) \leqslant \frac{1}{2} \cdot d(x, y)$, for all $x, y \in A^\infty$.*

(ii) *If $g_1, g_2 \in A^\infty$ are guards then so are $g_1 + g_2$, $g_1 \cdot x$ (for any $x \in A^\infty$), $g_1 \| g_2$, $g_1 \lfloor\!\lfloor g_2$, $g_1 | g_2$, $\partial_H(g_1)$.*

Proof. (i) ($\Leftarrow$) Assume g is not a guard. Then g has a finite branch all of whose edges are labeled with the atom τ. It follows from the definition of $(\cdot)_n$ and the axiom on left distributivity that the inequality $d(gx, gy) \leqslant \frac{1}{2} \cdot d(x, y)$, cannot be true for all $x, y \in A^\infty$, which is a contradiction.

$(\Rightarrow)$ If g has no finite branch then $gx = g$, for all $x \in A^\infty$, and hence $d(gx, gy) = 0$, for all $x, y \in A^\infty$. Assume that g has finite branches. Since in forming the product gx the process x can only be appended to finite branches of g it is clear that $d(gx, gy) \leqslant \frac{1}{2} \cdot d(x, y)$, for all $x, y \in A^\infty$.

(ii) This is straightforward by considering the graphs corresponding to the processes $g_1 + g_2$, $g_1 \cdot x$ (for any $x \in A^\infty$), $g_1 \| g_2$, $g_1 \mathbin{\|\!\_} g_2$, $g_1 | g_2$, $\partial_H(g_1)$ (see also Bergstra and Klop, 1986). ∎

Now it is possible to define the notion of guarded operator.

DEFINITION 5.3 (in the signature $+$, $\cdot$, $\|$, $\mathbin{\|\!\_}$, $|$, ∂_H, δ, τ, a $(a \in A)$). Suppose that $T(v_1, ..., v_n, p_1, ..., p_m)$ is an operator with variables $v_1, ..., v_n$ and parameters $p_1, ..., p_m \in A^\infty$. Call T **guarded** if the following conditions hold:

(i) $T \equiv p$, where $p \in A^\infty \cup \{p_1, ..., p_m\}$ is a guard.

(ii) $T \equiv pv_i$, where $p \in A^\infty \cup \{p_1, ..., p_m\}$ is a guard.

(iii) $T \equiv T_1 \cdot T_2$, where T_1 is guarded.

(iv) $T \equiv T_1 + T_2$ or $T \equiv T_1 \| T_2$ or $T \equiv T_1 \mathbin{\|\!\_} T_2$ or $T \equiv T_1 | T_2$, where both operators T_1, T_2 are guarded.

(v) $T \equiv \partial_H(T_1)$, where T_1 is guarded.

Remark. As in (Baeten, Bergstra and Klop, 1984) one might be tempted to call $T(v_1, ..., v_n, p_1, ..., p_m)$ guarded if for any occurrence of a variable v_i in T, the operator T has a subterm of the form $p \cdot T'$, where $p \in A^\infty \cup \{p_1, ..., p_m\}$ is a guard, and this occurrence of v_i occurs in T'. However, it is easy to show by induction on the construction of operators that every operator guarded in this sense will also be guarded in the sense of Definition 5.3.

LEMMA 5.4 (in the signature $+$, $\cdot$, $\|$, $\mathbin{\|\!\_}$, $|$, ∂_H, δ, τ, a $(a \in A)$). *For any guarded operator $T(v_1, ..., v_n, p_1, ..., p_m)$ with variables $v_1, ..., v_n$ and parameters $p_1, ..., p_m \in A^\infty$ and any $x_1, ..., x_n \in A^\infty$, $T(x_1, ..., x_n, p_1, ..., p_m)$ is a guard.*

Proof. The proof is by induction on the construction of T using part (ii) of Lemma 5.2. ∎

6. UNIQUENESS THEOREMS IN THE PROJECTIVE MODEL

The main lemma used in proving the uniqueness of the solutions of a system of guarded fixed point equations is given below.

