ON LIE ALGEBRA WEIGHT SYSTEMS FOR 3-GRAPHS

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Abstract. A 3-*graph* is a connected cubic graph such that each vertex is is equipped with a cyclic order of the edges incident with it. A *weight system* is a function f on the collection of 3-graphs which is *antisymmetric*: $f(H) = -f(G)$ if H arises from G by reversing the orientation at one of its vertices, and satisfies the IHX-equation. Key instances of weight systems are the functions $\varphi_{\mathfrak{g}}$ obtained from a metric Lie algebra $\mathfrak{g}$ by taking the structure tensor c of $\mathfrak{g}$ with respect to some orthonormal basis, decorating each vertex of the 3-graph by c, and contracting along the edges.

We give equations on values of any complex-valued weight system that characterize it as complex Lie algebra weight system. It also follows that if $f = \varphi_{\mathfrak{g}}$ for some complex metric Lie algebra $\mathfrak{g}$, then $f = \varphi_{\mathfrak{g}'}$ for some unique complex reductive metric Lie algebra $\mathfrak{g}'$. Basic tool throughout is geometric invariant theory.

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1. Introduction

A 3-*graph* is a connected, nonempty, cubic graph such that each vertex v is equipped with a cyclic order of the edges incident with v. Loops and multiple edges are allowed. Also the 'vertexless loop' $\bigcirc$ counts as 3-graph.

3-graphs come up in various branches of mathematics, under several names. They play an important role in studying the Vassiliev knot invariants (see [6]), and in this context the term 3-graph was introduced by Duzhin, Kaishev, and Chmutov [8]. We adopt this name as it is short and settled in [6]. 3-graphs also emerge in the related Chern-Simons topological field theory (Bar-Natan [2], Axelrod and Singer [1]). They are in one-to-one correspondence with cubic graphs that are cellularly embedded on a compact oriented surface, and hence, through graph duality, with triangulations of a compact oriented surface. Moreover, 3-graphs produce a generating set for the algebra of $O(n, \mathbb{C})$-invariant regular functions on the space $\Lambda^3\mathbb{C}^n$ of alternating 3-tensors, by Weyl's 'first fundamental theorem' of invariant theory [17].

For the Vassiliev knot invariants, 'weight systems' are pivotal. For 3-graphs they are defined as follows. Let $\mathcal{G}$ denote the collection of all 3-graphs, and call a function $f : \mathcal{G} \to \mathbb{C}$ a *weight system* if f satisfies the *AS-equation* (for antisymmetry): $f(H) = -f(G)$ whenever H arises from G by turning the cyclic order of the edges at one of the vertices of G, and the *IHX-equation*:

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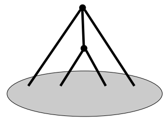
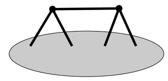
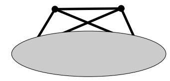
(1) $f($  $) \; = \; f($  $) \; - \; f($  $).$

Here the cyclic order of edges at any vertex is given by the clockwise order of edges at the vertex. The grey areas in (1) represent the remainder of the 3-graphs, the same in each of these areas.

Any C_3-invariant tensor $c = (c_{i,j,k})_{i,j,k=1}^n \in ((\mathbb{C}^n)^{\otimes 3})^{C_3}$ gives the *partition function* $p_c : \mathcal{G} \to \mathbb{C}$, defined by

$$p_c(G) := \sum_{\psi : E(G) \to [n]} \prod_{v \in V(G)} c_{\psi(e_1),\psi(e_2),\psi(e_3)} \tag{2}$$

for any 3-graph G, where, for any $v \in V(G)$, e_1, e_2, e_3 denote the edges incident with v, in cyclic order. (As usual, $[k] := \{1, \ldots, k\}$ for any $k \in \mathbb{Z}_+ = \{0, 1, 2 \ldots\}$. Moreover, if a group Γ acts on a set X, then X^Γ denotes the set of Γ-invariant elements of X.)

Note that (2) is invariant under orthonormal transformations of c, and that $p_c(\bigcirc) = n$. ($p_c(G)$ is the 'partition function' of the 'vertex model' c, in the sense of de la Harpe and Jones [11]. It may also be viewed as 'edge coloring model' as in Szegedy [15].)

An important class of weight systems is obtained as follows from the structure constants of metric Lie algebras, which roots in papers of Penrose [14] and Murphy [13], and the relevance for knot theory was pioneered by Bar-Natan [2,3] and Kontsevich [12]. (In this paper, all Lie algebras are finite-dimensional and complex.) A *metric Lie algebra* is a Lie algebra $\mathfrak{g}$ enriched with a nondegenerate symmetric bilinear form $\langle ., . \rangle$ which is *ad-invariant*, that is, $\langle [x, y], z \rangle = \langle x, [y, z] \rangle$ for all $x, y, z \in \mathfrak{g}$. If $b_1, \ldots, b_n$ is an orthonormal basis of $\mathfrak{g}$, the structure constants $(c_\mathfrak{g})_{i,j,k}$ of $\mathfrak{g}$ are characterized by $(c_\mathfrak{g})_{i,j,k} := \langle [b_i, b_j], b_k \rangle$ for $i, j, k = 1, \ldots, n$. Then $c_\mathfrak{g} \in (\mathfrak{g}^{\otimes 3})^{C_3}$, and $\varphi_\mathfrak{g} := p_{c_\mathfrak{g}}$ is a weight system. Indeed, the set of such structure constants $c_\mathfrak{g}$ of metric Lie algebras $\mathfrak{g}$ in n dimensions is equal to the affine variety $\mathcal{V}_n$ in $\Lambda^3\mathbb{C}^n$ determined by the quadratic equations

$$\sum_{a=1}^{n} (x_{i,j,a}x_{a,k,l} + x_{k,i,a}x_{a,j,l} + x_{j,k,a}x_{a,i,l}) = 0 \qquad \text{for } i, j, k, l = 1, \ldots, n. \tag{3}$$

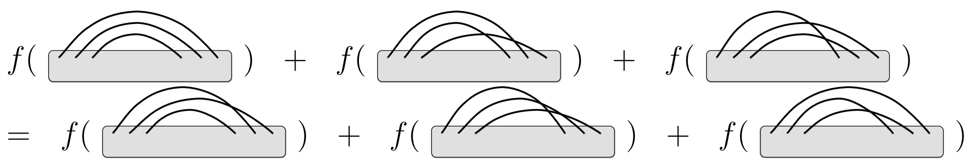
This directly implies that p_x satisfies the AS- and IHX-equations; that is, p_x is a weight system.

