Two-dimensional volume-frozen percolation: deconcentration and prevalence of mesoscopic clusters

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Abstract

Frozen percolation on the binary tree was introduced by Aldous [1] around fifteen years ago, inspired by sol-gel transitions. We investigate a version of the model on the triangular lattice, where connected components stop growing ("freeze") as soon as they contain at least N vertices, for some parameter $N \geq 1$.

This process has a substantially different behavior from the diameter-frozen process, studied in [32, 16]: in particular, we show that many (more and more as $N \to \infty$) frozen clusters surrounding the origin appear successively, each new cluster having a diameter much smaller than the previous one. This separation of scales is instrumental, and it helps to approximate the process in sufficiently large (as a function of N) finite domains by a Markov chain. This allows us to establish a deconcentration property for the sizes of the holes of the frozen clusters around the origin.

For the full-plane process, we then show that it can be coupled with the process in large finite domains, so that the deconcentration property also holds in this case. In particular, this implies that with high probability (as $N \to \infty$), the origin does not belong to a frozen cluster in the final configuration.

This work requires some new properties for near-critical percolation, which we develop along the way, and which are interesting in their own right: in particular, an asymptotic formula involving the percolation probability $\theta(p)$ as $p \searrow p_c$, and regularity properties for large holes in the infinite cluster. Volume-frozen percolation also gives insight into forest-fire processes, where lightning hits independently each tree with a small rate, and burns its entire connected component immediately.

Contents

1 Introduction

1.1 Frozen percolation

Frozen percolation is a growth process which was first introduced by Aldous [1] on the binary tree, motivated by sol-gel transitions [27]. Let us first define it informally, on an infinite simple graph $G = (V, E)$, where the vertices may be interpreted as particles. We start with all edges closed (i.e. all particles are isolated), and we try to turn them open independently of each other: at some random time τ_e uniform between 0 and 1, the edge $e \in E$ becomes open if and only if it connects two finite open connected components (otherwise it just stays closed). In other words, a connected components grows until it becomes infinite (i.e. it gelates), at which time it just stops growing: we say that it freezes, which explains the name of the process. Apart from sol-gel transitions, one may think of other interpretations, e.g. population dynamics (group formation), and pattern formation in general. There are (somewhat surprisingly at first sight) also interesting connections with (and potential applications to) forest-fire models.

The existence of the frozen percolation process is not clear at all. In [1], Aldous studies the case when G is the infinite 3-regular tree, as well as the case of the planted binary tree (where all vertices have degree 3, except the root vertex which has degree 1): using the tree structure, which allows for explicit computations, he shows that the frozen percolation process does exist in these two cases. However, Benjamini and Schramm noticed soon after Aldous' paper that such a process does not exist on the square lattice $\mathbb{Z}^2$ (see also Remark (i) after Theorem 1 in [31]).

In order to circumvent this non-existence issue, a "truncated" process was introduced in [32] by de Lima and two of the authors, where a connected component stops growing when it reaches a certain "size" N, where $N \geq 1$ is some parameter of the process. Formally, the original frozen percolation process corresponds to $N = \infty$, and one would like to understand what happens as $N \to \infty$, in view of the non-existence result.

When N is finite, "size" can have various meanings, and in [32], the size of a cluster is measured by its diameter. This diameter-frozen process was then further studied by the second author in [16], who established a precise description as $N \to \infty$, which, roughly speaking, can be summarized as follows. Let us fix some $K > 1$, and look at the process in a square of side length KN: only finitely many frozen clusters appear (the probability that there are more than k such clusters decays exponentially in k, uniformly in N), and they all freeze in a near-critical window around the percolation threshold p_c. In particular, it is shown that the frozen clusters all look like near-critical percolation clusters, with density converging to 0 as $N \to \infty$, and with high probability, the origin does not belong to a frozen cluster: in the final configuration, a typical point is on a macroscopic non-frozen cluster, i.e. a cluster with diameter of order N, but smaller than N.

The truncated process on a binary tree is studied in [33], where it is shown that the final configuration is completely different: a typical point is either on a frozen cluster (i.e. with diameter $\geq N$), or on a microscopic ones (with diameter $O(1)$), but one observes neither macroscopic non-frozen clusters, nor mesoscopic ones. Moreover, the way of measuring the size of a cluster does not really matter in this case: under suitable hypotheses, the process converges (in some weak sense) to Aldous' process as $N \to \infty$.

In the present paper, we go back to the case of a two-dimensional lattice, where we measure the size of a cluster by the number of vertices that it contains. Throughout the paper, we work with a site version of frozen percolation, on the (planar) triangular lattice $\mathbb{T}$ (we do this because site percolation on $\mathbb{T}$ is the planar percolation process for which the most precise results are known, as discussed below). This lattice has vertex set

$$V(\mathbb{T}) = \{x + ye^{\pi i/3} \in \mathbb{C} \,:\, x, y \in \mathbb{Z}\},$$

and edge set $E(\mathbb{T})$ obtained by connecting all pairs $u, v \in V(\mathbb{T})$ for which $\|u - v\|_2 = 1$. If $u, v \in V$ are connected by an edge, i.e. $(u, v) \in E(\mathbb{T})$, we say that u and v are neighbors, and we write $u \sim v$.

The independent site percolation process on $\mathbb{T}$ can be described as follows. We consider a family $(\tau_v)_{v \in V(\mathbb{T})}$ of i.i.d random variables, with uniform distribution on $[0, 1]$. For $p \in [0, 1]$, we say that a vertex v is p-black (resp. p-white) if $\tau_v \leq p$ (resp. $\tau > p$). Then, p-black and p-white vertices are distributed according to independent site percolation with parameter p, where vertices are independently black or white, with respective probabilities p and $1 - p$: we denote by $\mathbb{P}_p$ the corresponding probability measure. Vertices can be grouped into maximal connected components (clusters) of black sites and white sites, which defines a partition of $V(\mathbb{T})$. It is a celebrated result [13] that for all $p \leq p_c := 1/2$, there is a.s. no infinite p-black cluster, while for $p > p_c$, there exists a.s. a unique infinite p-black cluster. We refer the reader to [12] for an introduction to percolation theory.

Indirectly, our work relies on the conformal invariance property of critical percolation [25] and the SLE (Schramm-Loewner Evolution) technology [18, 19] (see also [35]). A key ingredient for the more refined results about the percolation phase transition is the construction of the scaling limit of near-critical percolation [11]. Note that our site version of frozen percolation (described below) is the exact analogue of the bond version on $\mathbb{Z}^2$: if the above-mentioned ingredients were available in the latter case, all our proofs would be applicable as well.

We can then define the volume-frozen percolation process itself, based on the same collection $(\tau_v)_{v \in V(\mathbb{T})}$. For a subset $A \subseteq V(\mathbb{T})$, its volume is the number of vertices that it contains, denoted by $|A|$. Let $G = (V, E)$ be a subgraph of $\mathbb{T}$, and $N \geq 1$ be a fixed parameter. At time $t = 0$, we set all the vertices in V to be white, and as time t evolves from 0 to 1, each vertex $v \in V$ can become black at time $t = \tau_v$ only: it is allowed to do so if and only if all the black clusters touching v have a volume strictly smaller than N (otherwise, v stays white until the end, i.e time $t = 1$). That is, black clusters keep growing until their volume gets at least N, when their growth is stopped: such a cluster is said to be frozen. We use the notation $\mathbb{P}_N^{(G)}$ for the corresponding probability measure, and we omit the graph G used when it is clear from the context. Note that this process is well-defined: it can be seen as a finite range interacting particle system, thus general theory [22] provides existence. Our results show in particular that the frozen sites (i.e. the sites belonging to a frozen cluster in the final configuration) vanish asymptotically.

Theorem 1.1. *For the volume-frozen percolation process on $\mathbb{T}$ with parameter $N \geq 1$,*

$$\mathbb{P}_N^{(\mathbb{T})}(0 \textit{ is frozen at time } 1) \underset{N \to \infty}{\longrightarrow} 0. \tag{1.1}$$

In fact, the proof of Theorem 1.1 provides a stronger result (a deconcentration property), which shows a substantial difference with the diameter-frozen model, as well as with Aldous' model on the tree. Let us consider two independent realizations of the cluster of 0 at time 1 in the frozen percolation process: if we let C_1 be the larger one, and C_2 the smaller one, then $\frac{|C_1|}{|C_2|} \to +\infty$ in probability as $N \to \infty$. Note that this property implies Theorem 1.1, since the ratio of the volumes of two frozen clusters is between $\frac{1}{2}$ and 2. It also shows that we only observe mesoscopic clusters: for every $M > 1$,

$$\mathbb{P}_N^{(\mathbb{T})}\left(M < |\mathcal{C}_1(0)| < \frac{N}{M}\right) \underset{N \to \infty}{\longrightarrow} 1.$$

The proof also shows that as $N \to \infty$, the number of frozen clusters surrounding the origin tends to ∞ in probability.

1.2 Exceptional scales in volume-frozen percolation

In [34], we showed the existence of a sequence of exceptional scales $m_k = m_k(N)$, $k \geq 1$, with $m_1(N) = \sqrt{N}$ and $m_{k+1}(N) \gg m_k(N)$ (as $N \to \infty$) for all $k \geq 1$.

Let us denote by $B_n := [-n, n]^2$ the ball of radius n around the origin in the L^∞ norm. The scales $(m_k(N))_{k \geq 1}$ are exceptional in the sense that if we consider the volume-frozen percolation process in B_m, for some $m = m(N)$, we get two very different behaviors according to whether m stays close to one of these scales or not. More precisely, we proved in [34] that the following dichotomy holds.

- If $m_k \ll m \ll m_{k+1}$ as $N \to \infty$ for some $k \geq 1$ (i.e. we start between two exceptional scales but far from them), then (w.h.p.) k successive frozen clusters appear around 0, at (random) times $p_k < p_{k-1} < \ldots < p_1$ (all strictly larger than $p_c = \frac{1}{2}$) such that $m_{i-1} \ll L(p_i) \ll m_i$ (where $L(p)$ is the characteristic length at p: see (2.3) below for a precise definition). Moreover, the cluster $\mathcal{C}_1(0)$ of the origin at time 1 satisfies $1 \ll |\mathcal{C}_1(0)| \ll N$: in other words, we only see mesoscopic (non-frozen) clusters.

- On the other hand, if $m \asymp m_k$ as $N \to \infty$ (for a given $k \geq 1$), then (w.h.p.) one of the following three situations occurs, each having a probability bounded away from 0: either there are $k - 1$ successive freezings, and $|\mathcal{C}_1(0)| < N$ but is $\asymp N$, or there are k successive freezings, and either $|\mathcal{C}_1(0)| \geq N$, or $|\mathcal{C}_1(0)| \asymp 1$. That is, we only observe macroscopic (frozen and non-frozen) and microscopic clusters.

Another significant difference is that in the first case, all the frozen clusters appear close to p_c, while in the second case, freezing can occur on the whole time interval $(p_c, 1)$ (as on the binary tree, but note that there are no macroscopic non-frozen clusters on the tree).

These exceptional scales clearly highlight the non-monotonicity of the process, which makes it quite challenging to study: we need to develop specific tools and ideas to study its dynamics. Note that the existence of these exceptional scales constitutes a big difference with diameter-frozen percolation [16]. For the diameter case, there is essentially one characteristic scale (N), and most frozen clusters leave holes which are too small for new frozen clusters to emerge, while for the volume case, most frozen clusters leave holes where new clusters can freeze.

Heuristically, we expect the resulting configuration in the full-plane process to correspond to the first case, i.e. $m_k \ll m \ll m_{k+1}$. However, we proceed in a different way: we first prove that even if we start close to m_k, the successive freezings create enough randomness as $k \to \infty$ (the successive freezing scales get deconcentrated), so that we end up far away from m_1. This yields in particular the following result.

Theorem 1.2. *For all $\varepsilon > 0$, there exists $l \geq 1$ such that for all $k \geq l$, the following holds: if $m(N) \in [m_k(N), m_{k+1}(N)]$ for N large enough, then*

$$\limsup_{N \to \infty} \mathbb{P}_N^{(B_{m(N)})}(0 \textit{ is frozen at time } 1) \leq \varepsilon.$$

This result is interesting in itself, but it is also an intermediate step to prove Theorem 1.1: for that, we "connect" the full-plane frozen percolation process with the process in large enough (as a function of N) domains. We actually need a more uniform result than Theorem 1.2, where boxes can be replaced by domains which are "sufficiently regular" (see Theorem 6.2 in Section 6.2 for a precise statement).

1.3 Organization of the paper

In the first three sections (Sections 2 to 4), we collect and develop all the tools from independent percolation which are used in our proofs. More specifically, we need results about the near-critical regime, close to the percolation threshold p_c.

In Section 2, we first discuss classical results, and we derive some consequences of these results. We then prove more involved properties, for which the scaling limit of near-critical percolation (see [11]) is needed. In particular, we establish an improved formula for the asymptotic behavior of the density $\theta(p)$ of the infinite cluster as $p \searrow p_c$.

A central object in our reasonings is the hole of the origin in the infinite cluster (in the supercritical regime $p > p_c$), and we study it further in Section 3, proving continuity (with respect to p) and regularity properties which are interesting in themselves. In particular, one of the difficulties is to rule out the existence of certain bottlenecks, which could perturb the future evolution of the process.

In Section 4, we discuss and extend several estimates (from [4]) on the volume of the largest connected component in a finite domain. These estimates are used repeatedly in our proofs, to obtain a good control on the successive freezing times.

We then turn to the frozen percolation process itself. We first study it in finite domains, before analyzing the full-plane process in Section 7.

In Section 5, we discuss the exceptional scales, and we introduce several chains associated with the frozen percolation process in a finite box. One of these chains is an exact Markov chain, and we prove a deconcentration property for it, using an abstract lemma obtained in Section 5.4.

This deconcentration property is then used in Section 6 to prove Theorem 1.2. Roughly speaking, we need to know that the number of frozen clusters surrounding the origin is sufficiently large: for instance, we can start with a box with side length $\gg m_k(N)$, for k large enough.

We then establish Theorem 1.1 in Section 7, i.e. the asymptotic absence of dust in the full-plane process. For that, we explain how to couple the process in $\mathbb{T}$ with the process in finite, large enough (as a function of N) domains, which allows us to use the results from the previous section. Finally, in Section 7.3, we briefly discuss the potential connection with two other natural processes.

2 Preliminary: near-critical percolation

Our proofs rely heavily on a precise description of independent percolation near criticality, i.e. on how this model behaves through its phase transition. We collect here all the results that are needed later, before turning to frozen percolation itself in the subsequent sections. After fixing notations in Section 2.1, we present properties which have by now become classical, in Section 2.2, and we derive a few consequences of these properties in Section 2.3. We then turn to more specific technical tools, in Sections 2.4, 2.5 and 2.6. The proofs of these results turn out to be more involved, relying on recent breakthroughs by Garban, Pete, and Schramm [10, 11], and they only hold for site percolation on the triangular lattice.

2.1 Notations

In what follows, a *path* is a sequence of vertices, where any two consecutive vertices are neighbors. Two vertices u and v are said to be connected, which we denote by $u \leftrightarrow v$, if there exists a path from u to v on $\mathbb{T}$ that consists of black sites only (we also consider white connections, but in this case, we always mention explicitly the color). Two subsets $A, B \subseteq V(\mathbb{T})$ are said to be connected if there exist $u \in A$ and $v \in B$ which are connected, and we write $A \leftrightarrow B$. For $p > p_c$, the unique infinite p-black cluster is denoted by $\mathcal{C}_\infty(p)$. We also write $v \leftrightarrow \infty$ for the event $v \in \mathcal{C}_\infty$, and we use the notation

$$\theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty)$$

for the density of $\mathcal{C}_\infty$.

For $A \subseteq \mathbb{T}$, we consider its inner boundary $\partial^{\text{in}} A$, which contains all the sites in A that are neighbor with a site in A^c, and its outer boundary $\partial^{\text{out}} A = \partial^{\text{in}}(A^c)$, which consists of all the sites in A^c neighbor to a site in A. Note that if A is a black cluster, then $\partial^{\text{in}} A$ and $\partial^{\text{out}} A$ consist of black and white sites, respectively.

For a rectangle $R = [x_1, x_2] \times [y_1, y_2]$ ($x_1 < x_2$, $y_1 < y_2$), we denote by $\mathcal{C}_H(R)$ (resp. $\mathcal{C}_V(R)$) the event that there exists a black path in R that connects the two vertical (resp. horizontal) sides of R. We write $\mathcal{C}^*_H(R)$ and $\mathcal{C}^*_V(R)$ for the analogous events with white paths.

For $0 < m < n$, we define the annulus

$$A_{m,n} := B_n \setminus B_m.$$

For $z \in \mathbb{C}$, we use the short-hand notations $B_r(z) = z + B_r$ and $A_{m,n}(z) = z + A_{m,n}$. For notational convenience, we also allow the value $n = \infty$, writing $A_{m,\infty}(z) = \mathbb{C} \setminus B_m(z)$. For $A = A_{m,n}(z)$, the event that there exists a black (resp. white) circuit in A, i.e. surrounding $B_m(z)$, is denoted by $\mathcal{O}(A)$ (resp. $\mathcal{O}^*(A)$), and we often use the outermost such black circuit in A, which we denote by $\mathcal{C}^{\text{out}}_A$ (we take $\mathcal{C}^{\text{out}}_A = \emptyset$ when such a circuit does not exist).

As often when studying near-critical percolation, the so-called arm events play a central role in our proofs. For $k \geq 1$ and $\sigma \in \mathfrak{S}_k := \{b, w\}^k$ (where we write b and w for black and white, respectively), we define the event $\mathcal{A}_\sigma(A)$ that there exist k disjoint paths γ_i ($1 \leq i \leq k$) in A, in counter-clockwise order, each connecting $B_m(z)$ to $\partial B_n(z)$, and such that γ_i has color σ_i for each i. We denote

$$\pi_\sigma(m, n) = \mathbb{P}_{p_c}\left(\mathcal{A}_\sigma\left(A_{m,n}\right)\right), \tag{2.1}$$

and we simply write $\pi_\sigma(n)$ for $\pi_\sigma(0, n)$ (for paths starting from a neighbor of the origin). For notational convenience, we write $\mathcal{A}_1$, $\mathcal{A}^*_1$, $\mathcal{A}_4$, and $\mathcal{A}_6$ in the cases when σ is (b), (w), $(bwbw)$, and $(bwwbww)$, respectively. Note that for an annulus A, $\mathcal{O}(A) = (\mathcal{A}^*_1(A))^c$. For notational convenience, we also write

$$\mathcal{O}(z; m, n) = \mathcal{O}(A_{m,n}(z)) \quad \text{and} \quad \mathcal{A}_\sigma(z; m, n) = \mathcal{A}_\sigma(A_{m,n}(z)), \tag{2.2}$$

where z is assumed to be 0 when it is omitted.

We define the characteristic length by

$$L(p) = \max\left\{n > 0 \,:\, \mathbb{P}_p\left(\mathcal{C}_H\left([0, n] \times [0, 2n]\right)\right) \geq 0.01\right\} \tag{2.3}$$

for $p < 1/2$, and by $L(1-p) = L(p)$ for $p > 1/2$. From the definition above, it is clear that $L(p)$ is piecewise linear, and not continuous. Thus we use a slightly different function $\tilde{L}$ defined as follows. For $p_d \in (0,1) \setminus \{p_c\}$ discontinuity points of L, we set $\tilde{L}(p_d) = L(p_d)$. Then we extend $\tilde{L}$ to $p \in (0,1)\setminus\{p_c\}$ by linear interpolation. The function $\tilde{L}$ has similar properties as L with the additional benefit of being continuous, which will come handy later. With a slight abuse of notation, in the following we write L for $\tilde{L}$.

2.2 Classical results

Here we collect some of the results in near-critical percolation which will be used throughout the paper.

(i) Exponential decay with respect to $L(p)$: there exist universal constants $c_1, c_2 > 0$ such that

$$\mathbb{P}_p\left(\mathcal{C}_V\left([0,2n] \times [0,n]\right)\right) \leq c_1 e^{-c_2 \frac{n}{L(p)}} \tag{2.4}$$

for all $p < 1/2$ (see Lemma 39 in [24]).

(ii) Extendability and quasi-multiplicativity of arm events at criticality: for all $k \geq 1$ and $\sigma \in \mathfrak{S}_k$, there are constants $c_1, c_2 > 0$ (depending on σ only) such that

$$c_1 \pi_\sigma(2n_1, n_2) \leq \pi_\sigma(n_1, n_2) \leq c_2 \pi_\sigma(n_1, 2n_2) \tag{2.5}$$

and

$$c_1 \pi_\sigma(n_1, n_3) \leq \pi_\sigma(n_1, n_2) \pi_\sigma(n_2, n_3) \leq c_2 \pi_\sigma(n_1, n_3) \tag{2.6}$$

for all $0 \leq n_1 \leq n_2 \leq n_3$ (see Propositions 16 and 17 in [24], respectively).

(iii) Arm events in near-critical regime: for every $k \geq 1$, $\sigma \in \mathfrak{S}_k$, let $\mathcal{A}_\sigma^{p,p'}$ denote the modification of the event $\mathcal{A}_\sigma$ where the black arms are p-black, and the white ones are p'-white. Then for all $\Lambda \geq 1$, there exist constants $c_1, c_2 > 0$ (depending on k, σ, and Λ) such that

$$c_1 \pi_\sigma(m,n) \leq \mathbb{P}_p(\mathcal{A}_\sigma^{p,p'}(m,n)) \leq c_2 \pi_\sigma(m,n) \tag{2.7}$$

for all $p, p' \in (0,1)$, and all $m, n \leq \Lambda L(p) \wedge L(p')$ (see Lemma 6.3 in [7], or Lemma 8.4 in [11]).

(iv) Lower and upper bounds on the 1-arm exponent: there exist universal constants $c_1, c_2, \eta > 0$ such that

$$c_1 \left(\frac{m}{n}\right)^{1/2} \leq \mathbb{P}_p(\mathcal{A}_1(m,n)) \leq c_2 \left(\frac{m}{n}\right)^{\eta} \tag{2.8}$$

for all $m, n > 0$ with $m < n < L(p)$. This implies that for all $k \geq 1$, $\sigma \in \mathfrak{S}_k$, and $\Lambda \geq 1$, there exist universal constants $c_3, \alpha > 0$ such that

$$\mathbb{P}(\mathcal{A}_\sigma^{p,p'}(m,n)) \leq c_3 \left(\frac{m}{n}\right)^{\alpha} \tag{2.9}$$

for all p, p', and $m, n \leq \Lambda L(p) \wedge L(p')$.

(v) Lower bound on the 4-arm exponent: there exist universal constants $c, \delta > 0$ such that

$$\mathbb{P}_p(\mathcal{A}_4(m,n)) \geq c \left(\frac{m}{n}\right)^{2-\delta} \tag{2.10}$$

for all $m, n > 0$ with $m < n < L(p)$ (this is a consequence of Theorem 24 (3) in [24]). In particular, there is a universal constant C' such that

$$\sum_{k=1}^{n} 2^{2k} \pi_4(2^k) \leq C' 2^{2n} \pi_4(2^n) \tag{2.11}$$

for all $n \geq 1$.

(vi) Upper bound on the 6-arm exponent: there exist universal constants $c, \delta > 0$ such that

$$\mathbb{P}_p(\mathcal{A}_6(m,n)) \leq c \left(\frac{m}{n}\right)^{2+\delta} \tag{2.12}$$

for all $m, n > 0$ with $m < n < L(p)$ (we refer the reader to Theorem 24 (3) in [24]).

(vii) Asymptotic equivalences: we have

$$\theta(p) \asymp \pi_1(L(p)) \tag{2.13}$$

as $p \searrow p_c$ (see Theorem 2 of [14], or (7.25) in [24]), and

$$|p - p_c| L(p)^2 \pi_4(L(p)) \asymp 1 \tag{2.14}$$

as $p \to p_c$ (see (4.5) in [14], or Proposition 34 of [24]).

2.3 Additional properties

Let us first give a definition.

Definition 2.1. *For $0 < a < b$, we consider all the horizontal and vertical rectangles of the form*

$$B_a(2ax) \cup B_a(2ax'), \quad \textit{with } x, x' \in B_{\lceil b/2a \rceil + 1},\ x \sim x'$$

(covering the ball B_{b+2a}), and we denote by $\mathcal{N}_p(a,b)$ the event that in each of these rectangles, there exists a p-black crossing in the long direction.

Note that $\mathcal{N}_p(a,b)$ implies the existence of a p-black cluster $\mathcal{N}$ which ensures that all the p-black clusters and all the p-white clusters that intersect B_b, except $\mathcal{N}$ itself, have a diameter at most $4a$. In the following, such a cluster $\mathcal{N}$ is called a *net*.

Lemma 2.2. *There exist universal constants $c_1, c_2 > 0$ such that: for all $0 < a < b$ and $p > p_c$,*

$$\mathbb{P}(\mathcal{N}_p(a,b)) \geq 1 - c_1 \left(\frac{b}{a}\right)^2 e^{-c_2 \frac{a}{L(p)}}. \tag{2.15}$$

Proof of Lemma 2.2. This follows immediately from the exponential decay property (2.4). □

We also derive the following lower bound, which is used in the proof of Proposition 7.2.

Lemma 2.3. *For all $\Lambda, \Lambda' \geq 1$, there exists a constant $c = c(\Lambda, \Lambda') > 0$ such that: for all $p, p' > p_c$ with $p < p'$ and $L(p') \geq \Lambda'^{-1} L(p)$, all $n \geq \Lambda^{-1} L(p)$,*

$$\mathbb{P}(0 \overset{p'}{\leftrightarrow} \infty,\ 0 \overset{p}{\not\leftrightarrow} \partial B_n) \geq c \frac{|p' - p|}{|p - p_c|} \theta(p). \tag{2.16}$$

Proof of Lemma 2.3. Since the left-hand side of (2.16) is increasing in n, we can assume that $n = \Lambda^{-1} L(p)$. We construct a sub-event of $\{0 \overset{p'}{\leftrightarrow} \infty,\ 0 \overset{p}{\not\leftrightarrow} \partial B_n\}$ for which the desired lower bound holds, as follows. We start with the events

$$E_1 := \{\text{there is a } p\text{-black circuit } \mathcal{C}_1 \text{ in } A_{\Lambda^{-1}L(p)/4, \Lambda^{-1}L(p)/2} \text{ s.t. } 0 \overset{p}{\leftrightarrow} \mathcal{C}_1\},$$

$$\text{and} \quad E_2 := \{\text{there is a } p\text{-black circuit } \mathcal{C}_2 \text{ in } A_{\Lambda^{-1}L(p), 2\Lambda^{-1}L(p)} \text{ s.t. } \mathcal{C}_2 \overset{p}{\leftrightarrow} \infty\}.$$

These two events are independent, and the Russo-Seymour-Welsh theorem implies that

$$\mathbb{P}(E_1)\mathbb{P}(E_2) \geq c_1\theta(p)$$

for some constant $c_1 = c_1(\Lambda) > 0$.

If we also introduce

$$\mathcal{W}_p := \{v \in A_{\Lambda^{-1}L(p)/2,\Lambda^{-1}L(p)} : v \text{ is } p\text{-white}, \partial v \overset{p}{\leftrightarrow} \partial B_{\Lambda^{-1}L(p)/4}, \partial v \overset{p}{\leftrightarrow} \partial B_{2\Lambda^{-1}L(p)}\}$$

(where we denote by ∂v the set of neighbors of v), then there exist constants $c_2, c_3 > 0$ (depending only on Λ) such that the event

$$E_3 := \{\text{there is a } p\text{-white circuit in } A_{\Lambda^{-1}L(p)/2,\Lambda^{-1}L(p)}\} \\ \cap \{|\mathcal{W}_p| \geq c_2 L(p)^2 \pi_4(L(p))\}$$

satisfies: for all $p > p_c$, $\mathbb{P}(E_3) \geq c_3$. This property follows from standard arguments, and we sketch a proof on Figure 2.1.

We now restrict ourselves to the event $E_1 \cap E_2$, we let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the inner- and outermost circuits appearing in the events E_1 and E_2, respectively, and we condition on the circuits $\mathcal{C}_1$ and $\mathcal{C}_2$, as well as on the configuration inside $\mathcal{C}_1$ and outside $\mathcal{C}_2$. The configuration between $\mathcal{C}_1$ and $\mathcal{C}_2$ is thus fresh, and we obtain

$$\begin{aligned}\mathbb{P}(E_1 \cap E_2 \cap E_3) &= \sum_{C_1,C_2} \mathbb{P}(E_3 \mid \mathcal{C}_1 = C_1,\, \mathcal{C}_2 = C_2)\mathbb{P}(\mathcal{C}_1 = C_1,\, \mathcal{C}_2 = C_2) \\ &\geq c_1\theta(p) \cdot c_3.\end{aligned}$$

Using the pivotal vertices produced by the event E_3, we deduce that for some $c_4 > 0$,

$$\begin{aligned}\mathbb{P}(0 \overset{p'}{\leftrightarrow} \infty,\, 0 \overset{p}{\not\leftrightarrow} \partial B_{\Lambda^{-1}L(p)}) &\geq c_4|p'-p|L(p)^2\pi_4(L(p))\mathbb{P}(E_1 \cap E_2 \cap E_3) \\ &\geq c_1c_3c_4|p'-p|L(p)^2\pi_4(L(p))\theta(p)\end{aligned}$$

(here, we use the fact that $|p'-p|L(p)^2\pi_4(L(p)) \leq c_5\frac{|p'-p|}{|p-p_c|} \leq c_6$ for some universal $c_5 > 0$, and $c_6 = c_6(\Lambda') > 0$, from (2.14) and the hypothesis on p and p'), which completes the proof of Lemma 2.3 (by applying again (2.14)). □

Note that Lemma 2.3 implies in particular the following: there exists a constant $c = c(\Lambda') > 0$ such that for all $p, p' > p_c$ with $p < p'$ and $L(p') \geq \Lambda'^{-1}L(p)$,

$$\theta(p') - \theta(p) \geq c\frac{|p'-p|}{|p-p_c|}\theta(p)$$

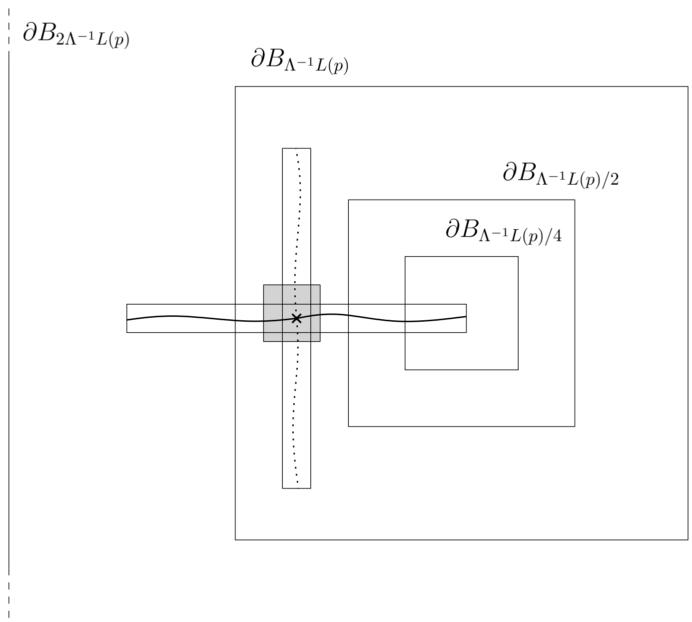

Figure 2.1: It follows from a second moment argument that with a probability $\geq c_3 > 0$, there exist $\asymp L(p)^2 \pi_4(L(p))$ vertices in the gray region with four well-separated arms, as depicted. By using RSW, we can then extend the two p-white arms into a p-white circuit in $A_{\Lambda^{-1}L(p)/2, \Lambda^{-1}L(p)}$.