LEMMA 6.1 (in the signature $+$, $\cdot$, $\|$, $\lfloor\!\lfloor$, $|$, ∂_H, δ, τ, a $(a \in A)$). *For any guarded operator* $T(v_1, ..., v_n, p_1, ..., p_m)$ *with variables* $v_1, ..., v_n$ *and parameters* $p_1, ..., p_m \in A^{\infty}$ *and any* $x_1, ..., x_n, y_1, ..., y_n \in A^{\infty}$,

$$d(T(x_1, ..., x_n, p_1, ..., p_m), T(y_1, ..., y_n, p_1, ..., p_m)) \leqslant \tfrac{1}{2} \cdot \max\{d(x_1, y_1), ..., d(x_n, y_n)\}.$$

Proof. Let x, y, p be abbreviations for the sequences $x_1, ..., x_n$, $y_1, ..., y_n$, $p_1, ..., p_m$, respectively. The proof is by induction on the construction of the operator T. The result is clear if T is of one of the forms p or pv_i, where p is a guard. If T is of one of the forms $T_1 + T_2$, $T_1 \| T_2$, $T_1 \lfloor\!\lfloor T_2$, $T_1 | T_2$ then by the definition of guardedness both T_1 and T_2 are guarded. Hence, it follows from the induction hypothesis that

$$\begin{aligned} d(T(x, p), T(y, p)) &\leqslant \max\{d(T_1(x, p), T_1(y, p)), d(T_2(x, p), T_2(y, p))\} \\ &= \tfrac{1}{2} \cdot \max\{d(x_1, y_1), ..., d(x_n, y_n)\}. \end{aligned}$$

The case $\partial_H(T_1)$ is similar. It remains to consider the case $T \equiv T_1 T_2$, where T_1 is (but T_2 does not have to be) guarded. Now it is clear that

$$\begin{aligned} d(T(x, p), T(y, p)) &= \max\{d(T_1(x, p)\, T_2(x, p), T_1(y, p)\, T_2(y, p))\} \\ &= d(g_1 u_1, g_2 u_2), \end{aligned}$$

where $g_1 = T_1(x, p)$, $u_1 = T_2(x, p)$, $g_2 = T_1(y, p)$, and $u_2 = T_1(y, p)$. However, Lemma 5.4 implies that both g_1, g_2 are guards. Hence, the result will follow from the following claim.

CLAIM. $d(g_1 u_1, g_2 u_2) \leqslant \max\{d(g_1, g_2), (\tfrac{1}{2}) \cdot d(u_1, u_2)\}$.

Assuming the claim, the remaining proof is easy. Indeed, using Lemma 2.4 and the induction hypothesis:

$$\begin{aligned} d(T(x, p), T(y, p)) &= d(g_1 u_1, g_2 u_2) \\ &\leqslant \max\{d(g_1, g_2), \tfrac{1}{2} d(u_1, u_2)\} \\ &= \max\{d(T_1(x, p), T_1(y, p)), \tfrac{1}{2} d(T_2(x, p), T_2(y, p))\} \\ &\leqslant \tfrac{1}{2} \max\{d(x_1, y_1), ..., d(x_n, y_n)\}. \end{aligned}$$

Proof of the Claim. Using part (i) of Lemma 5.2 and the fact that d is an ultrametric it is easy to show that

$$\begin{aligned} d(g_1 u_1, g_2 u_2) &\leqslant \max\{d(g_1 u_1, g_1 u_2), d(g_1 u_2, g_2 u_2)\} \\ &= \max\{d(g_1, g_2), \tfrac{1}{2} d(u_1, u_2)\}. \end{aligned}$$

This completes the proof of the lemma. ∎

THEOREM 6.2 (uniqueness theorem, in $+$, $\cdot$, $\|$, $\lfloor\!\lfloor$, $|$, ∂_H, δ, τ, a $(a \in A)$). *Let*

$$\Sigma = \{v_i = T_i(V_i, P_i): i \in I\}$$

be an arbitrary system of guarded fixed point equations such that each V_i (resp. P_i) is a finite set of variables (resp. parameters) in A^∞ such that it is true that

$$\{v_i: i \in I\} = \bigcup_{i \in I} \{V_i: i \in I\}.$$

Then Σ has a unique fixed point in $(A^\infty)^I$.

Proof. Consider the matric space $(E, D) = ((A^\infty)^I, D)$, where the metric D is defined as

$$D(X, Y) = \text{supremum}\{d(x_i, y_i): i \in I\},$$

for $X = (x_i: i \in I)$, $Y = (y_i: i \in I)$. The proof of the theorem uses the following claim.