As was shown by Bar-Natan [4], the functions $\varphi_{\mathfrak{sl}(n)}$ connect to basic graph theory properties like edge-colorability and planarity, and the four-color theorem can be expressed as a relation between the zeros and the degree of $\varphi_{\mathfrak{sl}(n)}(G)$ (as polynomial in n).

It is easy to construct a weight system that is not equal to $\varphi_\mathfrak{g}$ for any metric Lie algebra: just take a different Lie algebra for each number of vertices of the 3-graph G. More interestingly, Vogel [16] showed that there is a weight system that is no Lie algebra weight system even when restricted to 3-graphs with 34 vertices.

The AS and IHX conditions are '2-term' and '3-term' relations. We give a characterization of those weight systems that come from an n-dimensional Lie algebra by adding an '$(n+1)!$-term' relation. To illustrate it, for $n = 2$ it is the following 6-term relation:

(4)



To describe it in general, we need the following notion (cf. [6]). A *k-legged fixed diagram* is a graph with trivalent vertices, each equipped with a cyclic order of the edges incident with it, and moreover precisely k univalent vertices (called *legs*), labeled $1, \ldots, k$. (Connectivity is not required.) Let $\mathcal{F}_k$ denote the collection of all k-legged fixed diagrams.

So $\mathcal{F}_0$ is the collection of disjoint unions of 3-graphs. Any function f on 3-graphs can be extended to $\mathcal{F}_0$ by the 'multiplicativity rule' $f(G \sqcup H) = f(G)f(H)$, where $\sqcup$ denotes disjoint union, and setting $f(\emptyset) := 1$.

For $G, H \in \mathcal{F}_k$, let $G \cdot H$ be the graph in $\mathcal{F}_0$ obtained from the disjoint union of G and H by identifying the i-labeled legs in G and H and joining the two incident edges to one edge (thus forgetting these end vertices as vertex), for each $i = 1, \ldots, k$.

For $\pi \in S_k$, let P_π be the $2k$-legged fixed diagram consisting of k disjoint edges $e_1, \ldots, e_k$, where the ends of edge e_i are labeled i and $k + \pi(i)$ (for $i = 1, \ldots, k$). Then we call the following $k!$-term relation the *Δ_k-equation*:

$$\sum_{\pi \in S_k} \operatorname{sgn}(\pi) f(P_\pi \cdot H) = 0 \qquad \text{for each } 2k\text{-legged fixed diagram } H. \tag{5}$$

Theorem. *Let $f : \mathcal{G} \to \mathbb{C}$ be a weight system. Then there exists a complex reductive metric Lie algebra $\mathfrak{g}$ with $f = \varphi_{\mathfrak{g}}$ if and only if $f(\bigcirc) \in \mathbb{Z}_+$ and f satisfies the $\Delta_{f(\bigcirc)+1}$-equation. If $\mathfrak{g}$ exists, it is unique.*

Although (5) may look like a linear constraint separately for each fixed number of vertices of 3-graphs (as the AS- and IHX-equations are), it in fact interconnects 3-graphs with different numbers of vertices, since $P_\pi \cdot H$ can be a disjoint union of 3-graphs, taking multiplicativity of f as above. So (5) is a polynomial relation between f-values of 3-graphs. It describes f as common zero of a set of polynomials in the ring $\mathbb{C}[\mathcal{G}]$ formally generated by the collection $\mathcal{G}$ of 3-graphs (which are connected by definition), in which the disjoint union $\sqcup$ is taken as multiplication. We note that, for any $k \in \mathbb{Z}_+$, the Δ_k-equation implies that $f(\bigcirc)$ is a nonnegative integer strictly less than k (by taking $H := P_{\mathrm{id}}$, where id is the identity permutation in S_k).

The theorem implies that if $f = \varphi_{\mathfrak{g}}$ for some metric Lie algebra $\mathfrak{g}$, then $f = \varphi_{\mathfrak{g}}$ for a unique reductive metric Lie algebra $\mathfrak{g}$. Indeed, let $f = \varphi_{\mathfrak{g}}$ for some n-dimensional Lie algebra $\mathfrak{g}$. So $f = p_{c_{\mathfrak{g}}}$ and $f(\bigcirc) = n$. We show that f satisfies the Δ_{n+1}-equation.

Take a $2(n+1)$-legged fixed diagram H and consider formulas (2) and (5) for $f := p_{c_{\mathfrak{g}}}$. The summations over π and ψ can be interchanged. For each fixed $\psi : E(H) \to [n]$, we need to add up $\mathrm{sgn}(\pi)$ over all those $\pi \in S_{n+1}$ for which, for each $i \in [n+1]$, legs i and $k + \pi(i)$ of H have the same ψ-value. As $n < n+1$, there exist two legs $i, j \in [n+1]$ of H with the same ψ-value. Let $\sigma \in S_{n+1}$ be the transposition of i and j. Then we can pair up each π with $\pi\sigma$, and in the summation they cancel. So for each ψ, the sum is 0, and therefore the Δ_{n+1}-equation holds.

