The philosophy of deformations: introductory remarks and a guide to this volume

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1. Introduction. The idea of deformations

One of the more prominent, specifically modern, and pervasive trends in mathematics has to do with perturbations and deformations. Instead of studying one particular model, e.g. one differential equation, or one particular algebra of operators, one is as least as interested in families of these things, and the question of how various properties change as the object under consideration is varied. One reason of this is no doubt the modern emphasis on the tenuous relation (logically speaking) between a mathematical model and the phenomena it is designed to deal with. Thus, to paraphrase Arnol'd, when dealing with models intended to apply to the real world, the question soon arises of choosing those properties of the model which are not very sensitive to small changes in the model and which thus have a chance of representing some properties of the real process.

Intuitively a deformation of a mathematical object is a family of the same kind of objects depending on some parameter(s). Thus for example one could have a family of differential equations

$$\dot{x} = f(x,t,\lambda) \qquad (\mathrm{d}_\lambda)$$

depending on a real parameter λ, or for example a family of real three dimensional algebras defined by

$$A_\lambda = \mathbf{R}[X]/(X^3 - \lambda X).$$

Of course the parameter on which the family under consideration depends need not be one dimensional but can be a vector and it need not vary over a vectorspace but can also vary over various subsets of a vectorspace or over more general objects such as algebraic schemes. By viewing the object associated to the parameter(s) value λ as 'lying over λ' one obtains a 'fibre object' picture

$$\begin{array}{c} X \\ \downarrow \\ B \end{array}$$

where the fibre over $\lambda \in B$ is the object from the family labelled by λ. For example in the case of the family of differential equations the fibre over $\lambda \in B$ is the n-dimensional vector space $\mathbf{R}^n$ with a flow

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given by $\dot{x} = f(x,t,\lambda)$ and for the 3-dimensional algebra example the fibre over $\lambda \in B$ is the commutative algebra

$$\mathbf{R}[X]/(X^3 - \lambda X).$$

Thus for the geometric type categories such as topological spaces, schemes, manifolds with singularities, ..., a deformation is simply a (surjective) morphism with, however, special stress on aspects and questions which involve how the fibres, i.e. the inverse images of points b in B, vary with b.

There is no clear cut dividing line between *perturbations* and *deformations*, the subject of this book. Perhaps the difference can be indicated roughly by saying that perturbations consider unstructured neighborhoods of a given object: all nearby objects are considered on an equal footing, while deformation theory is concerned with the (detailed) structure of the set of (isomorphism) classes of objects of the kind under consideration and how they fit into (smooth) families. Also deformation theory and its applications are not necessarily concerned only with small neighborhoods. For example one of Poincaré's favourite techniques (the continuation method) consisted of imbedding the problem in a one-parameter family of problems depending on an auxiliary parameter s and to consider the solubility of the problem as s varies. This also makes it clear that deformation theoretic ideas have very old roots. Indeed, the idea of "moduli", originally the number of parameters on which a given kind of structure depends, goes back to Riemann, as do some other deformation theoretic ideas.

2. Deformation theoretic questions

Let us consider some typical deformation theoretic questions.

A first one, no doubt, is *rigidity*. Intuitively, an object X_0 is rigid if for every deformation X_t into which it fits, it is true that X_t is isomorphic to X_0. Depending on context this intuitive idea must be made precise in various ways. For instance in the theory of deformations of algebras one important way in which to make precise the idea of a deformation of an associative algebra A over a field k is as follows. A deformation of A is an associative algebra A_t over the power series ring $k[[t]]$ such that $A_0 = A_t \otimes_{k[[t]]} k$ is isomorphic to A. Two deformations A_t and A'_t are equivalent if A_t and A'_t are isomorphic as $k[[t]]$ algebras; and the trivial deformation is $A \otimes_k k[[t]]$. An algebra is rigid if every deformation is equivalent to the trivial one.