CLAIM. *(E, D) is a complete metric space.*

Proof of the Claim. This is in fact a rather general and well-known result: *a product of complete metric spaces with the "sup" metric, is complete.* But here is the proof, for the sake of completeness. Let $\{X_n\}$ be a Cauchy sequence and let $X_n = (x_{n,i}: i \in I)$, Given $\epsilon > 0$, let n_0 be an integer such that for all n, $m \geqslant n_0$,

$$D(X_n, X_m) = \text{supremum}\{d(x_{n,i}, x_{m,i}): i \in I\} \leqslant \varepsilon.$$

It follows that for each $i \in I$ the sequence $\{x_{n,i}\}$ is a Cauchy sequence in the metric space (A^∞, d). By completeness of this last metric space, for each $i \in I$ the sequence $\{x_{n,i}\}$ has a limit point, say x_i. Put $X = (x_i: i \in I)$. It is now easy to show that $\lim_{n \to \infty} X_n = X$, in the metric D, which proves the claim.

To finish the proof of the theorem notice that the function $T: E \to E$ defined by

$$X \to T(X) = (T_i(X_i, P_i): i \in I),$$

where for each $i \in I$, $X_i = \{x_k: v_k \in V_i\}$, is a contraction. Clearly, this last assertion is an immediate consequence of the definition of D and Lemma 6.1. This completes the proof of the theorem. ∎

It is essential to note that the *abstraction operator* τ_I is not included in the signature of the statement of Theorem 6.2, otherwise the theorem is false as the following example shows.

EXAMPLE 6.3 (C. A. R. Hoare, and J. W. Klop). Equation $x = a\tau_{\{a\}}(x)$ has more than one solution in $\{a, b\}^\infty$, e.g., any x of the form $x = ay$, where $y = b^n$, and $n \geqslant 1$.

The last part of this section will be dedicated to a proof of the converse of the uniqueness theorem.

Proof of Theorem 1.6. Without loss of generality it can be assumed that $\|$ does not occur in the operator T (this is because $x \| y = x \mathbin{⌊\!⌊} y + y \mathbin{⌊\!⌊} x$, in the absence of communication). After some rewritings using the axioms of process algebras it becomes clear that $T(x)$ must be the sum of terms of the forms:

(i) $a \cdot t_1(x)$,

(ii) $x \cdot t_2(x)$,

(iii) $x \mathbin{⌊\!⌊} t_3(x)$,

(iv) a,

(v) x,

where the atoms a in (i) and (iv) range over a certain finite subset B of A and the terms $t_1(x)$, $t_2(x)$, $t_3(x)$ are operators (and some category of summands might be missing from the sum above). The proof is naturally divided into two parts. The first part shows how to reduce the theorem to the case $B = \emptyset$. The second part handles the case $B = \emptyset$.

Part 1: *Reduction to* $B = \emptyset$. It can be assumed that $B \neq A$, since by the hypothesis of the theorem there is at least one atom in A which does not occur in T. Without loss of generality it can also be assumed that at least one type of terms among (ii), (iii), and (v) occurs as a summand of $T(x)$, otherwise the operator T is guarded. It is clear that for all $x \in A^\infty$,

$$(T(x))_1 = \sum_{a \in B} a + (x)_1.$$

Now it is easy to show by induction on n, that for all $x \in A^\infty$,

$$(T^n(x))_1 = \sum_{a \in B} a + (x)_1.$$

For any $q \in A_\omega$ let $l(q) = \lim_{n \to \infty} T^n(q)$ be the fixed point of $x = T(x)$ obtained from q by iterating the operator T (see Bergstra and Klop, 1982).

It is clear that for any $q \in A_\omega$,

$$(l(q))_1 = \sum_{a \in B} a + (q)_1.$$

It follows that in order to obtain two different fixed points of $x = T(x)$ it is enough to find two finite terms p, q such that the following two sums

$$\sum_{a \in B} a + (q)_1, \qquad \sum_{a \in B} a + (p)_1$$

are distinct. If B were nonempty then the above observation would imply that $x = T(x)$ has at least two solutions, namely $l(p)$, $l(q)$, where

$$p = \sum_{a \in B} a, \qquad q = b,$$

with $b \in A - B$, which is a contradiction. Hence, without loss of generality it can be assumed that B is empty.

