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Compact orbit spaces in Hilbert spaces and limits of edge-colouring models

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A R T I C L E   I N F O

A B S T R A C T

Let G be a group of orthogonal transformations of a real Hilbert space H. Let R and W be bounded G-stable subsets of H. Let $\|.\|_R$ be the seminorm on H defined by $\|x\|_R := \sup_{r \in R} |\langle r, x \rangle|$ for $x \in H$. We show that if W is weakly compact and the orbit space R^k/G is compact for each $k \in \mathbb{N}$, then the orbit space W/G is compact when W is equipped with the norm topology induced by $\|.\|_R$.

As a consequence we derive the existence of limits of edge-colouring models which answers a question posed by Lovász. It forms the edge-colouring counterpart of the graph limits of Lovász and Szegedy, which can be seen as limits of vertex-colouring models.

In the terminology of de la Harpe and Jones, vertex- and edge-colouring models are called 'spin models' and 'vertex models' respectively.

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1. Introduction

In a fundamental paper, Lovász and Szegedy [10] showed that the collection of simple graphs fits in a natural way in a compact metric space W that conveys several phenomena of extremal and probabilistic graph theory and of statistical mechanics. In particular, a limit behaviour of graphs can be derived.

The elements of W are called *graphons*, as generalization of graphs, but they can also be considered to generalize the vertex-colouring models, or 'spin models' in the sense of de la Harpe and Jones [2]. In this context, the partition functions of spin models form a compact space. In the present paper, we investigate to what extent edge-colouring models, or 'vertex models' in the terminology of [2],

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behave similarly. Indeed, the edge-colouring models form a dense subset in a compact space, and thus we obtain limits of edge-colouring models. This solves a problem posed by Lovász [8].

To obtain these results, we prove a general theorem on compact orbit spaces in Hilbert space, that applies to both vertex and edge-colouring models. This compactness theorem uses and extends theorems of Lovász and Szegedy [11] on Szemerédi-like regularity in Hilbert spaces.

For background on graph limits we also refer to the recent book of Lovász [9]. Partition functions of edge-colouring models with a finite number of states were characterized by Szegedy [14] and Draisma, Gijswijt, Lovász, Regts, and Schrijver [3].

2. Formulation of results

In this section we describe our results, giving proofs in Sections 2.1, 2.2, and 2.3. Throughout, for any Hilbert space H, $B(H)$ denotes the closed unit ball in H. Moreover, measures are Lebesgue measures.

Compact orbit spaces in Hilbert spaces. Let H be a real Hilbert space and let R be a bounded subset of H. Define a seminorm $\|.\|_R$ and a pseudometric[1] d_R on H by, for $x, y \in H$:

$$\|x\|_R := \sup_{r \in R} |\langle r, x \rangle| \quad \text{and} \quad d_R(x, y) := \|x - y\|_R. \tag{1}$$

In this paper, we use the topology induced by this pseudometric only if we explicitly mention it, otherwise we use the topology induced by the usual Hilbert norm $\|.\|$.

A subset W of H is called *weakly compact* if it is compact in the weak topology on H. Note that for any $W \subseteq H$ (cf. [7]):

$$W \text{ closed, bounded, and convex} \Rightarrow W \text{ weakly compact} \Rightarrow W \text{ bounded.} \tag{2}$$

Let G be a group acting on a topological space X. Then the *orbit space* X/G is the quotient space of X taking the orbits of G as classes. A subset Y of X is called *G-stable* if $g \cdot y \in Y$ for each $g \in G$ and $y \in Y$.