These arguments also yield the necessity of the condition in the theorem. We prove sufficiency in Section 2 and uniqueness in Section 3.

By the theorem, if the $\Delta_{f(\bigcirc)+1}$-equation holds, then there exist unique (up to permuting indices) simple Lie algebras $\mathfrak{g}_1, \ldots \mathfrak{g}_t$ and nonzero complex numbers $\lambda_1, \ldots, \lambda_t$ such that

$$f(G) = \sum_{i=1}^{t} \lambda_i^{\frac{1}{2}|V(G)|} \varphi_{\mathfrak{g}_i}(G) \tag{6}$$

for each 3-graph $G \neq \bigcirc$, taking the Killing forms as metrics. So any linear combination of 3-graphs 'detected' (to be nonzero) by a Lie algebra weight system, is detected by a simple Lie algebra weight system.

It can be proved that if $n := f(\bigcirc) \in \mathbb{Z}_+$, then the Δ_{n+1}-equation can be replaced by the equivalent condition that for each $k \in \mathbb{Z}_+$, the rank of the $\mathcal{F}_k \times \mathcal{F}_k$ matrix $C_{f,k} := (f(G \cdot H)_{G,H \in \mathcal{F}_k})$ is at most n^k. A weaker, but also equivalent condition is that there exists an $m \in \mathbb{Z}_+$ such that the rank of $C_{f,2(n+1)m}$ is less than the dimension of the space of all $\mathrm{GL}(d)$-invariant tensors in $\mathfrak{gl}(d)^{\otimes m}$, where $d := n^{n+1} + 1$.

Our proof is based on some basic theorems from invariant theory (Weyl's first and second fundamental theorem for the orthogonal group, and the unique closed orbit theorem; cf. [5],[10]), and roots in methods used in [7], [9], and [15]. For any $n \in \mathbb{Z}_+$ and any 3-graph G, let $p(G)$ be the regular function on $((\mathbb{C}^n)^{\otimes 3})^{C_3}$ defined by

$$p(G)(x) := p_x(G) \qquad \text{for } x \in ((\mathbb{C}^n)^{\otimes 3})^{C_3} \tag{7}$$

(cf. (2)). Then each $p(G)$ is $O(n,\mathbb{C})$-invariant, and the first fundamental theorem of invariant theory implies that the algebra of $O(n,\mathbb{C})$-invariant regular functions on $((\mathbb{C}^n)^{\otimes 3})^{C_3}$ is generated by $\{p(G) \mid G \text{ 3-graph}\}$.

Note that $O(n,\mathbb{C})$ acts naturally on the affine variety $\mathcal{V}_n$ defined by (3), and that for any two metrized Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ one trivially has: $\mathfrak{g} = \mathfrak{g}'$ if and only if $c_{\mathfrak{g}}$ and $c_{\mathfrak{g}'}$ belong to the same $O(n,\mathbb{C})$-orbit on $\mathcal{V}_n$. Moreover, the closed orbit theorem implies that $\varphi_{\mathfrak{g}} = \varphi_{\mathfrak{g}'}$ if and only if the closures of the orbits $O(n,\mathbb{C}) \cdot c_{\mathfrak{g}}$ and $O(n,\mathbb{C}) \cdot c_{\mathfrak{g}'}$ intersect; that is, if and only if they project to the same point in $\mathcal{V}_n // O(n,\mathbb{C})$.

The proof implies that a metric Lie algebra $\mathfrak{g}$ is reductive if and only if the orbit $O(\mathfrak{g}) \cdot c_{\mathfrak{g}}$ is closed. Hence, for each n, there is a one-to-one correspondence between the points in the orbit space $\mathcal{V}_n // O(n,\mathbb{C})$ and the n-dimensional complex reductive metric Lie algebras.

2. Proof of the theorem: existence of $\mathfrak{g}$

Let $f : \mathcal{G} \to \mathbb{C}$ satisfy the Δ_{n+1}-equation (5), where $n := f(\bigcirc) \in \mathbb{Z}_+$. As above, we extend f to the collection $\mathcal{F}_0$ of disjoint unions of 3-graphs by the rule that $f(\emptyset) = 1$ and $f(G \sqcup H) = f(G)f(H)$ for all $G, H \in \mathcal{F}_0$ (where $\sqcup$ denotes disjoint union). For any k, let $\mathbb{C}\mathcal{F}_k$ be the linear space of formal $\mathbb{C}$-linear combinations of elements of $\mathcal{F}_k$. Any (bi-)linear function on $\mathcal{F}_k$ can be extended (bi-)linearly to $\mathbb{C}\mathcal{F}_k$. Taking $\sqcup$ as product, $\mathbb{C}\mathcal{F}_0$ becomes an algebra (which is equal to $\mathbb{C}[\mathcal{G}]$ described above), and f becomes an algebra homomorphism $\mathbb{C}\mathcal{F}_0 \to \mathbb{C}$. Similarly, p (as defined in (7)) extends to an algebra homomorphism $\mathbb{C}\mathcal{F}_0 \to \mathcal{O}(((\mathbb{C}^n)^{\otimes 3})^{C_3})$. (As usual, $\mathcal{O}(\cdot)$ denotes the algebra of $\mathbb{C}$-valued regular functions on $\cdot$.)

Proposition 1. $\mathrm{Ker}(p) \subseteq \mathrm{Ker}(f)$.

Proof. Let $\gamma \in \mathbb{C}\mathcal{F}_0$ with $p(\gamma) = 0$. We prove that $f(\gamma) = 0$. As each homogeneous component of $p(\gamma)$ is 0, we can assume that γ is a linear combination of graphs in $\mathcal{F}_0$ that all have the same number of vertices, say k (which is necessarily even).