(by fixing one value of Λ, e.g. $\Lambda = 1$, and letting $n \to \infty$ in the right-hand side of (2.16)). It is also possible to derive a similar upper bound on $\theta(p') - \theta(p)$.

Lemma 2.4. *For all $\Lambda > 1$, there exists a constant $C = C(\Lambda) > 0$ such that: for all $p, p' > p_c$ with $p < p'$ and $L(p') \geq \Lambda^{-1} L(p)$, we have*

$$\theta(p') - \theta(p) \leq C \frac{|p' - p|}{|p - p_c|} \theta(p). \tag{2.17}$$

Proof of Lemma 2.4. Since this result is not used later in the paper, we postpone the proof to Appendix A.1. □

2.4 Asymptotics of π_σ

We now recall some results on the large scale behavior of arm events at criticality. We first remind that their probabilities are described asymptotically by critical exponents, whose values are known (except in the so-called monochromatic case, for $k \geq 2$ arms of the same color). The following result is due to Smirnov and Werner [26] (except for the case $k = 1$ [20], and for the existence in the $k \geq 2$ monochromatic case [2]). Its proof relies on the connection between critical percolation and SLE (Schramm-Loewner Evolution) processes with parameter 6, which uses the conformal invariance property of critical percolation (in the scaling limit) [25] and properties of SLE processes [18, 19].

Lemma 2.5. *For all $k \geq 1$ and $\sigma \in \mathfrak{S}_k$,*

$$\pi_\sigma(k, n) = n^{-\alpha_\sigma + o(1)} \quad \text{as } n \to \infty,$$

for some constant $\alpha_\sigma > 0$. Furthermore,

- *$\alpha_\sigma = \frac{5}{48}$ for $k = 1$,*
- *and $\alpha_\sigma = \frac{k^2-1}{12}$ for all $k \geq 2$ and $\sigma \in \mathfrak{S}_k$ containing both colors.*

This has the following consequence, known as a *ratio-limit theorem.*

Lemma 2.6 (Proposition 4.9 of [10]). *For all $k \geq 1$, $\sigma \in \mathfrak{S}_k$ and $\lambda > 1$,*

$$\lim_{n \to \infty} \frac{\pi_\sigma(k, \lambda n)}{\pi_\sigma(k, n)} = \lambda^{-\alpha_\sigma},$$

where α_σ is as in Lemma 2.5.

Actually, we make use of a slightly stronger version of this result: the above point-wise convergence holds locally uniformly in λ.

Lemma 2.7. *For all $k \geq 1$, $\sigma \in \mathfrak{S}_k$, $\Lambda > 1$ and $\varepsilon > 0$, there exists $K \geq 1$ such that: for all $n' > n \geq K$ with $\frac{n'}{n} \leq \Lambda$, we have*

$$\frac{\pi_\sigma(k, n')}{\pi_\sigma(k, n)} \cdot \left(\frac{n'}{n}\right)^{\alpha_\sigma} \in (1 - \varepsilon, 1 + \varepsilon),$$

where α_σ is as in Lemma 2.5.

Proof of Lemma 2.7. This is a rather immediate consequence of Lemma 2.6, and the fact that π_σ is decreasing in its second argument. Indeed, let us write $\Lambda = (1+\alpha)^m$, for some $\alpha > 0$ very small, depending on ε (a precise choice is made later). We can apply Lemma 2.6 for each $\lambda \in \mathcal{L}_m := \{(1+\alpha)^i \,:\, 1 \leq i \leq m\}$: there exists $K \geq 1$ large enough so that for all $n \geq K$ and $\lambda \in \mathcal{L}_m$,

$$\frac{\pi_\sigma(k, \lambda n)}{\pi_\sigma(k, n)} \cdot \lambda^{\alpha_\sigma} \in \left(1 - \frac{\varepsilon}{10}, 1 + \frac{\varepsilon}{10}\right). \tag{2.18}$$

Now, let us consider any $n \geq K$ and $n' \in (n, \Lambda n]$: there exists some $i \in \{1, \ldots, m\}$ for which $(1+\alpha)^i \leq \frac{n'}{n} \leq (1+\alpha)^{i+1}$, and the monotonicity of π_σ implies

$$\frac{\pi_\sigma(k, (1+\alpha)^{i+1} n)}{\pi_\sigma(k, n)} \leq \frac{\pi_\sigma(k, n')}{\pi_\sigma(k, n)} \leq \frac{\pi_\sigma(k, (1+\alpha)^{i} n)}{\pi_\sigma(k, n)}.$$

By combining this with (2.18), we obtain

$$\left(1 - \frac{\varepsilon}{10}\right) \left((1+\alpha)^{i+1}\right)^{-\alpha_\sigma} \leq \frac{\pi_\sigma(k, n')}{\pi_\sigma(k, n)} \leq \left(1 + \frac{\varepsilon}{10}\right) \left((1+\alpha)^{i}\right)^{-\alpha_\sigma},$$

and so

$$\left(1 - \frac{\varepsilon}{10}\right) (1+\alpha)^{-\alpha_\sigma} \left(\frac{n'}{n}\right)^{-\alpha_\sigma} \leq \frac{\pi_\sigma(k, n')}{\pi_\sigma(k, n)} \leq \left(1 + \frac{\varepsilon}{10}\right) (1+\alpha)^{\alpha_\sigma} \left(\frac{n'}{n}\right)^{-\alpha_\sigma}.$$

This yields the desired conclusion, by choosing α small enough: we need

$$\left(1 - \frac{\varepsilon}{10}\right) (1+\alpha)^{-\alpha_\sigma} \geq 1 - \varepsilon \quad \text{and} \quad \left(1 + \frac{\varepsilon}{10}\right) (1+\alpha)^{\alpha_\sigma} \leq 1 + \varepsilon.$$

□

2.5 Asymptotics of $\theta(p)$

In this section and the next one, we derive two more specific properties of near-critical percolation. To our knowledge, these results are new, and we believe that they are interesting in themselves. Their proofs are more involved, since they rely on the scaling limit of near-critical percolation [11] constructed by Garban, Pete and Schramm.

Our description of volume-frozen percolation relies on locating precisely the successive freezing times, for which we need to closely keep track of the value of θ. For that, it turns out that the asymptotic equivalence (2.13) is not good enough, and we make use of the strengthened version below.

Proposition 2.8. *There exists a constant $c_\theta \in (0, \infty)$ such that*

$$\frac{\theta(p)}{\pi_1(L(p))} \underset{p \searrow p_c}{\longrightarrow} c_\theta.$$

Before we dive into the proof, we extend our notations to accommodate the triangular lattice at different mesh sizes, as in [10, 11]. These new notations are used only in this section and the next one.

For $\eta > 0$, let $\mathbb{T}^\eta$ be the lattice with vertex set $\eta V(\mathbb{T})$ isomorphic to $\mathbb{T}$. For all the quantities defined so far, we add a superscript η to indicate the dependence on the mesh size. In particular, $\mathbb{P}_p^\eta$ refers to site percolation on $\mathbb{T}^\eta$ with parameter p. Note that

$$L^\eta(p) = \eta L(p), \quad \text{and} \quad \pi_j^\eta(a,b) = \pi_j(a\eta^{-1}, b\eta^{-1})$$

for all $j \in \{1,4\}$ and $0 < a < b$.

We make use of the following near-critical parameter scale: for $\lambda \in \mathbb{R}$, we set

$$p_\lambda(\eta) := p_c + \lambda \frac{\eta^2}{\pi_4^\eta(\eta,1)}. \tag{2.19}$$

We use the short-hand $\mathbb{P}^{\eta,\lambda} := \mathbb{P}^\eta_{p_\lambda(\eta)}$, and we extend the notation $\pi_j^\eta(a,b)$ by

$$\pi_j^{\eta,\lambda}(a,b) := \mathbb{P}^{\eta,\lambda}(\mathcal{A}_j(a,b))$$

for $j \in \{1,4\}$ and $0 < a < b$. Finally, we set (with a slight abuse of notation) $L^\eta(\lambda) := L^\eta(p_\lambda(\eta))$.

First, let us recall some results from [10] and [11].

Theorem 2.9 (Theorem 9.4 of [11])**.** *For any $\lambda \in \mathbb{R}$, the percolation model on $\mathbb{T}^\eta$ with parameter $p_\lambda(\eta)$ converges in distribution to the continuum near-critical percolation model as $\eta \to 0$ (in the quad-crossing topology).*

The above theorem immediately implies that for any fixed $\lambda \in \mathbb{R}$, the correlation lengths $L^\eta(p_\lambda(\eta))$ converge to the continuum correlation lengths $L^0(\lambda)$ as $\eta \to 0$. Moreover, it follows from Lemma 2.9 of [10] that for all $j \in \{1,4\}$ and $0 < a < b$,

$$\pi_j^{\eta,\lambda}(a,b) \underset{\eta\to 0}{\longrightarrow} \pi_j^{0,\lambda}(a,b). \tag{2.20}$$

Furthermore, we have the following result, where we denote by $\operatorname{sgn}(\lambda) := \frac{|\lambda|}{\lambda} \in \{-1,1\}$ the sign of $\lambda \neq 0$.

Theorem 2.10 (Theorem 10.3 and Corollary 10.5 of [11])**.** *For all $\lambda \in \mathbb{R} \setminus \{0\}$, $j \in \{1,4\}$ and $0 < a < b$,*

$$L^0(\lambda) = |\lambda|^{-4/3} L^0(1) \in (0,\infty),$$

$$\text{and} \quad \pi_j^{0,\lambda}(a,b) = \pi_j^{0,\operatorname{sgn}(\lambda)}(|\lambda|^{4/3}a, |\lambda|^{4/3}b).$$

In particular,

$$\pi_j^{0,1}(aL^0(1), bL^0(1)) = \pi_j^{0,\lambda}(aL^0(\lambda), bL^0(\lambda))$$

for all $\lambda > 0$ and $0 < a < b$.

Let us also remind that conformal invariance of critical percolation in the scaling limit (see Theorem 7 of [5]) implies that

$$\pi_j^0(a,b) = \pi_j^0(sa,sb) \tag{2.21}$$

for all $j \in \{1,4\}$, $0 < a < b$ and $s > 0$.

The results above rely on the following ratio-limit theorem.

Proposition 2.11. *For any fixed $a, r > 0$ and $\lambda \in \mathbb{R}$, there exists a constant $c_\lambda(r,a)$ such that*

$$\lim_{\eta \to 0} \frac{\pi_j^{\eta,\lambda}(\eta,r)}{\pi_j^{\eta,\lambda}(\eta,a)} = \lim_{\varepsilon \to 0} \frac{\pi_j^{0,\lambda}(\varepsilon,r)}{\pi_j^{0,\lambda}(\varepsilon,a)} = c_\lambda(r,a)$$

for all $j \in \{1,4\}$. In the case where $\lambda = 0$, one has $c_0(r,a) = (\frac{r}{a})^{-\alpha_j}$, with $\alpha_1 = \frac{5}{48}$ and $\alpha_4 = \frac{5}{4}$.

Proof of Proposition 2.11. The case $\lambda = 0$ coincides with Proposition 4.9 of [10]. The case $\lambda \neq 0$ follows from a combination of the proof of that proposition, and (2.20). □

The following result is a key lemma in [11].

Lemma 2.12 (Lemma 8.4 of [11])**.** *For all $\lambda \in \mathbb{R}$ and $j \in \{1,4\}$, there exist constants $0 < c < C < \infty$ and η_0 (depending on λ and j) such that: for all $\eta \leq \eta_0$,*

$$c \leq \frac{\pi_j^{\eta,\lambda}(\eta,1)}{\pi_j^{\eta}(\eta,1)} \leq C.$$

Before we proceed to the proof of Proposition 2.8, we need one more lemma.

Lemma 2.13. *For all $\lambda \in \mathbb{R}$ and $j \in \{1,4\}$, there exists a constant $C = C(\lambda, j) > 0$ such that for all $a \in (0,1)$, we have:*

$$\left| \frac{\pi_j^{\eta,\lambda}(\eta,a)}{\pi_j^{\eta}(\eta,a)} - 1 \right| \leq C a^2 \pi_4^{\eta}(a,1)$$

for all $\eta \leq \eta_0 = \eta_0(\lambda, j, a)$.

Proof of Lemma 2.13. We suppose that $\lambda > 0$ and $j = 1$, since the cases when $\lambda < 0$ or $j = 4$ can be treated in a similar way. We note that

$$\pi_j^{\eta,\lambda}(\eta, a) - \pi_j^{\eta}(\eta, a) = \mathbb{P}^{\eta}(\mathcal{B}),$$

where $\mathcal{B}$ is the event that there exists a $p_\lambda(\eta)$-black arm in $A_{\eta,a}$, but no p_c-black arm. If $\mathcal{B}$ occurs, there is a p_c-white circuit in $A_{\eta,a}$, so there exists a vertex which lies, at the same time, on a p_c-white circuit, and on a $p_\lambda(\eta)$-black arm: among the vertices having this property, let v be the one which is closest to the origin (if there are multiple choices, we pick one by using some deterministic procedure). This vertex v is then p_c-white and $p_\lambda(\eta)$-black, and we see four disjoint arms around v: two $p_\lambda(\eta)$-black and two p_c-white arms, starting from v and reaching a distance $d = d(v, \{0\} \cup \partial B_a)$.

In order to obtain an upper bound on $\mathbb{P}^{\eta}(\mathcal{B})$, we distinguish two cases, depending on the distance from v to the origin: we introduce the two sub-events

$$\mathcal{B}_1 := \left\{ d(0, v) \leq \frac{a}{2} \right\} \subseteq \mathcal{B} \quad \text{and} \quad \mathcal{B}_2 := \mathcal{B} \setminus \mathcal{B}_1.$$

We start by bounding the probability of $\mathcal{B}_1$. Let $i_{\max} := \lceil \log_2 \left(\frac{a}{2\eta}\right) \rceil$: by dividing the annulus $A_{\eta,a}$ into the dyadic annuli $A_i = A_{2^{i-1}\eta, 2^i\eta}$ $(1 \leq i \leq i_{\max})$, we obtain

$$\begin{aligned}
\mathbb{P}^{\eta}(\mathcal{B}_1) &\leq \sum_{i=1}^{i_{\max}} \mathbb{P}^{\eta}(v \in A_i) \\
&\leq |p_\lambda(\eta) - p_c| \sum_{i=1}^{i_{\max}} |A_i| \cdot \mathbb{P}^{\eta}_{p_\lambda(\eta)}\big(\mathcal{A}_1(\eta, 2^{i-2}\eta)\big) \cdot \mathbb{P}^{\eta}\big(\mathcal{A}_4^{p_\lambda(\eta), p_c}(\eta, 2^{i-1}\eta)\big) \\
&\qquad\qquad \cdot \mathbb{P}^{\eta}_{p_\lambda(\eta)}\big(\mathcal{A}_1(2^{i+1}\eta, a)\big) \\
&\leq C_1 |p_\lambda(\eta) - p_c| \pi_1^{\eta}(\eta, a) \sum_{i=1}^{i_{\max}} 2^{2i+2} \pi_4^{\eta}(\eta, 2^{i-1}\eta),
\end{aligned}$$

for some constant $C_1 = C_1(\lambda) > 0$ (using (2.9), (2.5) and (2.6)). Hence,

$$\begin{aligned}
\mathbb{P}^{\eta}(\mathcal{B}_1) &\leq C_2 |p_\lambda(\eta) - p_c| \pi_1^{\eta}(\eta, a) \left(\frac{a}{\eta}\right)^2 \pi_4^{\eta}(\eta, a) \\
&= C_2 \pi_1^{\eta}(\eta, a) a^2 \frac{\pi_4^{\eta}(\eta, a)}{\pi_4^{\eta}(\eta, 1)}
\end{aligned}$$

for some $C_2 = C_2(\lambda) > 0$, where we used (2.11), and then the definition of $p_\lambda(\eta)$ (2.19). Using finally (2.6), we obtain that for some $C_3 = C_3(\lambda) > 0$,

$$\mathbb{P}^{\eta}(\mathcal{B}_1) \leq C_3 \pi_1^{\eta}(\eta, a) a^2 \pi_4^{\eta}(a, 1). \tag{2.22}$$

We can bound the probability of $\mathcal{B}_2$ in a similar way, and obtain that

$$\mathbb{P}^\eta(\mathcal{B}_2) \leq C_4 \pi_1^\eta(\eta, a) a^2 \pi_4^\eta(a, 1), \tag{2.23}$$

where $C_4 = C_4(\lambda) > 0$. The computation in this case is slightly more complicated: when v is close to ∂B_a, we only have 4 short arms, but we get 3 long arms in a half plane, unless v is close to a corner, in which case we have 2 long arms in a quarter plane. Since the corresponding exponents $\alpha_3^{\mathrm{hp}} = 2$ (3 arms in a half plane) and $\alpha_2^{\mathrm{qp}} = 2$ (2 arms in a quarter plane) are larger than the 4-arm exponent α_4, the summations above can be adapted to this case. A combination of (2.22) and (2.23) then finishes the proof of Lemma 2.13. □

We can now obtain the following result.

Lemma 2.14. *For all $\lambda \in \mathbb{R}$, $j \in \{1, 4\}$ and $r > 0$, there exists a constant $c = c(r, \lambda, j) > 0$ such that*

$$\lim_{\eta \to 0} \frac{\pi_j^{\eta,\lambda}(\eta, r)}{\pi_j^\eta(\eta, r)} = \lim_{\varepsilon \to 0} \frac{\pi_j^{0,\lambda}(\varepsilon, r)}{\pi_j^0(\varepsilon, r)} = c. \tag{2.24}$$

Proof of Lemma 2.14. By Lemma 2.13, the ratio in the left-hand side of (2.24) is bounded away from 0 and ∞. We can rewrite it as

$$\frac{\pi_j^{\eta,\lambda}(\eta, r)}{\pi_j^{\eta,\lambda}(\eta, \varepsilon)} \cdot \frac{\pi_j^{\eta,\lambda}(\eta, \varepsilon)}{\pi_j^\eta(\eta, \varepsilon)} \cdot \left(\frac{\pi_j^\eta(\eta, r)}{\pi_j^\eta(\eta, \varepsilon)} \right)^{-1}$$

for some $\varepsilon \in (0, r \wedge 1)$. By Proposition 2.11, the first and third terms above converge to $c_\lambda(r, \varepsilon)$ and $c_0(r, \varepsilon)$ as $\eta \to 0$, respectively. On the other hand, it follows from Lemma 2.13 that the middle term is $1 + O(\varepsilon^\alpha)$, uniformly in η. By combining this with the boundedness property, we obtain that the ratio in the left-hand side of (2.24) is a Cauchy sequence in η, which finishes the proof of the lemma. □

For all $j \in \{1, 4\}$, we introduce

$$S_j(r) := \lim_{\eta \to 0} \frac{\pi_j^{\eta,1}(\eta, r)}{\pi_j^\eta(\eta, r)} = \lim_{\varepsilon \to 0} \frac{\pi_j^{0,1}(\varepsilon, r)}{\pi_j^0(\varepsilon, r)}. \tag{2.25}$$

Remark 2.15. *One can check that the functions S_j ($j \in \{1, 4\}$) are continuous on $(0, \infty)$.*

Corollary 2.16. *For all $\lambda, r > 0$ and $j \in \{1, 4\}$,*

$$\lim_{\eta \to 0} \frac{\pi_j^{\eta,\lambda}(\eta, 1)}{\pi_j^{\eta}(\eta, 1)} = \lim_{\varepsilon \to 0} \frac{\pi_j^{0,\lambda}(\varepsilon, 1)}{\pi_j^{0}(\varepsilon, 1)} = S_j\left(\frac{L^0(1)}{L^0(\lambda)} r\right) = S_j\big(\lambda^{4/3} r\big).$$

Proof of Corollary 2.16. This follows by combining Theorem 2.10, (2.21) and Lemma 2.14. □

Lemma 2.17. *There exists a universal constant $c_2 > 0$ such that: for all $\lambda > 0$,*

$$\lim_{\eta \to 0} \frac{\theta(p_\lambda(\eta))}{\pi_1^{\eta,\lambda}(\eta, L^\eta(\lambda))} = c_2. \tag{2.26}$$

Proof of Lemma 2.17. First, it follows from (2.13) that the ratio in the left-hand side of (2.26) is bounded away from 0 and ∞. We rewrite it as

$$\frac{\theta(p_\lambda(\eta))}{\pi_1^{\eta,\lambda}(\eta, L^\eta(\lambda))} = \frac{\theta(p_\lambda(\eta))}{\pi_1^{\eta,\lambda}(\eta, KL^\eta(\lambda))} \cdot \left(\frac{\pi_1^{\eta,\lambda}(\eta, KL^\eta(\lambda))}{\pi_1^{\eta,\lambda}(\eta, L^\eta(\lambda))}\right)^{-1}$$

for some $K > 1$. It then follows from (2.4) that the first term above is $1+o(1)$ as $K \to \infty$, uniformly for small η, while for every fixed K, the second term converges to $c(\lambda, K)$ as $\eta \to 0$. This implies the existence of the limit in the left-hand side of (2.26).

By combining Corollary 2.16 and Remark 2.15, we then obtain

$$\lim_{\eta \to 0} \frac{\pi_1^{\eta,\lambda}(\eta, KL^\eta(\lambda))}{\pi_1^{\eta,\lambda}(\eta, L^\eta(\lambda))} = \frac{S_1(KL^0(1))}{S_1(L^0(1))}.$$

Hence, the limit above is independent of λ, which completes the proof of Lemma 2.17. □

We are now ready to prove Proposition 2.8.

Proof of Proposition 2.8. Let $p > p_c$, and set $\eta = \eta(p) = \frac{L^0(1)}{L(p)}$. Note that $\eta(p) \to 0$ as $p \searrow p_c$. Further, we set $\lambda = \lambda(p)$ such that $p_{\lambda(p)}(\eta(p)) = p$. Hence $L^\eta(\lambda(p)) = L^0(1)$. Theorem 2.10 and the continuity of L^0 imply that $\lambda(p) \to 1$ as $p \searrow p_c$. Let $\varepsilon > 0$ and $p_0 > p_c$ such that $|\lambda(p) - 1| < \varepsilon$ for all $p \in (p_c, p_0)$. Then

$$\frac{\theta(p_{1-\varepsilon}(\eta))}{\pi_1^{\eta,1+\varepsilon}(\eta, L^\eta(1+\varepsilon))} \leq \frac{\theta(p)}{\pi_1^{\eta,\lambda(p)}(\eta, L^\eta(\lambda(p)))} \leq \frac{\theta(p_{1+\varepsilon}(\eta))}{\pi_1^{\eta,1-\varepsilon}(\eta, L^\eta(1-\varepsilon))}.$$

If we take the limit as $\eta \to 0$, Lemma 2.17 implies that the lower and the upper bounds become $c_2 \frac{S_1((1-\varepsilon)^{4/3})}{S_1((1+\varepsilon)^{4/3})}$ and $c_2 \frac{S_1((1+\varepsilon)^{4/3})}{S_1((1-\varepsilon)^{4/3})}$, respectively, where c_2 is as in Lemma 2.17. Since $\varepsilon > 0$ was arbitrary, the continuity of S_1 (see Remark 2.15) shows that

$$\lim_{p \searrow p_c} \frac{\theta(p)}{\pi_1^{\eta,\lambda(p)}(\eta, L^\eta(\lambda(p)))} = c. \tag{2.27}$$

In a similar way, we have

$$\lim_{p \searrow p_c} \frac{\pi_1^{\eta,\lambda(p)}(\eta, L^0(1))}{\pi_1^{\eta,1}(\eta, L^0(1))} = 1. \tag{2.28}$$

Now, recall that $L^\eta(\lambda(p)) = L^0(1)$, $L(p) = L^1(p) = \eta^{-1} L^\eta(p)$ and $\pi(a,b) = \pi^\eta(\eta a, \eta b)$. We can thus complete the proof of Proposition 2.8 by combining (2.27), (2.28) and Lemma 2.14. □

2.6 Asymptotic formula for L

In this section, we study the quantity $|p - p_c| L(p)^2 \pi_4(L(p))$ as $p \to p_c$. We already know from (2.14) that it is $\asymp 1$, and we now show that it has actually a limit.

Lemma 2.18. *There exists a constant $c > 0$ such that*

$$|p - p_c| L(p)^2 \pi_4(L(p)) \underset{p \searrow p_c}{\longrightarrow} c. \tag{2.29}$$

Proof of Lemma 2.18. We use the notations of Section 2.5, in particular we consider $\lambda, \eta > 0$ and $p_\lambda(\eta)$ as in (2.19). For $p = p_\lambda(\eta)$, the left-hand side of (2.29) is equal to

$$|p_\lambda(\eta) - p_c| L(p_\lambda(\eta))^2 \pi_4(L(p_\lambda(\eta))) = \left(\frac{L(p_\lambda(\eta))}{\eta^{-1}} \right)^2 \cdot \frac{\pi_4(L(p_\lambda(\eta)))}{\pi_4(\eta)} \cdot \lambda.$$

Let us now take the limits as $\eta \to 0$: the first term converges to $(L^0(\lambda))^2$ by Theorem 2.9, and the second term converges to $L^0(\lambda)^{-3/4}$, by the same theorem combined with Lemma 2.6 (ratio-limit theorem for π_4). Hence,

$$\lim_{\eta \searrow 0} |p_\lambda(\eta) - p_c| L(p_\lambda(\eta))^2 \pi_4(L(p_\lambda(\eta))) = (L^0(\lambda))^{5/4} \lambda,$$

which is constant in $\lambda > 0$ by Theorem 2.10. This, combined with arguments similar to the end of the proof of Proposition 2.8, finishes the proof of Lemma 2.18. □

In particular, we can compare precisely $L(p)$ and $L(p')$ when p and p' are close to p_c.

Lemma 2.19. *For all $\varepsilon > 0$ and $\Lambda > 1$, there exists $p_0 > p_c$ such that: for all $p, p' \in (p_c, p_0)$ with $\frac{L(p)}{L(p')} \in (\Lambda^{-1}, \Lambda)$, we have*

$$\left(\frac{L(p)}{L(p')}\right)^{-3/4} \cdot \left(\frac{p - p_c}{p' - p_c}\right)^{-1} \in (1 - \varepsilon, 1 + \varepsilon).$$

Proof of Lemma 2.19. This follows by combining Lemma 2.18 with Lemma 2.7. □

3 Holes in supercritical percolation

3.1 Definition and a-priori estimates

In the supercritical regime $p > p_c$, the unique infinite black cluster $\mathcal{C}_\infty(p)$ either contains the origin, or surrounds it (a.s.). In the latter case, the origin lies in a "hole": this geometric object plays an important role to study the successive freezings. In this section, we prove estimates on this hole, which is defined formally as follows.

Definition 3.1. *We call* hole of the origin *at time $p > p_c$, denoted by $\mathcal{H}(p)$, the connected component of 0 in $\mathbb{T} \setminus (\mathcal{C}_\infty(p) \cup \partial^{out} \mathcal{C}_\infty(p))$, i.e. when we remove the vertices which are neither in the infinite black cluster, nor neighbor of it. By convention, we take $\mathcal{H}(p) = \emptyset$ if 0 belongs to $\mathcal{C}_\infty(p) \cup \partial^{out} \mathcal{C}_\infty(p)$.*

Note that clearly, $\mathcal{H}(p) \supseteq \mathcal{H}(p')$ for $p_c < p < p'$. The reason why we remove an extra layer of white sites along the boundary of $\mathcal{C}_\infty(p)$ comes from the connection with frozen percolation, where white vertices along the boundary of a frozen cluster are not allowed to become black at a later time (see Definition 5.1 below).

We start with some easy a-priori estimates on $\mathcal{H}(p)$.

Lemma 3.2. *The following lower bounds hold for all $p > p_c$, where $\alpha, c > 0$ are some universal constants.*

(i) For all $\lambda \geq 1$,

$$\mathbb{P}\left(\frac{\mathcal{H}(p)}{L(p)} \subseteq B_\lambda\right) \geq 1 - e^{-\alpha\lambda}. \tag{3.1}$$

(ii) For all $\lambda \leq 1$*,*

$$\mathbb{P}\left(\frac{\mathcal{H}(p)}{L(p)} \supseteq B_\lambda\right) \geq 1 - c\pi_1\left(\lambda L(p), L(p)\right). \tag{3.2}$$

Proof of Lemma 3.2. (i) We have

$$\mathbb{P}\left(\frac{\mathcal{H}(p)}{L(p)} \subseteq B_\lambda\right) \geq \mathbb{P}_p\Big(\mathcal{O}\Big(\frac{\lambda}{2}L(p), \lambda L(p)\Big)\Big)\mathbb{P}_p\Big(\mathcal{A}_1\Big(\frac{\lambda}{2}L(p), \infty\Big)\Big)$$

(using the FKG inequality). The desired lower bound then follows from (2.4).

(ii) We can write

$$\mathbb{P}\left(\frac{\mathcal{H}(p)}{L(p)} \supseteq B_\lambda\right) \geq \mathbb{P}_p\big(\mathcal{O}^*(\lambda L(p), 2L(p))\big) = 1 - \mathbb{P}_p\big(\mathcal{A}_1(\lambda L(p), 2L(p))\big).$$

It then suffices to use that $\mathbb{P}_p\big(\mathcal{A}_1(\lambda L(p), 2L(p))\big) \asymp \pi_1(\lambda L(p), L(p))$ (from (2.7) and (2.5)). ☐

Based on the scaling limit of near-critical percolation, it is natural to expect that $\frac{\mathcal{H}(p)}{L(p)}$ converges in distribution as $p \searrow p_c$ (in a suitable topology), and so its volume $\frac{|\mathcal{H}(p)|}{L(p)^2}$ as well. However, the precise knowledge of the scaling limit is not needed for the proofs of Theorems 1.1 and 1.2: we only need to know that $\frac{|\mathcal{H}(p)|}{L(p)^2}$ does not fluctuate too much as $p \searrow p_c$. We use the following estimates, which are weaker and can be proved in an elementary way.

Lemma 3.3. *There exist universal constants* $\alpha_1, \alpha_2, c_1 > 0$ *such that the following bounds hold for all* $p > p_c$*.*

(i) For all $\lambda \geq 1$*,*

$$e^{-\alpha_1\sqrt{\lambda}} \leq \mathbb{P}\left(\frac{|\mathcal{H}(p)|}{L(p)^2} \geq \lambda\right) \leq e^{-\alpha_2\sqrt{\lambda}}, \tag{3.3}$$

$$\text{and} \quad e^{-\alpha_1\lambda} \leq \mathbb{P}\left(\frac{\mathcal{H}(p)}{L(p)} \supseteq B_\lambda\right) \leq e^{-\alpha_2\lambda}. \tag{3.4}$$

(ii) For all $\lambda \leq 1$*,*

$$c_1\pi_1\left(\sqrt{\lambda}L(p), 2L(p)\right) \leq \mathbb{P}\left(\frac{|\mathcal{H}(p)|}{L(p)^2} \leq \lambda\right) \leq \pi_1\left(\sqrt{\lambda}L(p), 2L(p)\right), \tag{3.5}$$

$$\text{and} \quad c_1\pi_1\left(\lambda L(p), 2L(p)\right) \leq \mathbb{P}\left(\frac{\mathcal{H}(p)}{L(p)} \subseteq B_\lambda\right) \leq \pi_1\left(\lambda L(p), 2L(p)\right). \tag{3.6}$$

Proposition 3.4. *There exists $\beta > 1$ such that: for all $\Lambda > 1$, for all $p, p' > p_c$ sufficiently close to p_c (depending on Λ), and all $\lambda \in [\Lambda^{-1}, \Lambda]$, one has*

$$\mathbb{P}\left(\frac{|\mathcal{H}(p')|}{L(p')^2} \geq \lambda\right) \geq \mathbb{P}\left(\frac{|\mathcal{H}(p)|}{L(p)^2} \geq \beta\lambda\right). \tag{3.7}$$

We first prove Lemma 3.3.