Our first main theorem (which we prove in Section 2.1) is:

Theorem 1. *Let H be a Hilbert space and let G be a group of orthogonal transformations of H. Let W and R be G-stable subsets of H, with W weakly compact and R^k/G compact for each k. Then $(W, d_R)/G$ is compact.*

Application to graph limits and vertex-colouring models. As a consequence of Theorem 1 we now first derive the compactness of the graphon space, which was proved by Lovász and Szegedy [11]. Let $H = L^2([0, 1]^2)$, the set of all square integrable functions $[0, 1]^2 \to \mathbb{R}$. Let R be the collection of functions $\chi^A \times \chi^B$, where A and B are measurable subsets of $[0, 1]$ and where χ^A and χ^B denote the incidence functions of A and B. Let $S_{[0,1]}$ be the group of measure space automorphisms of $[0, 1]$. The group $S_{[0,1]}$ acts naturally on H. Moreover, $R^k/S_{[0,1]}$ is compact for each k. (This can be derived from the fact that for each measurable $A \subseteq [0, 1]$ there is a $\pi \in S_{[0,1]}$ such that $\pi(A)$ is an interval up to a set of measure 0 (cf. [12]).)

Let W be the set of $[0, 1]$-valued functions $w \in H$ satisfying $w(x, y) = w(y, x)$ for all $x, y \in [0, 1]$. Then W is a closed bounded convex $S_{[0,1]}$-stable subset of H. So by Theorem 1, $(W, d_R)/S_{[0,1]}$ is compact. The elements of W are called *graphons*, where two elements w, w' of W are assumed to be the same graphon if $w' = g \cdot w$ for some $g \in S_{[0,1]}$. Therefore one may say that the graphon space is compact with respect to d_R.

In the context of de la Harpe and Jones [2], graphons can be considered as 'spin models' (with infinitely many states). For any $w \in W$, the *partition function* $\tau(w)$ of w is given by, for any graph F:

$$\tau(w)(F) := \int_{[0,1]^{V(F)}} \prod_{uv \in E(F)} w(x(u), x(v)) dx. \tag{3}$$

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[1] A *seminorm* is a norm except that nonzero elements may have norm 0. A *pseudometric* is a metric except that distinct points may have distance 0. One can turn a pseudometric space into a metric space by identifying points at distance 0, but for our purposes it is notationally easier and sufficient to maintain the original space. Notions like convergence pass easily over to pseudometric spaces, but limits need not be unique.

Let $\mathcal{F}$ denote the collection of simple graphs. Lovász and Szegedy [10] showed that $\tau : (W, d_R) \to \mathbb{R}^{\mathcal{F}}$ is continuous (here the restriction to simple graphs is necessary). Since $(W, d_R)/S_{[0,1]}$ is compact and since τ is $S_{[0,1]}$-invariant, the collection of functions $f : \mathcal{F} \to \mathbb{R}$ that are partition functions of graphons is compact. Hence each sequence $\tau(w_1), \tau(w_2), \ldots$ of partition functions of graphons $w_1, w_2, \ldots$, such that $\tau(w_i)(F)$ converges for each F, converges to the partition function $\tau(w)$ of some graphon w.

Since simple graphs can be considered as elements of W (by considering their adjacency matrix as element of W), this also gives a limit behaviour of simple graphs.

Application to edge-colouring models. We next show how Theorem 1 applies to the edge-colouring model (also called vertex model). For this, it will be convenient to use a different, but universal model of Hilbert space. Let C be a finite or infinite set, and consider the Hilbert space $H := \ell^2(C)$, the set of all functions $f : C \to \mathbb{R}$ with $\sum_{c \in C} f(c)^2 < \infty$, having norm $\|f\| := (\sum_{c \in C} f(c)^2)^{1/2}$. (Each Hilbert space is isomorphic to $\ell^2(C)$ for some C.) Following de la Harpe and Jones [2], any element of $\ell^2(C)$ is called an *edge-colouring model*, with *state set* (or *colour set*) C.