Let $\mathcal{M}$ be the collection of perfect matchings on $[3k]$. We can naturally identify $\mathcal{M}$ with the set of $3k$-legged fixed diagrams with no trivalent vertices and no copies of $\bigcirc$.

Let H be the the $3k$-legged fixed diagram with precisely k trivalent vertices $v_1, \ldots, v_k$ and $3k$ legs, where v_i is adjacent to legs $3i-2$, $3i-1$, $3i$, in order. (So H is the disjoint union of k copies of the tri-star $K_{1,3}$.) Then each graph in $\mathcal{F}_0$ with k vertices is equal to $M \cdot H$ for at least one $M \in \mathcal{M}$. Hence we can write

$$(8) \qquad \gamma = \sum_{M \in \mathcal{M}} \lambda(M) M \cdot H$$

for some $\lambda : \mathcal{M} \to \mathbb{C}$.

The symmetric group S_{3k} acts naturally on $\mathcal{F}_{3k}$ (by permuting leg-labels). Let Q be the group of permutations $\sigma \in S_{3k}$ with $H^\sigma = H$. (So Q is the wreath product of the cyclic group C_3 with S_k. It stabilizes the partition $\{\{3i-2, 3i-1, 3i\} \mid i = 1, \ldots, k\}$ of $[3k]$ and permutes each class in this partition cyclically.) Since $M^\sigma \cdot H = M \cdot H^{\sigma^{-1}} = M \cdot H$ for each $M \in \mathcal{M}$ and $\sigma \in Q$, we can assume that λ is invariant under the action of Q on $\mathcal{M}$.

Define linear functions F_M (for $M \in \mathcal{M}$) and F on $(\mathbb{C}^n)^{\otimes 3k}$ by

$$(9) \qquad F_M(a_1 \otimes \cdots \otimes a_{3k}) := \prod_{ij \in M} a_i^{\mathsf{T}} a_j \quad \text{and} \quad F := \sum_{M \in \mathcal{M}} \lambda(M) F_M,$$

for $a_1, \ldots, a_{3k} \in \mathbb{C}^n$. ($ij$ stands for the unordered pair $\{i, j\}$; so $ij = ji$.) Note that $F_M(x^{\otimes k}) = p(M \cdot H)(x)$ for any $x \in ((\mathbb{C}^n)^{\otimes 3})^{C_3}$. Hence $F(x^{\otimes k}) = p(\gamma)(x) = 0$. We show that this implies that $F = 0$.

Indeed, suppose $F(u_1 \otimes \cdots \otimes u_k) \neq 0$ for some $u_1, \ldots, u_k \in (\mathbb{C}^n)^{\otimes 3}$. Since F is Q-invariant (as λ is Q-invariant), we can assume that each u_i is C_3-invariant.

For $y \in \mathbb{C}^k$, define the C_3-invariant tensor $b_y := y_1 u_1 + \cdots + y_k u_k$. As F is Q-invariant, the coefficient of the monomial $y_1 \cdots y_k$ in the polynomial $F(b_y^{\otimes k})$ is equal to $k! \cdot F(u_1 \otimes \cdots \otimes u_k) \neq 0$. So the polynomial is nonzero, hence $F(b_y^{\otimes k}) \neq 0$ for some $y \in \mathbb{C}^k$, a contradiction. Therefore, $F = 0$.

Define for each multiset N of singletons and unordered pairs from $[3k]$ the monomial q_N on $S^2\mathbb{C}^{3k}$ (= the set of symmetric matrices in $\mathbb{C}^{3k \times 3k}$), and define moreover the polynomial q on $S^2\mathbb{C}^{3k}$ by:

$$q_N(X) := \prod_{ij \in N} X_{i,j} \quad \text{and} \quad q := \sum_{M \in \mathcal{M}} \lambda(M) q_M, \tag{10}$$

for $X = (X_{i,j}) \in S^2\mathbb{C}^{3k}$. Note that for each monomial μ on $S^2\mathbb{C}^{3k}$ there is a unique multiset N of singletons and unordered pairs from $[3k]$ with $\mu = q_N$. Now $F = 0$ implies

$$q(X) = 0 \text{ if } \operatorname{rank}(X) \leq n. \tag{11}$$

Indeed, if $\operatorname{rank}(X) \leq n$, then there exist $a_1, \ldots, a_{3k} \in \mathbb{C}^n$ such that $X_{i,j} = a_i^{\mathsf{T}} a_j$ for all $i, j = 1 \ldots, 3k$. By (9) and (10), $q(X) = F(a_1 \otimes \cdots \otimes a_{3k}) = 0$, proving (11).

By the second fundamental theorem of invariant theory (cf. [10] Theorem 12.2.12), (11) implies that q belongs to the ideal in $\mathcal{O}(S^2\mathbb{C}^{3k})$ generated by the $(n+1) \times (n+1)$ minors of $X \in S^2\mathbb{C}^{3k}$. That is, q is a linear combination of polynomials $\det(X_{I,J}) q_N(X)$, where $I, J \subseteq [3k]$ with $|I| = |J| = n+1$ and where N is a multiset of singletons and unordered pairs from $[3k]$. Here $X_{I,J}$ denotes the $I \times J$ submatrix of X.

Now such triples I, J, N occur in two kinds: (1) those with $I \cap J = \emptyset$ and N a perfect matching on $[3k] \setminus (I \cup J)$, in which case *all* monomials occurring in $\det(X_{I,J}) q_N(X)$ are equal to q_M for some $M \in \mathcal{M}$; and (2) all other triples I, J, N, in which case *none* of the monomials occurring in $\det(X_{I,J}) q_N(X)$ is equal to q_M for some $M \in \mathcal{M}$. Since $q(X)$ consists completely of monomials q_M with $M \in \mathcal{M}$, we can ignore all triples of kind (2), and conclude that $q(X)$ is a linear combination of $\det(X_{I,J}) q_N(X)$ with I, J, N of kind (1).