Proof of Lemma 3.3. We only prove (3.3) and (3.5), since (3.4) and (3.6) follow in similar ways.

(i) Let us consider $\lambda \geq 1$. For the lower bound, we note that

$$\mathbb{P}\big(|\mathcal{H}(p)| \geq \lambda L(p)^2\big) \geq \mathbb{P}_p\left(\mathcal{O}^*\left(\frac{\sqrt{\lambda}}{2}L(p), \infty\right)\right),$$

which is at least $e^{-\alpha_1\sqrt{\lambda}}$, from (2.4).

Let us now turn to the upper bound. If $\mathcal{H}(p)$ has a volume at least $\lambda L(p)^2$, then $\partial^{\mathrm{out}}\mathcal{H}(p)$, which is a white circuit surrounding 0, must contain one site at a distance at least $\frac{\sqrt{\lambda}}{4}L(p)$ from 0. This implies

$$\begin{aligned}\mathbb{P}\big(|\mathcal{H}(p)| &\geq \lambda L(p)^2\big) \\ &\leq \mathbb{P}_p\left(\mathcal{A}_1^*\left(\frac{\sqrt{\lambda}}{4}L(p), \frac{\sqrt{\lambda}}{2}L(p)\right)\right) + \mathbb{P}_p\left(\mathcal{O}^*\left(\frac{\sqrt{\lambda}}{4}L(p), \infty\right)\right),\end{aligned}$$

which is at most $e^{-\alpha_1\sqrt{\lambda}}$, using once again (2.4).

(ii) We now consider $\lambda \leq 1$. The upper bound follows from the observation that if $|\mathcal{H}(p)| \leq \lambda L(p)^2$, then the infinite black cluster intersects $B_{\sqrt{\lambda}L(p)}$, so

$$\mathbb{P}\big(|\mathcal{H}(p)| \leq \lambda L(p)^2\big) \leq \mathbb{P}_p\left(\mathcal{A}_1\left(\frac{\sqrt{\lambda}}{2}L(p), L(p)\right)\right).$$

For the lower bound, we note that

$$\begin{aligned}\mathbb{P}\big(|\mathcal{H}(p)| \leq \lambda L(p)^2\big) &\geq \mathbb{P}_p\left(\mathcal{O}\left(\frac{\sqrt{\lambda}}{4}L(p), \frac{\sqrt{\lambda}}{2}L(p)\right) \cap \mathcal{A}_1\left(\frac{\sqrt{\lambda}}{4}L(p), \infty\right)\right) \\ &\geq \tilde{c}_1 \mathbb{P}_p\left(\mathcal{A}_1\left(\frac{\sqrt{\lambda}}{4}L(p), \infty\right)\right)\end{aligned}$$

for some universal constant $\tilde{c}_1 > 0$ (using the FKG inequality, and then RSW). We can now use the existence of $\tilde{c}_2 > 0$ such that

$$\mathbb{P}_p\left(\mathcal{A}_1\left(L(p), \infty\right)\right) \geq \tilde{c}_2$$

(this is a direct consequence of (2.4)): we have

$$\mathbb{P}_p\bigg(\mathcal{A}_1\bigg(\frac{\sqrt{\lambda}}{4}L(p),\infty\bigg)\bigg) \geq \tilde{c}_3\mathbb{P}_p\left(\mathcal{A}_1\left(L(p),\infty\right)\right)\mathbb{P}_p\bigg(\mathcal{A}_1\bigg(\frac{\sqrt{\lambda}}{4}L(p),2L(p)\bigg)\bigg)$$
$$\geq \tilde{c}_3\tilde{c}_2\mathbb{P}_{1/2}\bigg(\mathcal{A}_1\bigg(\frac{\sqrt{\lambda}}{4}L(p),2L(p)\bigg)\bigg)$$

(applying once again RSW, in the annulus $A_{L(p),2L(p)}$), which allows us to conclude. □

These bounds can now be used to prove Proposition 3.4.

Proof of Proposition 3.4. We first prove the following claim: there exist $\tilde{\beta} > 1$, and $0 < \Lambda_1 < \Lambda_2$, such that for all $\lambda > 0$ with $\lambda \notin (\Lambda_1, \Lambda_2)$, for all $p, p' > p_c$ sufficiently close to p_c (depending on λ),

$$\mathbb{P}\left(\frac{|\mathcal{H}(p')|}{L(p')^2} \geq \lambda\right) \geq \mathbb{P}\left(\frac{|\mathcal{H}(p)|}{L(p)^2} \geq \tilde{\beta}\lambda\right). \tag{3.8}$$

For $\lambda \geq 1$, applying successively the two bounds of (3.3) yields

$$\mathbb{P}\left(\frac{|\mathcal{H}(p')|}{L(p')^2} \geq \lambda\right) \geq e^{-\alpha_1\sqrt{\lambda}} = e^{-\alpha_2\frac{\alpha_1}{\alpha_2}\sqrt{\lambda}} \geq \mathbb{P}\left(\frac{|\mathcal{H}(p)|}{L(p)^2} \geq \frac{\alpha_1^2}{\alpha_2^2}\lambda\right), \tag{3.9}$$

which proves the claim for $\lambda \geq \Lambda_2$, with $\Lambda_2 = 1$ and $\tilde{\beta} = \frac{\alpha_1^2}{\alpha_2^2}$. Let us now turn to small values of λ. Using the lower bound provided by (3.5), we get: for $\lambda \leq 1$,

$$\mathbb{P}\left(\frac{|\mathcal{H}(p)|}{L(p)^2} \leq \lambda\right) \geq c_1\pi_1\left(\sqrt{\lambda}L(p),2L(p)\right)$$
$$\geq c_1\tilde{c}_1\pi_1\left(\sqrt{\lambda}L(p'),2L(p')\right).$$

Here, we used the ratio-limit theorem (Lemma 2.6): we need $L(p)$ and $L(p')$ to be sufficiently large, i.e. p and p' to be close enough to p_c, depending on λ. It then follows from (2.7) that

$$\pi_1\left(\sqrt{\lambda}L(p'),2L(p')\right) \geq \tilde{c}_2\mathbb{P}_{p'}\left(\mathcal{A}_1\left(\sqrt{\lambda}L(p'),2L(p')\right)\right).$$

We can then write, for $\varepsilon > 0$ small enough,

$$c_1\tilde{c}_1\tilde{c}_2\mathbb{P}_{p'}\left(\mathcal{A}_1\left(\sqrt{\lambda}L(p'),2L(p')\right)\right) \geq \mathbb{P}_{p'}\left(\mathcal{A}_1\left(\sqrt{\varepsilon\lambda}L(p'),2L(p')\right)\right)$$

(using (2.6), (2.7) and (2.8)), which is at least $\mathbb{P}\left(\frac{|\mathcal{H}(p')|}{L(p')^2} \leq \varepsilon\lambda\right)$ (from the upper bound in (3.5)). This completes the proof of the claim for $\lambda \leq \Lambda_1 = \varepsilon$, with $\tilde{\beta} = \frac{1}{\varepsilon}$.

We now use the claim to prove the proposition itself. Let us define

$$\lambda_i = \left(\frac{\Lambda_2}{\Lambda_1}\right)^{i-1} \Lambda_1, \quad i \in \mathbb{Z}$$

(so that $\lambda_i = \Lambda_i$ for $i = 1, 2$). Noting that for every $i \in \mathbb{Z}$, $\lambda_i \notin (\Lambda_1, \Lambda_2)$, we deduce from the claim that for all $p, p' > p_c$ close enough to p_c (depending on i),

$$\mathbb{P}\left(\frac{|\mathcal{H}(p')|}{L(p')^2} \geq \lambda_i\right) \geq \mathbb{P}\left(\frac{|\mathcal{H}(p)|}{L(p)^2} \geq \tilde{\beta}\lambda_i\right). \tag{3.10}$$

Moreover, the same conclusion holds for all $p, p' > p_c$ close enough to p_c (depending on Λ) and all $i \in \mathbb{Z}$ such that $\lambda_i \in \left[\Lambda^{-1}, \frac{\Lambda_2}{\Lambda_1}\Lambda\right]$ simultaneously. Indeed, there are only finitely many such values of i, since $\lambda_i \to 0$ and $+\infty$ as $i \to +\infty$ and $-\infty$, respectively. Now, let us consider $\lambda \in [\Lambda^{-1}, \Lambda]$: there exists $i \in \mathbb{Z}$ such that $\lambda \in [\lambda_i, \lambda_{i+1}]$, and for all $p, p' > p_c$ close enough to p_c (depending on Λ),

$$\mathbb{P}\left(\frac{|\mathcal{H}(p')|}{L(p')^2} \geq \lambda\right) \geq \mathbb{P}\left(\frac{|\mathcal{H}(p')|}{L(p')^2} \geq \lambda_{i+1}\right) \geq \mathbb{P}\left(\frac{|\mathcal{H}(p)|}{L(p)^2} \geq \tilde{\beta}\lambda_{i+1}\right),$$

using $\lambda \leq \lambda_{i+1}$, and then (3.10) (note that $\lambda_{i+1} \in \left[\Lambda^{-1}, \frac{\Lambda_2}{\Lambda_1}\Lambda\right]$). We can now write

$$\begin{aligned}\mathbb{P}\left(\frac{|\mathcal{H}(p)|}{L(p)^2} \geq \tilde{\beta}\lambda_{i+1}\right) &= \mathbb{P}\left(\frac{|\mathcal{H}(p)|}{L(p)^2} \geq \tilde{\beta}\frac{\lambda_{i+1}}{\lambda}\lambda\right) \\ &\geq \mathbb{P}\left(\frac{|\mathcal{H}(p)|}{L(p)^2} \geq \tilde{\beta}\frac{\Lambda_2}{\Lambda_1}\lambda\right),\end{aligned}$$

which completes the proof of Proposition 3.4, with $\beta = \tilde{\beta}\frac{\Lambda_2}{\Lambda_1}$. □

3.2 Approximable domains

In this section, we introduce a regularity property for domains, which plays an important role when studying successive (nested) frozen clusters, allowing one to describe the frozen percolation process in an iterative manner. Roughly speaking, this property says that the domain can be approximated

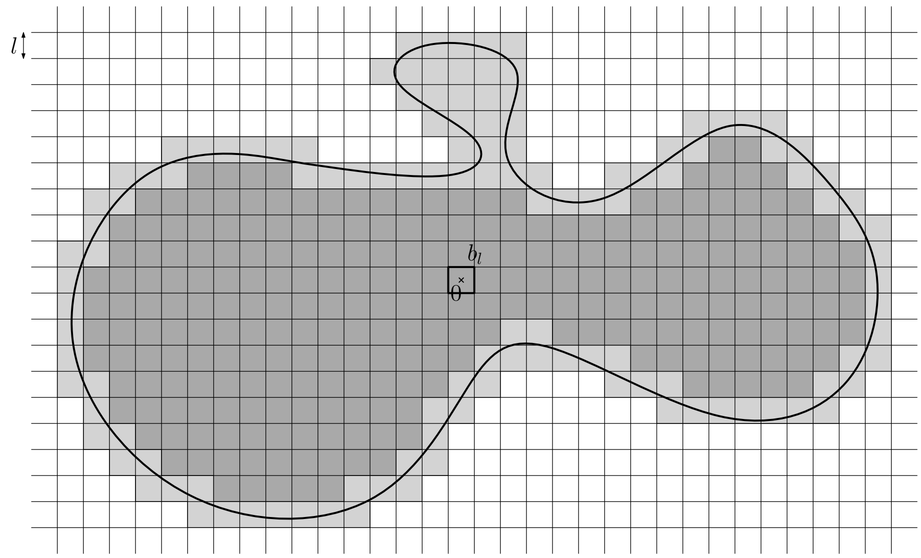

Figure 3.1: The inner and outer approximations of a domain Λ: $\Lambda^{\mathrm{int}(l)}$ and $\Lambda^{\mathrm{ext}(l)} \setminus \Lambda^{\mathrm{int}(l)}$ are in dark and light gray, respectively.

by a union of small squares, and we first prove in Section 3.3 that it is satisfied by percolation holes. We then use it to establish a continuity property for $|\mathcal{H}(p)|$ (Section 3.4). We explain later, in Section 4.1, that it can be used to predict the volume of the largest connected component in a domain.

Let us now give a formal definition. First, we need to introduce some notation. For every $l > 0$, we consider a partition of the plane into squares of side length l, such that 0 is the center of one of these squares:

$$\mathbb{C} = \bigsqcup_{k_1, k_2 \in \mathbb{Z}} \big((k_1 l, k_2 l) + b_l \big),$$

with $b_l = \big[-\frac{l}{2}, \frac{l}{2} \big)^2$. These squares are called *l-blocks*. Each l-block has four neighbors, and this notion of adjacency gives rise to connected components of l-blocks.

For a connected domain $\Lambda \subseteq \mathbb{C}$, we introduce the following inner and outer approximations by l-blocks (see Figure 3.1).

- We consider the collection of l-blocks which are entirely contained in Λ. These l-blocks can be grouped into connected components, and we denote by $\Lambda^{\mathrm{int}(l)}$ the union of all l-blocks in the connected component of b_l. By convention, we take $\Lambda^{\mathrm{int}(l)} = \emptyset$ if b_l is not contained in Λ.

- We denote by $\Lambda^{\mathrm{ext}(l)}$ the union of all l-blocks that intersect Λ.

Definition 3.5. *Let $\Lambda \subseteq \mathbb{C}$ be a bounded and simply connected domain. For $l > 0$ and $\eta \in (0,1)$, we say that Λ is* (l,η)-approximable *if*

(i) $b_l \subseteq \Lambda$,

(ii) and $\left|\Lambda^{ext(l)} \setminus \Lambda^{int(l)}\right| < \eta|\Lambda|$.

Clearly, $\Lambda^{\mathrm{int}(l)} \subseteq \Lambda \subseteq \Lambda^{\mathrm{ext}(l)}$, so this property implies in particular that

$$|\Lambda^{\mathrm{int}(l)}| > (1-\eta)|\Lambda| \quad \text{and} \quad |\Lambda^{\mathrm{ext}(l)}| < (1+\eta)|\Lambda|.$$

We also define the *t-shrinking* of a domain Λ (for $t > 0$) as

$$\big\{z : d(z, \mathbb{C} \setminus \Lambda) \geq t\big\}, \tag{3.11}$$

where d is the distance induced by the ∞ norm on $\mathbb{C}$. In other words, it is the complement of the t-neighborhood (for the distance d) of $\mathbb{C} \setminus \Lambda$. This notion is used in the particular case when Λ is a union of l-blocks, as depicted on Figure 3.2. In such a situation, for $\varepsilon \in (0,1)$, we denote by $\Lambda^l_{(\varepsilon)}$ the (εl)-shrinking of Λ. The value of l is most often clear from the context, in which case we drop the superscript for notational convenience. For future use, let us note that for all $\varepsilon \in (0, \frac{1}{4})$ (and uniformly in l),

$$(1-4\varepsilon)|\Lambda| \leq |\Lambda^l_{(\varepsilon)}| \leq |\Lambda|. \tag{3.12}$$

3.3 Approximability of $\mathcal{H}(p)$

We now prove that for α small enough, with high probability, $\mathcal{H}(p)$ can be approximated by using squares of side length $\alpha L(p)$.

Lemma 3.6. *For all $\varepsilon, \eta > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and $\alpha = \alpha(\varepsilon,\eta) > 0$ such that: for all $p \in (p_c, p_c + \delta)$,*

$$\mathbb{P}\big(\mathcal{H}(p) \text{ is } (\alpha L(p), \eta)\text{-approximable}\big) > 1 - \varepsilon. \tag{3.13}$$

Proof of Lemma 3.6. Let us fix $\varepsilon > 0$. First, we can deduce from Lemma 3.2 the existence of $\tilde{\delta} > 0$ and $0 < c_1 < c_2$ such that: for all $p \in (p_c, p_c + \tilde{\delta})$,

$$\mathbb{P}\left(B_{c_1 L(p)} \subseteq \mathcal{H}(p) \subseteq B_{c_2 L(p)}\right) > 1 - \frac{\varepsilon}{2}. \tag{3.14}$$

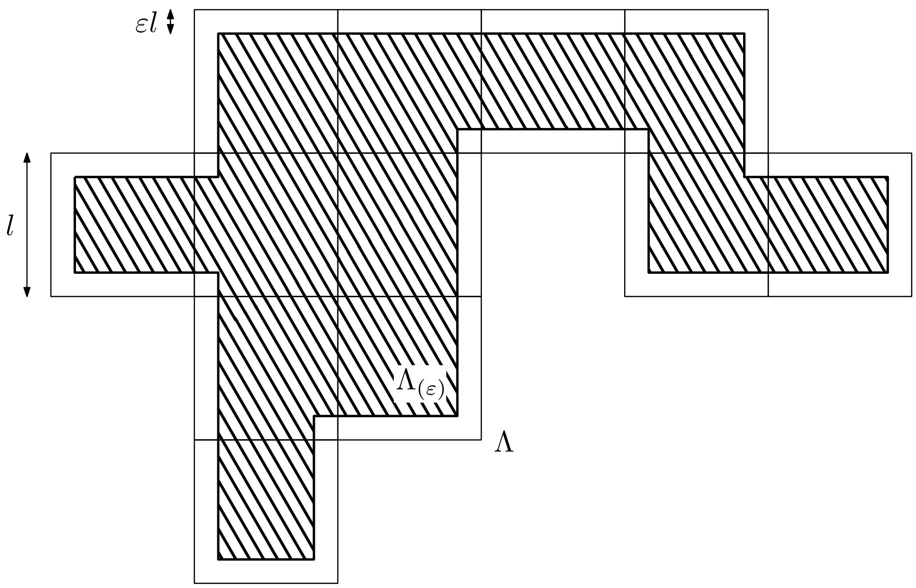

Figure 3.2: This figure depicts a union of l-blocks Λ, and its (εl)-shrinking $\Lambda^l_{(\varepsilon)} = \Lambda_{(\varepsilon)}$.

In what follows, we consider $p \in (p_c, p_c + \tilde{\delta})$, and we assume that the event in (3.14), that we denote by A, holds. We also take some small $\alpha > 0$, explaining later how to choose it appropriately. We want to derive upper bounds on the volume of $\mathcal{H}(p)^{\text{ext}(\alpha L(p))} \setminus \mathcal{H}(p)^{\text{int}(\alpha L(p))}$, which is a union of $(\alpha L(p))$-blocks. By definition of the inner and outer approximations, it can be decomposed as

$$\mathcal{H}(p)^{\text{ext}(\alpha L(p))} \setminus \mathcal{H}(p)^{\text{int}(\alpha L(p))} = \Lambda_1 \cup \Lambda_2,$$

where Λ_1 is the union of blocks that intersect $\partial^{\text{out}}\mathcal{H}(p)$, and Λ_2 is the union of blocks which are entirely contained in $\mathcal{H}(p)$, but not connected to $b_{\alpha L(p)}$ inside $\mathcal{H}(p)$.

On the one hand, each block $b = z + b_{\alpha L(p)}$ in Λ_1 is connected to $z + \partial B_{c_1 L(p)}$ by three arms: two white arms and one black arm, obtained by following $\partial^{\text{out}}\mathcal{H}(p)$ and $\partial^{\text{in}}\mathcal{H}(p)$, respectively (since $B_{c_1 L(p)} \subseteq \mathcal{H}(p)$). Denoting $\sigma = (wwb)$, this has a probability

$$\mathbb{P}_p \left(\mathcal{A}_\sigma \left(\alpha L(p), c_1 L(p) \right) \right) \leq \left(\frac{\alpha L(p)}{c_1 L(p)} \right)^\mu = \left(\frac{\alpha}{c_1} \right)^\mu,$$

for some $\mu > 0$ (using (2.9)). We deduce

$$\mathbb{E}_p\big[|\Lambda_1|\mathbb{1}_A\big] \leq c_3 \left(\frac{2c_2 L(p)}{\alpha L(p)}\right)^2 \left(\frac{\alpha}{c_1}\right)^{\mu} (\alpha L(p))^2 = c_4 \alpha^{\mu} L(p)^2. \tag{3.15}$$

On the other hand, Λ_2 consists of connected components of $(\alpha L(p))$-blocks which are contained in $\mathcal{H}(p)$. It follows from a max-flow min-cut argument that each such component is disconnected from $B_{c_1 L(p)}$ by at most $3\alpha L(p)$ vertices. If we assume such a cut to be minimal, we can form a circuit around the component by following the cut and part of the boundary of $\mathcal{H}(p)$. This observation allows us to partition the components in Λ_2, so that each element of the partition is disconnected by a cut. Then, each cut produces an $(\alpha L(p))$-block of the form $b = z + b_{\alpha L(p)}$ (in an injective way), so that b is connected by six arms (with colors $\tilde{\sigma} = (wwbwwb)$) to $z + \partial B_{\sqrt{r}\alpha L(p)}$, where r is the number of blocks in the group, and there exist also three arms (for the same reason as before, with colors $\sigma = (wwb)$) in $A_{\sqrt{r}\alpha L(p), c_1 L(p)}(z)$. The area of such a group is the area "lost through b", denoted by $\Lambda_2(b)$. We can write $\Lambda_2 = \sum_b \Lambda_2(b)$, where the sum ranges over all $(\alpha L(p))$-blocks contained in $B_{c_2 L(p)}$ (recall that the event A is assumed to hold). It follows from (2.12) (and (2.9) again) that

$$\begin{aligned}
&\mathbb{E}_p\big[|\Lambda_2(b)|\mathbb{1}_A\big] \\
&\quad \leq c_5 (\alpha L(p))^2 \sum_{r=1}^{(2c_2/\alpha)^2} \mathbb{P}_p\left(\mathcal{A}_{\tilde{\sigma}}\left(\alpha L(p), \sqrt{r}\alpha L(p)\right)\right) \mathbb{P}_p\left(\mathcal{A}_{\sigma}\left(\sqrt{r}\alpha L(p), c_1 L(p)\right)\right) \\
&\quad \leq c_5 (\alpha L(p))^2 \sum_{r=1}^{(2c_2/\alpha)^2} \left(\frac{\alpha L(p)}{\sqrt{r}\alpha L(p)}\right)^{2+\delta} \left(\frac{\sqrt{r}\alpha L(p)}{c_1 L(p)}\right)^{\mu} \\
&\quad = c_6 (\alpha L(p))^2 \alpha^{\mu} \sum_{r=1}^{(2c_2/\alpha)^2} r^{-1-\frac{\delta}{2}+\frac{\mu}{2}} \\
&\quad \leq c_7 (\alpha L(p))^2 \alpha^{\mu}
\end{aligned}$$

(we may choose $\mu < \delta$ in the beginning, without loss of generality). We thus obtain

$$\mathbb{E}_p\big[|\Lambda_2|\mathbb{1}_A\big] \leq c_8 \left(\frac{2c_2 L(p)}{\alpha L(p)}\right)^2 \alpha^{\mu} (\alpha L(p))^2 = c_9 \alpha^{\mu} L(p)^2. \tag{3.16}$$

Finally, we can write

$$\mathbb{P}_p\big(|\mathcal{H}(p)^{\mathrm{ext}(\alpha L(p))} \setminus \mathcal{H}(p)^{\mathrm{int}(\alpha L(p))}| \geq \eta |\mathcal{H}(p)|\big)$$

$$
\begin{aligned}
&\le \mathbb{P}_p\big(A \cap \{|\Lambda_1| + |\Lambda_2| \ge \eta|\mathcal{H}(p)|\}\big) + \mathbb{P}_p(A^c) \\
&\le \mathbb{P}_p\big(A \cap \{|\Lambda_1| + |\Lambda_2| \ge \eta(2c_1)^2 L(p)^2\}\big) + \frac{\varepsilon}{2},
\end{aligned}
$$

and the first term can be bounded with Markov's inequality, using (3.15) and (3.16). □

3.4 Continuity property for $\mathcal{H}(p)$

We now establish a continuity property for the volume of $\mathcal{H}(p)$, based on the approximability property.

Lemma 3.7. *For all $\varepsilon > 0$, there exist $\alpha, \delta > 0$ such that: for all $p, p' \in (p_c, p_c + \delta)$ with $p < p'$ and $\frac{L(p)}{L(p')} < 1 + \delta$, one has*

(i) $\mathcal{H}(p)$ is $(\alpha L(p), \varepsilon)$-approximable

(ii) and $(\mathcal{H}(p)^{int(\alpha L(p))})_{(\varepsilon)} \subseteq \mathcal{H}(p') \subseteq \mathcal{H}(p)$

with probability $> 1 - \varepsilon$.

This lemma implies directly the following continuity result for $|\mathcal{H}(p)|$.

Corollary 3.8. *For all $\varepsilon > 0$, there exists $\delta > 0$ such that: for all $p, p' \in (p_c, p_c + \delta)$ with $p < p'$ and $\frac{L(p)}{L(p')} < 1 + \delta$, one has*

$$
\mathbb{P}\left(\frac{|\mathcal{H}(p)|}{|\mathcal{H}(p')|} < 1 + \varepsilon\right) > 1 - \varepsilon. \tag{3.17}
$$

Proof of Corollary 3.8 from Lemma 3.7. For any $\varepsilon > 0$, let us assume that properties (i) and (ii) from Lemma 3.7 are satisfied for some α, p, p'. Using successively (ii) and (3.12), we obtain

$$
|\mathcal{H}(p')| \ge \big|(\mathcal{H}(p)^{\mathrm{int}(\alpha L(p))})_{(\varepsilon)}\big| \ge (1 - 4\varepsilon)\big|\mathcal{H}(p)^{\mathrm{int}(\alpha L(p))}\big|.
$$

From the $(\alpha L(p), \varepsilon)$-approximability of $\mathcal{H}(p)$ (property (i)), we can then deduce

$$
|\mathcal{H}(p')| \ge (1 - 4\varepsilon)(1 - \varepsilon)|\mathcal{H}(p)|,
$$

which allows us to conclude. □

Proof of Lemma 3.7. **Step 1.** We first show that every vertex on $\partial^{\mathrm{out}}\mathcal{H}(p')$ is close to $\partial^{\mathrm{out}}\mathcal{H}(p)$. We establish the following claim: for all $\varepsilon, \eta > 0$, there

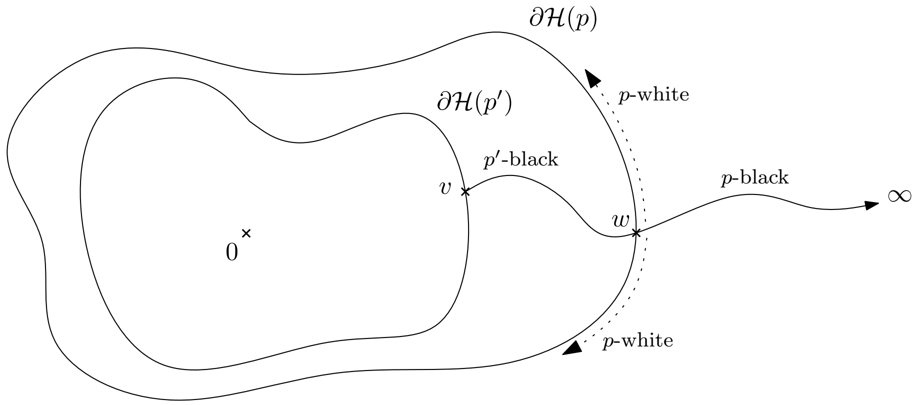

Figure 3.3: The four arm configuration appearing in the proof of Lemma 3.7.

exists $\delta > 0$ such that: for all $p, p' \in (p_c, p_c+\delta)$ with $p < p'$ and $\frac{L(p)}{L(p')} < 1+\delta$, one has

$$\mathbb{P}\left(\exists v \in \partial^{\text{out}}\mathcal{H}(p') \text{ with } d(v, \partial^{\text{in}}\mathcal{H}(p)) \geq \eta L(p)\right) < \varepsilon. \tag{3.18}$$

Let us fix $\varepsilon > 0$. It is enough to prove the claim for all η smaller than some value (depending on ε), so we may assume (using the a-priori bounds provided by Lemma 3.2) that η satisfies: for all p close enough to p_c,

$$\mathbb{P}\left(B_{\eta L(p)} \subseteq \mathcal{H}(p) \subseteq B_{\eta^{-1}L(p)}\right) > 1 - \frac{\varepsilon}{2}. \tag{3.19}$$

Now, suppose there is a vertex $v \in \partial^{\text{out}}\mathcal{H}(p')$ with $d(v, \partial^{\text{in}}\mathcal{H}(p)) \geq \eta L(p)$. It follows from the definition of $\mathcal{H}(p')$ that there exists an infinite p'-black path starting from v. Since $\mathcal{H}(p') \subseteq \mathcal{H}(p)$, this path intersects the circuit $\partial^{\text{in}}\mathcal{H}(p)$, and we call w the first such intersection point, which is thus p'-black and p-white.

If we now assume that the event in (3.19) holds, we can find four arms starting from neighbors of w (see Figure 3.3) to $w + \partial B_{\eta L(p)}$: two p'-black arms (using the infinite path starting from v), and two p-white ones (following $\partial^{\text{out}}\mathcal{H}(p)$ in two directions). For each vertex w, these two properties (being p'-black and p-white, and having four arms to distance $\eta L(p)$) have a probability $\leq c_1(p'-p)\pi_4(\eta L(p))$ (using (2.7)). Since there are at most $c_2(\eta^{-1}L(p))^2$ choices for w, we deduce (combined with (3.19)) that

$$\begin{aligned}\mathbb{P}\big(&\exists v \in \partial^{\text{out}}\mathcal{H}(p') \text{ with } d(v, \partial^{\text{in}}\mathcal{H}(p)) \geq \eta L(p)\big)\\ &< \frac{\varepsilon}{2} + c_3\eta^{-2}L(p)^2(p'-p)\pi_4(\eta L(p))\end{aligned}$$

$$< \frac{\varepsilon}{2} + c_4 \eta^{-2} (\pi_4(\eta L(p), L(p)))^{-1} \frac{p'-p}{p-p_c} \left[L(p)^2 (p-p_c) \pi_4(L(p)) \right]$$

(using (2.6)). We know that $\eta^{-2}(\pi_4(\eta L(p), L(p)))^{-1} \leq c_5 \eta^{-4}$, and $L(p)^2(p-p_c)\pi_4(L(p)) \asymp 1$ (see (2.14)), so the desired probability is $< \varepsilon$ for $\frac{p'-p}{p-p_c}$ small enough (depending on η), i.e. $\frac{L(p)}{L(p')}$ sufficiently close to 1 (using now Lemma 2.19).