Define for each $k = 0, 1, \ldots$:

$$H_k := \ell^2(C^k). \tag{4}$$

The tensor power $\ell^2(C)^{\otimes k}$ embeds naturally in $\ell^2(C^k)$, and the orthogonal group $O(H)$ of H acts naturally on H_k. Define moreover

$$R_k := \{r_1 \otimes \cdots \otimes r_k \mid r_1, \ldots, r_k \in B(H)\} \subseteq H_k. \tag{5}$$

Again, let $\mathcal{F}$ be the collection of simple graphs. As usual, $H_k^{S_k}$ denotes the set of elements of H_k that are invariant under the natural action of S_k on H_k. Define the function

$$\pi : \prod_{k=0}^{\infty} H_k^{S_k} \to \mathbb{R}^{\mathcal{F}} \text{ by } \pi(h)(F) := \sum_{\phi : E(F) \to C} \prod_{v \in V(F)} h_{\deg(v)}(\phi(\delta(v))) \tag{6}$$

for $h = (h_k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}} H_k^{S_k}$ and $F \in \mathcal{F}$. Here $\deg(v)$ denotes the degree of v. Moreover, $\delta(v)$ is the set of edges incident with v, in some arbitrary order, say $e_1, \ldots, e_k$, with $k := \deg(v)$. Then $\phi(\delta(v)) := (\phi(e_1), \ldots, \phi(e_k))$ belongs to C^k. (So $\phi(\delta(v))$ may be viewed as the set of colours 'seen' from v.) For (6), the order chosen is irrelevant, as h_k is S_k-invariant.

The function $\pi(h) : \mathcal{F} \to \mathbb{R}$ can be considered as the *partition function* of the edge-colouring model h. It is not difficult to see that π is well-defined, and continuous if we take the usual Hilbert metric on each H_k, even if we replace $\mathcal{F}$ be the collection of all graphs without loops (cf. (14)). For simple graphs it remains continuous on $\prod_k B_k$ where $B_k := B(H_k^{S_k})$ if we replace for each k the Hilbert metric on B_k by d_{R_k}:

Theorem 2. π *is continuous on* $\prod_{k \in \mathbb{N}} (B_k, d_{R_k})$.

This is proved in Section 2.2, while in Section 2.3 we derive from Theorem 1:

Theorem 3. $(\prod_{k=0}^{\infty} (B_k, d_{R_k}))/O(H)$ *is compact.*

Now π is $O(H)$-invariant. This follows from the facts that $\ell^2(C^k)$ is the completion of $\ell^2(C)^{\otimes k}$ and that $O(H)$-invariance is direct if each h_k belongs to $\ell^2(C)^{\otimes k}$. Hence Theorem 3 implies:

Corollary 3a. *The image* $\pi(\prod_k B_k)$ *of* π *is compact.*

This implies:

Corollary 3b. *Let* $h^1, h^2, \ldots \in \prod_k B_k$ *be such that for each simple graph F, $\pi(h^i)(F)$ converges as $i \to \infty$. Then there exists $h \in \prod_k B_k$ such that for each simple graph F, $\lim_{i \to \infty} \pi(h^i)(F) = \pi(h)(F)$.*

As $\ell^2(C')$ embeds naturally in $\ell^2(C)$ if $C' \subseteq C$, all edge-colouring models with any finite number of states embed in $\ell^2(C)$ if C is infinite. So the corollary also describes a limit behaviour of finite-state edge-colouring models, albeit that the limit may have infinitely many states.

The corollary holds more generally for sequences in $\prod_k \lambda_k B_k$, for any fixed sequence $\lambda_0, \lambda_1, \ldots \in \mathbb{R}$.

We do not know if the quotient function $\pi / \sim : (\prod_k B_k) / \sim \to \mathbb{R}^{\mathcal{F}}$ is one-to-one, where $\sim$ is the equivalence relation on $\prod_k B_k$ with $h \sim h'$ whenever h' belongs to the closure of the $O(H)$-orbit of h. (For finite C and $\mathcal{F}$ replaced by the set of all graphs, this was proved in [13].) The analogous result for vertex-colouring models (i.e., graph limits) was proved by Borgs, Chayes, Lovász, Sós, and Vesztergombi [1].

