Formations of monoids, congruences, and formal languages

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Abstract

The main goal in this paper is to use a dual equivalence in automata theory started in [RBBCL13] and developed in [BBCLR14] to prove a general version of the Eilenberg-type theorem presented in [BBPSE12]. Our principal results confirm the existence of a bijective correspondence between formations of (non-necessarily finite) monoids, that is, classes of monoids closed under taking epimorphic images and finite subdirect products, with formations of languages, which are classes of (non-necessarily regular) formal languages closed under coequational properties. Applications to non-r-disjunctive languages are given.

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1   Introduction

An important result in the algebraic study of formal languages and automata is Eilenberg's variety theorem establishing a one-to-one correspondence between varieties of regular languages, which are classes of regular languages closed under Boolean operations, derivatives, and preimages under monoid morphisms, and varieties of finite monoids, which are classes of finite monoids closed under finite products, submonoids, and homomorphic images. At the heart of this result lie the characterisation of varieties of regular languages by their syntactic monoids and the closure properties of the corresponding classes of finite monoids.

Several extensions of Eilenberg's theorem, obtained by replacing monoids by other algebraic structures or modifying closure properties on the definition of variety of languages, are known in the literature. In this context, we mention a local version of Eilenberg's theorem proved by Gehrke, Grigorieff, and Pin [GGP08] working with a fixed finite alphabet and considering only regular languages on it, and the extension of this result to the level of an abstract duality of categories by Adámek, Milius, Myers, and Urbat [AMMU14].

Another further step in this research programme is to replace varieties of finite monoids by the more general notion of formation, that is, a class of finite monoids closed under taking epimorphic images and finite subdirect products.

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Formations of finite groups are important for a better understanding of the structure of finite groups, and the more general notion of formation of algebraic structures, introduced and studied by Shemetkov and Skiba in [SS89], plays a central role in universal algebra. Therefore it seems quite natural to seek an Eilenberg type theorem establishing a connection between formations of finite monoids and formations of regular languages, which are classes of regular languages closed under Boolean operations and derivatives with a weaker property on the closure under inverse monoid morphism. This was established in [BBPSE12]. The weaker closure conditions for formations lead to more possibilities than for varieties as more general classes of languages can be described and understood.

Our principal aim here is to extend the main theorem of [BBPSE12] to the level of general monoids. Our results are motivated by the significant role played by formations of non-necessarily finite groups in the structural study of the groups and some interesting families of non-regular languages that have recently appeared in the literature.

The main contribution of this paper is an Eilenberg type theorem which bijectively relates formations of non-necessarily finite monoids with formations of non-necessarily regular languages. This result is the most general correspondence known to us. We use a completely different approach, based on a dual equivalence in automata started in [RBBCL13] and developed in [BBCLR14]. This dual equivalence relates two special classes of automata: on the one hand, the class of quotients of the initial automaton A^* with respect to a congruence relation $C \subseteq A^* \times A^*$; and on the other hand, the classes of preformations of languages, which are subautomata of the final automaton 2^{A^*} that are complete atomic Boolean algebras closed under derivatives. This result is ultimately based on the description of two important functors on automata, free and cofree, defined upon equations and coequations, respectively. The coalgebraic approach used in this result adds expressiveness to our treatment and it highlights the fundamental role of duality in algebraic automata theory. Furthermore, this dual equivalence generalises a recent line of work which uses a Stone-like duality as a tool for proving the correspondence between local varieties of regular languages and local pseudovarieties of monoids [GGP08]. This result is called *local* Eilenberg's theorem in [AMMU14].

Our approach depends heavily on the notion of a formation of congruences, which is an assignment that maps every alphabet A to a filter on the set of all congruences on A^* closed under taking kernels of monoid epimorphisms. We prove that there is a bijective correspondence between formations of monoids and formations of congruences (Theorem 20), and formations of languages are in a one-to-one correspondence with formations of congruences (Theorem 23).

We end the paper by showing an example of an application to relatively disjunctive languages. The generalised disjunctive languages have been considered by some authors in the literature, such as, Guo, Reis, and Thierrin ([GRT88, ZGS14, RS78]). A language L is relatively disjunctive if there exists a dense language intersecting finitely many times on each class of the syntactic congruence associated to L. It has been shown in [LSG08] that this condition is equivalent to L having a non relatively regular syntactic monoid, that is, a monoid not containing a finite ideal. In this paper, we prove that the set of all non-r-disjunctive languages is a formation of languages and consequently, we see that it is Boolean algebra closed under derivatives.

We have done our best to make the paper self-contained and so we present in Sections 2 and 3 the results on automata theory proved in [RBBCL13] and [BBCLR14] which are fundamental in the proofs of our main theorems. Section 4 covers various topics of formation theory. Section 5 contains our main results, Theorem 20 and Theorem 23. Finally, in Sec-

tion 6 we present an example of an application of our main theorems to relatively disjunctive languages.