There are both local and global aspects to rigidity though the local ones have received far more attention. Thus in a geometric setting of, say, a deformation $\pi: X \to B$ of manifolds with a diffeomorphism on them, parameterized by a topological space B, it may very well be the case that for every $b \in B$ the fibres $\pi^{-1}(b)$ are isomorphic geometric objects, i.e. isomorphic discrete dynamical systems in this case, without it being the case that $\pi: X \to B$ is isomorphic to the trivial deformation $X_0 \times B \to B$. This depends on whether the isomorphisms $\phi_b: X_b \tilde{\to} X_0$ can be chosen in such a way that they depend continuously on $b \in B$. The example of (locally trivial) vectorbundles shows that this need not be the case. The matter is related to the distinction between coarse and fine moduli spaces in algebraic geometry.

For discrete dynamical systems and differential equations on a manifold M (local) rigidity is something like structural stability, the property which says that nearby systems have "the same" phase portrait. (Where, of course, there are several meanings which can be given to the phrase "the same".)

Typically, deformations come in various guises, ranging from infinitesimal ones (= possible deformation directions), to formal ones, to true families. For algebras e.g. an infinitesimal deformation A_ϵ is an algebra over $k[\epsilon]/\epsilon^2$; a (one dimensional) formal deformation would be an algebra over the power series ring $k[[t]]$ and a true family could be an algebra over the polynomials $k[t]$ or, an intermediate case which makes sense e.g. when $k = \mathbf{C}$ or $\mathbf{R}$, an algebra defined over the rings of convergent power series $\mathbf{R}\{\{t\}\}$ or $\mathbf{C}\{\{t\}\}$.

Typically infinitesimal deformations are classified by a suitable 2-nd cohomology group like

$H^2(A,A)$ and extending these to formal deformations involves various cohomological obstructions. The step from formal deformations to something like true (local and global) families may involve a variety of ideas and techniques among which are Artin approximation and sheaf theory.

One way to think about deformation theory is as an attempt to help classify all objects of a given kind (up to some suitable notion of isomorphism); the simplest kinds of classification problems yield countable lists of nonisomorphic objects such that each object is isomorphic to one of them ("finite problems"; finite because, given a suitable notion of dimension, in each dimension there will be but finitely many isomorphism classes); the simple finite dimensional Lie algebras over $\mathbf{C}$ are a nice and famous example; the next type yields a countable number of nice (smooth) finite dimensional families of nonisomorphic objects ("tame problems"); everything else is "wild", which does not mean impossible to handle. Both deformation theory and moduli theory have much to do with finding these continuous families and with determining how in a given family the isomorphism classes of objects vary.

A final central question of deformation theory concerns what may happen to the automorphisms (i.e. symmetries) of an object during a deformation. Experimenting with geometric figures in the plane like squares and rectangles one rapidly gets the feeling that the following could be true. Given an object, then, for a sufficiently small deformation, the symmetry group of the deformed object can only be smaller or equal to the symmetry of the original object. And, indeed, there are theorems to this effect. One of them is as follows. Suppose that the objects we are trying to classify form a smooth manifold M. Suppose further that the notion of isomorphism corresponds to an action of a compact Lie group G on M. The symmetry of an object $m \in M$ is then the isotropy subgroup $G_m = \{g \in G: gm = m\}$. In this case the "diminishing symmetry" result holds: for m' sufficiently close to m, G_m is larger than $G_{m'}$, which here means that G_m contains $G_{m'}$ up to conjugacy.

In general, the theorem is definitely not true as the example of three dimensional associative unital algebras over $\mathbf{R}$ shows. In that case, it turns out, there are two competing symmetry groups: a design (or accidental) one which tends to diminish during a deformation (i.e. large in special cases, small generically) and a generic one which tends to grow during a deformation (i.e. large generically, small in special cases).

There is certainly still much left to do regarding the behaviour of $\mathrm{Aut}(M)$ during a deformation.

It is also clear from the above that matters of equivariance (under a group action) and deformation theory are not unrelated.

3. Why deformations are important

There are quite a number of different reasons which make deformation theoretic ideas important in modern mathematics. Some of the more striking can be indicated as follows.