2.1. Proof of Theorem 1

In this section we give a proof of Theorem 1.

Proposition 1. *Let H be a Hilbert space and let R, $W \subseteq H$ with R bounded and W weakly compact. Then (W, d_R) is complete.*

Proof. Let $a_1, a_2, \ldots \in W$ be a Cauchy sequence with respect to d_R. We must show that it has a limit in W, with respect to d_R. As W is weakly compact, it has the Bolzano–Weierstrass property; that is, there is a point a such that each weak neighbourhood of a contains infinitely many terms of $a_1, a_2, \ldots$. Then a is a required limit, that is, $\lim_{n \to \infty} d_R(a_n, a) = 0$. For choose $\varepsilon > 0$. As $a_1, a_2, \ldots$ is a Cauchy sequence with respect to d_R, there is a p such that $d_R(a_n, a_m) < \frac{1}{2}\varepsilon$ for all $n, m \geq p$. For each $r \in R$, as each weak neighbourhood of a contains infinitely many terms of $a_1, a_2, \ldots$, there is an $m \geq p$ with $|\langle r, a_m - a \rangle| < \frac{1}{2}\varepsilon$. This implies for each $n \geq p$:

$$|\langle r, a_n - a \rangle| \leq |\langle r, a_n - a_m \rangle| + |\langle r, a_m - a \rangle| < \varepsilon. \tag{7}$$

So $d_R(a_n, a) \leq \varepsilon$ if $n \geq p$. ∎

Let G be a group acting on a pseudometric space (X, d) that leaves d invariant. Define a pseudometric d/G on X by, for $x, y \in X$:

$$(d/G)(x, y) = \inf_{g \in G} d(x, g \cdot y). \tag{8}$$

As d is G-invariant, $(d/G)(x, y)$ is equal to the distance of the G-orbits $G \cdot x$ and $G \cdot y$. Any two points x, y on the same G-orbit have $(d/G)(x, y) = 0$. If we identify points of $(X, d/G)$ that are on the same orbit, the topological space obtained is homeomorphic to the orbit space $(X, d)/G$ of the topological space (X, d). Note that being compact or not is invariant under such identifications.

Proposition 2. *If (X, d) is a complete metric space, then $(X, d/G)$ is complete.*

Proof. Let $a_1, a_2, \ldots \in X$ be a Cauchy sequence with respect to d/G. Then it has a subsequence $b_1, b_2, \ldots$ such that $(d/G)(b_k, b_{k+1}) < 2^{-k}$ for all k. Let $g_1 = 1 \in G$. If $g_k \in G$ has been found, let $g_{k+1} \in G$ be such that $d(g_k b_k, g_{k+1} b_{k+1}) < 2^{-k}$. Then $g_1 b_1, g_2 b_2, \ldots$ is a Cauchy sequence with respect to d. Hence it has a limit, say b. Then $\lim_{k \to \infty}(d/G)(b_k, b) = 0$, implying $\lim_{n \to \infty}(d/G)(a_n, b) = 0$. ∎

Let H be a Hilbert space and let $R \subseteq H$. For any $k \geq 0$, define

$$Q_k = \{\lambda_1 r_1 + \cdots + \lambda_k r_k \mid r_1, \ldots, r_k \in R, \lambda_1, \ldots, \lambda_k \in [-1, +1]\}. \tag{9}$$

For any pseudometric d, let $B_d(Z, \varepsilon)$ denote the set of points at distance at most ε from Z. The following is a form of 'weak Szemerédi regularity'. (cf. Lemma 4.1 of Lovász and Szegedy [11], extending results of Frieze and Kannan [6] and Fernandez de la Vega, Kannan, Karpinski, and Vempala [5]—the method is essentially that of [6]):