Step 2. We now complete the proof. Let us fix $\varepsilon > 0$, it follows from Step 1 and Lemma 3.6 that we can find $\alpha, \delta > 0$ small enough such that: for all $p, p' \in (p_c, p_c + \delta)$ with $p < p'$ and $\frac{L(p)}{L(p')} < 1 + \delta$, one has

(i) $B_{\frac{\alpha}{2} L(p)} \subseteq \mathcal{H}(p')$,

(ii) for all $v \in \partial^{\text{out}} \mathcal{H}(p')$, $d(v, \partial^{\text{in}} \mathcal{H}(p)) < \varepsilon \alpha L(p)$,

(iii) and $\mathcal{H}(p)$ is $(\alpha L(p), \varepsilon)$-approximable

with probability $> 1 - \varepsilon$. It then suffices to observe that

$$(\mathcal{H}(p)^{\text{int}(\alpha L(p))})_{(\varepsilon)} \subseteq \mathcal{H}(p')$$

follows from properties (i) and (ii). First, $\mathcal{H}(p)^{\text{int}(\alpha L(p))}$ is a connected component of $(\alpha L(p))$-blocks, and it is easy to convince oneself that its $(\varepsilon \alpha L(p))$-shrinking is connected as well (as in the example of Figure 3.2), e.g. by induction (me may assume that $\varepsilon < \frac{1}{2}$). We know from (i) that the block $b_{\alpha L(p)}$ is in $\mathcal{H}(p)^{\text{int}(\alpha L(p))}$, and that it is contained in $\mathcal{H}(p')$, so it is surrounded by the circuit $\partial^{\text{out}} \mathcal{H}(p')$. Hence, this circuit either completely surrounds the connected set $(\mathcal{H}(p)^{\text{int}(\alpha L(p))})_{(\varepsilon)}$, in which case the desired conclusion follows, or it intersects $(\mathcal{H}(p)^{\text{int}(\alpha L(p))})_{(\varepsilon)}$. But this second possibility cannot occur, since a vertex $v \in \partial^{\text{out}} \mathcal{H}(p') \cap (\mathcal{H}(p)^{\text{int}(\alpha L(p))})_{(\varepsilon)}$ would satisfy

$$d(v, \partial^{\text{in}} \mathcal{H}(p)) \geq d(v, (\mathcal{H}(p)^{\text{int}(\alpha L(p))})^c) \geq \varepsilon \alpha L(p)$$

(by definition of the shrinking), which contradicts (ii). □

4 Volume estimates

4.1 Largest clusters in an approximable domain

In order to study volume-frozen percolation, we need estimates on the volume of the largest black cluster inside a connected subset $\Lambda \subseteq \mathbb{T}$. Typically, these estimates are used in the case when Λ is a hole left by earlier freezing events.

More precisely, we can look at the percolation configuration in such a Λ, at time p: among all the black connected components, we denote by $\mathcal{C}_\Lambda^{\max}(p)$ the one with largest volume. Note that there may be several such components, but we can just choose one of them according to some deterministic rule (using for instance an ordering of the vertices).

Several properties of $\mathcal{C}_{B_n}^{\max}(p)$ were established in [4], in particular that it has a volume $\approx |B_n|\theta(p)$ if $n \gg L(p)$. We now explain how to extend this property to more general domains. The approximability property turns out to play an important role here.

Lemma 4.1. *For all $\varepsilon > 0$ and $C \geq 1$, there exists $\mu > 0$ such that: for all $p > p_c$, all n with $\frac{L(p)}{n} < \mu$, and all sets Λ of one of the two types*

- *$\Lambda = (\tilde{\Lambda})_{(\beta)}$, where $\beta \in [0, \frac{1}{3}]$ and $\tilde{\Lambda}$ is a connected component of $\leq C$ n-blocks containing b_n,*
- *or Λ is an $(n, \frac{\varepsilon}{2})$-approximable set with $B_n \subseteq \Lambda \subseteq B_{Cn}$,*

the following three properties are satisfied, with probability $> 1 - \varepsilon$:

(i) the largest p-black cluster in Λ, i.e. $\mathcal{C}_\Lambda^{max}(p)$, satisfies

$$\left| \frac{|\mathcal{C}_\Lambda^{max}(p)|}{\theta(p)|\Lambda|} - 1 \right| < \varepsilon,$$

(ii) this cluster contains a circuit in $A_{\frac{n}{8},\frac{n}{4}}$ which is connected to ∞ by a p-black path,

(iii) and all other p-black clusters in Λ have a volume at most $\varepsilon\theta(p)|\Lambda|$.

Note that property (ii) ensures that the hole around the origin in $\mathcal{C}_\Lambda^{\max}(p)$ coincides with $\mathcal{H}(p)$.

Proof of Lemma 4.1. We can follow essentially the proofs of [4]. For the convenience of the reader, we explain in Appendix A.2 which adaptations are needed in our particular setting. $\square$

For percolation in a hole $\mathcal{H}(p)$ created by the infinite cluster, Lemma 4.1 has the following direct consequence, which is used later to analyze frozen percolation.

Corollary 4.2. *For all $\varepsilon > 0$, there exist $\mu, \delta > 0$ such that: for all $p \in (p_c, p_c + \delta)$, all $p' > p$ with $\frac{L(p')}{L(p)} < \mu$, one has* *(i) the largest p'-black cluster in $\mathcal{H}(p)$, i.e. $\mathcal{C}^{max}_{\mathcal{H}(p)}(p')$, satisfies*

$$\left| \frac{|\mathcal{C}^{max}_{\mathcal{H}(p)}(p')|}{\theta(p')|\mathcal{H}(p)|} - 1 \right| < \varepsilon,$$

(ii) it contains a circuit in $A_{\sqrt{\mu}L(p),2\sqrt{\mu}L(p)}$ which is connected to ∞ by a p'-black path,

(iii) and all other p'-black clusters in $\mathcal{H}(p)$ have a volume at most $\varepsilon\theta(p')|\mathcal{H}(p)|$,

with probability $> 1 - \varepsilon$.

In order to prove Corollary 4.2, we need to define stopping sets. They play the role of stopping times in our situation, allowing us to study iteratively the frozen percolation process.

Definition 4.3. *Consider some set $\mathfrak{S}$, and a process $X = (X(s))_{s\in\mathfrak{S}}$ indexed by $\mathfrak{S}$. A random subset $\mathcal{S} \subseteq \mathfrak{S}$ is called a* stopping set *for X if it satisfies the following property:*

$$\text{for all } S \subseteq \mathfrak{S}, \quad \{\mathcal{S} = S\} \in \sigma(X(s) \,:\, s \in \mathfrak{S} \setminus S).$$

Stopping sets can be seen as a generalization of stopping times. For instance, if $X = (X_n)_{n\in\mathbb{Z}}$ is a discrete-time stochastic process and τ is an $(\mathcal{F}_n)_{n\in\mathbb{Z}}$-stopping time, where $\mathcal{F}_n = \sigma(X_m,\, m \le n)$, then

$$\mathcal{S}_\tau = [\tau + 1, +\infty) \cap \mathbb{Z}$$

is a stopping set for X. When studying percolation, the following stopping sets are often used.

- In a rectangle R, we can consider the lowest horizontal crossing γ, and $R^+(\gamma)$ the set of vertices located above γ (if such a crossing does not exist, we simply take $R^+(\gamma) = \emptyset$). Then $R^+(\gamma)$ is a stopping set.

- Similarly, if A is annulus, and $\mathcal{C}$ is the outermost black circuit in A, then the set $\mathrm{int}(\mathcal{C})$ of vertices inside it, i.e. in the finite connected component of $\mathbb{T} \setminus \mathcal{C}$, is a stopping set (again, we take $\mathrm{int}(\mathcal{C}) = \emptyset$ if such a circuit does not exist).

For a simply-connected domain Λ, the following stopping set turns out to be very useful in our proofs. We consider the percolation process with parameter p in Λ, and we look at the set $\mathcal{C}_\Lambda(p)$ of all black vertices inside

Λ which are connected to $\partial\Lambda$: if we remove this set, together with its outer boundary (which consists of white sites), we obtain as a "hole of the origin" the set

$$\mathcal{H}^{(\Lambda)}(p) := \text{connected component of } 0 \text{ in } \mathbb{T} \setminus (\partial^{\text{in}}\Lambda \cup \mathcal{C}_\Lambda(p) \cup \partial^{\text{out}}\mathcal{C}_\Lambda(p)), \tag{4.1}$$

which we take $= \emptyset$ if 0 belongs to $\mathcal{C}_\Lambda(p) \cup \partial^{\text{out}}\mathcal{C}_\Lambda(p)$. Note that $\mathcal{H}^{(\Lambda)}(p)$ is a stopping set if Λ is given, or if Λ itself is a stopping set. This property of "explorability from outside" makes it a useful substitute of $\mathcal{H}(p)$.

Remark 4.4. *Let us observe that there exists $\alpha > 0$ with the following property: for all $p > p_c$ and $\lambda \geq 1$, for all simply-connected domain Λ,*

$$B_{\lambda L(p)} \subseteq \Lambda \implies \mathbb{P}\big(\mathcal{H}^{(\Lambda)}(p) = \mathcal{H}(p)\big) \geq 1 - e^{-\alpha\lambda}. \tag{4.2}$$

Indeed, we note that $\mathcal{H}(p) \subseteq B_{\lambda L(p)}$ implies the existence of a p-black circuit in $B_{\lambda L(p)}$ which surrounds 0, and which is connected to ∞. In particular, if $B_{\lambda L(p)} \subseteq \Lambda$, this circuit is contained in Λ, and it is connected to its boundary, so that $\mathcal{H}^{(\Lambda)}(p)$ and $\mathcal{H}(p)$ coincide in this case. Finally, Lemma 3.2 (i) implies that for some $\alpha > 0$,

$$\mathbb{P}\big(\mathcal{H}(p) \subseteq B_{\lambda L(p)}\big) \geq 1 - e^{-\alpha\lambda}.$$

We are now in a position to prove Corollary 4.2, about the largest black cluster in a percolation hole.

Proof of Corollary 4.2. Using Lemma 3.2 (i), we can choose c_0 so that

$$\mathbb{P}\left(\mathcal{H}(p) \subseteq B_{c_0 L(p)}\right) \geq 1 - \frac{\varepsilon}{3}. \tag{4.3}$$

We also know from Lemma 3.6 that $\mathcal{H}(p)$ is $(\alpha L(p), \frac{\varepsilon}{2})$-approximable with probability $\geq 1 - \frac{\varepsilon}{3}$, for α small enough.

As noted in the previous remark, if the event in (4.3) occurs, we have $\mathcal{H}(p) = \mathcal{H}^{(\Lambda)}(p)$ for $\Lambda = B_{c_0 L(p)}$. Since $\mathcal{H}^{(\Lambda)}(p)$ is a stopping set, we can condition on a realization H of it, and we can suppose that H is $(\alpha L(p), \frac{\varepsilon}{2})$-approximable, with $H \subseteq B_{c_0 L(p)}$. We are thus in a position to apply Lemma 4.1, with $n = \alpha L(p)$ and $C = c_0 \alpha^{-1}$. $\square$

4.2 Tail estimates and moment bounds

We first mention a tail estimate on $\left|\mathcal{C}_\Lambda^{\max}\right|$. This estimate is needed only in the case of boxes, and we can apply directly a result from [4].

Lemma 4.5. *There exist universal constants $c_1, c_2, X > 0$ such that for all $p > p_c$, $n \geq L(p)$, and $x \geq X$,*

$$\mathbb{P}_p \left(\left| \mathcal{C}_{B_n}^{max} \right| \geq x n^2 \theta(p) \right) \leq c_1 e^{-c_2 x \frac{n^2}{L(p)^2}}. \tag{4.4}$$

Proof of Lemma 4.5. It follows from Proposition 4.3 (iii) in [4] that for all $p > p_c$, $n \geq L(p)$ and $x \geq 0$,

$$\mathbb{P}_p \left(\left| \mathcal{C}_{B_n}^{\max} \right| \geq x n^2 \theta(p) \right) \leq c_1' \frac{n^2}{L(p)^2} e^{-c_2' x \frac{n^2}{L(p)^2} + c_3' \frac{n^2}{L(p)^2}}$$

(in that result, we have $s(L(p)) = (2L(p))^2 \pi_1(L(p)) \asymp L(p)^2 \theta(p)$, using (2.13)). Hence, if we take $X > 0$ so that $-c_2' X + c_3' = -\frac{c_2'}{2} X$, we obtain: for all $x \geq X$,

$$\mathbb{P}_p \left(\left| \mathcal{C}_{B_n}^{\max} \right| \geq x n^2 \theta(p) \right) \leq c_1' \frac{n^2}{L(p)^2} e^{-\frac{c_2'}{2} x \frac{n^2}{L(p)^2}},$$

which is $\leq c_4' e^{-\frac{c_2'}{4} x \frac{n^2}{L(p)^2}}$ for some universal constant c_4' large enough. □

We now derive moment bounds for the random variables

$$\mathcal{V}_n(z) := \left| \{ v \in B_n(z) \, : \, v \leftrightarrow \partial B_{2n}(z) \} \right| \tag{4.5}$$

(when $z = 0$, we simply write $\mathcal{V}_n$).

Lemma 4.6. *There exists a universal constant $C > 0$ such that for all $p > p_c$ and $n \geq L(p)$,*

$$\text{for all } m \geq 1, \quad \mathbb{E}_p \left[(\mathcal{V}_n)^m \right] \leq m! \left(C n^2 \theta(p) \right)^m. \tag{4.6}$$

Proof of Lemma 4.6. It follows from Lemma 4.2 in [4] (with $d = 2$) that for all $p > p_c$,

$$\text{for all } m \geq 1, \quad \mathbb{E}_p \left[(\mathcal{V}_{L(p)})^m \right] \leq m! \left(C' L(p)^2 \theta(p) \right)^m, \tag{4.7}$$

where $C' > 0$ is a universal constant (using also that $\pi_1(L(p)) \asymp \theta(p)$, from (2.13)). For $n > L(p)$, we can cover B_n with $k = \left\lceil \frac{n}{L(p)} \right\rceil^2 \leq 4 \frac{n^2}{L(p)^2}$ (possibly overlapping) boxes of the form $z_i + B_{L(p)} \subseteq B_n$ ($1 \leq i \leq k$), and write

$$\mathcal{V}_n \leq \sum_{i=1}^{k} \mathcal{V}_{L(p)}(z_i).$$

Minkowski's inequality then implies that for all $m \geq 1$,

$$\mathbb{E}_p\big[(\mathcal{V}_n)^m\big] \leq \mathbb{E}_p\left[\left(\sum_{i=1}^{k} \mathcal{V}_{L(p)}(z_i)\right)^m\right] \leq k^m \mathbb{E}_p\big[(\mathcal{V}_{L(p)})^m\big],$$

since each $\mathcal{V}_{L(p)}(z_i) \overset{(d)}{=} \mathcal{V}_{L(p)}$. Combined with (4.7), this yields the desired result. ☐

These bounds are used in Section 7, in combination with the following form of Bernstein's inequality.

Lemma 4.7. *Let $(X_i)_{1 \leq i \leq n}$ ($n \geq 1$) be independent real-valued random variables, satisfying: for all $1 \leq i \leq n$,*

$$\mathbb{E}[X_i] = 0 \quad \textit{and} \quad \mathbb{E}\big[|X_i|^m\big] \leq m! M^{m-2} \frac{\sigma_i^2}{2} \quad \textit{for all } m \geq 2,$$

for some $M > 0$ and $(\sigma_i)_{1 \leq i \leq n}$. Then for all $y \geq 0$,

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} X_i\right| \geq y\right) \leq 2e^{-\frac{1}{2}\frac{y^2}{\sigma^2 + My}}, \quad \textit{where } \sigma^2 = \sum_{i=1}^{n} \sigma_i^2. \tag{4.8}$$

Proof of Lemma 4.7. This follows from an application of Markov's inequality to the random variable $e^{\lambda \sum_{i=1}^{n} X_i}$, for a well-chosen value of the parameter $\lambda > 0$ (here, $\lambda = \frac{y}{\sigma^2 + My}$). We refer the reader to the proof of (7) in [3] for more details. ☐

Finally, let us state a consequence of Lemma 4.6, which is needed in Section 4.3.

Lemma 4.8. *There exists a constant $c_0 > 0$ satisfying the following property. For all $\varepsilon > 0$, there exists $X \geq 1$ (depending only on ε) such that for all $p > p_c$ and $n \geq XL(p)$,*

$$\mathbb{P}_p\big(\mathcal{V}_n \geq c_0 n^2 \theta(p)\big) \geq 1 - \varepsilon.$$

Proof of Lemma 4.8. Let us consider $n = xL(p)$, with $x \geq 1$. If we denote $E_1 := \mathcal{N}\big(\frac{\sqrt{x}L(p)}{8}, 2n\big)$ (recall Definition 2.1 for nets), it follows from Lemma 2.2 that

$$\mathbb{P}_p(E_1) \geq 1 - c_1 \left(\frac{2n \cdot 8}{\sqrt{x}L(p)}\right)^2 e^{-c_2 \frac{\sqrt{x}L(p)}{8L(p)}} = 1 - c_3 x e^{-c_4 \sqrt{x}},$$

which is $\geq 1 - \frac{\varepsilon}{2}$ for all $x \geq X_1 = X_1(\varepsilon)$.

Now, let us consider $k = \left\lfloor \frac{n}{3\sqrt{x}L(p)} \right\rfloor^2 \geq c_1 x$ disjoint boxes of the form $z_i + B_{2\sqrt{x}L(p)} \subseteq B_n$ ($1 \leq i \leq k$). In each of them, we have

$$\mathbb{E}_p[\mathcal{V}_{\sqrt{x}L(p)}(z_i)] \asymp xL(p)^2\theta(p)$$

(indeed, (2.4) implies that $\mathbb{P}_p(0 \leftrightarrow \partial B_n) \asymp \theta(p)$ for $n \geq L(p)$), and

$$\mathbb{E}_p[(\mathcal{V}_{\sqrt{x}L(p)}(z_i))^2] \leq c_2(xL(p)^2\theta(p))^2$$

(using (4.6) with $m = 2$), we can thus deduce from a second-moment argument that

$$\mathbb{P}_p(\mathcal{V}_{\sqrt{x}L(p)}(z_i) \geq c_3 xL(p)^2\theta(p)) \geq c_3,$$

for some universal constant $c_3 > 0$ small enough. If we call

$$E_2 := \left\{ \left| \left\{ 1 \leq i \leq k \ : \ \mathcal{V}_{\sqrt{x}L(p)}(z_i) \geq c_3 xL(p)^2\theta(p) \right\} \right| \geq \frac{c_3}{2} k \right\},$$

then $\mathbb{P}_p(E_2) \geq 1 - \frac{\varepsilon}{2}$ for all $x \geq X_2 = X_2(\varepsilon)$ (using that the $k \geq c_1 x$ boxes are disjoint). We observe

$$\mathcal{V}_n \geq \mathbb{1}_{E_1} \left(\sum_{1 \leq i \leq k} \mathcal{V}_{\sqrt{x}L(p)}(z_i) \right),$$

so on the event $E_1 \cap E_2$ (which occurs with probability $\geq 1 - \varepsilon$ if $x \geq \max(X_1, X_2)$), we have

$$\mathcal{V}_n \geq \left(\frac{c_3}{2} c_1 x \right) \cdot \left(c_3 xL(p)^2\theta(p) \right) = c_7 n^2 \theta(p).$$

□

4.3 Nice circuits

In Section 7, when we explain how to couple the full-plane process with the process in finite domains, the following quantity plays an important role. If $\mathcal{C}$ is a circuit, recall that we denote by $\mathrm{int}(\mathcal{C})$ the set of vertices inside it. We introduce

$$X_p^{\mathcal{C}} := \left| \mathcal{I}^{\mathcal{C}}(p) \right|, \quad \text{where } \mathcal{I}^{\mathcal{C}}(p) := \{ v \in \mathrm{int}(\mathcal{C}) \ : \ v \overset{p}{\leftrightarrow} \mathcal{C} \}.$$

We can obtain good estimates on this quantity when $\mathcal{C}$ is well-behaved, which occurs with high probability if $\mathcal{C}$ is obtained as the outermost black circuit $\mathcal{C}_A^{\mathrm{out}}$ in an annulus A. Before stating precise results, we introduce a notation for quantiles.

Definition 4.9. *For a real-valued random variable X and $\varepsilon \in (0,1)$, we denote by $\underline{Q}_\varepsilon(X)$ and $\overline{Q}_\varepsilon(X)$ the (resp.)* lower *and* upper ε-quantiles *of X, defined as*

$$\underline{Q}_\varepsilon(X) := \inf\{x \in \mathbb{R} \,:\, \mathbb{P}(X \leq x) \geq \varepsilon\} \tag{4.9}$$

$$\text{and} \quad \overline{Q}_\varepsilon(X) := \sup\{x \in \mathbb{R} \,:\, \mathbb{P}(X \geq x) \geq \varepsilon\} \tag{4.10}$$

(so that $\mathbb{P}(X < \underline{Q}_\varepsilon(X)) \leq \varepsilon$ and $\mathbb{P}(X > \overline{Q}_\varepsilon(X)) \leq \varepsilon$).

Definition 4.10. *For $p > p_c$ and $C > 0$, we say that a circuit $\mathcal{C}$ is (p, C)-*nice *if $f_p(\mathcal{C}) \leq Cn^2\theta(p)$, where $n = \mathrm{diam}(\mathcal{C})$ and*

$$f_p(\mathcal{C}) := \sum_{i=1}^{\lceil \log_2 L(p) \rceil - 1} \left|\{v \in \mathrm{int}(\mathcal{C}) \,:\, 2^{i-1} \leq d(v, \mathcal{C}) < 2^i\}\right| \cdot \pi_1(2^{i-1}).$$

Lemma 4.11. *For all $\varepsilon > 0$, there exists a constant $C > 0$ (depending only on ε) such that: for all $p > p_c$ and $n \geq L(p)$,*

$$\mathbb{P}_p\Big(\mathcal{C}^{out}_{A_{n/2,n}} \text{ exists and is not } (p, C)\text{-nice}\Big) \leq \varepsilon.$$

Proof of Lemma 4.11. We denote by E_0 the event that there exists a p-black circuit in $A = A_{n/2,n}$, and by $\mathcal{C} = \mathcal{C}^{\mathrm{out}}_A$ the outermost such circuit when it exists (otherwise, $\mathcal{C} = \emptyset$ by convention). We have

$$\mathbb{E}\big[f_p(\mathcal{C})\big] = \sum_{i=1}^{i_{\max}-1} \mathbb{E}\Big[\left|\{v \in \mathrm{int}(\mathcal{C}) \,:\, 2^{i-1} \leq d(v, \mathcal{C}) < 2^i\}\right|\Big] \cdot \pi_1(2^{i-1}),$$

and if v is within a distance $2^i < L(p)$ from $\mathcal{C}$, then there exist two black arms in the annulus $A_{2^i, L(p)}(v)$ (coming from the black circuit $\mathcal{C}$). Hence, if we denote $\pi_2 = \pi_{bw}$,

$$\mathbb{E}\big[f_p(\mathcal{C})\big] \leq C'_1 n^2 \sum_{i=1}^{i_{\max}-1} \pi_2(2^i, L(p))\pi_1(2^{i-1}) \tag{4.11}$$

(using (2.7)). We have $\pi_1(L(p)) \asymp \pi_1(2^i)\pi_1(2^i, L(p))$ (from quasi-multiplicativity (2.6)), and $\pi_1(2^i) \asymp \pi_1(2^{i-1})$ (from (2.5)), so

$$\mathbb{E}\big[f_p(\mathcal{C})\big] \leq C'_2 n^2 \pi_1(L(p)) \sum_{i=1}^{i_{\max}-1} \frac{\pi_2(2^i, L(p))}{\pi_1(2^i, L(p))}. \tag{4.12}$$

Since $\pi_1(L(p)) \asymp \theta(p)$ (from (2.13)), and

$$\frac{\pi_2(2^i, L(p))}{\pi_1(2^i, L(p))} \leq \left(\frac{2^i}{L(p)}\right)^\eta$$

for some $\eta > 0$ (this follows from the BK inequality, and the a-priori bound (2.8)), we finally obtain

$$\mathbb{E}\big[f_p(\mathcal{C})\big] \leq C_3' n^2 \theta(p). \tag{4.13}$$

It then follows from Markov's inequality that there exists a constant $C_4' = C_4'(\varepsilon)$ such that for all $p > p_c$ and $n \geq L(p)$,

$$\mathbb{P}\big(f_p(\mathcal{C}) \geq C_4' n^2 \theta(p)\big) \leq \varepsilon.$$

□

We can now state the result which is used in Section 7.

Lemma 4.12. *For all $\varepsilon > 0$ and $C > 0$, there exist constants $\underline{c}, \overline{c} > 0$ and $X \geq 1$ (depending only on ε and C) such that the following property holds. For all $p > p_c$ and $n \geq XL(p)$, if we have a finite collection $(\mathcal{C}^z)_{z \in Z}$ of (p, C)-nice circuits ($Z \subseteq V(\mathbb{T})$) with disjoint interiors ($\mathrm{int}(\mathcal{C}^z) \cap \mathrm{int}(\mathcal{C}^{z'}) = \emptyset$ for all $z \neq z'$), and such that $\mathcal{C}^z \subseteq A_{n/2,n}(z)$, then: for all $p' \geq p$,*

$$\underline{c}|Z| n^2 \theta(p') \leq \underline{Q}_\varepsilon\left(\sum_{z \in Z} X_{p'}^{\mathcal{C}^z}\right) \leq \overline{Q}_\varepsilon\left(\sum_{z \in Z} X_{p'}^{\mathcal{C}^z}\right) \leq \overline{c}|Z| n^2 \theta(p').$$

Proof of Lemma 4.12. In what follows, we consider an arbitrary $p' \geq p$ (so that in particular, $L(p') \leq L(p)$).

Let us consider $c_0 > 0$ from Lemma 4.8, as well as $X \geq 1$ associated with $\frac{\varepsilon}{2}$:

$$\mathbb{P}_{p'}\big(\mathcal{V}_n \geq c_0 n^2 \theta(p')\big) \geq 1 - \frac{\varepsilon}{2}$$

if $n \geq XL(p')$. Since $X^{\mathcal{C}^z} \geq \mathcal{V}_{n/2}(z)$ for every $z \in Z$ (from the definition (4.5) of $\mathcal{V}$), we have: for all $n \geq XL(p)$,

$$\mathbb{P}\big(X_{p'}^{\mathcal{C}^z} \geq c_0 n^2 \theta(p')\big) \geq 1 - \frac{\varepsilon}{2}.$$

Since the random variables $(X_{p'}^{\mathcal{C}^z})_{z \in Z}$ are independent by assumption (the circuits have disjoint interiors), we deduce the existence of $\underline{c} = \underline{c}(\varepsilon)$ such that

$$\mathbb{P}\left(\sum_{z \in Z} X_{p'}^{\mathcal{C}^z} \geq \underline{c}|Z| n^2 \theta(p')\right) > 1 - \varepsilon$$

(by distinguishing the two cases $|Z|$ small, and $|Z|$ large enough). This finally implies

$$\underline{Q}_\varepsilon\left(\sum_{z\in Z} X_{p'}^{\mathcal{C}^z}\right) \geq \underline{c}|Z|n^2\theta(p').$$

In order to estimate the upper quantile of $\sum_{z\in Z} X_{p'}^{\mathcal{C}^z}$, let us fix some $z \in Z$, and write $\mathcal{C} = \mathcal{C}^z$. We subdivide the vertices in $\mathrm{int}(\mathcal{C})$ according to their distance to $\mathcal{C}$: if we denote $i_{\max} := \lceil \log_2 L(p) \rceil$ and $i'_{\max} := \lceil \log_2 L(p') \rceil$, we have

$$\begin{aligned}\mathbb{E}\big[X_{p'}^{\mathcal{C}}\big] \leq \sum_{i=1}^{i'_{\max}-1} \big|\{v \in \mathrm{int}(\mathcal{C}) \ : \ 2^{i-1} \leq d(v,\mathcal{C}) < 2^i\}\big| \cdot \mathbb{P}_{p'}(0 \leftrightarrow \partial B_{2^{i-1}}) \\ + C_2 n^2 \mathbb{P}_{p'}(0 \leftrightarrow \partial B_{2^{i'_{\max}-1}}),\end{aligned}$$

for some universal constant $C_2 > 0$. Using that $\mathbb{P}_{p'}(0 \leftrightarrow \partial B_{2^{i-1}}) \asymp \pi_1(2^{i-1})$ and $\mathbb{P}_{p'}(0 \leftrightarrow \partial B_{2^{i'_{\max}-1}}) \asymp \mathbb{P}_{p'}(0 \leftrightarrow \partial B_{L(p')}) \asymp \theta(p')$, we obtain

$$\begin{aligned}\mathbb{E}\big[X_{p'}^{\mathcal{C}}\big] &\leq C_3 \sum_{i=1}^{i'_{\max}-1} \big|\{v \in \mathrm{int}(\mathcal{C}) \ : \ 2^{i-1} \leq d(v,\mathcal{C}) < 2^i\}\big| \cdot \pi_1(2^{i-1}) \\ &\qquad + C_4 n^2 \theta(p'), \\ &\leq C_3 f_p(\mathcal{C}) + C_4 n^2 \theta(p'). \end{aligned} \tag{4.14}$$

For the last inequality, we replaced $i'_{\max}$ by $i_{\max}$ in the summation, which we can do since $L(p') \leq L(p)$. Using that $f_p(\mathcal{C}) \leq C(2n)^2\theta(p)$ (since $\mathcal{C}$ is (p,C)-nice), we deduce from (4.14) that

$$\mathbb{E}\big[X_{p'}^{\mathcal{C}}\big] \leq C_5 n^2(\theta(p) + \theta(p')) \leq C_6 n^2 \theta(p')$$

(since $p' \geq p$). Hence,

$$\mathbb{E}\left[\sum_{z\in Z} X_{p'}^{\mathcal{C}^z}\right] \leq C_6 |Z| n^2 \theta(p').$$

Using Markov's inequality, we deduce the existence of a constant $\overline{c} = \overline{c}(\varepsilon)$ such that

$$\overline{Q}_\varepsilon\left(\sum_{z\in Z} X_{p'}^{\mathcal{C}^z}\right) \leq \overline{c}|Z|n^2\theta(p'),$$

which completes the proof of Lemma 4.12. □

5 Deconcentration argument

5.1 Frozen percolation: notations

We now go back to frozen percolation. Recall that $\mathbb{P}_N^{(G)}$ refers to volume-frozen percolation with parameter $N \geq 1$ on a graph $G = (V, E)$. The set of frozen sites at time p is denoted by $\mathcal{F}^{(G)}(p)$, and we simply write $\mathcal{F}(p)$ when G is clear from the context. Let us also stress that $(\tau_v)_{v \in V(\mathbb{T})}$ provides a natural coupling of the processes on various subgraphs of $\mathbb{T}$.

In a similar way as for the hole $\mathcal{H}(p)$ in $\mathcal{C}_\infty(p)$ (Definition 3.1), we define the hole of the origin in the frozen percolation process, replacing $\mathcal{C}_\infty(p)$ by the set of frozen sites at time p.