For any $M \in \mathcal{M}$, define $\Gamma(q_M) := M \cdot H$, and extend Γ linearly to linear combinations of the q_M for $M \in \mathcal{M}$. Then $\Gamma(q) = \gamma$. Moreover, for each I, J, N of kind (1), by the Δ_{n+1}-equation for f, $f(\Gamma(\det(X_{I,J}) q_N(X))) = 0$. As γ is a linear combination of elements $\Gamma(\det(X_{I,J}) q_N(X))$, we have $f(\gamma) = 0$, as required. ∎

By this proposition, there exists a linear function $\Phi : p(\mathbb{C}\mathcal{F}_0) \to \mathbb{C}$ such that $\Phi \circ p = f$. Then Φ is an algebra homomorphism, since for $G, H \in \mathcal{F}_0$ one has $\Phi(p(G)p(H)) = \Phi(p(G \sqcup H)) = f(G \sqcup H) = f(G)f(H) = \Phi(p(G))\Phi(p(H))$.

By the first fundamental theorem of invariant theory,

$$\mathcal{O}(((\mathbb{C}^n)^{\otimes 3})^{C_3})^{O(n)} = p(\mathbb{C}\mathcal{F}_0) \tag{12}$$

(setting $O(n) := O(n, \mathbb{C})$). So Φ is an algebra homomorphism $\mathcal{O}(((\mathbb{C}^n)^{\otimes 3})^{C_3})^{O(n)} \to$

$\mathbb{C}$. Hence the affine $O(n)$-variety

$$\mathcal{V} := \{x \in ((\mathbb{C}^n)^{\otimes 3})^{C_3} \mid q(x) = \Phi(q) \text{ for each } q \in \mathcal{O}(((\mathbb{C}^n)^{\otimes 3})^{C_3})^{O(n)}\} \tag{13}$$

is nonempty (as $O(n)$ is reductive). By (12) and by substituting $q = p(G)$ in (13),

$$\mathcal{V} := \{x \in ((\mathbb{C}^n)^{\otimes 3})^{C_3} \mid p_x = f\}. \tag{14}$$

Hence as $\mathcal{V} \neq \emptyset$ there exists $c \in ((\mathbb{C}^n)^{\otimes 3})^{C_3}$ with $p_c = f$. We choose c such that the orbit $O(n) \cdot c$ is a closed. This is possible by the unique closed orbit theorem (cf. Brion [5]), which also implies that c is contained in each nonempty $O(n)$-invariant closed subset of $\mathcal{V}$.

Then c gives the required Lie algebra:

Proposition 2. *$c = c_{\mathfrak{g}}$ for some complex reductive metric Lie algebra $\mathfrak{g}$.*

Proof. We extend $p(G)$ to a function $\widehat{p}$ on fixed diagrams as follows. For each k and $G \in \mathcal{F}_k$, let $\widehat{p}(G) : ((\mathbb{C}^n)^{\otimes 3})^{C_3} \to (\mathbb{C}^n)^{\otimes k}$ be defined by

$$\widehat{p}(G)(x) := \sum_{\psi: E(G) \to [n]} \Big(\prod_{v \in V_3(G)} x_{\psi(e_1),\psi(e_2),\psi(e_3)} \Big) \bigotimes_{j=1}^{k} b_{\psi(\varepsilon_j)} \tag{15}$$

for $x \in ((\mathbb{C}^n)^{\otimes 3})^{C_3}$, where $V_3(G)$ is the set of trivalent vertices of G, e_1, e_2, e_3 are the edges incident with v, in order, and ε_j is the edge incident with leg labeled j (for $j = 1, \ldots, k$). Moreover, $b_1, \ldots, b_n$ is the standard basis of $\mathbb{C}^n$.

Then for all $G, H \in \mathcal{F}_k$ and $x \in ((\mathbb{C}^n)^{\otimes 3})^{C_3}$,

$$\widehat{p}(G)(x) \cdot \widehat{p}(H)(x) = p(G \cdot H)(x), \tag{16}$$

where $\cdot$ denotes the standard inner product on $(\mathbb{C}^n)^{\otimes k}$.

Claim. For each k and $\tau \in \mathbb{C}\mathcal{F}_k$, if $f(\tau \cdot H) = 0$ for each $H \in \mathcal{F}_k$, then $\widehat{p}(\tau)(c) = 0$.

Proof. As $\widehat{p}(\tau)$ is $O(n)$-equivariant, it suffices to show that $\widehat{p}(\tau)$ has a zero x in $\mathcal{V}$, since then the $O(n)$-stable closed set $\{x \in \mathcal{V} \mid \widehat{p}(\tau)(x) = 0\}$ is nonempty, and hence must contain c.

Suppose that no such zero exists. Then the functions $\widehat{p}(\tau)$ and $p(G) - f(G)$ (for $G \in \mathcal{G}$) have no common zero. Hence, by the Nullstellensatz, there exist regular functions $s : ((\mathbb{C}^n)^{\otimes 3})^{C_3} \to (\mathbb{C}^n)^{\otimes k}$ and $g_1, \ldots, g_t : ((\mathbb{C}^n)^{\otimes 3})^{C_3} \to \mathbb{C}$, and $G_1, \ldots, G_t \in \mathcal{G}$ such that

$$\widehat{p}(\tau)(x) \cdot s(x) + \sum_{i=1}^{t} (p(G_i)(x) - f(G_i)) g_i(x) = 1 \tag{17}$$