Proposition 3. *If $R \subseteq B(H)$, then for each $k \geq 1$:*

$$B(H) \subseteq B_{d_R}(Q_k, 1/\sqrt{k}). \tag{10}$$

Proof. Choose $a \in B(H)$ and set $a_0 := a$. If, for some $i \geq 0$, a_i has been found, and $d_R(a_i, 0) > 1/\sqrt{k}$, choose $r \in R$ with $|\langle r, a_i \rangle| > 1/\sqrt{k}$. Define $a_{i+1} := a_i - \langle r, a_i \rangle r$. Then, with induction,

$$\|a_{i+1}\|^2 = \|a_i\|^2 - \langle r, a_i \rangle^2 (2 - \|r\|^2) \leq \|a_i\|^2 - \langle r, a_i \rangle^2 \leq \|a_i\|^2 - 1/k \leq 1 - (i+1)/k. \quad (11)$$

Moreover, as $\langle r, a_i \rangle \in [-1, +1]$, we know by induction that $a - a_i \in Q_i$.

By (11), as $\|a_{i+1}\|^2 \geq 0$, the process terminates for some $i \leq k$. For this i one has $d_R(a_i, 0) \leq 1/\sqrt{k}$. Hence, as $Q_i \subseteq Q_k$, $d_R(a, Q_k) \leq d_R(a, Q_i) \leq d_R(a, a - a_i) = d_R(a_i, 0) \leq 1/\sqrt{k}$. $\blacksquare$

Proof of Theorem 1. Observe that R is bounded as R/G is compact. So we can assume that $R, W \subseteq B(H)$.

By Propositions 1 and 2, $(W, d_R/G)$ is complete. So it suffices to show that $(W, d_R/G)$ is totally bounded; that is, for each $\varepsilon > 0$, W can be covered by finitely many d_R/G-balls of radius ε (cf. [4]).

Set $k := \lceil 4/\varepsilon^2 \rceil$. As R^k/G is compact, Q_k/G is compact (since the function $R^k \times [-1, 1]^k \to Q_k$ mapping $(r_1, \ldots, r_k, \lambda_1, \ldots, \lambda_k)$ to $\lambda_1 r_1 + \cdots + \lambda_k r_k$ is continuous, surjective, and G-equivariant). Hence (as $d_R \leq d_2$) $(Q_k, d_R)/G$ is compact, and so $(Q_k, d_R/G)$ is compact. Therefore, $Q_k \subseteq B_{d_R/G}(F, 1/\sqrt{k})$ for some finite F. Then with Proposition 3,

$$W \subseteq B(H) \subseteq B_{d_R}(Q_k, 1/\sqrt{k}) \subseteq B_{d_R/G}(Q_k, 1/\sqrt{k})$$

$$\subseteq B_{d_R/G}(F, 2/\sqrt{k}) \subseteq B_{d_R/G}(F, \varepsilon). \quad \blacksquare \quad (12)$$

2.2. Proof of Theorem 2

For any graph F, define a function

$$\pi_F : \prod_{v \in V(F)} B_{\deg(v)} \to \mathbb{R} \text{ by } \pi_F(h) := \sum_{\phi:E(F) \to C} \prod_{v \in V(F)} h_v(\phi(\delta(v))) \quad (13)$$

for $h = (h_v)_{v \in V(F)} \in \prod_{v \in V(F)} B_{\deg(v)}$. (The sum in (13) converges, as follows from (14) below.)

Proposition 4. If F is a simple graph, then π_F is continuous on $\prod_{v \in V(F)} (B_{\deg(v)}, d_{R_{\deg(v)}})$.