Definition 5.1. *For a subgraph G of $\mathbb{T}$, we denote by $\bar{\mathcal{H}}^{(G)}(p)$ the connected component of the origin in $G \setminus (\mathcal{F}(p) \cup \partial^{out}\mathcal{F}(p))$ (and we take $\bar{\mathcal{H}}^{(G)}(p) = \emptyset$ if 0 belongs to $\mathcal{F}(p) \cup \partial^{out}\mathcal{F}(p)$).*

By analogy, $\bar{\mathcal{H}}^{(G)}(p)$ is also called hole of the origin, in the frozen percolation process. However, note that it does not need to be a hole in the geometric sense, i.e. surrounded by one frozen cluster.

Remark 5.2. *Here, the (natural) rule that sites adjacent to a frozen cluster remain white forever, is crucial. The frozen percolation process would behave very differently if such sites were allowed to become black at a later time (and form new connected components).*

5.2 Exceptional scales

Heuristically, if we consider a box with volume $\simeq K^2$, then Lemma 4.1 implies that for $p > p_c$, a giant connected component arises, with volume $\simeq \theta(p)K^2$ (and all the other components are tiny). Hence, for volume-frozen percolation in this box, we expect the first freezing event to occur at a time p such that $\theta(p)K^2 \simeq N$, i.e. (using Proposition 2.8)

$$c_\theta \pi_1(L(p))K^2 \simeq N.$$

This freezing event then leaves holes with diameter of order $L(p)$, so that $L(p)$ can be seen as the next scale in the process.

This informal explanation leads us to define $\psi_N(K) := K'$ via the equation $c_\theta \pi_1(K')K^2 \simeq N$. More precisely, for all $N \geq 1$ and K large enough (so that $c_\theta K^2 > N$), we introduce

$$\psi_N(K) := \sup\{K' \,:\, c_\theta \pi_1(K')K^2 \geq N\}. \tag{5.1}$$

We also use $\psi_N^{-1}(K') := \inf\{K : \psi_N(K) \geq K'\}$.

We can now define inductively the sequence of exceptional scales $(m_k(N))_{k\geq 0}$ by: $m_0 = 1$, and for all $k \geq 0$,

$$m_{k+1}(N) = \psi_N^{-1}(m_k(N)). \tag{5.2}$$

It follows easily from the definitions and the monotonicity of π_1 that

$$m_{k+1} = \left\lceil \left(\frac{N}{c_\theta \pi_1(m_k)} \right)^{1/2} \right\rceil, \tag{5.3}$$

and that $(m_k(N))_{k\geq 0}$ is non-decreasing for every fixed $N \geq 1$. Note also that $m_1(N) \sim c_0\sqrt{N}$ as $N \to \infty$, for some constant $c_0 > 0$.

By using Lemma 2.5, we can see that each m_k follows a power law: $m_k(N) = N^{\delta_k + o(1)}$ as $N \to \infty$, where the sequence of exponents $(\delta_k)_{k\geq 0}$ satisfies

$$\delta_0 = 0, \quad \text{and} \;\; \delta_{k+1} = \frac{1}{2} + \frac{5}{96}\delta_k \;\; (k \geq 0). \tag{5.4}$$

Note that this sequence is strictly increasing, and that it converges to $\delta_\infty = \frac{48}{91}$.

It is natural to introduce the (approximate) fixed point of ψ_N:

$$m_\infty(N) := \sup\{m : c_\theta \pi_1(m) m^2 \leq N\} \tag{5.5}$$

(note that if we consider critical percolation in a box of volume m^2, the quantity $\pi_1(m)m^2$ gives the order of magnitude for the volume of the largest connected components). Lemma 2.5 implies that $m_\infty(N) = N^{\delta_\infty + o(1)}$ as $N \to \infty$, where $\delta_\infty = \frac{48}{91}$ is the exponent found previously. The following observation is useful later.

Lemma 5.3. *There exist universal constants $c, \eta > 0$ such that: for all $N \geq 1$, all $K \leq m_\infty(N)$,*

$$\frac{\psi_N(K)}{K} \leq c \left(\frac{K}{m_\infty} \right)^\eta. \tag{5.6}$$

Proof of Lemma 5.3. We know from the definitions of $\psi_N(K)$ (5.1) and m_∞ (5.5) that $c_\theta \pi_1(\psi_N(K))K^2 \geq N \geq c_\theta \pi_1(m_\infty) m_\infty^2$, so

$$\frac{\pi_1(K)}{\pi_1(\psi_N(K))} = \frac{c_\theta \pi_1(K) K^2}{c_\theta \pi_1(\psi_N(K)) K^2} \leq \frac{c_\theta \pi_1(K) K^2}{c_\theta \pi_1(m_\infty) m_\infty^2}. \tag{5.7}$$

It follows from (2.8) that

$$\frac{\pi_1(K)}{\pi_1(\psi_N(K))} \geq c_1 \left(\frac{\psi_N(K)}{K}\right)^{1/2} \tag{5.8}$$

and

$$\frac{c_\theta \pi_1(K) K^2}{c_\theta \pi_1(m_\infty) m_\infty^2} \leq c_2 \left(\frac{K}{m_\infty}\right)^{3/2} \tag{5.9}$$

for some $c_1, c_2 > 0$. The desired result then follows by combining (5.7), (5.8), and (5.9). $\square$

This lemma implies that if $\tilde{m}(N) \ll m_\infty(N)$ as $N \to \infty$, then $\psi_N(\tilde{m}) \ll \tilde{m}$. It holds in particular for $\tilde{m} = m_k$ ($k \geq 0$), since $m_k(N) = N^{\delta_k + o(1)}$, with $\delta_k < \delta_\infty$.

Remark 5.4. *Even if our definition of exceptional scales and the one in [34] (let us call it $(m'_k(N))_{k \geq 0}$) differ slightly, they are equivalent in the following sense: for every $k \geq 1$, $m_k(N) \asymp m'_k(N)$ as $N \to \infty$. In particular, the results below also apply with this modified definition.*

Finally, we define the corresponding times by: for $k \in \mathbb{N} \cup \{\infty\}$,

$$q_k(N) := \sup\{p > p_c \,:\, L(p) \geq m_k(N)\}. \tag{5.10}$$

Our analysis focuses on the time window $[q_\infty, q_1]$, q_1 being roughly the time when the last frozen clusters may appear.

Let us now recall the main results from [34] about the scales $(m_k)_{k \geq 1}$, showing that they indeed play a particular role. The first theorem corresponds to the case when one starts with a box of side length of order m_k, for some fixed $k \geq 1$.

Theorem 5.5 ([34], Theorem 1)**.** *Let $k \geq 2$ be fixed. For every $C \geq 1$, every function $\tilde{m}(N)$ that satisfies*

$$C^{-1} m_k(N) \leq \tilde{m}(N) \leq C m_k(N) \tag{5.11}$$

for N large enough, we have

$$\liminf_{N \to \infty} \mathbb{P}_N^{(B_{\tilde{m}(N)})}(0 \text{ is frozen at time } 1) > 0. \tag{5.12}$$

The second theorem deals with the case when one starts far from the exceptional scales.

Theorem 5.6 ([34], Theorem 2)**.** *For every integer $k \geq 0$ and every $\varepsilon > 0$, there exists a constant $C = C(k, \varepsilon) \geq 1$ such that: for every function $\tilde{m}(N)$ that satisfies*

$$Cm_k(N) \leq \tilde{m}(N) \leq C^{-1} m_{k+1}(N) \tag{5.13}$$

for N large enough, we have

$$\limsup_{N\to\infty} \mathbb{P}_N^{(B_{\tilde{m}(N)})}(0 \text{ is frozen at time } 1) \leq \varepsilon. \tag{5.14}$$

These two results were proved by induction, and for that, we established some slightly stronger versions that we now state. For a circuit γ, we denote by $\mathcal{D}(\gamma) \subseteq \mathbb{T}$ the domain that it encloses. For any $0 < n_1 < n_2$, we introduce

- $\Gamma_N(n_1, n_2) = \{$for every circuit γ in A_{n_1,n_2}, for the process in $\mathcal{D}(\gamma)$ with parameter N, 0 is frozen$\}$,
- and $\tilde{\Gamma}_N(n_1, n_2) = \{$there exists a circuit γ in A_{n_1,n_2} such that for the process in $\mathcal{D}(\gamma)$ with parameter N, 0 is frozen$\}$.

Here, we use the natural coupling for the frozen percolation processes in various subgraphs of $\mathbb{T}$.

Proposition 5.7 ([34], Proposition 2)**.** *For any $k \geq 2$, and $0 < C_1 < C_2$, we have*

$$\liminf_{N\to\infty} \mathbb{P}(\Gamma_N(C_1 m_k(N), C_2 m_k(N))) > 0. \tag{5.15}$$

This result also holds for $k = 1$ under the extra condition that $C_1 > C_0$, where $C_0 > 0$ is a universal constant.

Proposition 5.8 ([34], Proposition 3)**.** *Let $k \geq 0$, $\varepsilon > 0$, and $0 < C_1 < C_2$. Then there exists a constant $C = C(k, \varepsilon, C_1, C_2)$ such that: for every function $\tilde{m}(N)$ that satisfies*

$$Cm_k(N) \leq C_1\tilde{m}(N) \leq C_2\tilde{m}(N) \leq C^{-1} m_{k+1}(N) \tag{5.16}$$

for N large enough, we have

$$\limsup_{N\to\infty} \mathbb{P}(\tilde{\Gamma}_N(C_1\tilde{m}(N), C_2\tilde{m}(N))) \leq \varepsilon. \tag{5.17}$$

For future use, let us note that Proposition 5.8 can be formulated in the following way, which may look stronger at first sight. For all $k \geq 0$, $\varepsilon > 0$, and $0 < C_1 < C_2$, there exist C and N_0 such that: for all $N \geq N_0$, all $\tilde{m}$ with $Cm_k(N) \leq C_1\tilde{m} \leq C_2\tilde{m} \leq C^{-1} m_{k+1}(N)$, we have

$$\mathbb{P}(\tilde{\Gamma}_N(C_1\tilde{m}, C_2\tilde{m})) \leq \varepsilon.$$

Remark 5.9. *Even if we are not using it later, we would like to mention that with small adjustments to the proofs of Propositions 5.7 and 5.8, we can also get some information on the size of the final cluster $\mathcal{C}_1(0)$ of the origin. For any $0 < n_1 < n_2$ and $M \geq 1$, let us introduce*

- $\Gamma_N^{(M)}(n_1, n_2) = \{$*for every circuit γ in A_{n_1,n_2}, for the process in $\mathcal{D}(\gamma)$ with parameter N, $|\mathcal{C}_1(0)| \notin \big(M, \frac{N}{M}\big)$*$\}$,
- *and* $\tilde{\Gamma}_N^{(M)}(n_1, n_2) = \{$*for every circuit γ in A_{n_1,n_2}, for the process in $\mathcal{D}(\gamma)$ with parameter N, $|\mathcal{C}_1(0)| \in \big(M, \frac{N}{M}\big)$*$\}$.

We can then distinguish the same two cases as before.

- *For all $k \geq 2$, $\varepsilon > 0$, and $0 < C_1 < C_2$, there exists $M > 1$ such that:*

  $$\liminf_{N\to\infty} \mathbb{P}(\Gamma_N^{(M)}(C_1 m_k(N), C_2 m_k(N))) \geq 1 - \varepsilon.$$

  Moreover, we can also show that each of the three cases $|\mathcal{C}_1(0)| \leq M$ ($\mathcal{C}_1(0)$ is microscopic), $|\mathcal{C}_1(0)| \in \big[\frac{N}{M}, N\big)$ (macroscopic and non-frozen), and $|\mathcal{C}_1(0)| \geq N$ (macroscopic and frozen) has a probability bounded away from 0 as $N \to \infty$.

- *For all $k \geq 0$, $\varepsilon > 0$, $0 < C_1 < C_2$, and $M > 1$, there exists a constant $C = C(k, \varepsilon, C_1, C_2, M)$ such that: for every function $\tilde{m}(N)$ that satisfies*

  $$C m_k(N) \leq C_1 \tilde{m}(N) \leq C_2 \tilde{m}(N) \leq C^{-1} m_{k+1}(N) \tag{5.18}$$

  for N large enough, we have

  $$\liminf_{N\to\infty} \mathbb{P}(\tilde{\Gamma}_N^{(M)}(C_1 \tilde{m}(N), C_2 \tilde{m}(N))) \geq 1 - \varepsilon.$$

5.3 Associated chains

We now present several chains related to frozen percolation in a simply connected, bounded domain $\Lambda \subseteq \mathbb{T}$. For all $p > p_c$, we denote by μ_p^{vol} the distribution of $\frac{|\mathcal{H}(p)|}{L(p)^2}$. In the following definitions, some value of the parameter $N \geq 1$ is fixed.

(i) First, we can consider the sequence of successive holes around 0 for the frozen percolation process in Λ.

   - We start with $\Lambda_0 = \Lambda$.

- Given Λ_i ($i = 0, \dots, k-1$), p_{i+1} is the time of the first freezing event for the frozen percolation process in Λ_i,
- and $\Lambda_{i+1} = \bar{\mathcal{H}}^{(\Lambda_i)}(p_{i+1})$.

(ii) If we are also given an initial scale $K > 0$, we can define the (deterministic) sequence $(K_i)_{0 \leq i \leq k}$ by:

$$K_0 = K, \quad \text{and} \quad K_{i+1} = \psi_N(K_i) \quad (i = 0, \dots, k-1).$$

We think of them as reference scales, at which the successive freezing events typically occur, as explained in Section 5.2.

(iii) Following the same heuristic explanation as in the beginning of Section 5.2, we expect the first freezing event in a domain Λ^* to occur at a time p^* such that

$$c_\theta \pi_1(L(p^*))|\Lambda^*| \simeq N \simeq c_\theta \pi_1(\psi_N(K))K^2,$$

and so (using (2.7))

$$\frac{L(p^*)}{\psi_N(K)} \simeq \left(\frac{K^2}{|\Lambda^*|}\right)^{-48/5}.$$

Moreover, the frozen percolation hole created in this way should look like $\mathcal{H}^{(\Lambda^*)}(p^*) \simeq \mathcal{H}(p^*)$. This leads us to introduce the following sequences of (random) sets $(\Lambda_i^*)_{0 \leq i \leq k}$ and (random) times $(p_i^*)_{1 \leq i \leq k}$. We expect them to approximate the real process in Λ, which we prove rigorously in Section 6.

- We start with $\Lambda_0^* = \Lambda$.
- Given Λ_i^* ($i = 0, \dots, k-1$), p_{i+1}^* is defined by

$$\frac{L(p_{i+1}^*)}{K_{i+1}} = \left(\frac{|\Lambda_i^*|}{K_i^2}\right)^{48/5}, \tag{5.19}$$

- and then, we take $\Lambda_{i+1}^* = \mathcal{H}(p_{i+1}^*)$.

(iv) We also introduce the chain $(p_i^{**})_{1 \leq i \leq k}$, defined by taking $\Lambda_0^{**} = \Lambda$ and

$$\frac{L(p_{i+1}^{**})}{K_{i+1}} = \left(\frac{|\Lambda_i^{**}|}{K_i^2}\right)^{48/5}$$

(as for $(p_i^*)_{1 \leq i \leq k}$), but $\Lambda_{i+1}^{**} = \mathcal{H}^{(\Lambda_i^{**})}(p_{i+1}^{**})$.

(v) Finally, we use a slight modification of the chain $(p_i^*)_{1\leq i\leq k}$. We can write

$$\frac{L(p_{i+1}^*)}{K_{i+1}} = \left(\frac{|\Lambda_i^*|}{L(p_i^*)^2}\right)^{48/5} \left(\frac{L(p_i^*)}{K_i}\right)^{96/5}$$

(with $L(p_0^*) = K_0$ by convention), which suggests to define a chain $(\tilde{p}_i)_{0\leq i\leq k}$ by

$$\frac{L(\tilde{p}_{i+1})}{K_{i+1}} = \tilde{\alpha}_i^{48/5} \left(\frac{L(\tilde{p}_i)}{K_i}\right)^{96/5}, \tag{5.20}$$

where $\tilde{\alpha}_i$ has distribution $\mu_{\tilde{p}_i}^{\mathrm{vol}}$ (with $L(\tilde{p}_0) = K_0$ as well).

This last chain $(\tilde{p}_i)_{0\leq i\leq k}$ is exactly a Markov chain, which makes it more convenient to work with. In particular, we start by proving deconcentration for this chain, in Section 5.5, based on an abstract result established in Section 5.4. Moreover, it is easy to see that it behaves, essentially, in the same way as $(p_i^*)_{0\leq i\leq k}$ and $(p_i^{**})_{0\leq i\leq k}$, as we explain now.

We denote by d_{TV} the total variation distance between two distributions, and with a slight abuse of notation, we also talk about the total variation distance between two random variables X and Y (defined as the distance between their respective distributions).

Lemma 5.10. *For all $k \geq 1$ and $\varepsilon > 0$, there exist $M_0, N_0 \geq 1$ such that: for all $N \geq N_0$, for all $p \in (p_c, q_{k+1}(N))$ with $L(p) \leq m_\infty(N)/M_0$,*

$$d_{TV}\big(L(p_k^*), L(\tilde{p}_k)\big) \leq \varepsilon \quad \text{and} \quad d_{TV}\big(L(p_k^{**}), L(\tilde{p}_k)\big) \leq \varepsilon.$$

Proof of Lemma 5.10. Lemma 5.3 ensures that by choosing M_0 large enough, we are in a position to use Remark 4.4 repeatedly: for each $i = 0, \ldots, k-1$, if $\Lambda_i^* = \Lambda_i^{**}$ and $p_i^* = p_i^{**}$, then $p_{i+1}^* = p_{i+1}^{**}$ (from the definition) and Remark 4.4 implies that for N large enough, $\Lambda_{i+1}^* = \Lambda_{i+1}^{**}$ with probability at least $1 - \frac{\varepsilon}{2k}$. We deduce

$$\mathbb{P}\big(\forall i \in \{0, \ldots, k\},\ p_i^* = p_i^{**} \text{ and } \Lambda_i^* = \Lambda_i^{**}\big) \geq 1 - \frac{\varepsilon}{2}. \tag{5.21}$$

We can then compare $(p_i^{**})_{1\leq i\leq k}$ and $(\tilde{p}_i)_{1\leq i\leq k}$ by using that the (Λ_i^{**}) are stopping sets, which allows us to successively "refresh" the configuration inside them. Given $(p_j^{**})_{1\leq j\leq i}$ and $(\Lambda_j^{**})_{1\leq j\leq i-1}$, the total variation distance between $\Lambda_i^{**} = \mathcal{H}^{(\Lambda_{i-1}^{**})}(p_i^{**})$ and $\tilde{\mathcal{H}}(p_i^{**})$ (obtained on an independent percolation configuration) is thus at most $\frac{\varepsilon}{2k}$ (using again Remark 4.4: note that

(4.2) provides in particular an upper bound on the total variation distance between $\mathcal{H}^{(\Lambda)}(p)$ and $\mathcal{H}(p)$), and so

$$d_{TV}(p_{i+1}^{**}, \tilde{p}_{i+1}) \leq d_{TV}(p_i^{**}, \tilde{p}_i) + \frac{\varepsilon}{2k}.$$

Combined with (5.21), this yields the desired result. □

5.4 Abstract deconcentration result

We now establish a general result that provides deconcentration for functions of independent random variables: we give a simple sufficient condition on such functions to ensure that they are spread out, i.e. that they cannot be concentrated on small intervals. This lemma is instrumental in our proof.

Let us mention that for sums of independent random variables, a result due to Le Cam can be applied (see [21], and (B) in [9]). This deconcentration result is used in [30], to show that for two-dimensional critical percolation in a box, there exist macroscopic gaps between the sizes of the largest clusters. In some cases, it is even possible to obtain CLT-type results by uncovering a renewal structure. In particular, McLeish's CLT for martingale differences [23] is used in [15] (for "critical" first-passage percolation in two dimensions – the proofs also apply for the maximal number of disjoint open circuits surrounding the origin in 2D percolation at criticality), [37] (for the number of open clusters in a box with side length n, for bond percolation on $\mathbb{Z}^d$, $d \geq 2$, for any parameter $p \in (0,1)$) and [36] (for winding angles of arms in 2D critical percolation). CLT-type results are also obtained in [6], for 2D invasion percolation, based on mixing properties. However, none of these techniques seems to be directly applicable in our setting.

We first introduce some notations. We use $\Omega_n = \{0,1\}^n$, and for $\omega \in \Omega_n$ and $1 \leq i \leq n$, we denote by $\omega^{(i)}$ (resp. $\omega_{(i)}$) the configuration that coincide with ω except at index i, where it is equal to 1 (resp. 0). We also write $|\omega| = |\{i \in \{1, \dots, n\} : \omega_i = 1\}|$.

Let us consider a family of independent Bernoulli-distributed random variables $(Y_i)_{i\geq 1}$, with corresponding parameters $p_i \in (0,1)$ (i.e. for each i, $\mathbb{P}(Y_i = 1) = p_i$ and $\mathbb{P}(Y_i = 0) = 1 - p_i$), and a sequence of functions $f_n : \Omega_n \to \mathbb{R}$.

Lemma 5.11. *Assume that there exists $\varepsilon > 0$ such that $p_i \in (\varepsilon, 1-\varepsilon)$ for all $i \geq 1$, and that for all $n \geq 1$, f_n satisfies*

$$\text{for all } 1 \leq i \leq n, \quad \nabla_i f_n \geq 1 \tag{5.22}$$

(i.e. for every $\omega \in \Omega_n$*,* $f_n(\omega^{(i)}) \geq f_n(\omega_{(i)}) + 1$*). Then there exists a constant* $c = c(\varepsilon) \in (0, \infty)$ *such that: for all* $N \geq 1$*, every interval* $I \subseteq \mathbb{R}$*,*

$$\mathbb{P}(f_n(Y_1, \ldots, Y_N) \in I) \leq \frac{c}{N^{1/2}}(|I| + 1),$$

where we denote by $|I|$ *the length of* I*.*

Remark 5.12. *This result provides an upper bound on the* Lévy concentration function *of the random variable* $X = f_n(Y_1, \ldots, Y_n)$*, defined by*

$$Q_X(\lambda) := \sup_{x \in \mathbb{R}} \mathbb{P}(X \in [x, x + \lambda]).$$

The proof of Lemma 5.11 is based on the following construction.

Lemma 5.13. *Let* $N \geq 1$*, and denote* $\omega = (Y_1, \ldots, Y_N)$*. One can construct a sequence* $\tilde{w}_0 = (0, \ldots, 0), \tilde{w}_1, \ldots, \tilde{w}_N = (1, \ldots, 1)$ *such that*

- *for every* $i \in \{0, \ldots, N-1\}$*,* $\tilde{w}_{i+1}$ *can be obtained from* $\tilde{w}_i$ *by switching one coordinate from* 0 *to* 1 *(so that each* $\tilde{w}_i$ *has exactly* i *coordinates equal to* 1*),*
- *for every* $i \in \{0, \ldots, N\}$*,* $\tilde{w}_i$ *has the same distribution as* ω *conditioned on* $|\omega| = i$*.*

Proof of Lemma 5.13. As we explained, Lemma 5.11 is used in Section 5.5 to show deconcentration for the chain $(L(\tilde{p}_i))_{0 \leq i \leq k}$, and for this application, we only need the case where $p_i = \frac{1}{2}$ for all $i \in \{1, \ldots, N\}$.

When all the parameters $(p_i)_{1 \leq i \leq N}$ are equal, the construction of $(\tilde{w}_i)_{0 \leq i \leq N}$ is straightforward: indeed, given $\tilde{w}_i$, we can produce $\tilde{w}_{i+1}$ by considering the $n - i$ coordinates which are equal to 0, choose one of them uniformly at random, and switch it to 1. We do not need the general case, which seems to be trickier. Nevertheless, since we find Lemma 5.11 interesting in itself, we provide a proof of Lemma 5.13 in Appendix A.3. $\square$

Proof of Lemma 5.11. We use the coupling provided by Lemma 5.13: for every interval $I \subseteq \mathbb{R}$,

$$\begin{aligned}
\mathbb{P}(f(\omega) \in I) &= \sum_{i=0}^{N} \mathbb{P}(f(\omega) \in I \mid |\omega| = i)\mathbb{P}(|\omega| = i) \\
&= \sum_{i=0}^{N} \mathbb{P}(f(\tilde{\omega}_i) \in I)\mathbb{P}(|\omega| = i).
\end{aligned}$$

We can now use that $\mathbb{P}(|\omega| = i) \leq \frac{c}{N^{1/2}}$, where $c = c(\varepsilon) < \infty$ depends only on ε: we obtain

$$\mathbb{P}(f(\omega) \in I) \leq \frac{c}{N^{1/2}} \sum_{i=0}^{N} \tilde{\mathbb{E}}\left[\mathbb{1}_{f(\tilde{\omega}_i) \in I}\right] = \frac{c}{N^{1/2}} \tilde{\mathbb{E}}\left[\sum_{i=0}^{N} \mathbb{1}_{f(\tilde{\omega}_i) \in I}\right].$$

It then suffices to observe that

$$\sum_{i=0}^{N} \mathbb{1}_{f(\tilde{\omega}_i) \in I} \leq |I| + 1,$$

from our assumption on f. □

5.5 Deconcentration for $(L(\tilde{p}_i))_{0 \leq i \leq k}$

We now obtain deconcentration for the Markov chain $(L(\tilde{p}_i))_{0 \leq i \leq k}$ by applying the abstract result from the previous section, Lemma 5.11.

Proposition 5.14. *For all $\varepsilon > 0$ and $\lambda > 1$, there exist $k_0, N_0 \geq 1$ such that: for all $k \geq k_0$, for all $N \geq N_0$, for all $p \in (p_c, q_{k+1}(N))$,*

$$\sup_{y>0} \mathbb{P}\big(L(\tilde{p}_k) \in (y, \lambda y)\big) < \varepsilon. \tag{5.23}$$

Proof of Proposition 5.14. Let us consider $\varepsilon > 0$ and $\lambda > 1$, and take $k \geq 1$ (we explain later how to choose it). In order to use Lemma 5.11, we describe the process $(L(\tilde{p}_i))_{0 \leq i \leq k}$ in terms of i.i.d. random variables $(U_i)_{0 \leq i \leq k-1}$ uniformly distributed on the interval $(0, 1)$, as we explain now. For $p > p_c$ and $u \in (0, 1)$, we introduce the lower u-quantile

$$q(p, u) := \underline{Q}_u\left(\frac{|\mathcal{H}(p)|}{L(p)^2}\right)$$

(recall Definition 4.9). It follows from (4.9) that if U is a random variable uniform on $(0, 1)$, then $q(p, U)$ has distribution μ_p^{vol}, so that in the definition (5.20) of $(\tilde{p}_i)_{0 \leq i \leq k}$, we can use the representation

$$\tilde{\alpha}_i = q(\tilde{p}_i, U_i) \quad (0 \leq i \leq k - 1). \tag{5.24}$$

Note that $\tilde{\alpha}_i$ is thus a function of $U_0, \ldots, U_i$.

From the upper bounds in (3.3) and (3.5), we can find Λ large enough (depending on k) such that if U is uniformly distributed on $(0, 1)$, we have: for all $p > p_c$,

$$\mathbb{P}\left(q(p, U) \notin [\Lambda^{-1}, \Lambda]\right) \leq \frac{\varepsilon}{10k}. \tag{5.25}$$

In particular, for all $p > p_c$ and $u \in \left(\frac{\varepsilon}{10k}, 1 - \frac{\varepsilon}{10k}\right)$, $q(p,u) \in [\Lambda^{-1}, \Lambda]$. We thus introduce the event

$$G_1 := \left\{\forall i \in \{0, \dots, k-1\},\ U_i \in \left(\frac{\varepsilon}{10k}, 1 - \frac{\varepsilon}{10k}\right)\right\}, \tag{5.26}$$

which satisfies $\mathbb{P}(G_1) \geq 1 - \frac{\varepsilon}{5}$. We can also take N large enough so that if the event G_1 holds (which we assume from now on), then the $(\tilde{p}_i)_{0 \leq i \leq k}$ are sufficiently close to p_c (with respect to our particular choice of Λ) to allow us to apply Proposition 3.4 to each of them. In particular, this implies that in (5.24), the combined effect of $U_0, \dots, U_{i-1}$ on $\tilde{\alpha}_i$ (through $\tilde{p}_i$) is not very large: a multiplicative factor between β^{-1} and β, where β is as in Proposition 3.4.

By iterating the definition (5.20) of $(\tilde{p}_i)_{0 \leq i \leq k}$, we obtain

$$\frac{L(\tilde{p}_k)}{K_k} = \prod_{i=0}^{k-1} \tilde{\alpha}_i^{\delta_i}, \quad \text{where } \delta_i = \frac{1}{2}\left(\frac{96}{5}\right)^{k-i}. \tag{5.27}$$

We can now make the following key observation. For some given $U_0, \dots, U_{k-1}$, let us assume that we change exactly one of them, say U_i (in such a way that G_1 still holds), so that

(i) $\tilde{\alpha}_i$ is multiplied by a factor at least 2β,

(ii) then $\tilde{\alpha}_0, \dots, \tilde{\alpha}_{i-1}$ are not affected (indeed, they depend only on $U_0, \dots, U_{i-1}$),

(iii) and $\tilde{\alpha}_{i+1}, \dots, \tilde{\alpha}_{k-1}$ are each changed by a factor between β^{-1} and β (using the previous observation).

Since the exponent δ_i of $\tilde{\alpha}_i$ in (5.27) satisfies

$$\frac{\delta_i}{\sum_{j=i+1}^{k-1} \delta_j} \geq \frac{1}{\sum_{j=1}^{\infty} (\frac{5}{96})^j} = \frac{91}{5},$$

these three properties together imply that $L(\tilde{p}_k)$ gets multiplied by a factor at least $2^{48/5}$.

We use this observation to apply Lemma 5.11, as follows. First, we can choose $\delta > 0$ small enough so that for all $p > p_c$,

$$q(p, 1-\delta) \geq 2\beta \cdot q\left(p, \frac{1}{2}\right)$$

(using the lower bound in (3.3)). We also introduce a modification of $(U_i)_{0 \leq i \leq k-1}$: for $i \in \{0, \dots, k-1\}$,

$$\tilde{U}_i := \begin{cases} U_i - \frac{1}{2} & \text{if } U_i \geq 1 - \delta, \\ U_i & \text{otherwise.} \end{cases}$$

Note that in order to prove our result, we can condition on $(\tilde{U}_i)_{0\leq i\leq k-1}$, and prove that deconcentration holds in this case.