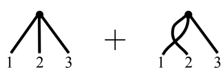
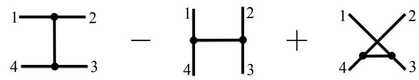
for all $x \in ((\mathbb{C}^n)^{\otimes 3})^{C_3}$. Now $\widehat{p}(\tau)$ and $p(G_1), \ldots, p(G_t)$ are $O(n)$-equivariant. Hence, by applying the Reynolds operator, we can assume that also s and $g_1, \ldots, g_k$ are $O(n)$-

equivariant. Then by the first fundamental theorem of invariant theory, $s = \widehat{p}(\beta)$ for some $\beta \in \mathbb{C}\mathcal{F}_k$, and $g_i = p(\gamma_i)$ for some $\gamma_i \in \mathbb{C}\mathcal{F}_0$, for $i = 1, \ldots, t$. This gives, with (16),

$$(18) \qquad 1 = \widehat{p}(\tau)(c) \cdot \widehat{p}(\beta)(c) + \sum_{i=1}^{t}(p(G_i)(c) - f(G_i))p(\gamma_i)(c) = p(\tau \cdot \beta)(c) + \sum_{i=1}^{t}(p(G_i)(c) - f(G_i))p(\gamma_i)(c) = f(\tau \cdot \beta) + \sum_{i=1}^{t}(f(G_i) - f(G_i))f(\gamma_i) = 0,$$

a contradiction, proving the Claim. □

Let AS $\in \mathbb{C}\mathcal{F}_3$ and IHX $\in \mathbb{C}\mathcal{F}_4$ be extracted from the the AS- and IHX-equations; that is,

(19) AS :=  , IHX :=  .

As f is a weight system, $f(\text{AS} \cdot H) = 0$ for each $H \in \mathcal{F}_3$ and $f(\text{IHX} \cdot H) = 0$ for each $H \in \mathcal{F}_4$. Hence the Claim implies that $\widehat{p}(\text{AS})(c) = 0$ and $\widehat{p}(\text{IHX})(c) = 0$. Therefore, $c = c_{\mathfrak{g}}$ for some metric Lie algebra $\mathfrak{g}$ (cf. (3)).

We show that $\mathfrak{g}$ is reductive. For this it suffices to show that the orthogonal complement $Z(\mathfrak{g})^\perp$ of the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is semisimple (as then $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = 0$, so $Z(\mathfrak{g})$ is nondegenerate).

Suppose to the contrary that $Z(\mathfrak{g})^\perp$ contains a nonzero abelian ideal I. We can assume that I is a minimal nonzero ideal. Then $I \subseteq I^\perp$, since, by the minimality of I, either $[\mathfrak{g}, I] = 0$, hence $I \subseteq Z(\mathfrak{g}) \subseteq I^\perp$, or $[\mathfrak{g}, I] = I$, hence $\langle I, I\rangle = \langle [\mathfrak{g}, I], I\rangle = \langle \mathfrak{g}, [I, I]\rangle = 0$.

So $I + Z(\mathfrak{g}) \subseteq I^\perp$. This implies that we can choose a subspace A of $I^\perp$ with $I \cap A = 0$ and $I + A = I^\perp$ such that

$$(20) \qquad Z(\mathfrak{g}) = (I \cap Z(\mathfrak{g})) + (A \cap Z(\mathfrak{g})).$$

Then A is nondegenerate, since $A \cap A^\perp = A \cap I^\perp \cap A^\perp = A \cap (I + A)^\perp = A \cap I = 0$.

So also $A^\perp$ is nondegenerate. As $I \subseteq A^\perp$ and $\dim(A^\perp) = 2\dim(I)$, there exists a self-orthogonal subspace C of $A^\perp$ with $I \cap C = 0$ and $I + C = A^\perp$. Then $\dim(C) = \dim(I)$ (as $\dim(A) = n - 2\dim(I)$).

Now define, for any nonzero $\alpha \in \mathbb{C}$, $\varphi_\alpha : \mathfrak{g} \to \mathfrak{g}$ by

$$(21) \qquad \varphi_\alpha(x) = \begin{cases} \alpha^{-1}x & \text{if } x \in I, \\ \alpha x & \text{if } x \in C, \\ x & \text{if } x \in A. \end{cases}$$

So $\varphi_\alpha \in O(n)$.

Let π_A denote the orthogonal projection $\mathfrak{g} \to A$. Then

$$\lim_{\alpha \to 0} c_{\mathfrak{g}} \cdot \varphi_\alpha^{\otimes 3} = c_{\mathfrak{g}} \cdot \pi_A^{\otimes 3}, \tag{22}$$

where $\cdot$ denotes the standard inner product on $(\mathbb{C}^n)^{\otimes 3}$. To prove (22), choose $x, y, z \in I \cup C \cup A$. If $x, y, z \in A$, then for each nonzero $\alpha \in \mathbb{C}$:

$$c_{\mathfrak{g}} \cdot \varphi_\alpha^{\otimes 3}(x \otimes y \otimes z) = c_{\mathfrak{g}} \cdot (x \otimes y \otimes z) = c_{\mathfrak{g}} \cdot \pi_A^{\otimes 3}(x \otimes y \otimes z). \tag{23}$$

If not all of x, y, z belong to A, let k be the number of x, y, z belonging to I minus the number of x, y, z belonging to C. Then

$$\lim_{\alpha \to 0} c_{\mathfrak{g}} \cdot \varphi_\alpha^{\otimes 3}(x \otimes y \otimes z) = \lim_{\alpha \to 0} \alpha^{-k} c_{\mathfrak{g}} \cdot (x \otimes y \otimes z) = 0. \tag{24}$$

The last equality follows from the fact that if $k \geq 0$, then we may assume (by symmetry) that $x \in I$ and $z \notin C$, so $z \in I + A = I^\perp$. Hence $c_{\mathfrak{g}} \cdot (x \otimes y \otimes z) = \langle [x, y], z \rangle = 0$, as $[x, y] \in I$.