Part 2: *Handling the case* $B = \emptyset$. If A had at least two atoms, say a, b, then by the above observation $l(a)$ and $l(b)$ would be distinct solutions of $x = T(x)$, which is also a contradiction. As a consequence, it can be assumed that A consists of a single atom, say a, and the operator $T(x)$ is atom-free (i.e., the atom a does not occur in T). The operator T is a finite sum of terms of the form (ii), (iii), and (v) (since $B = \emptyset$ terms of type (i) and (iv) drop out). It will be shown that in fact the single summand x (i.e., of type (v)) cannot occur in T. Indeed, assume otherwise. Since $A = \{a\}$, it is clear that for all $y \in A^\infty$, $(y)_1 = a$. Moreover, $(T(x))_2$ equals a sum consisting of summands of the form

$$(x)_2,\ ((x)_2(t_2(x))_1)_2 = ((x)_2 a)_2, \qquad ((x)_2 \mathbin{\|\!\_} (t_3(x))_1)_2 = ((x)_2 \mathbin{\|\!\_} a)_2.$$

Using this, and the fact that $(x)_2$ is a summand of $(T(x))_2$, it follows that for $p = a^2$ and $q = a + a^2$ one can prove by induction on n that

$$(T^n(p))_2 = a^2, \qquad (T^n(q))_2 = a + a^2.$$

Since $(l(p))_2 = a^2$, $(l(q))_2 = a + a^2$, $x = T(x)$ has at least two solutions, namely $l(a^2)$, $l(a + a^2)$, which is a contradiction.

It follows that $T(x)$ is a sum of operators of the form (ii) or (iii). In this case it will be shown that the unique solution of $x = T(x)$ must be a^ω. Hence, there is a guarded operator $S(x)$, namely $S(x) = ax$, such that $x = T(x)$ and $x = S(x)$ have exactly the same fixed point. Let s be any fixed point of T, i.e., $s = T(s)$. It will be shown by induction on n that $(s)_n = a^n$.

Since a is the unique atom in A, the result is clear for $n=1$. Assume the result is true for $n>0$. Then $(T(x))_{n+1}$ is a sum of terms of the form

$$((x)_{n+1}(t_2(x))_{n+1}, \qquad ((x)_{n+1} \mathbin{|\!\!\lfloor} (t_3(x))_n)_{n+1}.$$

In particular, since s is a solution of $x=T(x)$, $(s)_{n+1}$ is a sum of terms of the form

$$((s)_{n+1}(t_2(s))_n)_{n+1}, \qquad ((s)_{n+1} \mathbin{|\!\!\lfloor} (t_3(s))_n)_{n+1},$$

and consequently all branches in the tree representation of the process s must be infinite (here, use is made of the fact that x is not one of the summands in the representation of $T(x)$ as a sum). By induction hypothesis $(s)_n = a^n$. However, it is easy to show that every tree t in $\{a\}_\omega$ all of whose branches have length bigger than n must satisfy $(t)_{n+1} = a^{n+1}$. In particular, $(s)_{n+1} = a^{n+1}$, and the inductive proof of the claim is complete. This completes the proof of Theorem 1.6. ∎

An immediate consequence of the proof of Theorem 1.6 is the following.

COROLLARY 6.4 (in the signature $+$, $\cdot$, $\|$, $\mathbin{|\!\!\lfloor}$, a $(a \in A)$). *If the operator $T(x)$ is atom-free and the equation $x=T(x)$ has a unique solution in $\{a\}^\infty$ then its unique solution must be a^ω.*

EXAMPLE 6.5. Some examples of atom-free polynomial operators with unique solutions in $\{a\}^\infty$ are: $x=x^n$, with $n>1$, $x=x\|x$, etc. If in addition, the atom δ were included in the signature then Corollary 6.4 would be false, since δ would be a second solution of $x=x^n$.

7. DISCUSSION AND OPEN PROBLEMS

The proofs of Theorems 1.3, 1.4, and 1.5 are signature-free. In fact they depend only on the statements of Theorems 1.1 and 1.2 and the topological properties of A^∞. However, the proof of Theorem 1.1 is rather combinatorial in nature. It would be useful if one could prove Theorem 1.1 using Theorem 2.1 (ii) and an appropriate fixed point theorem for the Cantor set, because the proof would be topological in nature and hence very likely extendible to bigger signatures. Theorem 1.4 does not seem to be the most general result one might hope to prove. For example, it is not known if the theorem is true for infinite systems with an infinite alphabet A.