Proof. We first prove the following. For any k, any $h \in H_k^{S_k}$, and any $c_1, \ldots, c_l \in C$ with $l \leq k$, let $h(c_1, \ldots, c_l)$ be the element of $H_{k-l}^{S_{k-l}}$ with $h(c_1, \ldots, c_l)(c_{l+1}, \ldots, c_k) = h(c_1, \ldots, c_k)$ for all $c_{l+1}, \ldots, c_k \in C$. Let $k_1, \ldots, k_n \in \mathbb{N}$, let $h_i \in H_{k_i}^{S_{k_i}}$ for $i = 1, \ldots, n$, and let $F = ([n], E)$ be a graph with $\deg(i) \leq k_i$ for $i = 1, \ldots, n$. Then

$$\sum_{\phi:E \to C} \prod_{v \in [n]} \|h_v(\phi(\delta(v)))\| \leq \prod_{v \in [n]} \|h_v\|. \quad (14)$$

We prove this by induction on $|E|$, the case $E = \emptyset$ being trivial. Let $|E| \geq 1$, and choose an edge $ab \in E$. Set $E' := E \setminus \{ab\}$ and $\delta'(v) := \delta(v) \setminus \{ab\}$ for each $v \in V(F)$. Then

$$\sum_{\phi:E \to C} \prod_{v \in [n]} \|h_v(\phi(\delta(v)))\|$$

$$= \sum_{\phi:E' \to C} \sum_{c \in C} \|h_a(\phi(\delta'(a)), c)\| \|h_b(\phi(\delta'(b)), c)\| \prod_{\substack{v \in [n] \\ v \neq a, b}} \|h_v(\phi(\delta(v)))\|$$

$$\leq \sum_{\phi:E' \to C} \|h_a(\phi(\delta'(a)))\| \|h_b(\phi(\delta'(b)))\| \prod_{\substack{v \in [n] \\ v \neq a, b}} \|h_v(\phi(\delta(v)))\| \leq \prod_{v \in [n]} \|h_v\|, \quad (15)$$

where the inequalities follow from Cauchy–Schwarz and induction, respectively. This proves (14).

We next prove that for each $h = (h_v)_{v \in V(F)} \in \prod_{v \in V(F)} H_{\deg(v)}$ and each $u \in V(F)$:

$$\pi_F(h) \leq \|h_u\|_{R_{\deg(u)}} \prod_{\substack{v \in V(F) \\ v \neq u}} \|h_v\|. \tag{16}$$

To see this, let $N(u)$ be the set of neighbours of u, $F' := F - u$, and $\delta'(v) := \delta(v) \setminus \delta(u)$ for $v \in V(F) \setminus \{u\}$. Then

$$\pi_F(h) = \sum_{\phi : E(F) \to C} \prod_{v \in V(F)} h_v(\phi(\delta_F(v)))$$

$$= \sum_{\phi : E(F') \to C} \left\langle \bigotimes_{v \in N(u)} h_v(\phi(\delta'(v))), h_u \right\rangle \prod_{v \in V(F') \setminus N(u)} h_v(\phi(\delta(v)))$$

$$\leq \sum_{\phi : E(F') \to C} \|h_u\|_{R_{\deg(u)}} \prod_{v \in V(F')} \|h_v(\phi(\delta'(v)))\| \leq \|h_u\|_{R_{\deg(u)}} \prod_{v \in V(F')} \|h_v\|, \tag{17}$$

where the inequalities follow from the definition of $\|.\|_{R_{\deg(u)}}$ and from (14) (applied to F'), respectively. This proves (16).

Now let $g, h \in \prod_{v \in V(F)} B_{\deg(v)}$. We can assume that $V(F) = [n]$. For $u = 1, \ldots, n$, define $p^u \in \prod_{i \in [n]} B_{\deg(i)}$ by $p_i^u := g_i$ if $i < u$, $p_u^u := g_u - h_u$, and $p_i^u := h_i$ if $i > u$. Moreover, for $u = 0, \ldots, n$, define $q^u \in \prod_{i \in [n]} B_{\deg(i)}$ by $q_i^u := g_i$ if $i \leq u$ and $q_i^u := h_i$ if $i > u$. So $q^n = g$ and $q^0 = h$. By the multilinearity of π_F we have $\pi_F(q^u) - \pi_F(q^{u-1}) = \pi_F(p^u)$. Hence by (16) we have the following, proving the theorem,