We can take k large enough so that with probability at least $\frac{\varepsilon}{5}$, the number l of indices $i \in \{0,\dots,k-1\}$ such that $\tilde{U}_i \in \left(\frac{1}{2}-\delta,\frac{1}{2}\right)$ (i.e. $U_i \in \left(\frac{1}{2}-\delta,\frac{1}{2}\right) \cup (1-\delta,1)$) is at least δk: let us call G_2 this event, and assume that it occurs. We can list the corresponding indices as $i_1,\dots,i_l$. We then define, for all $j \in \{1,\dots,l\}$,

$$Y_j := \begin{cases} 1 & \text{if } U_{i_j} \geq 1-\delta \text{ (so that } U_{i_j} = \tilde{U}_{i_j} + \frac{1}{2}\text{)}, \\ 0 & \text{otherwise (in this case, } U_{i_j} = \tilde{U}_{i_j}\text{)}. \end{cases}$$

Since we assumed $(\tilde{U}_i)_{0\leq i\leq k-1}$ to be given, (5.27) allows us to see $L(\tilde{p}_k)$ as a function $g(Y_1,\dots,Y_l)$, and the $(Y_j)_{1\leq j\leq l}$ are independent Bernoulli$(\frac{1}{2})$ distributed. We are thus in a position to apply Lemma 5.11, to the function

$$f(Y_1,\dots,Y_l) := \ln g(Y_1,\dots,Y_l)$$

(note that the previous observation ensures: for all $j \in \{1,\dots l\}$, $\nabla_j f \geq \ln(2^{48/5}) > 1$), and we obtain

$$\begin{aligned} \mathbb{P}\big(g(Y_1,\dots,Y_l) \in (y,\lambda y)\big) &= \mathbb{P}\big(f(Y_1,\dots,Y_l) \in (\ln y, \ln y + \ln \lambda)\big) \\ &\leq \frac{c}{l^{1/2}}(\ln \lambda + 1), \end{aligned}$$

where c is a universal constant. We deduce, using $l \geq \delta k$: for all $y \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}\big(L(\tilde{p}_k) \in (y,\lambda y)\big) &\leq \frac{c}{(\delta k)^{1/2}}(\ln \lambda + 1) + \mathbb{P}(G_1^c) + \mathbb{P}(G_2^c) \\ &\leq \frac{c}{(\delta k)^{1/2}}(\ln \lambda + 1) + \frac{\varepsilon}{5} + \frac{\varepsilon}{5}, \end{aligned}$$

which is smaller than ε for k large enough. This completes the proof of Proposition 5.14. □

By combining Proposition 5.14 with Lemma 5.10, we can deduce immediately a deconcentration result for $L(p_k^*)$ and $L(p_k^{**})$, which is used in the next section (and we can now forget about the chain $(\tilde{p}_i)_{0\leq i\leq k}$).

Corollary 5.15. *For all $\varepsilon > 0$ and $\lambda > 1$, there exists k_0 such that the following property holds. For all $k \geq k_0$, there exist $M_0, N_0 \geq 1$ such that: for all $N \geq N_0$, for all $p \in (p_c, q_{k+1}(N))$ with $L(p) \leq m_\infty(N)/M_0$,*

$$\sup_{y>0} \mathbb{P}\big(L(p_k^*) \in (y,\lambda y)\big) < \varepsilon \quad \textit{and} \quad \sup_{y>0} \mathbb{P}\big(L(p_k^{**}) \in (y,\lambda y)\big) < \varepsilon. \tag{5.28}$$

6 Frozen percolation in finite boxes

6.1 Iteration lemma

We establish now an iteration lemma for frozen percolation, that allows us to compare the real frozen percolation process to the chain (p_i^*), for which we proved deconcentration in Section 5.5.

Before stating the lemma, we need to indroduce some terminology. Let us consider $n \geq 1$, and a partition $\{1, \dots, n\} = I \sqcup J \sqcup K$. A function $f : x = (x_1, \dots, x_n) \in \mathbb{R}^n \mapsto f(x) \in \mathbb{R}$ is said to be *small when* $(x_j)_{j \in J}$ *are small and* $(x_k)_{k \in K}$ *are large* if: for all $\varepsilon > 0$, all $(x_i)_{i \in I}$, there exists $C > 1$ such that whenever all $j \in J$ satisfy $x_j < C^{-1}$ and all $k \in K$ satisfy $x_k > C$, one has $f(x) < \varepsilon$.

Lemma 6.1. *Let $l \geq 3$, $N \geq 1$, $K \in [m_2(N), m_l(N)]$, $c_2 > c_1 > \alpha > 0$, $\eta > 0$, and $\beta \in (0, \frac{1}{10})$. Further, let Λ be a simply-connected $(\alpha K, \eta)$-approximable set, with $B_{c_1 K} \subseteq \Lambda \subseteq B_{c_2 K}$. Let $0 < c_{1,new} < c_{2,new}$, and $K_{new} := \psi_N(K)$, i.e. such that*

$$c_\theta K^2 \pi_1(K_{new}) \simeq N, \tag{6.1}$$

where c_θ is the constant appearing in Proposition 2.8. Then there exist $\alpha_{new} > 0$, $\eta_{new} > 0$ (small if η is small, and N is large), $\beta_{new} > 0$ (small if β and η are small, and N is large), and a simply connected stopping set Λ_{new} such that with probability $> 1 - \varepsilon$ (where ε is small if $\eta, c_{1,new}, \beta$ are small, and $N, c_{2,new}$ are large), the following three properties hold.

(i) Λ_{new} is $(\alpha_{new} K_{new}, \eta_{new})$-approximable, and

$$B_{c_{1,new} K_{new}} \subseteq \Lambda_{new} \subseteq B_{c_{2,new} K_{new}}.$$

(ii) For every simply connected $\tilde{\Lambda}$ with $(\Lambda^{int(\alpha K)})_{(\beta)} \subseteq \tilde{\Lambda} \subseteq \Lambda$, the first freezing event for the frozen percolation process in $\tilde{\Lambda}$ leaves a hole $\tilde{\Lambda}^F$ around 0 that satisfies

$$(\Lambda_{new}^{int(\alpha_{new} K_{new})})_{(\beta_{new})} \subseteq \tilde{\Lambda}^F \subseteq \Lambda_{new}.$$

Moreover, if we consider the modified frozen percolation process in $\tilde{\Lambda}$, where clusters not touching $\partial \tilde{\Lambda}$ do not freeze (i.e. they keep growing even if their volume is $\geq N$), the first freezing event leaves the same hole $\tilde{\Lambda}^F$ mentioned above.

(iii) For each $\tilde{\Lambda}$ with $(\Lambda^{int(\alpha K)})_{(\beta)} \subseteq \tilde{\Lambda} \subseteq \Lambda$, if we define p^{**} by

$$\frac{L(p^{**})}{K_{new}} = \left(\frac{|\tilde{\Lambda}|}{K^2}\right)^{48/5}, \tag{6.2}$$

then $\tilde{\Lambda}^{**} := \mathcal{H}^{(\tilde{\Lambda})}(p^{**})$ satisfies

$$(\Lambda_{new}^{int(\alpha_{new}K_{new})})_{(\beta_{new})} \subseteq \tilde{\Lambda}^{**} \subseteq \Lambda_{new}$$

for N large enough.

This result makes rigorous the heuristic explanation from Section 5.2. Indeed, the quantity K represents a rough estimate for the diameter of Λ, and we prove that $K_{\text{new}} = \psi_N(K)$ corresponds to the next scale in the process, after the first freezing event occurs.

Proof of Lemma 6.1. (i) Let us consider $N, K, c_1, c_2, \alpha, \eta, \beta$ and Λ as in the statement, and also the associated constant K^{new}. Let $\varepsilon > 0$. Let us take some $\delta \in (0, \frac{1}{4})$, and define p^- and p^+ by

$$|\Lambda| \cdot \theta(p^-) = N(1-\delta), \tag{6.3}$$

and

$$\left|(\Lambda^{\text{int}(\alpha K)})_{(\beta)}\right| \cdot \theta(p^+) = N(1+\delta). \tag{6.4}$$

We start by making a few observations on the various scales involved.

- Since $|\Lambda| \asymp K^2$ and $\theta(p^-) \asymp \pi_1(L(p^-))$, it follows from the definitions of K_{new} (6.1) and p^- (6.3) that $\pi_1(K_{\text{new}}) \asymp \pi_1(L(p^-))$, and so

$$L(p^-) \asymp K_{\text{new}}. \tag{6.5}$$

- The assumption that $K \leq m_l(N)$ implies (using Lemma 5.3) that

$$K_{\text{new}} = \psi_N(K) \ll K \quad \text{as } N \to \infty. \tag{6.6}$$

- From the definitions of p^- (6.3) and p^+ (6.4), we can write

$$1 \geq \frac{\theta(p^-)}{\theta(p^+)} = \frac{1-\delta}{1+\delta} \cdot \frac{\left|(\Lambda^{\text{int}(\alpha K)})_{(\beta)}\right|}{|\Lambda|} \geq \frac{1-\delta}{1+\delta} \cdot (1-4\beta)(1-\eta), \tag{6.7}$$

using (3.12) and the $(\alpha K, \eta)$-approximability of Λ. Proposition 2.8 and Lemma 2.7 imply

$$\frac{\theta(p^-)}{\theta(p^+)} \le (1+\delta)\frac{\pi_1(L(p^-))}{\pi_1(L(p^+))} \le (1+\delta)^2 \left(\frac{L(p^-)}{L(p^+)}\right)^{-5/48} \tag{6.8}$$

if N is large enough so that p^- and p^+ are sufficiently close to p_c. We deduce

$$1 \le \frac{L(p^-)}{L(p^+)} \le \left[\frac{1-\delta}{(1+\delta)^3}(1-4\beta)(1-\eta)\right]^{-48/5}. \tag{6.9}$$

For any given $\beta', \eta' > 0$, Lemmas 3.6 and 3.7 imply that if δ, β, η are sufficiently small, and N is sufficiently large, then there exists $\alpha' > 0$ such that: with probability $> 1 - \varepsilon$,

- $\mathcal{H}(p^-)$ is $(\alpha' L(p^-), \eta')$-approximable (in particular, it contains the block $b_{\alpha' L(p^-)}$),
- and $(\mathcal{H}(p^-)^{\mathrm{int}(\alpha' L(p^-))})_{(\beta')} \subseteq \mathcal{H}(p^+)$ (using (6.9)).

Now, we can consider, for the percolation process with parameter p^- in Λ, the set

$$\Lambda_{\mathrm{new}} = \mathcal{H}^{(\Lambda)}(p^-)$$

(recall the definition in (4.1)), which is a stopping set. Moreover, it coincides with $\mathcal{H}(p^-)$ with probability $> 1 - \varepsilon$, since $L(p^-) \asymp K_{\mathrm{new}} \ll K$ (from (6.5) and (6.6)). The a-priori bounds from Lemma 3.2 imply the existence of $0 < c_3 < c_4$ such that: for N large enough,

$$B_{c_3 K_{\mathrm{new}}} \subseteq \mathcal{H}(p^-) \subseteq B_{c_4 K_{\mathrm{new}}} \tag{6.10}$$

with probability $> 1 - \varepsilon$ (using (6.5)). We also note that if $\mathcal{H}(p^-)$ is $(\alpha' L(p^-), \eta')$-approximable, then it is also $(\alpha_{\mathrm{new}} K_{\mathrm{new}}, \eta')$-approximable, with $\alpha_{\mathrm{new}} = \alpha' \frac{L(p^-)}{K_{\mathrm{new}}}$, and $\alpha_{\mathrm{new}} \asymp \alpha'$ (using again (6.5)). Hence, our choice of Λ_{new} satisfies the desired properties with probability $> 1 - 3\varepsilon$, if we choose α_{new} as indicated, $\beta_{\mathrm{new}} = \beta'$, $c_{1,\mathrm{new}} = c_3$, $c_{2,\mathrm{new}} = c_4$, and $\eta_{\mathrm{new}} = \eta'$, which completes the proof of (i).

(ii) Let us now turn to the second property, and consider a simply connected domain $\tilde{\Lambda}$ with $(\Lambda^{\mathrm{int}(\alpha K)})_{(\beta)} \subseteq \tilde{\Lambda} \subseteq \Lambda$. Since Λ is $(\alpha K, \eta)$-approximable, with probability $> 1 - \varepsilon$, the largest p^--black cluster in Λ has volume at most

$$(1+\varepsilon)\theta(p^-)|\Lambda| = (1+\varepsilon)(1-\delta)N$$

(using the definition of p^- (6.3)), which is $< N$ if ε is small enough so that $(1+\varepsilon)(1-\delta) < 1$. Indeed, we can apply Lemma 4.1 (i), for η sufficiently small and N sufficiently large (so that $\frac{L(p^-)}{\alpha K}$ is sufficiently small). Moreover, we know from the same lemma (part (ii)) that this cluster contains a p^--black circuit in $A_{\frac{1}{8}\alpha K, \frac{1}{4}\alpha K}$ which is connected to ∞ by a p^--black path (hence, the hole around 0 in this cluster is $\mathcal{H}(p^-)$).

Similarly, Lemma 4.1 (i) implies that, with probability $> 1-\varepsilon$, the largest p^+-black cluster in $(\Lambda^{\text{int}(\alpha K)})_{(\beta)}$ has volume at least

$$(1-\varepsilon)\theta(p^+)\left|(\Lambda^{\text{int}(\alpha K)})_{(\beta)}\right| = (1-\varepsilon)(1+\delta)N$$

(using the definition of p^+ (6.4)), which is $> N$ for ε sufficiently small. Also, the same lemma (part (ii)) ensures that this cluster has a p^+-black circuit in $A_{\frac{1}{8}\alpha K, \frac{1}{4}\alpha K}$ which is connected to ∞ by a p^+-black path. Note that this cluster also contains the previously mentioned p^--black circuit.

Now, let us introduce the time $\tilde{p}$ of the first freezing event in $\tilde{\Lambda}$. The previous observations directly imply that

$$p^- < \tilde{p} < p^+.$$

We consider $\mathcal{C}^{\max}_{(\Lambda^{\text{int}(\alpha K)})_{(\beta)}}$: since $p^- < \tilde{p}$, $\mathcal{C}^{\max}_{(\Lambda^{\text{int}(\alpha K)})_{(\beta)}}(\tilde{p})$ either contains $\mathcal{C}^{\max}_{(\Lambda^{\text{int}(\alpha K)})_{(\beta)}}(p^-)$, or it is disjoint from it. But this latter case cannot occur, since it would imply the existence of two $\tilde{p}$-black clusters with volume close to (or larger than) N, using again Lemma 4.1 (i), and hence

- either two p^+-black clusters with volume close to (or larger than) N,
- or one p^+-black cluster with volume close to (or larger than) $2N$,

which contradicts part (iii) of the same lemma. Hence, $\mathcal{C}^{\max}_{(\Lambda^{\text{int}(\alpha K)})_{(\beta)}}(\tilde{p}) \supseteq \mathcal{C}^{\max}_{(\Lambda^{\text{int}(\alpha K)})_{(\beta)}}(p^-)$, and this latter cluster also contains the previously-mentioned circuit (again by Lemma 4.1 (ii)). Finally, let us note that

$$\left|\mathcal{C}^{\max}_{\tilde{\Lambda}}(\tilde{p})\right| \geq \left|\mathcal{C}^{\max}_{(\Lambda^{\text{int}(\alpha K)})_{(\beta)}}(\tilde{p})\right|,$$

and these two clusters cannot be disjoint, for the same reasons as before. We deduce $\mathcal{C}^{\max}_{\tilde{\Lambda}}(\tilde{p}) \supseteq \mathcal{C}^{\max}_{(\Lambda^{\text{int}(\alpha K)})_{(\beta)}}(\tilde{p})$. In particular, $\mathcal{C}^{\max}_{\tilde{\Lambda}}(\tilde{p})$ contains the earlier p^--black circuit, which is connected by a p^--black path to ∞, so the freezing event in $\tilde{\Lambda}$ (at time $\tilde{p}$) leaves a hole $\tilde{\Lambda}^F$ around 0 which is contained in $\mathcal{H}(p^-)$, and contains $\mathcal{H}(p^+)$. This completes the proof of (ii), since $\mathcal{H}(p^-) = \Lambda_{\text{new}}$, and we know that

$$\mathcal{H}(p^+) \supseteq (\mathcal{H}(p^-)^{\text{int}(\alpha' L(p^-))})_{(\beta')} = (\Lambda_{\text{new}}{}^{\text{int}(\alpha_{\text{new}} K_{\text{new}})})_{(\beta_{\text{new}})} \tag{6.11}$$

(recall that $\beta' = \beta_{\text{new}}$).

(iii) Even if this step is similar to the previous one in flavor, it follows from completely different reasons (it is essentially a deterministic statement). Let us consider $\tilde{\Lambda}$ as in the statement. Since we have chosen $\Lambda_{\text{new}} = \mathcal{H}(p^-)$, and $(\Lambda_{\text{new}}^{\text{int}(\alpha_{\text{new}} K_{\text{new}})})_{(\beta_{\text{new}})} \subseteq \mathcal{H}(p^+)$ (from (6.11)), it suffices to prove that $p^- < p^{**} < p^+$.

It follows from Proposition 2.8 that

$$\pi_1(L(p^-)) < \left(1 + \frac{\delta}{2}\right)(c_\theta)^{-1}\theta(p^-)$$

for N large enough. We then obtain from the definitions of K_{new} (6.1) and p^- (6.3) that

$$\frac{\pi_1(L(p^-))}{\pi_1(K_{\text{new}})} < \left(1 - \frac{\delta}{2}\right)\frac{K^2}{|\Lambda|}.$$

This implies, with Lemma 2.7, that

$$\frac{L(p^-)}{K_{\text{new}}} > \left(\left(1 - \frac{\delta}{4}\right)\frac{K^2}{|\Lambda|}\right)^{-48/5} \tag{6.12}$$

for N large enough, which yields

$$\frac{L(p^-)}{K_{\text{new}}} > \left(\frac{|\Lambda|}{K^2}\right)^{48/5} \geq \left(\frac{|\tilde{\Lambda}|}{K^2}\right)^{48/5} = \frac{L(p^{**})}{K_{\text{new}}} \tag{6.13}$$

(using (6.2)). Hence, $p^- < p^{**}$.

In a completely similar way, we can get from the definitions of K_{new} (6.1) and p^+ (6.4), combined with Proposition 2.8 and Lemma 2.7, that

$$\frac{L(p^+)}{K_{\text{new}}} < \left(\frac{|(\Lambda^{\text{int}(\alpha K)})_{(\beta)}|}{K^2}\right)^{48/5} \leq \left(\frac{|\tilde{\Lambda}|}{K^2}\right)^{48/5} = \frac{L(p^{**})}{K_{\text{new}}}, \tag{6.14}$$

and so $p^+ > p^{**}$, which completes the proof of (iii). $\square$

6.2 Proof of Theorem 1.2

In this section, we establish Theorem 1.2, for frozen percolation in boxes of side length between $m_k(N)$ and $m_{k+1}(N)$, with k large. We actually prove the stronger result below, which is useful then to study the full-plane process, i.e. to derive Theorem 1.1, which we explain in the next section.

Theorem 6.2. *For all $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that: for all $c_2 > c_1 > \alpha > 0$, there exists $k_0 \geq 1$ such that for all $k \geq k_0$, the following property holds. For all sufficiently large N, all $K \in (m_{k+2}(N), m_{k+5}(N))$, and all simply connected $(\alpha K, \eta)$-approximable sets Λ with $B_{c_1 K} \subseteq \Lambda \subseteq B_{c_2 K}$, we have*

$$\mathbb{P}_N^{(\Lambda)}(0 \textit{ is frozen at time } 1) < \varepsilon.$$

Note that this result clearly implies Theorem 1.2.

Proof of Theorem 6.2. Let us consider the various chains associated with the domain Λ and the initial scale K, as explained in Section 5.3.

- $(\Lambda_i)_{0 \leq i \leq k}$ is the sequence of successive holes around 0 for the frozen percolation process in Λ, with $(p_i)_{0 \leq i \leq k}$ the corresponding freezing times.
- $(K_i)_{0 \leq i \leq k}$ is a deterministic sequence of scales.
- $(\Lambda_i^{**})_{0 \leq i \leq k}$ and $(p_i^{**})_{0 \leq i \leq k}$ are two sequences, of random sets and random times, respectively, that are used to approximate the real process.

Lemma 3.2 implies the existence of a constant $c > 0$ such that

$$\mathbb{P}\left(B_{c^{-1}L(p_k^{**})} \subseteq \Lambda_k^{**} \subseteq B_{cL(p_k^{**})}\right) > 1 - \varepsilon. \tag{6.15}$$

The deconcentration result for $L(p_k^{**})$ (Corollary 5.15) implies the following. For every $\kappa > 1$, we can find k_0 large enough so that: for all $k \geq k_0$, for all N sufficiently large (depending on k), with probability $> 1 - \varepsilon$,

$$L(p_k^{**}) \in \mathcal{I}_\kappa := \big(m_1(N), m_6(N)\big) \setminus \left(\bigcup_{i=1}^{6} \big(\kappa^{-1} m_i(N), \kappa m_i(N)\big)\right) \tag{6.16}$$

(i.e. $L(p_k^{**})$ is between $m_1(N)$ and $m_6(N)$, but at least a factor κ different from each of the exceptional scales $m_1(N), m_2(N), \ldots, m_6(N)$). Together with (6.15), this implies (for the same k, and for all N large enough):

$$\mathbb{P}\big(\exists L \in \mathcal{I}_\kappa \text{ such that } B_L \subseteq \Lambda_k^{**} \subseteq B_{\tilde{c}L}\big) > 1 - \varepsilon, \tag{6.17}$$

with $\tilde{c} = c^2$.

By applying iteratively Lemma 6.1, we can then compare the chain $(\Lambda_i)_{1 \le i \le k}$ with $(\Lambda_i^{**})_{1 \le i \le k}$. Indeed, we can condition successively on the Λ_i^{**} ($1 \le i \le k$) and treat them as deterministic sets, since they are stopping sets (note that the claims of Lemma 6.1 all concern the configuration inside, *not* outside, a given domain Λ). Hence, we obtain that for η sufficiently small: for all N large enough, with probability $> 1 - \varepsilon$, Λ_i is "close" to Λ_i^{**} for all $i = 1, \ldots, k$. In particular, with probability $> 1 - 2\varepsilon$, Λ_k satisfies the same property as Λ_k^{**} in (6.17), but with κ and $\tilde{c}$ replaced by $\frac{\kappa}{2}$ and $2\tilde{c}$, respectively.

We are now in a position to use the results from [34] about the behavior of frozen percolation in finite boxes, recalled in Section 5.2. More precisely, we can apply Proposition 5.8 (see also the reformulated version, just below it), corresponding to the case when we start between two consecutive exceptional scales, but away from them (with $m_1, \ldots, m_6$). Indeed, $\tilde{c}$ is a universal constant, and we can take κ as large as we want, which completes the proof. □

7 Full-plane process

7.1 Coupling with approximable domains

In this section, we explain how Theorem 6.2 can be used to prove the corresponding result for the full-plane process, namely Theorem 1.1. For that, we show Proposition 7.2, that allows us to couple the full-plane process to the chains introduced in Section 5.3, for finite domains.

The following notation is used repeatedly in this section. For $N \ge 1$ and $p \in (p_c, 1]$ with $L(p) \ge \sqrt{N}$, we set $\widehat{p} = \widehat{p}(p, N) \in (p_c, 1]$ to be the solution of

$$L(p)^2 \theta(\widehat{p}) = N, \tag{7.1}$$

and we extend this notation recursively by setting $\widehat{p}_1 = \widehat{p}$, and $\widehat{p}_k = \widehat{(\widehat{p}_{k-1})}$.

As we explained in the beginning of Section 5.2, we expect $\widehat{p}$ to be roughly the time when the first frozen cluster appears for the frozen percolation process in $B_{L(p)}$. Note that Lemma 5.3 can be rephrased in the following way.

Lemma 7.1. *There exist constants $c, \alpha > 0$ such that: for all $N \ge 1$ and $p \in (q_\infty, q_1]$,*

$$\frac{L(p)}{L(\widehat{p})} \ge c \left(\frac{m_\infty}{L(p)} \right)^\alpha.$$

For later use, we also note that there exist constants $\tilde{c}, \tilde{\alpha} > 0$ such that: for all $p > q_\infty$ for which $\widehat{\widehat{p}}$ is well-defined, we have

$$\frac{L(\widehat{p})}{L(\widehat{\widehat{p}})} \leq \tilde{c} \left(\frac{L(p)}{L(\widehat{p})} \right)^{\tilde{\alpha}}. \tag{7.2}$$

Indeed, we can write

$$\left(\frac{L(p)}{L(\widehat{p})} \right)^2 = \frac{\theta(\widehat{\widehat{p}})}{\theta(\widehat{p})} \geq c_1 \frac{\pi_1(L(\widehat{\widehat{p}}))}{\pi_1(L(\widehat{p}))} \geq c_2 \big(\pi_1(L(\widehat{p}), L(\widehat{\widehat{p}}))\big)^{-1} \geq c_3 \left(\frac{L(\widehat{p})}{L(\widehat{\widehat{p}})} \right)^{\alpha'},$$

using successively (7.1), (2.13), (2.6) and (2.8).

Proposition 7.2. *For all $\varepsilon, \eta > 0$, there exist $c_2 > c_1 > \alpha > 0$, $M_0 > 0$ and $N_0 \geq 1$ such that: for all $N \geq N_0$ and $p \in (p_c, q_3(N))$ with $L(p) \leq m_\infty(N)/M_0$, we can find a simply connected stopping set $\Lambda^\#$ that satisfies the following two properties with probability at least $1 - \varepsilon$.*

(i) $\Lambda^\# = \bar{\mathcal{H}}^{(\mathbb{T})}(p^\#)$ for some $p^\# \in (p, \widehat{\widehat{\widehat{p}}})$.

(ii) $\Lambda^\#$ is $(\alpha L(p^\#), \eta)$-approximable, with $B_{c_1 L(p^\#)} \subseteq \Lambda \subseteq B_{c_2 L(p^\#)}$.

Before proving this result in the next section, we explain how to combine it with Theorem 6.2 and obtain Theorem 1.1.

Proof of Theorem 1.1. Let us consider some $\varepsilon > 0$ arbitrary, and $\eta = \eta(\varepsilon) > 0$ associated with it by Theorem 6.2. For this choice of ε and η, Proposition 7.2 then produces $c_2 > c_1 > \alpha > 0$, $M_0 > 0$ and $N_0 \geq 1$. We know from Theorem 6.2 that for these specific values α, c_1 and c_2, we can find $k \geq 1$ and $N_1 \geq N_0$ large enough such that: for all $N \geq N_1$, all $K \in (m_{k+2}(N), m_{k+5}(N))$, and all simply connected $(\alpha K, \eta)$-approximable stopping sets Λ with $B_{c_1 K} \subseteq \Lambda \subseteq B_{c_2 K}$, we have

$$\mathbb{P}_N^{(\Lambda)}(0 \text{ is frozen at time } 1) < \varepsilon. \tag{7.3}$$

In particular, for $p = q_{k+5}(N)$, we have $L(p) \leq m_\infty(N)/M_0$ for all $N \geq N_2$ (for some $N_2 \geq N_1$ large enough), so Proposition 7.2 provides us with $\Lambda^\#$ and $p^\#$ which satisfy, with probability $> 1 - \varepsilon$:

(i) $\Lambda^\# = \bar{\mathcal{H}}^{(\mathbb{T})}(p^\#)$,

(ii) $K = L(p^\#) \in (m_{k+2}(N), m_{k+5}(N))$ (since $p^\# \in (p, \widehat{\widehat{\widehat{p}}})$),

(iii) and $\Lambda^{\#}$ is a simply connected $(\alpha K, \eta)$-approximable stopping set, with $B_{c_1 K} \subseteq \Lambda \subseteq B_{c_2 K}$.

For all $N \geq N_2$, we can thus obtain from (7.3) (with such a pair $(\Lambda^{\#}, K)$, using (ii) and (iii)) that

$$\mathbb{P}_N^{(\Lambda^{\#})}(0 \text{ is frozen at time } 1) < \varepsilon,$$

which finally completes the proof, since $\Lambda^{\#} = \bar{\mathcal{H}}^{(\mathbb{T})}(p^{\#})$:

$$\mathbb{P}_N^{(\mathbb{T})}(0 \text{ is frozen at time } 1) < 2\varepsilon.$$

□

7.2 Proof of Proposition 7.2

Proof of Proposition 7.2. Let us consider some $\varepsilon > 0$, and assume, without loss of generality, that $\varepsilon < 1$. We also consider some large constant $M > 0$, that we specify later, and $p > p_c$ such that $L(p) \leq m_\infty / M$. We use this control over $\frac{L(p)}{m_\infty}$ only via Lemma 7.1, which implies that the ratio $L(p)/L(\widehat{p})$ can be made arbitrarily large by choosing M large enough.

By (2.8), we can set $\mu = \mu(\varepsilon) \in (0, 1)$ small enough so that: for all $p > p_c$,

$$\mathbb{P}\big(\mathcal{O}^*(\mu L(p), L(p)) \cap \mathcal{O}(\mu^2 L(p), \mu L(p)) \text{ holds at time } p\big) \geq 1 - \frac{\varepsilon}{100}. \quad (7.4)$$

We denote the event in (7.4) by $E(p)$.

Step 1. Let us fix some large $K \geq 1$ (we explain later how to choose it). We first prove that soon after p (we have to wait for at most one freezing), we can find a time p^* when the hole of the origin is large compared to the correlation length $L(p^*)$ at that time. Intuitively, imagine a flea jumping on $(q_\infty, 1]$: when it is at position q, it jumps to $\widehat{q}$. If we take q_1 and q_2 so that $q_2 - q_1$ is smaller than the length of a jump of the flea in $[q_1, q_2]$, then no matter where the flea starts, it will not get close to both q_1 and q_2. That is, a frozen circuit surrounding the origin cannot emerge close to both times q_1 and q_2, which implies that at time q_1 or q_2, the hole of the origin is large compared to the correlation length. Let us turn to a precise proof.

We first introduce the outermost p-black circuit $\mathcal{C}$ in $A_{\mu^2 L(p), \mu L(p)}$, taking $\mathcal{C} = \partial B_{\mu^2 L(p)}$ if such a circuit does not exist. Let p' be the first time that a vertex on $\mathcal{C}$ freezes for the modified frozen percolation process in $\mathbb{T}$

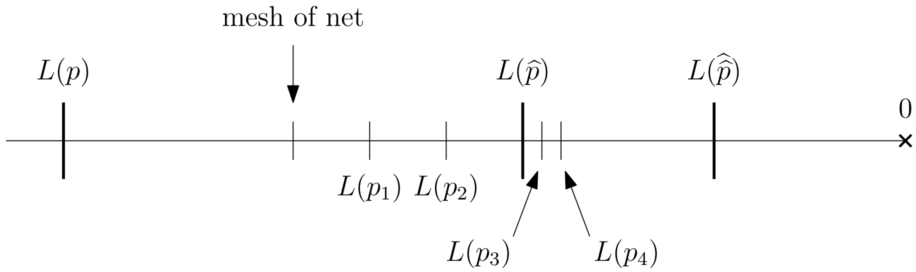

Figure 7.1: This figure presents, in a schematic way, the various scales involved in the proof of Proposition 7.2 (Steps 1 and 2).

where clusters still freeze as soon as they reach volume $\geq N$, *unless* they are included in $\mathrm{int}(\mathcal{C})$, in which case they keep growing as long as they do not contain a vertex of $\mathcal{C}$ (i.e. the clusters which are strictly inside $\mathcal{C}$ are allowed to grow after they reach volume N, until they intersect $\mathcal{C}$). Let us stress that this modified process is used only in this step and the next one (and not later in the proof), to ensure that p' has the right measurability property.