3.1. Robustness matters. Start with a given object, e.g. a dynamical system given by a differential equation $\dot{x} = f(x)$. But of course much more highly structured objects can also be considered in this way. Are perhaps all nearby objects necessarily isomorphic? This leads e.g. to the idea of "structural stability" (Thom) of dynamical systems, and the philosophy that only structurably stable systems (objects) are admissible as models for real phenomena. There are, however, in the dynamical systems world not enough structurally stable objects (they are not dense), and that means more refined questions must be faced as to how the objects in question can change, and whether these changes can be controlled/understood. I.e. more work for deformation theoretic ideas. Bifurcation theory fits into this general framework. Given an equation $F(x,\lambda) = 0$ or a differential equation $\dot{x} = G(x,t,\lambda)$, depending on a (vector) parameter λ, bifurcation theory studies how the set of solutions changes as λ varies (and what quantities remain invariant).

Much related are questions of "finite determinacy". Consider e.g. a dynamical system $\dot{x} = f(x)$. Under what conditions is the vector field $f(x)$ determined (up to isomorphism) by a finite chunk of its power series development.

3.2. Invariants. What properties of a given object remain invariant under which changes in the parameters. And if a change, say in symmetry, must take place, can anything be said about what change will take place (Example; in many cases broken symmetry must lead to a new symmetry group which is an isotropy subgroup of the original one, cf. also above). This idea led first of all to "homotopy" invariants: properties which remain invariant under all continuous deformations. In the right context the number of solutions of an equation (counted right) is such an invariant and this is course is the underlying fact at the basis of the Poincaré continuation method. A modern refinement is to deform an equation continuously until a trivially solvable equation is obtained, and then to deform back, this time taking the solution along. (The socalled homotopy or continuation methods of solving equations; a "hot" topic which has been the topic of several conferences in the last few years.) In physics "homotopy" invariants make their appearance in gauge field theories and condensed matter theory as instantons, topological charges, topological defects, kinks, etc..

Immediately related to the question of invariants (and the question of moduli) are canonical forms. Can one find a (nice) family of objects of the given kind such that each object of the given kind is isomorphic to precisely one of that family. The Jordan canonical form for square matrices is a nice and important example. For many problems only the isomorphism class of an object is important and then canonical forms can come in very handily for calculation or to verify a statement or conjecture. The next question involves continuous canonical forms: can one find a canonical form $A \mapsto c(A)$ which is continuous with respect to the parameters on which A depends. For instance the Jordan canonical form is not continuous and that makes it (sometimes) a dangerous tool to use in (numerical) calculations.

3.3. Unraveling complicated (highly singular) structures. It is something of a fact (or axiom?) that the most interesting models and structures tend to be very special: lots of things coincide, many special relations hold between various parameters; often there is more symmetry than is usual for general models of the class under consideration, and often there are complicated singularities (the interesting physics happens at the singularities; phase transitions e.g.). In such a situation deformation theory is a highly successful tool to pull the model apart and see what the elementary, say, singularities are, which in this model are piled on top of each other and to study how they come together. A very, very down to earth example of the phenomenon is provided by the example of the binomial coefficients (or more generally multinomial coefficients) in the formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum (\text{words in } a \text{ and } b \text{ of length } n)$$

where $\binom{n}{k}$ is understood - or explained - as the number of different words of length n in the letters a and b such that a occurs k times.) Here it is the special relation $ab = ba$ which makes the formula "difficult". Combinatorics abounds with examples where such "pulling apart by deformations" is a fruitful insight generating procedure. From this point of view a nontrivial part of the theory of generating series and practically all of "q-series" theory (both in combinatorics and in special function theory with applications in quantum mechanics) can be placed within the framework of deformation theory.

3.4. Deformations as a defining tool. Many entities in mathematics and related sciences cannot be obtained directly but must instead be defined by a deformation or approximation procedure. A most down to earth example concerns the definition of integrals, say, the integral of a function, which is defined by approximating the function by a step function or, in words, by deforming the graph of a function to a piecewise linear one. Another example concerns the number of intersection points of two plane algebraic curves. As is very often the case it is important for applications to know what the phrase means also in highly singular cases such as the case that the two curves coincide (selfintersection number). The answer is to jiggle one of the curves w.r.t. the other and define the intersection number as the "generic" intersection number of the jiggled curve with the other one. The notion of the degree of mapping arises in a similar way as a precisation of "average number of points with the same image under the mapping". Indeed, when one stops to think about it, most interesting entities in mathematics involve some such idea.