Theorem 1.6 is only an attempt to justify the fact that guarded equations are the only ones which have unique fixed points. However, it is not known if the result is true for systems of arbitrarily many equations or even in bigger signatures. The proof of Theorem 1.6 given here is combinatorial in nature and hence its direct extension to arbitrary systems would most likely be quite complex. It might be possible, however, to give a proof using results from the theory of metric spaces (see Dugundji and Granas, 1982). For example, the following interesting result about fixed points of polynomial operators can be shown using the partial converse of Banach's contraction principle stated in (Bessaga, 1959): let T be a polynomial operator; if for all $n>0$, $T^n(x)$ has at most one fixed point (notice that no existence of the fixed point of $T^n(x)$ is asserted) then $T(x)$ must have exactly one fixed point. A direct, combinatorial proof of this result along the lines of the proof of Theorem 1.6 would seem almost hopeless. A similar result can also be proved using the deeper theorem given in (Janos, 1967).

RECEIVED June 18, 1986; ACCEPTED May 1987

REFERENCES

ARNOLD, A. AND NIVAT, M. (1980), "The metric space of infinite trees: Algebraic and topological properties," *Fund. Inform.* **3**, No. 4 445–476.

BAETEN, J., BERGSTRA, J. A., AND KLOP, J. W. (1985), "On the Consistency of Koomen's Fair Abstraction Rule, Department of Computer Science Technical Report, CS-R8511, Centre of Mathematics and Computer Science, Amsterdam, May, 1985; *Theoret. Computer Science*, **51** Nos. 1/2 (1987).

DE BAKKER, J. W. AND ZUCKER, J. I. (1982), Denotational semantics of concurrency, *in* "Proceedings, 14th Sympos. Theory of Comput.," 153–158.

DE BAKKER, J. W., AND ZUCKER, J. I. (1982), Processes and the operational semantics of concurrency, *Inform. and Control* Vol. **54**, Nos. 1/2, 70–120.

BERGSTRA, J. A., AND KLOP, J. W. (1986), Algebra of communicating processes, *in* "Proceedings, CWI Sympos. on Math. and Comput. Sci." (J. W. de Bakker, M. Hazenwinkel and J. K. Lenstra, Eds.)

BERGSTRA, J. A., AND KLOP, J. W. (1982), "Fixed Point Semantics in Process Algebras," Department of Computer Science Technical Report IW 206/82, Math. Centrum, Amsterdam, Nov.

BESSAGA, C. (1959), "On the converse of the Banach fixed point principle," *Colloq. Math.* **7**, 41–43.

CHANG, C. C. AND KEISLER, H. J. (1973), "Model Theory," North-Holland, Amsterdam.

DUGUNDJI, J. (1966), "Topology," Allyn & Bacon, Rockledge, N.J.

DUGUNDJI, J., AND GRANAS, A. (1982), "Fixed Point Theory," Vol. I, PWN, Warszaw.

DIEUDONNÉ, J. (1968), "Élements d'Analyse, Tom 1," Gauthier-Villars, Paris.

VAN GLABBEEK, R. J. (1987), Bounded nondeterminism and the approximation principle in process algebra, *in* "Proceedings, 4th Annual Symposium on Theoretical Aspects of Computer Science, Passau, W. Germany, Feb. 1987" (F. J. Brandenburg, G. Vidal-Naquet, and M. Wirsing, Eds.), Springer-Verlag, Berlin/New York.

HOARE, C. A. R. (1985), "Communicating Sequential Process," Prentice-Hall, Englewood Cliffs, NJ.

JANOS, L., (1967), A converse of Banach's contraction theorem, *Proc. Amer. Math.* **18**, 287–289.

LLOYD, J., (1984), "Foundations of Logic Programming," Springer-Verlag, New York/Berlin.

MILNER, R. (1980), "A Calculus of Communicating Systems," Lecture Notes in Computer Science Vol. 92, Springer-Verlag, Berlin/New York.

RINOW, W. (1975), "Topologie," VEB Deutscher Verlag der Wissenschaften, Berlin.

ROUNDS, W. C. (1985), Applications of topology to semantics of communicating processes, *in* "Seminar in Concurrency," Lecture Notes in Computer Science, Vol. 197, pp. 360–372, Springer-Verlag, Berlin/New York.