We define the (deterministic) times p_1 and p_2 by

$$L(p_1) = L(p)^{1/3} L(\widehat{p})^{2/3} \quad \text{and} \quad L(p_2) = L(p)^{1/6} L(\widehat{p})^{5/6}. \tag{7.5}$$

Note that they satisfy $p < p_1 < p_2 < \widehat{p}$ (since $L(\widehat{p}) < L(p)$). In a similar way, we also introduce, for $\beta = \frac{1}{2\tilde{\alpha}} > 0$ (where $\tilde{\alpha}$ is the universal constant from (7.2)), the times p_3 and p_4 such that

$$L(p_3) = L(\widehat{p})^{1-\beta} L(\widehat{\widehat{p}})^{\beta} \quad \text{and} \quad L(p_4) = L(\widehat{p})^{1-2\beta} L(\widehat{\widehat{p}})^{2\beta}, \tag{7.6}$$

which satisfy $\widehat{p} < p_3 < p_4 < \widehat{\widehat{p}}$. For the convenience of the reader, we summarize on Figure 7.1 the different scales that we use.

We now consider $\Lambda^* = \mathcal{H}^{(\overline{\mathrm{int}\,\mathcal{C}})}(p')$, and we distinguish two cases, depending on whether $p' \leq p_1$ or $p' > p_1$.

- Case a: if $p' \leq p_1$, we take $p^* = p_2$.
- Case b: if $p' > p_1$, we take $p^* = p_4$.

In this way, we have produced a pair (p^*, Λ^*) such that Λ^* is a stopping set: we can thus condition on it, and treat it as a deterministic set. Moreover, we can also condition on p^*, which leaves unaffected the configuration in Λ^*.

Step 2. We now prove that there exist M and N large enough such that with probability $\geq 1 - \frac{\varepsilon}{10}$, the following three properties hold (recall that M was introduced in the beginning of the proof, so that $L(p) \leq m_\infty / M$).

(i) $B_{KL(p^*)} \subseteq \Lambda^*$,

(ii) $\Lambda^* = \bar{\mathcal{H}}^{(\mathbb{T})}(p^*)$,

(iii) and $\Lambda^* \cap \mathcal{F}(p^*) = \emptyset$ (i.e. Λ^* does not contain any frozen cluster at time p^*).

Let us stress that in property (ii) above, the notation $\bar{\mathcal{H}}^{(\mathbb{T})}$ refers to the original frozen percolation process (not the modified one).

First, let us note that

$$\mathbb{P}_{p_2}\left(\left|\mathcal{C}^{\max}_{B_{L(p)}}\right| \geq N\right) = \mathbb{P}_{p_2}\left(\left|\mathcal{C}^{\max}_{B_{L(p)}}\right| \geq x L(p)^2 \theta(p_2)\right),$$

with

$$x = \frac{N}{L(p)^2 \theta(p_2)} = \frac{\theta(\widehat{p})}{\theta(p_2)} \geq c_1 \frac{\pi_1(L(\widehat{p}))}{\pi_1(L(p_2))} \geq c_2 \left(\frac{L(p)}{L(\widehat{p})}\right)^{\alpha/6}$$

(using successively the definition of $\widehat{p}$ (7.1), (2.13), (2.8), and the definition of p_2 (7.5)). By Lemma 7.1, this last lower bound can be made arbitrarily large by choosing M large enough, so there exist constants $M_1 = M_1(\varepsilon)$ and $N_1 = N_1(\varepsilon)$ such that: for all $M \geq M_1$ and $N \geq N_1$, x is large enough so that we can apply (4.4), and

$$\mathbb{P}_{p_2}\left(\left|\mathcal{C}^{\max}_{B_{L(p)}}\right| \geq N\right) \leq c_3 e^{-c_4 \left(\frac{L(p)}{L(\widehat{p})}\right)^{\alpha/6}} \leq \frac{\varepsilon}{100}. \tag{7.7}$$

We now assume that the event $E_1 := \{|\mathcal{C}^{\max}_{B_{L(p)}}(p_2)| < N\}$ occurs.

Let us assume that $E_2 := E(p)$ also holds, which has a probability at least $1 - \frac{\varepsilon}{100}$ (using (7.4)). In particular, it implies that $\mathcal{C}$ exists: we claim that at time p, neither $\mathcal{C}$ nor anything inside it is frozen, with high probability. For that, let us set

$$E_3 := \left\{\mathcal{F}(p) \cap \overline{\mathrm{int}(\mathcal{C})} = \emptyset\right\},$$

where $\overline{\mathrm{int}(\mathcal{C})} = \mathcal{C} \cup \mathrm{int}(\mathcal{C})$. Note that $\mathcal{C}$ is protected by the p-white circuit in $A_{\mu L(p), L(p)}$ from frozen clusters outside it: we thus have

$$\mathbb{P}_N(E_2 \cap E_3^c) \leq \mathbb{P}_p\left(\left|\mathcal{C}^{\max}_{B_{L(p)}}\right| \geq N\right) \leq \frac{\varepsilon}{100}$$

(using (7.7), since $p < p_2$).

It follows that $\mathbb{P}_N(E_2 \cap E_3) \geq 1 - 2 \cdot \frac{\varepsilon}{100}$: we now assume that this event holds, so that in particular $p' > p$, and we examine the two cases introduced earlier.

- **Case a:** $p < p' \leq p_1$ and $p^* = p_2$. First, we have

$$\mu^2 L(p) > L(p_1) > \mu L(p_1) > K L(p_2) \tag{7.8}$$

for all $M \geq M_2 = M_2(\varepsilon, K)$ and $N \geq N_2 = N_2(\varepsilon, K)$ (using again Lemma 7.1, and (7.5)). We know from (7.7) that no cluster with volume $\geq N$ can emerge before time p_2 in $\overline{\text{int}(\mathcal{C})}$. Hence, $\mathcal{C}$ freezes at time p' in the frozen percolation process (which coincides with the modified process). Moreover, let us assume that the event $E_4 := E(p_1)$ occurs: $\mathbb{P}(E_4) \geq 1 - \frac{\varepsilon}{100}$ (from (7.4)), and since $p' \leq p_1$, the white circuit in $A_{\mu L(p_1), L(p_1)}$ is also present at time p'. Hence, no vertex in $B_{\mu L(p_1)}$ can freeze at time p', and the freezing at time p' leaves a hole $\bar{\mathcal{H}}^{(\mathbb{T})}(p') \subseteq B_{L(p)}$ in which no cluster with volume $\geq N$ can emerge before time p_2. This implies that

$$\bar{\mathcal{H}}^{(\mathbb{T})}(p_2) = \bar{\mathcal{H}}^{(\mathbb{T})}(p') = \Lambda^* \supseteq B_{\mu L(p_1)}$$

on the intersection of the events above. Using (7.8), we obtain that

$$\mathbb{P}_N(B_{KL(p^*)} \nsubseteq \bar{\mathcal{H}}^{(\mathbb{T})}(p^*),\ p' \leq p_1) \leq 4 \cdot \frac{\varepsilon}{100}$$

for all $M \geq \max(M_1, M_2)$ and $N \geq \max(N_1, N_2)$. We have thus checked properties (i), (ii) and (iii) in this case.

- **Case b:** $p' > p_1$ and $p^* = p_4$. In this case, we use the intermediate scale

$$\lambda = L(p)^{1/2} L(\widehat{p})^{1/2}$$

(which, intuitively, corresponds to a time strictly between p and p_1), and the event

$$E_5 := \mathcal{N}_{p_1}(\lambda/4, L(p))$$

(recall Definition 2.1 for nets). We know from Lemma 2.2 that

$$\mathbb{P}(E_5) \geq 1 - C_1 \left(\frac{L(p)}{\lambda/4} \right)^2 e^{-C_2 \frac{\lambda/4}{L(p_1)}} \geq 1 - C_3 \frac{L(p)}{L(\widehat{p})} e^{-C_4 \left(\frac{L(p)}{L(\widehat{p})} \right)^{1/6}},$$

for some universal constants $C_i > 0$ ($1 \leq i \leq 4$). Hence, there exist $M_3 = M_3(\varepsilon)$ and $N_3 = N_3(\varepsilon)$ such that: for all $M \geq M_3$ and $N \geq N_3$,

$$\mathbb{P}(E_5) \geq 1 - \frac{\varepsilon}{100}$$

(using Lemma 7.1 once again). In particular, it implies that with a probability $\geq 1 - \frac{\varepsilon}{100}$, there exists a p_1-black net inside $\overline{\mathrm{int}(\mathcal{C})}$ which is connected to $\mathcal{C}$, and which leaves holes with diameter $\leq \lambda$. Let us denote by E_6 the event that such a net exists, and a cluster with volume $\geq N$ that is not connected to $\mathcal{C}$ emerges in the time interval $(p_1, p_4]$. Because of the existence of a net at time p_1, any such cluster has to appear in one of the $k \leq C_5 \big(\frac{L(p)}{\lambda}\big)^2$ holes, each having a diameter $\leq \lambda$. We deduce

$$\mathbb{P}(E_6) \leq C_5 \Big(\frac{L(p)}{\lambda}\Big)^2 \mathbb{P}_{p_4} \left(\left| \mathcal{C}_{B_\lambda}^{\max} \right| \geq N \right),$$

which is $\leq \frac{\varepsilon}{100}$ for all $M \geq M_4$ and $N \geq N_4$: indeed, we can proceed as for (7.7), as we explain now. For that, we write

$$\mathbb{P}_{p_4} \left(\left| \mathcal{C}_{B_\lambda}^{\max} \right| \geq N \right) = \mathbb{P}_{p_4} \left(\left| \mathcal{C}_{B_\lambda}^{\max} \right| \geq x \lambda^2 \theta(p_4) \right),$$

with

$$x = \frac{N}{\lambda^2 \theta(p_4)} = \frac{L(p)}{L(\widehat{p})} \cdot \frac{\theta(\widehat{p})}{\theta(p_4)},$$

and there exist universal constants $c_i > 0$ $(1 \leq i \leq 4)$ such that

$$\frac{\theta(\widehat{p})}{\theta(p_4)} \geq c_1 \frac{\pi_1(L(\widehat{p}))}{\pi_1(L(p_4))} \geq c_2 \pi_1(L(p_4), L(\widehat{p})) \geq c_3 \bigg(\frac{L(p_4)}{L(\widehat{p})} \bigg)^{1/2} = c_3 \bigg(\frac{L(\widehat{\widehat{p}})}{L(\widehat{p})} \bigg)^{\beta}$$

(using (2.13), (2.6), (2.8) and the definition of p_4 (7.6)), which yields

$$x \geq c_4 \frac{L(p)}{L(\widehat{p})} \cdot \bigg(\frac{L(p)}{L(\widehat{p})} \bigg)^{-\beta \tilde{\alpha}} = c_4 \bigg(\frac{L(p)}{L(\widehat{p})} \bigg)^{1/2}$$

(this follows from (7.2) and our particular choice of β): we are thus in a position to combine Lemma 7.1 and (4.4).

On the other hand, with high probability, something has to freeze before time p_3 in $\overline{\mathrm{int}(\mathcal{C})}$. Indeed, we know from Lemma 4.1 that

$$\mathbb{P}_{p_3} \left(\left| \mathcal{C}_{B_{\mu^2 L(p)}}^{\max} \right| \geq \Big(1 - \frac{\varepsilon}{100} \Big) \theta(p_3) \left| B_{\mu^2 L(p)} \right| \right) \geq 1 - \frac{\varepsilon}{100}$$

as soon as $\frac{L(p_3)}{\mu^2 L(p)}$ is small enough, and we have

$$\frac{(1 - \frac{\varepsilon}{100}) \theta(p_3) \left| B_{\mu^2 L(p)} \right|}{N} \geq C_6 \frac{\theta(p_3) \big(\mu^2 L(p)\big)^2}{N} = C_6 \mu^4 \frac{\theta(p_3)}{\theta(\widehat{p})}$$

for some universal constant $C_6 > 0$ (using the definition of $\widehat{p}$ (7.1)), which is ≥ 1 for all $M \geq M_5$ and $N \geq N_5$ (thanks to Lemma 7.1

again). Hence, the only possible scenario is as follows: the connected component that contains $\mathcal{C}$ and the net at time p_1 freezes at time $p' \leq p_3$, and when it freezes, it leaves holes in which no other clusters with volume $\geq N$ can emerge before time p_4. In particular, $\Lambda^* = \bar{\mathcal{H}}^{(\mathbb{T})}(p') = \bar{\mathcal{H}}^{(\mathbb{T})}(p^*)$. We can then conclude the claims announced in the beginning of Step 2 by using the event $E_7 = E(p_3)$, which has a probability $\geq 1 - \frac{\varepsilon}{100}$, and ensures that $B_{\mu L(p_3)} \subseteq \Lambda^*$: indeed, $\mu L(p_3) > KL(p_4)$ for all $M \geq M_6$ and $N \geq N_6$.

Step 3. We now use the big hole created at time p^*. We consider

- $\tilde{p}^+ := \inf\{t \geq p^* \ : \ $ there exists a t-black cluster in Λ^* which has $\geq N$ vertices, intersects $\partial B_{KL(p^*)/2}$, and contains a circuit surrounding 0 that is included in $B_{KL(p^*)/2}\}$,
- and $\Lambda^+ := \mathcal{H}^{(B_{KL(p^*)/2})}(\tilde{p}^+)$ (so that Λ^+ is the hole of the origin in the cluster from the definition of $\tilde{p}^+$).

By construction, Λ^+ is a stopping set, and we have to prove that it has the desired properties. Throughout the proof, we use the intermediate scale $\gamma = \sqrt{K} L(p^*)$. If we set

$$E_1 := \mathcal{N}_p(\gamma/4, KL(p^*)),$$

Lemma 2.2 implies that

$$\mathbb{P}(E_1) \geq 1 - C_1 K e^{-C_2 \sqrt{K}}$$

for some suitable universal constants $C_1, C_2 > 0$. In particular, there exists a constant $K_1 = K_1(\varepsilon)$ such that for all $K \geq K_1$, this lower bound is at least $1 - \frac{\varepsilon}{100}$. We now restrict ourselves to this event E_1.

There exists a p^*-black circuit in $A_{\gamma/2,\gamma}(2\gamma x)$ for each $x \in \mathcal{N}_K$, where

$$\mathcal{N}_K := \mathbb{Z}[i] \cap B_{KL(p^*)/4\gamma} = \mathbb{Z}[i] \cap B_{\sqrt{K}/4}.$$

Let $\mathcal{C}^x$ denote the outermost such circuit. Note that all these circuits are connected by p^*-black paths (inside the net).

For $x \in \mathcal{N}_K$, we consider

$$X_t^x = X_t^{\mathcal{C}^x} = \left|\{v \in \mathrm{int}(\mathcal{C}^x) \ : \ v \overset{t}{\leftrightarrow} \mathcal{C}^x\}\right|$$

(recall the notation from Section 4.3). In the following, we define several random times in terms of $(X_t^x)_{x \in \mathcal{N}_K}$, we thus restrict ourselves to the x for

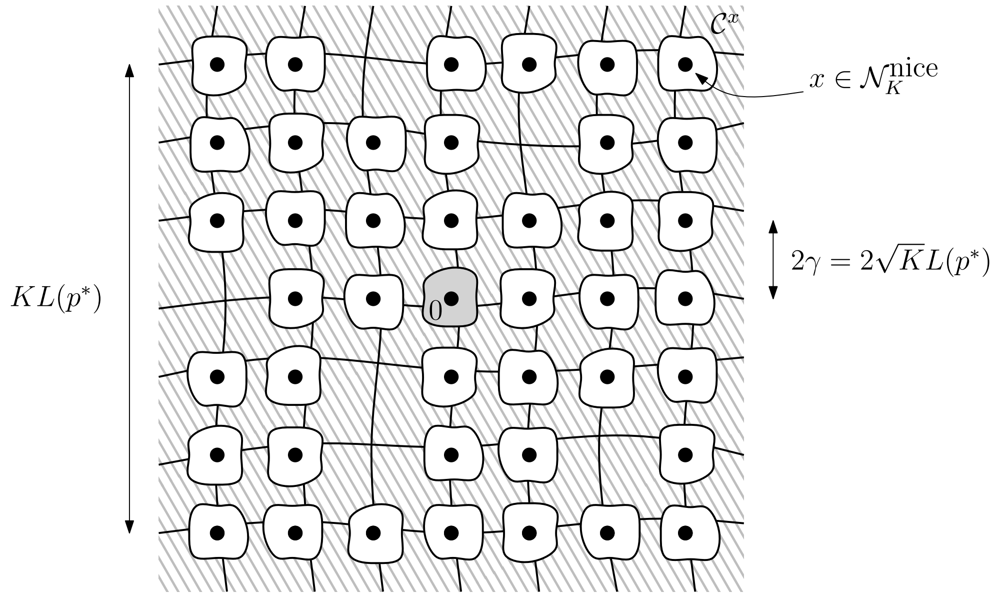

Figure 7.2: This figure depicts the construction used in the proof of Proposition 7.2. We consider the independent random variables $X_t^x = X_t^{\mathcal{C}^x}$, for $x \in \mathcal{N}_K^{\text{nice}}$ (i.e. such that the corresponding circuit $\mathcal{C}^x$ is nice), and we condition on the configuration outside: Y_t counts the number of vertices connected to at least one $(\mathcal{C}^x)_{x \in \mathcal{N}_K^{\text{nice}}}$ (including the vertices of the circuits themselves).

which we have a good control on the quantiles. More precisely, we know from Lemma 4.11 that there exists a constant $C_3 > 0$ such that for each $x \in \mathcal{N}_K$,

$$\mathbb{P}\big(\mathcal{C}^x \text{ is not } (p^*, C_3)\text{-nice}, \ E_1\big) \leq \frac{\varepsilon}{100}. \tag{7.9}$$

Note that the events

$$\big\{\mathcal{C}^x \text{ exists, and it is } (p^*, C_3)\text{-nice}\big\}$$

are independent, for $x \in \mathcal{N}_K$. We define the set

$$\mathcal{N}_K^{\text{nice}} := \{x \in \mathcal{N}_K \ : \ \mathcal{C}^x \text{ is } (p^*, C_3)\text{-nice}\}.$$

Further, we write

$$E_2 := \left\{0 \in \mathcal{N}_K^{\text{nice}}, \ |\mathcal{N}_K^{\text{nice}}| \geq \frac{K}{8}\right\} \quad \text{and} \quad E := E_1 \cap E_2.$$

Since $\varepsilon/100 < 1/2$, we deduce from Hoeffding's inequality and (7.9) the existence of $K_2 = K_2(\varepsilon)$ such that: for all $K \geq K_2$,

$$\mathbb{P}(E_1 \cap E_2^c) \leq \frac{\varepsilon}{100}.$$

Finally, let Y_t denote the number of vertices, in the frozen percolation process at time $t \geq p^*$ in Λ^*, which are either on one of the circuits $(\mathcal{C}^x)_{x \in \mathcal{N}_K^{\text{nice}}}$, or outside these circuits and connected to at least one of them. We set

$$p^+ := \inf \Big\{ t \geq p^* \ : \ \sum_{x \in \mathcal{N}_K^{\text{nice}}} X_t^x + Y_t \geq N \Big\}.$$

We also define the random times

$$\underline{p}^+ := \inf \Big\{ t \geq p^* \ : \ \overline{Q}_{\varepsilon/100}(X_t^0) + \sum_{x \in \mathcal{N}_K^{\text{nice}} \setminus \{0\}} X_t^x + Y_t \geq N \Big\} \quad (7.10)$$

$$\text{and} \quad \overline{p}^+ := \inf \Big\{ t \geq p^* \ : \ \underline{Q}_{\varepsilon/100}(X_t^0) + \sum_{x \in \mathcal{N}_K^{\text{nice}} \setminus \{0\}} X_t^x + Y_t \geq N \Big\}, \quad (7.11)$$

where we isolate X_t^0 by considering its quantiles (recall the notation for quantiles from Section 4.3). Further, let

$$\mathcal{S}_0 := V(\mathbb{T}) \setminus \text{int}(\mathcal{C}^0) \quad \text{and} \quad \mathcal{S}_1 := V(\mathbb{T}) \setminus \bigcup_{x \in \mathcal{N}_K^{\text{nice}}} \text{int}(\mathcal{C}^x).$$

For later use, we note that by definition,

- $(X_t^x)_{x \in \mathcal{N}_K^{\text{nice}} \setminus \{0\}}$ are measurable functions of $(\tau_v)_{v \in \mathcal{S}_0 \setminus \mathcal{S}_1}$,
- Y_t is a measurable function of $(\tau_v)_{v \in \mathcal{S}_1}$,
- and $\underline{p}^+$, $\overline{p}^+$ are measurable functions of $(\tau_v)_{v \in \mathcal{S}_0}$.

We condition on $\mathcal{S}_1$ and $(\tau_v)_{v \in \mathcal{S}_1}$ from now on. Under this conditioning, the function Y_t becomes deterministic, while the processes $(X_t^x)_{t \geq p}$ are independent for $x \in \mathcal{N}_K^{\text{nice}}$.

Step 4. We prove $p^* \leq \underline{p}^+ \leq p^+ \leq \overline{p}^+ \leq \widehat{p^*}$. For that, let us first introduce two rough bounds on p^+: we set

$$\underline{p} := \inf \Big\{ t \geq p^* \ : \ \overline{Q}_{\varepsilon/100}(X_t^0) + \overline{Q}_{\varepsilon/100}\Big(\sum_{x \in \mathcal{N}_K^{\text{nice}} \setminus \{0\}} X_t^x \Big) + Y_t \geq N \Big\},$$

and $\quad \overline{p} := \inf\left\{ t \geq p^* \;:\; \underline{Q}_{\varepsilon/100}(X_t^0) + \underline{Q}_{\varepsilon/100}\left(\sum_{x\in\mathcal{N}_K^{\text{nice}}\setminus\{0\}} X_t^x \right) + Y_t \geq N \right\}.$

Note that it follows from the definitions that $\underline{p}$, $\overline{p}$ are measurable functions of $(\tau_v)_{v\in\mathcal{S}_1}$.

Since we clearly have $p^* \leq \underline{p}$, we prove that $\underline{p} \leq \underline{p}^+ \leq p^+ \leq \overline{p}^+ \leq \overline{p}$ with probability at least $1-\varepsilon/5$, and we then show $\overline{p} \leq \widehat{p^*}$ separately. Out of the first four inequalities, we only prove that $p^+ \leq \overline{p}^+$ with probability at least $1-\varepsilon/20$, since the other inequalities can be established in a similar way. For that, we argue by contradiction, assuming that $p^+ > \overline{p}^+$: then,

$$\underline{Q}_{\varepsilon/100}(X_{\overline{p}^+}^0) + \sum_{x\in\mathcal{N}_K^{\text{nice}}\setminus\{0\}} X_{\overline{p}^+}^x + Y_{\overline{p}^+} \geq N > X_{\overline{p}^+}^0 + \sum_{x\in\mathcal{N}_K^{\text{nice}}\setminus\{0\}} X_{\overline{p}^+}^x + Y_{\overline{p}^+},$$

so in particular,

$$\underline{Q}_{\varepsilon/100}(X_{\overline{p}^+}^0) > X_{\overline{p}^+}^0.$$

Since the process $(X_t^0)_{t\geq p^*}$ is conditionally independent of $(\sum_{x\in\mathcal{N}_K^{\text{nice}}\setminus\{0\}} X_t^x + Y_t)_{t\geq p^*}$, and thus of $\overline{p}^+$, the above has a probability at most $\varepsilon/100$.

We now prove $\overline{p} \leq \widehat{p^*}$. For that, we set $\widehat{p}^{(K)}$ via

$$K^{3/2}L(p)^2\theta(\widehat{p}^{(K)}) = N. \tag{7.12}$$

For $K \geq 1$, the monotonicity of θ implies that $\widehat{p}^{(K)} \leq \widehat{p}$, it is thus enough to prove that

$$\text{for all } K \text{ large enough,} \quad \overline{p} \leq \widehat{p}^{(K)} \text{ with probability } \geq 1 - \frac{\varepsilon}{20}, \tag{7.13}$$

which we do now (this slightly stronger result is used in the next step). Let us also note that for some constants M_7 and N_7 depending only on K, we have: for all $p > p_c$ with $L(p) \leq m_\infty(N)/M_7$, and all $N \geq N_7$,

$$\widehat{p}^{(K)} \geq p^*. \tag{7.14}$$

Indeed, for every fixed K, it follows from (7.1) and (7.12) that $\theta(\widehat{p^*}) \asymp \theta(\widehat{p}^{(K)})$, so $L(\widehat{p^*}) \asymp L(\widehat{p}^{(K)})$ (by (2.13) and (2.8)), and we can use Lemma 7.1.

Recall that C_3 was chosen according to Lemma 4.11, and $\mathcal{C}^0$ is (p^*, C_3)-nice on E. Since $\mathcal{C}^0 \subseteq A_{\gamma/2,\gamma}$, with $\gamma = \sqrt{K}L(p^*)$, we obtain from Lemma 4.12 that for some $\underline{c}_3 > 0$,

$$\underline{Q}_{\varepsilon/100}(X_{\widehat{p}^{(K)}}^0) \geq \underline{c}_3(\sqrt{K}L(p^*))^2\theta(\widehat{p}^{(K)}),$$

and similarly,

$$\underline{Q}_{\varepsilon/100}\Bigg(\sum_{x\in\mathcal{N}_K^{\mathrm{nice}}\setminus\{0\}} X^x_{\widehat{p}^{(K)}}\Bigg) \geq \underline{c}_3|\mathcal{N}_K^{\mathrm{nice}}\setminus\{0\}|(\sqrt{K}L(p^*))^2\theta(\widehat{p}^{(K)}).$$

Since $|\mathcal{N}_K^{\mathrm{nice}}| \geq \frac{K}{8}$ on the event E, we deduce

$$\begin{aligned}\underline{Q}_{\varepsilon/100}(X^0_{\widehat{p}^{(K)}}) + \underline{Q}_{\varepsilon/100}\Bigg(\sum_{x\in\mathcal{N}_K^{\mathrm{nice}}\setminus\{0\}} X^x_{\widehat{p}^{(K)}}\Bigg) + Y_{\widehat{p}^{(K)}} &\geq \frac{\underline{c}_3}{8}K^2L(p^*)^2\theta(\widehat{p}^{(K)}),\\ &\geq \frac{\underline{c}_3}{8}K^{1/2}N\end{aligned}$$

(using (7.12)), which is $\geq N$ for all $K \geq K_3 = K_3(\varepsilon)$. Hence, we get that for all $K \geq K_3$, $\overline{p} \leq \widehat{p}^{(K)}$, which completes the proof of (7.13), and thus of Step 4.

Step 5. We show that with high probability,

(i) $\tilde{p}^+ = p^+$ (i.e. $\tilde{p}^+$ is the time when the structure consisting of the circuits $(\mathcal{C}^x)_{x\in\mathcal{N}_K^{\mathrm{nice}}}$ freezes),

(ii) $\Lambda^+ = \bar{\mathcal{H}}^{(\mathbb{T})}(p^+)$,

(iii) and $\mathcal{H}^{(B_{\gamma/8})}(\overline{p}^+) \subseteq \Lambda^+ \subseteq \mathcal{H}^{(B_{\gamma/8})}(\underline{p}^+)$.

Recall that $p^+ \leq \widehat{p}^{(K)}$ for all K large enough: in a similar way as in Step 2 (Case b), we see that if, apart from the net from Step 3, no other cluster intersecting $B_{KL(p^*)/2}$ reaches volume N before time $\widehat{p}^{(K)}$, then $\tilde{p}^+ = p^+$. Hence,

$$\mathbb{P}_N\big(\tilde{p}^+ \neq p^+,\, p^+ \leq \widehat{p}^{(K)},\, E\big) \leq K\cdot\mathbb{P}_{\widehat{p}^{(K)}}\Big(\big|\mathcal{C}^{\max}_{B_\gamma}\big| \geq N\Big). \tag{7.15}$$

It then follows from (4.4) (with $x = K^{1/2}$) and (7.12) that

$$\mathbb{P}_{\widehat{p}^{(K)}}\Big(\big|\mathcal{C}^{\max}_{B_\gamma}\big| \geq N\Big) \leq c_1 e^{-c_2K^{1/2}\frac{\gamma^2}{L(\widehat{p}^{(K)})^2}} = c_1e^{-c_2K^{3/2}\frac{L(p^*)^2}{L(\widehat{p}^{(K)})^2}} \leq c_1e^{-c_2K^{3/2}},$$

since $L(\widehat{p}^{(K)}) \leq L(p^*)$ (from (7.14)). The upper bound in (7.15) is thus $\leq \frac{\varepsilon}{100}$ for all $K \geq K_4(\varepsilon)$, which shows properties (i) and (ii). Since $\underline{p}^+ \leq p^+ \leq \overline{p}^+$, we also have

$$\mathcal{H}^{(B_{2\gamma})}(\overline{p}^+) \subseteq \Lambda^+ \subseteq \mathcal{H}^{(B_{2\gamma})}(\underline{p}^+).$$

Further, let

$$E_3 := \big\{\text{there is a } p^*\text{-black circuit } \mathcal{C} \text{ in } A_{\sqrt[4]{K}L(p^*),\sqrt{K}L(p^*)} \text{ s.t. } \mathcal{C} \overset{p^*}{\leftrightarrow} \infty\big\}.$$

We have that for all $K \geq K_5 = K_5(\varepsilon)$, $\mathbb{P}(E_3) \geq 1 - \frac{\varepsilon}{100}$ (from (2.4)), and E_3 implies in particular that

$$\text{for all } t \geq p^*, \quad \mathcal{H}^{(B_{2\gamma})}(t) = \mathcal{H}^{(B_{\gamma/8})}(t).$$

By using this observation at times $\overline{p}^+$ and $\underline{p}^+$, we finally get property (iii).

We are now almost in a position to conclude: indeed, note that since $\overline{p}^+$ and $\underline{p}^+$ are measurable functions of $(\tau_v)_{v\in\mathbb{T}\setminus B_{\gamma/2}}$, we can apply Lemma 3.7 to $\mathcal{H}^{(B_{\gamma/8})}(\overline{p}^+)$ and $\mathcal{H}^{(B_{\gamma/8})}(\underline{p}^+)$ to deduce that Λ^+ has the desired properties, if we know that $L(\underline{p}^+)/L(\overline{p}^+)$ can be made arbitrarily close to 1. Hence, there only remains to prove this property, which we do in a last step.