3.5. Relative objects and "devissage". Suppose we are involved with some geometric category such as, e.g., a category of algebraic geometric schemes. A deformation here is simply a (surjective) morphism $X \to B$ (or a scheme over B). A powerful technique to prove theorems is now as follows. Prove the result for B and for the objects which can occur as fibres; prove a relative or family version (for the relative object X over B), and finally using all this prove the desired result for X.

3.6. Other aspects. It is stimulating and interesting to note that a systematic study of the possible deformations of a theory has predictive power. It tells us about the potential other theories of which our present one is a limiting case. Thus, a study of the deformations of the Galilei group would have turned up the Poincaré-Lorentz group and the possibility of special relativity (indeed almost inevitably). I owe this remark to Moshe Flato.

Also in many ways the "universal deformation of an object" (such things often exist) is a rather nicer, and more maniable gadget than the object itself. Also more regular/beautiful, a general tendency of "universal" objects. Still another application of deformation ideas occurs when it desired to construct objects with certain desired properties. Often an example of such an object is at hand but it does not have all the desired properties. One good way to proceed is then to try to deform the object systematically and observe what changes and what remains invariant.

3.7. Some statistics of deformation theory. It will be clear from the above that "deformation theoretic ideas" occur all over the mathematical sciences and in fact form an integral part of our general scientific patterns of though. And indeed examples of applications range from descriptions of the possible failure modes of control systems, through bifurcation theory, symmetry breaking and pattern formation phenomena, to the interrelations between classical physics and quantum and relativity theory.

When I first started thinking, now 5 years ago, about something like a summer school on deformation theoretic ideas, I did an exploratory computer search which yielded some 4300 papers in the mathematical literature with deformation(s) in the title and some 1350 had been published $\geqslant$ 1980. They were scattered all through the various classification schemes involved.

No doubt it would be an extremely fruitful and rewarding (and challenging) task to bring together all these manifestations of "deformation philosophy"; to bring together, and get to communicate suitable representatives of each school, and to record the resulting synthesis in some 10 or 15 volumes (my estimate). One then should also include the very much related ideas of singular perturbations, sensitivity analysis and resolution of singularities.

4. What is present and what is absent

Let me conclude with a few brief and incomplete remarks on what is absent and what is present in the present volume.

It has by now become clear that 'deformations of algebras' are central to the whole topic. And these are well represented. First of all by means of the very large and fundamental paper of Gerstenhaber-Shack on 'Algebraic cohomology and deformation theory', and second via a number of papers on deformations of some especially important algebras such as the Lie algebra of vectorfields on the line (Fialowski), current algebras (Roger) and triangular algebras (Gerstenhaber-Shack).

As indicated above a complete theory of deformations includes a study of infinitesimals deformations. It is thus not totally surprising that an environment in which infinitesimal really exist (i.e. nonstandard analysis) could be useful. This is indeed the case; more, these nonstandard techniques are a powerful tool (Goze, Ancochea-Bermudez). For instance this yields rigid Lie algebras of which the rigidity can not be proved via the fact that the suitable 2-nd cohomology group is zero.

Though now central, deformations of algebras were not the first systematic theory of deformations. That honor belongs to the theory of deformations of more geometric objects and structures, particularly the powerful and beautiful theory of deformations of geometric structures developed by Kodaira and Don Spencer and his school. This also is well represented (two related papers by Gasqui-Goldschmidt, a long expository paper by Hermann, and a thought-provoking paper by Pommaret). This theory of deformations of geometric objects and pseudogroups served as the inspiring example for the theory of deformations of algebras. But despite many formal similarities, particularly with respect to cohomological tools, a real deeper interrelation was long undiscovered. This however is now also present, cf. again the large paper of Gerstenhaber-Shack and also Pommaret and the paper by Rochberg which discusses relations between deformations of Riemann surfaces and of associated Banach algebras of functions.