This proves (22). Hence, as $O(n) \cdot c_{\mathfrak{g}}$ is closed, there exists $\varphi \in O(n)$ such that

$$c_{\mathfrak{g}} \cdot \pi_A^{\otimes 3} = c_{\mathfrak{g}} \cdot \varphi^{\otimes 3}. \tag{25}$$

This implies

$$\varphi(I + C + Z(\mathfrak{g})) \subseteq Z(\mathfrak{g}). \tag{26}$$

To see this, by (20), $I + C + Z(\mathfrak{g}) = I + C + (A \cap Z(\mathfrak{g}))$. Now choose $x \in I \cup C \cup (A \cap Z(\mathfrak{g}))$. Then for all $y, z \in \mathfrak{g}$, using (25):

$$\langle [\varphi(x), \varphi(y)], \varphi(z) \rangle = \langle [\pi_A(x), \pi_A(y)], \pi_A(z) \rangle = 0. \tag{27}$$

Indeed, if $x \in I + C = A^\perp$ then $\pi_A(x) = 0$. If $x \in A \cap Z(\mathfrak{g})$, then $\pi_A(x) = x$, and (27) follows as $x \in Z(\mathfrak{g})$. As (27) holds for all $y, z \in \mathfrak{g}$, $\varphi(x)$ belongs to $Z(\mathfrak{g})$.

So we have (26), which implies $\dim(I + C + Z(\mathfrak{g})) \leq \dim(Z(\mathfrak{g}))$, so $C \subseteq Z(\mathfrak{g})$, hence $C = 0$, as $C \cap Z(\mathfrak{g}) \subseteq C \cap (I + A) = 0$. Therefore, $\dim(I) = \dim(C) = 0$, contradicting $I \neq 0$. Concluding, $\mathfrak{g}$ is reductive. ∎

3. Proof of the theorem: uniqueness of $\mathfrak{g}$

We first show uniqueness if the metrics are the Killing forms.

Proposition 3. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be complex semisimple Lie algebras with their Killing forms as metrics. If $\varphi_{\mathfrak{g}} = \varphi_{\mathfrak{g}'}$ then $\mathfrak{g} = \mathfrak{g}'$.*

Proof. As $\varphi_{\mathfrak{g}} = \varphi_{\mathfrak{g}'}$, we know $\dim(\mathfrak{g}) = \varphi_{\mathfrak{g}}(\bigcirc) = \varphi_{\mathfrak{g}'}(\bigcirc) = \dim(\mathfrak{g}')$. Let $\mathfrak{h}$ and $\mathfrak{h}'$ be real compact forms in $\mathfrak{g}$ and $\mathfrak{g}'$ respectively. Since the Killing forms are negative definite on $\mathfrak{h}$ and $\mathfrak{h}'$, we can assume that the inner product spaces underlying $\mathfrak{h}$ and $\mathfrak{h}'$ both are $\mathbb{R}^n$ with standard negative definite inner product, and that $c_{\mathfrak{g}}$ and $c_{\mathfrak{g}'}$ belong to $((\mathbb{R}^n)^{\otimes 3})^{C_3}$.

Suppose $\mathfrak{g} \neq \mathfrak{g}'$. Hence the orbits $O(n,\mathbb{R}) \cdot c_{\mathfrak{g}}$ and $O(n,\mathbb{R}) \cdot c_{\mathfrak{g}'}$ are disjoint compact subsets of $((\mathbb{R}^n)^{\otimes 3})^{C_3}$. By the Stone-Weierstrass theorem, there exists a real-valued polynomial q on $((\mathbb{R}^n)^{\otimes 3})^{C_3}$ such that $q(x) \leq 0$ for each $x \in O(n,\mathbb{R}) \cdot c_{\mathfrak{g}}$ and $q(x) \geq 1$ for each $x \in O(n,\mathbb{R}) \cdot c_{\mathfrak{g}'}$. Applying the Reynolds operator, we may assume that q is $O(n,\mathbb{R})$-invariant. By the first fundamental theorem of invariant theory, q belongs to the algebra generated by $\{p(G) \mid G \text{ 3-graph}\}$. However, $p(G)(c_{\mathfrak{g}}) = \varphi_{\mathfrak{g}}(G) = \varphi_{\mathfrak{g}'}(G) = p(G)(c_{\mathfrak{g}'})$ for each 3-graph G. So $q(c_{\mathfrak{g}}) = q(c_{\mathfrak{g}'})$, contradicting $q(c_{\mathfrak{g}}) \leq 0$ and $q(c_{\mathfrak{g}'}) \geq 1$. □

For each complex metric Lie algebra $\mathfrak{g}$ of positive dimension, define

$$\varphi'_{\mathfrak{g}} := \frac{1}{\dim(\mathfrak{g})}\varphi_{\mathfrak{g}}. \tag{28}$$

From Proposition 3 we derive the next proposition.

Proposition 4. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be complex simple metric Lie algebras. If $\varphi'_{\mathfrak{g}} = \varphi'_{\mathfrak{g}'}$ then $\mathfrak{g} = \mathfrak{g}'$.*

Proof. Let B and B' denote the bilinear forms associated with $\mathfrak{g}$ and $\mathfrak{g}'$, respectively, and let K and K' be the Killing forms of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Since $\mathfrak{g}$ and $\mathfrak{g}'$ are simple, there are nonzero $\alpha, \alpha' \in \mathbb{C}$ such that $B = \alpha K$ and $B' = \alpha' K'$. Then

$$\varphi_{\mathfrak{g},B}(\ominus) = \alpha^{-1}\varphi_{\mathfrak{g},K}(\ominus) = -\alpha^{-1}K^{\otimes 3}(c_{\mathfrak{g},K}, c_{\mathfrak{g},K}) = \alpha^{-1}\dim(\mathfrak{g}), \tag{29}$$

and similarly $\varphi_{\mathfrak{g}',B'}(\ominus) = \alpha'^{-1}\dim(\mathfrak{g}')$. Since $\varphi'_{\mathfrak{g},B}(\ominus) = \varphi'_{\mathfrak{g}',B'}(\ominus)$, this implies $\alpha = \alpha'$. So $\varphi'_{\mathfrak{g},K} = \varphi'_{\mathfrak{g}',K'}$, hence we can assume that $\alpha = 1$, so $B = K$ and $B' = K'$.