$$\pi_F(g) - \pi_F(h) = \sum_{u=1}^{n} (\pi_F(q^u) - \pi_F(q^{u-1})) = \sum_{u=1}^{n} \pi_F(p^u) \leq \sum_{u=1}^{n} \|p_u^u\|_{R_{\deg(u)}}$$

$$= \sum_{u=1}^{n} \|g_u - h_u\|_{R_{\deg(u)}}. \quad \blacksquare \tag{18}$$

Proof of Theorem 2. For each $F \in \mathcal{F}$, the function $\psi : \prod_{k \in \mathbb{N}} B_k \to \prod_{v \in V(F)} B_{\deg(v)}$ mapping $(h_k)_{k \in \mathbb{N}}$ to $(h_{\deg(v)})_{v \in V(F)}$ is continuous. As $\pi(.)(F) = \pi_F(\psi(.))$, the theorem follows from Proposition 4. $\quad \blacksquare$

2.3. Proof of Theorem 3

We first show:

Proposition 5. *Let $(X_1, \delta_1), (X_2, \delta_2), \ldots$ be complete metric spaces and let G be a group acting on each X_k, leaving δ_k invariant $(k = 1, 2, \ldots)$. Then $(\prod_{k=1}^{\infty} X_k)/G$ is compact if and only if $(\prod_{k=1}^{t} X_k)/G$ is compact for each t.*

Proof. Necessity being direct, we show sufficiency. We can assume that space X_k has diameter at most $1/k$. Let $A := \prod_{k=1}^{\infty} X_k$, and let d be the supremum metric on A. Then d is G-invariant and $\prod_{k=1}^{\infty} (X_k, \delta_k)$ is G-homeomorphic with (A, d). So it suffices to show that $(A, d)/G$ is compact.

As each (X_k, δ_k) is complete, (A, d) is complete (cf., e.g., [4, Theorem XIV.2.5]). Hence, by Proposition 2, $(A, d/G)$ is complete. So it suffices to prove that $(A, d/G)$ is totally bounded; that is, for each $\varepsilon > 0$, A can be covered by finitely many d/G-balls of radius ε.

Set $t := \lfloor \varepsilon^{-1} \rfloor$. Let $B := \prod_{k=1}^{t} X_k$ and $C := \prod_{k=t+1}^{\infty} X_k$, with supremum metrics d_B and d_C respectively. As B/G is compact (by assumption), it can be covered by finitely many d_B/G-balls of radius ε. As C has diameter at most $1/(t + 1) \leq \varepsilon$, $A = B \times C$ can be covered by finitely many d/G-balls of radius ε. $\quad \blacksquare$

Proof of Theorem 3. As each (B_k, d_{R_k}) is complete (Proposition 1), by Proposition 5 it suffices to show that for each t, $(\prod_{k=0}^{t}(B_k, d_{R_k}))/O(H)$ is compact. Consider the Hilbert space $\prod_{k=0}^{t} H_k$, and let $W := \prod_{k=0}^{t} B_k$ and $R := \prod_{k=0}^{t} R_k$. Then the identity function on W is a homeomorphism from (W, d_R) to $\prod_{k=0}^{t}(B_k, d_{R_k})$. So it suffices to show that $(W, d_R)/O(H)$ is compact. Now for each n, $R^n/O(H)$ is compact, as it is a continuous image of $B(H)^m/O(H)$, with $m := n(1 + 2 + \cdots + t)$. The latter space is compact, as it is a continuous image of the compact space $B(\mathbb{R}^m)^m$ (assuming $|C| = \infty$, otherwise $B(H)$ itself is compact). So by Theorem 1, $(W, d_R)/O(H)$ is compact. ∎

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