Three other papers (Johnson, Jarosz, Christensen) are also concerned with perturbations of operator and function algebras particularly with questions of rigidity under small perturbations, a notion which here takes on added meaning (compared to the case of algebras) because of the presence of metric aspects.

The large and important topic of canonical forms, changes of variables and invariants for (meromorphic) differential equations is present in the form of a long and important paper of Babbitt and Varadarajan. It is pleasing to note that also in this setting algebraic and group theoretic ideas (in the form of deformations of matrices over rings) play an important role.

The last systematic group of papers (Lichnérowicz, de Wilde-Lecomte, Melotte) is also concerned with deformations of function algebras. They concern - among other things - the socalled $\star$-product or deformation theoretic approach to quantum mechanics, and with some 115 pages, comprising two expository survey papers and a smaller more technical one, this topic is also well represented.

Let me elaborate slightly on this topic because of the very recent extra impetus this approach received by means of the topic "quantum groups". A classical dynamical system on a manifold M is given by a symplectic structure on M plus a Hamiltonian. The symplectic structure on M defines a Poisson Lie algebra structure on $C(M)$, the differentiable functions on M, making $C(M)$ into a commutative Poisson algebra, i.e. a commutative algebra A with a second Lie-algebra multiplication $(f,g) \mapsto \{f,g\}$ which satisfies $\{fg,h\} = f\{f,h\}g$. And this is all that is needed to write down the dynamics $\frac{df}{dt} = \{f,H\}$ where H is the Hamiltonian. A quantization of A is now a non-commutative algebra A_t over $\mathbf{R}[[t]]$ such that $A_0 = A$ and $\{f(0),g(0)\}t = f(t)g(t) - g(t)f(t) \bmod t^2$. Thus the quantized object A_t, which is basically simply an associative algebra, is really a simpler object than its classical limit, the Poisson algebra A.

In case M is a Lie group this leads to quantum groups which are essentially noncommutative and noncommutative Hopf algebras. (Twisted matrix groups are examples, and these in turn appear to link up with q-orthogonal polynomials and q-special functions, and definitely link up with Hecke algebras which are deformed group algebras (of Weyl groups, or, Coxeter groups).)

Let me also say a few words about what is missing. There is practically nothing (except incidentally) on the very large and important topic of the (algebraic-geometric) theory of deformations of singularities resolutions of singularities, and the (algebraic-geometric and several complex variables) theory of moduli and such things as Teichmüller spaces. And, consequently, the roles of moduli spaces in string theory and such things as family index theorems are not even mentioned.

Another largely missing topic is that of isospectral deformations (of matrices and operators; an isospectral deformation of an operator Q is a family of operators Q_t such that $\text{spec}(Q_t) = \text{spec}(Q)$ for all t), Lax equations $\dot{Q} = [P, Q]$, isomonodromy deformations, and the much related and vast theory of integrable systems and soliton equations. Except tangentially in the form of Calogero's remarkable "remarkable matrix" paper.

I have already mentioned bifurcation theory which studies how the set of solutions of a dynamical system $\dot{x} = G(x, \lambda)$ varies in dependence on λ. There are other aspects of dynamical systems which can be studied in dependance on parameters. For example for discrete dynamical systems $f_\mu \colon X \to X$ one can study the behaviour of the orbits x, $f_\mu(x)$, $f_\mu(f_\mu(x))$, $f_\mu(f_\mu(f_\mu(x)))$,... in dependance on μ. This is (part of) the theory of deterministic chaos and of universality phenomena of iterated maps another substantial field that is largely missing except for a stimulating paper by Vilela Mendes.

Finally there is nothing about the subject of deformations, or bendings, of convex surfaces, and deformations of metrics on topological and Banach spaces. Both are topics falling within the philosophy of deformations as do isotopies and homotopies in differential topology. On the other hand all these topics seem to be some steps further removed from the central topic of this volume: the deformations of algebras and of structures (such as various geometric ones) whose deformations are adequately reflected in terms of corresponding deformations of suitable algebras of functions.

As already remarked, the total field of deformations is a very large one and could not possibly be surveyed adequately in a volume the size of this one. Thus the present volume merely presents the current state-of-the-art as regards deformations of algebras and geometric structures and their interrelations.