Now let $\widetilde{\mathfrak{g}}$ be the direct sum of $\dim(\mathfrak{g}')$ copies of $\mathfrak{g}$. Similarly, let $\widetilde{\mathfrak{g}'}$ be the direct sum of $\dim(\mathfrak{g})$ copies of $\mathfrak{g}'$. So $\dim \widetilde{\mathfrak{g}} = \dim \widetilde{\mathfrak{g}'}$, and for each 3-graph G, as $\varphi'_{\mathfrak{g}} = \varphi'_{\mathfrak{g}'}$ and as G is connected:

$$\varphi_{\widetilde{\mathfrak{g}}}(G) = \dim(\mathfrak{g}')\varphi_{\mathfrak{g}}(G) = \dim(\mathfrak{g})\varphi_{\mathfrak{g}'}(G) = \varphi_{\widetilde{\mathfrak{g}'}}(G). \tag{30}$$

Hence by Proposition 3, $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}'}$, and so $\mathfrak{g} = \mathfrak{g}'$. □

We note that also if $\mathfrak{g}$ is a complex 1-dimensional metric Lie algebra and $\mathfrak{g}'$ is a complex simple metric Lie algebra, then $\varphi'_{\mathfrak{g}} \neq \varphi'_{\mathfrak{g}'}$, since $\varphi'_{\mathfrak{g}}(\ominus) = 0$ while $\varphi'_{\mathfrak{g}'}(\ominus) \neq 0$. This and Proposition 4 is used to prove the last proposition, which settles the theorem.

Proposition 5. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be complex reductive metric Lie algebras. If $\varphi_{\mathfrak{g}} = \varphi_{\mathfrak{g}'}$ then $\mathfrak{g} = \mathfrak{g}'$.*

Proof. As $\mathfrak{g}$ and $\mathfrak{g}'$ are reductive, we can write

$$\mathfrak{g} = \bigoplus_{i=1}^{m} \mathfrak{g}_i \text{ and } \mathfrak{g}' = \bigoplus_{j=1}^{m'} \mathfrak{g}'_j, \tag{31}$$

where each $\mathfrak{g}_i$ and $\mathfrak{g}'_j$ is either simple or 1-dimensional. Then, since 3-graphs are connected,

$$(32)\qquad \sum_{i=1}^{m} \varphi_{\mathfrak{g}_i} = \varphi_{\mathfrak{g}} = \varphi_{\mathfrak{g}'} = \sum_{j=1}^{m'} \varphi_{\mathfrak{g}'_j}.$$

So we can assume that $\mathfrak{g}_i \neq \mathfrak{g}'_j$ for all $i \in [m]$ and $j \in [m']$. Hence, by Proposition 4 and the remark thereafter, there exist finitely many 3-graphs $G_1, \ldots, G_k$ such that for all $i \in [m]$ and $j \in [m']$ there exists $t \in [k]$ with $\varphi'_{\mathfrak{g}_i}(G_t) \neq \varphi'_{\mathfrak{g}'_j}(G_t)$. That is, for each $i \in [m]$ and $j \in [m']$, the following vectors $\mathbf{y}_i$, $\mathbf{z}_j \in \mathbb{C}^k$:

$$(33)\qquad \mathbf{y}_i := (\varphi'_{\mathfrak{g}_i}(G_1), \ldots, \varphi'_{\mathfrak{g}_i}(G_k)) \quad \text{and} \quad \mathbf{z}_j := (\varphi'_{\mathfrak{g}'_j}(G_1), \ldots, \varphi'_{\mathfrak{g}'_j}(G_k))$$

are distinct. So there exists a polynomial $q \in \mathbb{C}[x_1, \ldots, x_k]$ such that $q(\mathbf{y}_i) = 0$ for each $i = 1, \ldots, m$ and $q(\mathbf{z}_j) = 1$ for each $j = 1, \ldots, m'$. Now set $\gamma := q(G_1, \ldots, G_k)$, taking formal linear sums of 3-graphs and applying the following composition of 3-graphs G and H as product ([8]): take the disjoint union of G and H, choose an edge uv of G and an edge $u'v'$ of H, and replace them by uu' and vv'. Let F be the 3-graph thus arising. Then for any complex simple or 1-dimensional metric Lie algebra $\mathfrak{g}$: $\varphi'_{\mathfrak{g}}(F) = \varphi'_{\mathfrak{g}}(G)\varphi'_{\mathfrak{g}}(H)$, independently of the choice of uv and $u'v'$ (see Proposition 7.18 in [6]).

We extend each $\varphi'_{\mathfrak{g}_i}$ and $\varphi'_{\mathfrak{g}'_j}$ linearly to γ. Then $\varphi'_{\mathfrak{g}_i}(\gamma) = q(\mathbf{y}_i) = 0$ for each $i = 1, \ldots, m$ while $\varphi'_{\mathfrak{g}'_j}(\gamma) = q(\mathbf{z}_j) = 1$ for each $j = 1, \ldots, m'$. Hence $\varphi_{\mathfrak{g}_i}(\gamma) = 0$ for each $i = 1, \ldots, m$ and $\varphi_{\mathfrak{g}'_j}(\gamma) = \dim(\mathfrak{g}'_j)$ for each $j = 1, \ldots, m'$. Therefore, by (32), $m' = 0$. Similarly, $m = 0$. ∎

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