Step 6. We now fix an arbitrary $\delta > 0$, and we bound the probability of $\big\{\frac{L(\underline{p}^+)}{L(\overline{p}^+)} > 1 + \delta\big\}$. First, we show that for the rough lower and upper bounds $\underline{p}$ and $\overline{p}$, $L(\underline{p})$ and $L(\overline{p})$ are comparable. It follows from the definitions of $\underline{p}$ and $\overline{p}$ that

$$\lim_{t\nearrow\overline{p}} \Bigg(\underline{Q}_{\varepsilon/100}(X_t^0) + \underline{Q}_{\varepsilon/100}\Bigg(\sum_{x\in\mathcal{N}_K^{\text{nice}}\setminus\{0\}} X_t^x\Bigg) + Y_t\Bigg) \leq N$$
$$\leq \lim_{t\searrow\underline{p}} \Bigg(\overline{Q}_{\varepsilon/100}(X_t^0) + \overline{Q}_{\varepsilon/100}\Bigg(\sum_{x\in\mathcal{N}_K^{\text{nice}}\setminus\{0\}} X_t^x\Bigg) + Y_t\Bigg). \quad (7.16)$$

From the same reasoning as in the end of Step 3, we obtain that in the left-hand side of (7.16),

$$\lim_{t\nearrow\overline{p}} \Bigg(\underline{Q}_{\varepsilon/100}(X_t^0) + \underline{Q}_{\varepsilon/100}\Bigg(\sum_{x\in\mathcal{N}_K^{\text{nice}}\setminus\{0\}} X_t^x\Bigg)\Bigg) \geq \underline{c}_3|\mathcal{N}_K^{\text{nice}}|KL(p)^2\theta(\overline{p})$$

(using the continuity of θ at $\overline{p}$), and a similar upper bound holds for the right-hand side of (7.16), with $\underline{c}_3$ replaced by $\overline{c}_3$. These bounds, combined with (7.16), show that

$$\underline{c}_3|\mathcal{N}_K^{\text{nice}}|KL(p)^2\theta(\overline{p}) \leq \overline{c}_3|\mathcal{N}_K^{\text{nice}}|KL(p)^2\theta(\underline{p}). \quad (7.17)$$

Combined with (2.13), this shows the existence of a constant $C_4 = C_4(\varepsilon)$ such that

$$L(\underline{p}) \leq C_4 L(\overline{p}). \quad (7.18)$$

Now, let t_i be defined by

$$\text{for all } i \in \{0, \dots, 2\log_{1+\delta} C_4\}, \quad L(t_i) = (1+\delta)^{-i/2} L(\underline{p}).$$

We argue by contradiction: if we assume that $\frac{L(\underline{p}^+)}{L(\overline{p}^+)} > 1+\delta$, then there exists an $i \in \mathcal{I}_\delta := \{0, \dots, 2\log_{1+\delta} C_4 - 1\}$ for which $\underline{p}^+ < t_i$ and $\overline{p}^+ > t_{i+1}$ (using the rough bound $\underline{p} \le \underline{p}^+ \le \overline{p}^+ \le \overline{p}$). For this i, the definitions of $\underline{p}^+$ (7.10) and $\overline{p}^+$ (7.11) imply

$$\begin{aligned} \overline{Q}_{\varepsilon/100}(X^0_{t_i}) + & \sum_{x \in \mathcal{N}_K^{\text{nice}} \setminus \{0\}} X^x_{t_i} + Y_{t_i} \ge N \\ & > \underline{Q}_{\varepsilon/100}(X^0_{t_{i+1}}) + \sum_{x \in \mathcal{N}_K^{\text{nice}} \setminus \{0\}} X^x_{t_{i+1}} + Y_{t_{i+1}}, \end{aligned}$$

from which we deduce

$$\begin{aligned} \sum_{x \in \mathcal{N}_K^{\text{nice}} \setminus \{0\}} \big(X^x_{t_{i+1}} - X^x_{t_i}\big) &\le \big(Y_{t_i} - Y_{t_{i+1}}\big) + \big(\overline{Q}_{\varepsilon/100}(X^0_{t_i}) - \underline{Q}_{\varepsilon/100}(X^0_{t_{i+1}})\big) \\ &\le \overline{Q}_{\varepsilon/100}(X^0_{\overline{p}}) \\ &\le C_5 K L(p)^2 \theta(\overline{p}), \end{aligned} \tag{7.19}$$

for some $C_5 = C_5(\varepsilon) > 0$.

As it turns out, it is easier to work with a slightly different collection of random variables: we set

$$\begin{aligned} Z^x_i &:= \big|\{v \in B_{\gamma/10}(2\gamma x) \,:\, v \overset{t_{i+1}}{\leftrightarrow} \mathcal{C}^x,\ v \overset{t_i}{\not\leftrightarrow} \partial B_{\gamma/10}(v)\}\big| \\ &\le X^x_{t_{i+1}} - X^x_{t_i}, \end{aligned}$$

and it follows from (7.19) that it is enough to bound, for each $i \in \mathcal{I}_\delta$, the probability of

$$\sum_{x \in \mathcal{N}_K^{\text{nice}} \setminus \{0\}} Z^x_i \le C_5 K L(p)^2 \theta(\overline{p}).$$

Note that

$$\begin{aligned} \mathbb{E}\big[Z^x_i \,|\, \mathcal{C}^x\big] &\ge \sum_{v \in B_{\gamma/10}(2\gamma x)} \mathbb{P}\big(v \overset{t_{i+1}}{\leftrightarrow} \mathcal{C}^x,\ v \overset{t_i}{\not\leftrightarrow} \partial B_{\gamma/10}(v) \,|\, \mathcal{C}^x\big) \\ &\ge \frac{1}{100} K L(p)^2 \mathbb{P}\big(0 \overset{t_{i+1}}{\leftrightarrow} \infty,\ 0 \overset{t_i}{\not\leftrightarrow} \partial B_{\gamma/10}\big) \\ &\ge C_6 K L(p)^2 \frac{|t_{i+1} - t_i|}{|t_i - p_c|} \theta(t_i) \end{aligned} \tag{7.20}$$

$$\geq C_7 K L(p)^2 \theta(\overline{p}), \tag{7.21}$$

for some suitable universal constants C_6 and $C_7 = C_7(\delta)$ (using Lemma 2.3 in (7.20), and Lemma 2.19 in (7.21), combined with the definition of t_i).

Let us fix $i \in \mathcal{I}_\delta$, and consider $(Z_i^x)_{x \in \mathcal{N}_K^{\text{nice}} \setminus \{0\}}$. Since $Z_i^x \leq \mathcal{V}_{\gamma/10}(2\gamma x)$ (recall the definition of $\mathcal{V}_n$ in (4.5)), Lemma 4.6 provides the moment bound

$$\begin{aligned}\mathbb{E}\big[(Z_i^x)^m\big] \leq \mathbb{E}\big[(\mathcal{V}_{\gamma/10}(2\gamma x))^m\big] &\leq C_8^m m! (\gamma^2 \theta(t_{i+1}))^m \\ &\leq C_8^m m! (K L(p)^2 \theta(\overline{p}))^m\end{aligned}$$

for some universal constant $C_8 > 0$. This shows that (on the event E) we can apply Bernstein's inequality (Lemma 4.7) to the centered random variables $(Z_i^x - \mathbb{E}[Z_i^x])_{x \in \mathcal{N}_K^{\text{nice}} \setminus \{0\}}$, with

$$n = \big|\mathcal{N}_K^{\text{nice}} \setminus \{0\}\big| \asymp K, \quad M = C_9 K L(p)^2 \theta(\overline{p}), \quad \sigma_x^2 = M^2$$

(for some constant $C_9 = C_9(\delta)$ large enough), and

$$y = C_5 K L(p)^2 \theta(\overline{p}) - \mathbb{E}\Bigg[\sum_{x \in \mathcal{N}_K^{\text{nice}} \setminus \{0\}} Z_i^x\Bigg].$$

Noting that $|y| \asymp KM$ (since $\mathbb{E}[Z_i^x] \asymp M$ for every $x \in \mathcal{N}_K^{\text{nice}}$, from (7.21)), we obtain: for each $i \in \mathcal{I}_\delta$,

$$\mathbb{P}\Bigg(\sum_{x \in \mathcal{N}_K^{\text{nice}} \setminus \{0\}} Z_i^x \leq C_5 K L(p)^2 \theta(\overline{p})\Bigg) \leq 2 e^{-C_{10} K}$$

for some $C_{10} = C_{10}(\delta)$. Hence,

$$\begin{aligned}\mathbb{P}\bigg(\frac{L(p^+)}{L(\overline{p}^+)} > 1 + \delta\bigg) &\leq \mathbb{P}\Bigg(\exists i \in \mathcal{I}_\delta \ : \sum_{x \in \mathcal{N}_K^{\text{nice}} \setminus \{0\}} \big(X_{t_{i+1}}^x - X_{t_i}^x\big) \leq C_5 K L(p)^2 \theta(\overline{p})\Bigg) \\ &\leq \mathbb{P}\Bigg(\exists i \in \mathcal{I}_\delta \ : \sum_{x \in \mathcal{N}_K^{\text{nice}} \setminus \{0\}} Z_i^x \leq C_5 K L(p)^2 \theta(\overline{p})\Bigg) \\ &\leq 4 (\log_{1+\delta} C_4) e^{-C_{10} K}\end{aligned}$$

(using $|\mathcal{I}_\delta| \leq 2 \log_{1+\delta} C_4$), which is $\leq 1 - \varepsilon/100$ for all $K \geq K_6(\varepsilon)$. Hence, if we set $K = \max_{1 \leq i \leq 6} K_i$, and then $M = \max_{1 \leq i \leq 7} M_i(K, \varepsilon)$ and $N = \max_{1 \leq i \leq 7} N_i(K, \varepsilon)$, all the desired bounds hold, which completes the proof of Proposition 7.2.

□

7.3 Concluding remark: related processes

In this last section, we briefly and informally indicate the robustness of our methods, by considering some other interesting models for which a similar behavior as for volume-frozen percolation, and analogs of Theorems 1.1 and 1.2, can be expected. We discuss in particular two closely related processes (the proof of existence requires substantial work: see [8]). For these two processes, all vertices are initially white, and they can turn black according to some Poisson process of "births", with intensity 1. We also use a second, independent, Poisson process of "lightnings", with a small rate $\varepsilon_N > 0$: each vertex is hit by lightning at a rate ε_N, independently of the other vertices. To fix ideas, let us take $\varepsilon_N = N^{-\alpha}$, for some $\alpha > 0$.

We can first introduce a modified volume-frozen percolation process, where a black connected component freezes when one of its vertices is hit by lightning (so that the rate at which a cluster freezes is proportional to its volume). As a starting point, we can look for a similar separation of scales as in our previous volume-frozen percolation process. Here and further in this section, we make the usual translation $p(t) = 1 - e^{-t}$, and we define t_c as the solution of $p(t_c) = p_c$. We also write $L(t)$ for $L(p(t))$, and similarly for $\theta(t)$.

Heuristically, the recursion formula (7.1) should be replaced by

$$\varepsilon_N |\widehat{t} - t_c| L(t)^2 \theta(\widehat{t}) \asymp 1,$$

where $L(t)^2\theta(\widehat{t})$ corresponds to the volume of the "giant" connected component in a box with side length $L(t)$, and $\varepsilon_N|\widehat{t} - t|$ is the probability for any given vertex to be hit between times t and $\widehat{t}$, which we replace by $\varepsilon_N|\widehat{t} - t_c|$ (since we look for the property $|\widehat{t} - t_c| \gg |t - t_c|$).

A quick computation then yields a sequence of exceptional scales

$$m_k^{(\alpha)}(N) = N^{\delta_k^{(\alpha)} + o(1)} \quad \text{as } N \to \infty$$

(and corresponding times $q_k^{(\alpha)}(N) = t_c + N^{-\frac{3}{4}\delta_k^{(\alpha)} + o(1)}$), where the sequence of exponents $(\delta_k^{(\alpha)})_{k \geq 0}$ satisfies

$$\delta_0^{(\alpha)} = 0, \quad \text{and } \delta_{k+1}^{(\alpha)} = \frac{\alpha}{2} + \frac{41}{96}\delta_k^{(\alpha)} \quad (k \geq 0).$$

This sequence is strictly increasing, and it converges to $\delta_\infty^{(\alpha)} = \frac{48}{55}\alpha$. We then have a separation of scales, i.e. $L(\widehat{t}) \gg L(t)$, for all $t > t_c$ such that $L(t) \ll m_\infty^{(\alpha)}(N)$, as in Lemma 7.1 (where $m_\infty^{(\alpha)}(N) = N^{\delta_\infty^{(\alpha)} + o(1)}$ as $N \to \infty$).

In particular, we can consider a net with mesh $\gg L(\widehat{t})$ and $\ll L(t)$ at an intermediate time between t and $\widehat{t}$ (as we did, for instance, in Step 2 of the proof of Proposition 7.2), and the next freezing time coincides with the freezing time of this net (w.h.p.). In particular, the next hole looks like $\mathcal{H}(t^{\#})$, for some random $t^{\#}$ such that $|t^{\#} - t_c|$ is comparable to $|\widehat{t} - t_c|$.

We can also consider the forest fire process obtained from the same Poisson processes (of births and of lightnings), where a black connected component "burns", i.e. all its vertices become white, when one of its vertices is hit (and may later become black again according to the Poisson process of births). During a first non-trivial stage of the process (immediately after t_c), the sequence of holes should be approximately the same (as $N \to \infty$) as in the previous modified frozen percolation process. Indeed, the recent work [17] for self-destructive percolation [28] indicates that it takes a positive time $\delta > 0$ for macroscopic connections outside the hole to reappear, and the next burning event occurs much before that time. In particular, this suggests the existence (hinted in [29]) of a $\delta > 0$ for which: w.h.p. (as $N \to \infty$), the origin does not burn on the time interval $[0, t_c + \delta]$.

A Appendix: additional proofs

A.1 Proof of Lemma 2.4

Proof of Lemma 2.4. We consider Λ, p and p' as in the statement, and write

$$\theta(p') - \theta(p) = \mathbb{P}(\mathcal{B}),$$

where $\mathcal{B} := \{0 \overset{p'}{\leftrightarrow} \infty,\ 0 \overset{p}{\not\leftrightarrow} \infty\}$. Let us assume that this event occurs, which implies that there exists a p-white circuit surrounding 0, as well as a p'-black infinite path starting from 0. We can thus introduce the closest vertex v from the origin which lies on both a p-white circuit surrounding 0, and a p'-black path from 0 to ∞ (when there are multiple choices, we just pick one in some deterministic way). Note that locally around v, we see four disjoint arms: two p-white arms (coming from the p-white circuit), and two p'-black arms (from the p'-black path to ∞).

We now distinguish two cases, depending on the distance from v to the origin: we introduce the events

$$\mathcal{B}_1 := \{d(0, v) \leq L(p)\} \quad \text{and} \quad \mathcal{B}_2 := \{d(0, v) > L(p)\}.$$

We start by bounding the probability of $\mathcal{B}_1$. Let $i_{\max} := \lceil \log_2 L(p) \rceil$: by dividing the annulus $A_{1,L(p)}$ into the dyadic annuli $A_i = A_{2^{i-1},2^i}$ ($1 \leq i \leq$

$i_{\max}$), we obtain

$$\begin{aligned}
\mathbb{P}(\mathcal{B}_1) &= \sum_{i=1}^{i_{\max}} \mathbb{P}(v \in A_i) \\
&\leq |p'-p| \sum_{i=1}^{i_{\max}} |A_i| \mathbb{P}(0 \overset{p'}{\leftrightarrow} \partial B_{2^{i-2}}) \mathbb{P}(\mathcal{A}_4^{p',p}(1, 2^{i-1})) \mathbb{P}(\partial B_{2^{i+1}} \overset{p'}{\leftrightarrow} \infty) \\
&\leq C_1 |p'-p| \sum_{i=1}^{i_{\max}} 2^{2i+2} \pi_1(2^{i-2}) \pi_4(2^{i-1}) \pi_1(2^{i+1}, 4L(p)) \qquad \text{(A.1)} \\
&\leq C_2 |p'-p| \pi_1(L(p)) \sum_{i=0}^{i_{\max}-1} 2^{2i} \pi_4(2^i) \qquad \text{(A.2)} \\
&\leq C_3 |p'-p| L(p)^2 \pi_4(L(p)) \theta(p) \qquad \text{(A.3)}
\end{aligned}$$

for some constants $C_j = C_j(\Lambda) > 0$ ($j = 1, 2, 3$) (in (A.1) we used (2.9), in (A.2) we used a combination of (2.5), (2.6) and (2.13), while we used (2.11) in (A.3)).

Let us turn to $\mathbb{P}(\mathcal{B}_2)$. If we now divide $\mathbb{T} \setminus B_{L(p)}$ into the annuli $A'_i = A_{2^i L(p), 2^{i+1} L(p)}$ ($i \geq 0$), we obtain

$$\begin{aligned}
\mathbb{P}(\mathcal{B}_2) &= \sum_{i \geq 0} \mathbb{P}(v \in A'_i) \\
&\leq |p'-p| \sum_{i \geq 0} |A'_i| \mathbb{P}(0 \overset{p}{\leftrightarrow} \partial B_{L(p)/2}) \mathbb{P}(\mathcal{A}_4^{p,p'}(1, L(p)/2)) \\
&\qquad \mathbb{P}(\partial B_{L(p)/2}(v) \overset{p-\text{white}}{\leftrightarrow} \partial B_{2^i L(p)}(v)) \\
&\leq C_4 |p'-p| L(p)^2 \pi_4(L(p)/2) \theta(p) \sum_{i \geq 0} 2^{2i} \exp(-c_2 2^i) \qquad \text{(A.4)} \\
&\leq C_5 |p'-p| L(p)^2 \pi_4(L(p)) \theta(p) \sum_{i \ geq 0} 2^{2i} \exp(-c_2 2^i) \qquad \text{(A.5)} \\
&\leq C_6 |p'-p| L(p)^2 \pi_4(L(p)) \theta(p) \qquad \text{(A.6)}
\end{aligned}$$

for some constants $C_j = C_j(\Lambda)$ ($j = 4, 5, 6$), and c_2 as in (2.4) (in (A.4), we used (2.5) combined with (2.13) and (2.4), while we used (2.5) in (A.5)). Lemma 2.4 then follows, by combining (A.3) and (A.6). □

A.2 Proof of Lemma 4.1

We use the fact that $\chi^{\mathrm{fin}}(p) := \mathbb{E}_p\big[|\mathcal{C}(0)| \,;\, |\mathcal{C}(0)| < \infty\big]$ and $\chi^{\mathrm{cov}}(p) := \sum_{v \in \mathbb{T}} \mathrm{Cov}_p\big(\mathbb{1}_{0 \leftrightarrow \infty}, \mathbb{1}_{v \leftrightarrow \infty}\big)$ satisfy

$$\chi^{\mathrm{fin}}(p),\ \chi^{\mathrm{cov}}(p) \leq c_1 L(p)^2 \theta(p)^2 \tag{A.7}$$

for all $p > p_c$ (where $c_1 > 0$ is a universal constant), which is a consequence of (2.4) (see Section 6.4 in [4]).

Let us introduce some more notation, used only in this section. For a connected subset Λ of $\mathbb{T}$, the connected components inside Λ (at time p) can be listed by decreasing volume as $(\mathcal{C}^{(i)}_{\Lambda,\infty})_{i \geq 1}$ and $(\mathcal{C}^{(i)}_{\Lambda,<\infty})_{i \geq 1}$, according to whether they are included in the infinite cluster $\mathcal{C}_\infty(p)$ or not, respectively. Clearly, $\mathcal{C}^{\max}_{\Lambda}$ coincides with either $\mathcal{C}^{(1)}_{\Lambda,\infty}$ or $\mathcal{C}^{(1)}_{\Lambda,<\infty}$, so in particular $\big|\mathcal{C}^{\max}_{\Lambda}\big| \leq \big|\mathcal{C}^{(1)}_{\Lambda,\infty}\big| + \big|\mathcal{C}^{(1)}_{\Lambda,<\infty}\big|$. Note also that

$$\big|\mathcal{C}_\infty \cap \Lambda\big| = \sum_{i \geq 1} \big|\mathcal{C}^{(i)}_{\Lambda,\infty}\big|. \tag{A.8}$$

Lemma A.1. *For some universal constant $c_1 > 0$, we have*

(i) $\mathbb{E}_p\Big[\big|\mathcal{C}^{(1)}_{\Lambda,\infty}\big|\Big] \leq |\Lambda| \theta(p)$,

(ii) and $\mathbb{E}_p\Big[\big|\mathcal{C}^{(1)}_{\Lambda,<\infty}\big|\Big] \leq c_1 |\Lambda|^{1/2} L(p) \theta(p)$.

Proof of Lemma A.1. (i) It follows immediately from (A.8) that

$$\big|\mathcal{C}^{(1)}_{\Lambda,\infty}\big| \leq \big|\mathcal{C}_\infty \cap \Lambda\big| = \sum_{v \in \Lambda} \mathbb{1}_{v \leftrightarrow \infty},$$

and we can conclude by taking the expectation of both sides.

(ii) If we introduce $t_\Lambda := |\Lambda|^{1/2} L(p) \theta(p)$, we can write

$$\begin{aligned}
\mathbb{E}_p\Big[\big|\mathcal{C}^{(1)}_{\Lambda,<\infty}\big|\Big] &\leq t_\Lambda + \mathbb{E}_p\Big[\big|\mathcal{C}^{(1)}_{\Lambda,<\infty}\big| \,;\, \big|\mathcal{C}^{(1)}_{\Lambda,<\infty}\big| \geq t_\Lambda\Big] \\
&\leq t_\Lambda + \sum_{v \in \Lambda} \mathbb{P}_p\Big(\big|\mathcal{C}(v)\big| = \big|\mathcal{C}^{(1)}_{\Lambda,<\infty}\big|,\ \big|\mathcal{C}(v)\big| \geq t_\Lambda,\ v \not\leftrightarrow \infty\Big) \\
&\leq t_\Lambda + |\Lambda| \mathbb{P}_p\Big(\big|\mathcal{C}(0)\big| \geq t_\Lambda,\ 0 \not\leftrightarrow \infty\Big).
\end{aligned}$$

We can then conclude by noting that

$$\mathbb{P}_p\Big(\big|\mathcal{C}(0)\big| \geq t_\Lambda,\ 0 \not\leftrightarrow \infty\Big) \leq \frac{\chi^{\mathrm{fin}}(p)}{t_\Lambda} \leq \frac{c_1 L(p)^2 \theta(p)^2}{|\Lambda|^{1/2} L(p) \theta(p)},$$

using successively the definition of χ^{fin}, and (A.7). □

We are now in a position to prove the main lemma.

Proof of Lemma 4.1. First, we observe that Lemma A.1 implies that

$$\mathbb{E}_p\Big[\big|\mathcal{C}^{(1)}_{\Lambda,<\infty}\big|\Big] \leq c_1(|\Lambda|\theta(p))\frac{L(p)}{|\Lambda|^{1/2}},$$

and in both cases, $|\Lambda| \geq n^2$ (since Λ contains b_n), so

$$\frac{L(p)}{|\Lambda|^{1/2}} \leq \frac{L(p)}{n} \leq \mu.$$

Using Markov's inequality, we obtain that

$$\mathbb{P}_p\Big(\big|\mathcal{C}^{(1)}_{\Lambda,<\infty}\big| \geq \varepsilon|\Lambda|\theta(p)\Big) \leq \frac{c_1\mu}{\varepsilon^{-1}} \leq \frac{\varepsilon}{10}$$

for μ small enough. We can thus restrict our attention to $\mathcal{C}^{(1)}_{\Lambda,\infty}$ and $\mathcal{C}^{(2)}_{\Lambda,\infty}$.

Let us consider $\big|\mathcal{C}_\infty \cap \Lambda\big|$: we already noted that $\mathbb{E}_p\Big[\big|\mathcal{C}_\infty \cap \Lambda\big|\Big] = |\Lambda|\theta(p)$, and we have

$$\begin{aligned}\mathrm{Var}_p\Big(\big|\mathcal{C}_\infty \cap \Lambda\big|\Big) &= \sum_{v,w\in\Lambda} \mathrm{Cov}_p\big(\mathbb{1}_{v\leftrightarrow\infty}, \mathbb{1}_{w\leftrightarrow\infty}\big)\\ &\leq \sum_{v\in\Lambda}\sum_{w\in\mathbb{T}} \mathrm{Cov}_p\big(\mathbb{1}_{v\leftrightarrow\infty}, \mathbb{1}_{w\leftrightarrow\infty}\big) = |\Lambda|\chi^{\mathrm{cov}}(p).\end{aligned}$$

Using (A.7), we obtain

$$\mathrm{Var}_p\Big(\big|\mathcal{C}_\infty \cap \Lambda\big|\Big) \leq c_1(|\Lambda|\theta(p))^2\left(\frac{L(p)}{|\Lambda|^{1/2}}\right)^2, \tag{A.9}$$

and Chebyshev's inequality implies that for μ small enough,

$$\mathbb{P}_p\left(\left(1-\frac{\varepsilon}{10}\right)|\Lambda|\theta(p) \leq \big|\mathcal{C}_\infty \cap \Lambda\big| \leq \left(1+\frac{\varepsilon}{10}\right)|\Lambda|\theta(p)\right) \geq 1-\frac{\varepsilon}{10}, \tag{A.10}$$

which gives the desired upper bound for $\big|\mathcal{C}^{(1)}_{\Lambda,\infty}\big|$ (using (A.8)).

Now, we need to distinguish the two cases for Λ. We first consider $\Lambda = (\tilde{\Lambda})_{(\beta)}$, where $\beta \in (0, \frac{1}{3})$ and $\tilde{\Lambda}$ is a connected component of $\leq C$ n-blocks that contains b_n. We consider all the horizontal rectangles of the form $[i\mu^{1/2}n, (i+2)\mu^{1/2}n] \times [j\mu^{1/2}n, (j+1)\mu^{1/2}n]$, and all the vertical rectangles of the form $[i\mu^{1/2}n, (i+1)\mu^{1/2}n] \times [j\mu^{1/2}n, (j+2)\mu^{1/2}n]$ (i, j integers), which are entirely contained in Λ. The probability that all of them have a p-black crossing in the long direction is at least

$$1 - c_1(\mu^{-1/2})^2 C e^{-c_2 \mu^{1/2} n / L(p)} \geq 1 - c_1 C \mu^{-1} e^{-c_2 \mu^{-1/2}}$$

for some constants $c_1, c_2 > 0$ (using (2.4)), which is at least $1 - \frac{\varepsilon}{10}$ for μ small enough. Let us assume that it is indeed the case, so that the crossings form a net that covers the sub-domain $\Lambda' = (\tilde{\Lambda})_{(\beta + 3\mu^{1/2})}$. We note that all vertices in $\mathcal{C}_\infty \cap \Lambda'$ are connected by the net inside Λ, so that $\left|\mathcal{C}^{(1)}_{\Lambda,\infty}\right| \geq \left|\mathcal{C}_\infty \cap \Lambda'\right|$. Moreover, for the same reason as for (A.10), with probability at least $1 - \frac{\varepsilon}{10}$,

$$\left|\mathcal{C}_\infty \cap \Lambda'\right| \geq \left(1 - \frac{\varepsilon}{10}\right) |\Lambda'| \theta(p) \geq \left(1 - \frac{\varepsilon}{10}\right) (1 - 12 \cdot 3\mu^{1/2}) |\Lambda| \theta(p) \geq \left(1 - \frac{\varepsilon}{5}\right) |\Lambda| \theta(p) \tag{A.11}$$

(for μ small enough). This gives the desired lower bound for $\left|\mathcal{C}^{(1)}_{\Lambda,\infty}\right|$, and we can then get an upper bound on $\left|\mathcal{C}^{(2)}_{\Lambda,\infty}\right|$ from (A.8):

$$\left|\mathcal{C}^{(2)}_{\Lambda,\infty}\right| \leq \left|\mathcal{C}_\infty \cap \Lambda\right| - \left|\mathcal{C}^{(1)}_{\Lambda,\infty}\right| \leq \left(1 + \frac{\varepsilon}{10}\right) |\Lambda| \theta(p) - \left(1 - \frac{\varepsilon}{5}\right) |\Lambda| \theta(p)$$

with probability at least $1 - \frac{\varepsilon}{5}$ (using (A.10) and (A.11)). Finally, the net provides a circuit as desired, which is connected to ∞ with high probability (using once again (2.4)).

In the case when Λ is an $(n, \frac{\varepsilon}{2})$-approximable set with $B_n \subseteq \Lambda \subseteq B_{Cn}$, we proceed in the same way, by introducing $\Lambda' = (\Lambda^{\mathrm{int}(n)})_{(3\mu^{1/2})}$ (note that $\Lambda^{\mathrm{int}(n)}$ consists of at most C^2 n-blocks). □

A.3 Proof of Lemma 5.13

Proof of Lemma 5.13. Let us denote $x_i = \frac{p_i}{1 - p_i}$. For notational convenience, we identify ω with the subset $\{i \in \{1, \ldots, N\} : \omega_i = 1\}$. For every $S \subseteq \{1, \ldots, N\}$,

$$\mathbb{P}(\omega = S) = \prod_{i \in S} p_i \cdot \prod_{i \in S^c} (1 - p_i) = \prod_{i=1}^{N} (1 - p_i) \cdot \sigma_S,$$

with $\sigma_S := \prod_{i \in S} x_i$. Hence, we want to ensure that for every S with $|S| = n$,

$$\mathbb{P}(\tilde{w}_n = S) = \frac{\sigma_S}{\Sigma_n}, \quad \text{with } \Sigma_n = \sum_{\substack{S \subseteq \{1, \ldots, N\} \\ |S| = n}} \sigma_S$$

(where $\Sigma_0 = 1$ by convention). We claim that the desired coupling can be obtained with the following transition probabilities: for every S with $|S| = n < N$, every $j \in S^c$,

$$p_{S,S\cup\{j\}} = \frac{x_j}{\Sigma_{n+1}} \sum_{\substack{T:j\notin T\\|T|=n}} \frac{\sigma_T}{n+1-|S\cap T|} = \frac{1}{\Sigma_{n+1}} \sum_{\substack{T:j\in T\\|T|=n+1}} \frac{\sigma_T}{n+1-|S\cap T|}.$$

Since the summand in the last expression is $\leq \sigma_T$, it is clear that $p_{S,S\cup\{j\}} \in [0,1]$. One also has

$$\begin{aligned}\sum_{j\in S^c} p_{S,S\cup\{j\}} &= \frac{1}{\Sigma_{n+1}} \sum_{j\in S^c} \sum_{\substack{T:j\in T\\|T|=n+1}} \frac{\sigma_T}{n+1-|S\cap T|}\\ &= \frac{1}{\Sigma_{n+1}} \sum_{T:|T|=n+1} \sum_{j\in S^c\cap T} \frac{\sigma_T}{n+1-|S\cap T|}\\ &= \frac{\Sigma_{n+1}}{\Sigma_{n+1}} = 1,\end{aligned}$$

as desired. Finally, there only remains to check that for every $0 \leq i \leq N$, we obtain the right distribution for $\tilde{w}_i$. We proceed by induction over i: this clearly holds for $i = 0$, and let us assume that it holds for some $0 \leq i < N$. Then for every $T \subseteq \{1, \ldots, N\}$ with $|T| = i+1$,

$$\begin{aligned}\mathbb{P}(\tilde{w}_{i+1} = T) &= \sum_{j\in T} \mathbb{P}(\tilde{w}_i = T \setminus \{j\}) p_{T\setminus\{j\},T}\\ &= \sum_{j\in T} \frac{\sigma_{T\setminus\{j\}}}{\Sigma_i} \frac{x_j}{\Sigma_{i+1}} \sum_{\substack{U:j\notin U\\|U|=i}} \frac{\sigma_U}{i+1-|(T\setminus\{j\})\cap U|},\end{aligned}$$

using the induction hypothesis. Since $\sigma_{T\setminus\{j\}} x_j = \sigma_T$, we obtain

$$\begin{aligned}\mathbb{P}(\tilde{w}_{i+1} = T) &= \frac{\sigma_T}{\Sigma_{i+1}\Sigma_i} \sum_{U:|U|=i} \sum_{j\in T\cap U^c} \frac{\sigma_U}{i+1-|T\cap U|}\\ &= \frac{\sigma_T}{\Sigma_{i+1}\Sigma_i} \Sigma_i = \frac{\sigma_T}{\Sigma_{i+1}},\end{aligned}$$

which completes the proof of Lemma 5.13. □

Acknowledgements

We thank Gábor Pete for his help with Proposition 2.8. We also thank him and Christophe Garban for an early access to an expanded version of [11].

Part of this work was done at NYU Abu Dhabi, the Institut Henri Poincaré in Paris, and the Isaac Newton Institute in Cambridge, and we thank these institutions for their support and their hospitality.

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