## 2   Preliminaries

Some results on automata

An *automaton* is a pair (X, α) consisting of a possibly infinite set X of states and a transition function $\alpha \colon X \to X^A$, with inputs from an alphabet A. In pictures we use the following notation:

$$x \xrightarrow{\ a\ } y \quad \Leftrightarrow \quad \alpha(x)(a) = y$$
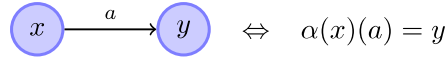
We also write $x_a = \alpha(x)(a)$ and, more generally,

$$x_\varepsilon = x, \qquad x_{wa} = \alpha(x_w)(a).$$

An automaton can have an *initial state* $x \in X$, here represented by a function $x \colon 1 \to X$, where $1 = \{0\}$. We call a triple (X, x, α) a *pointed* automaton. In pictures we use an entering arrow to indicate that a state is initial. An automaton can also be coloured by means of a colouring function $c \colon X \to 2$ using as set of colours $2 = \{0, 1\}$. We call a state x *accepting* (or *final*) if $c(x) = 1$. We call a triple (X, c, α) a *coloured* automaton. In pictures we use a double circle to indicate that a state is accepting. We call a 4-tuple (X, x, c, α) a *pointed and coloured* automaton. For instance, in the following automaton over $A = \{a, b\}$,
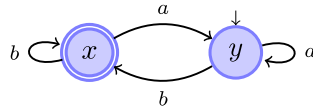
the state x is accepting and the state y is initial.

A function $h \colon X \to Y$ is a *homomorphism* between the automata (X, α) and (Y, β) if for each word $w \in A$, $h(x_w) = h(x)_w$. An *epimorphism* is a homomorphism that is surjective, a *monomorphism* is a homomorphism that is injective, and, finally, an *isomorphism* is homomorphism that is bijective. A homomorphism of pointed automata moreover respects initial states. Conversely, a homomorphism of coloured automata respects colours. If $X \subseteq Y$ and h is subset inclusion, then we call X a *subautomaton* of Y (respectively a *pointed* and a *coloured subautomaton*). For an automaton (X, α) and $x \in X$, the *subautomaton generated by* x, denoted by $\langle x \rangle \subseteq X$, is the smallest subset of X that contains x and is closed under transitions. We call a relation $R \subseteq X \times Y$ a *bisimulation of automata* if for all $(x, y) \in X \times Y$

$$(x, y) \in R \quad \Rightarrow \quad \forall a \in A, \ (x_a, y_a) \in R.$$

A bisimulation $E \subseteq X \times X$ is called a bisimulation *on* X. If E is an equivalence relation, then we call it a *bisimulation equivalence*. The quotient map of a bisimulation equivalence on X is an epimorphism of automata $q \colon X \to X/E$. For a homomorphism $h \colon X \to Y$, $\mathsf{ker}(h)$ is a bisimulation equivalence on X and $\mathsf{im}(h)$ is a subautomaton of Y and, moreover, $X/\mathsf{ker}(h)$ is isomorphic to $\mathsf{im}(h)$.

The set A^* forms a pointed automaton $(A^*, \varepsilon, \sigma)$ with initial state ε and transition function σ defined by concatenation, that is $\sigma(w)(a) = wa$. It is *initial* in the following sense: for

3

each given automaton (X, α) and every choice of initial state $x \colon 1 \to X$, it induces a unique homomorphism $r_x \colon (A^*, \sigma) \to (X, \alpha)$, given by $r_x(w) = x_w$, that makes the following diagram commute:

$$
\begin{array}{ccc}
1 & \xrightarrow{\;\;x\;\;} & \\
{\scriptstyle \varepsilon}\downarrow & & \searrow \\
(A^*, \sigma) & \xrightarrow[r_{\mathsf{x}}]{} & (X, \alpha)
\end{array}
$$

The function r_x maps a word w to the state x_w reached from the initial state x on input w and is therefore called the *reachability* map for (X, x, α).

The set 2^{A^*} of languages forms a coloured automaton $(2^{A^*}, \varepsilon?, \tau)$ with colouring function $\varepsilon?$ defined by $\varepsilon?(L) = 1$ if and only if $\varepsilon \in L$, and transition function τ defined by right derivation, that is $\tau(L)(a) = L_a$, where

$$
L_a = \{w \in A^* \mid aw \in L\}.
$$

Left derivation is defined analogously. It is *final* in the following sense: for each given automaton (X, α) and every choice of colouring function $c \colon X \to 2$, it induces a unique homomorphism $o_c \colon (X, \alpha) \to (2^{A^*}, \tau)$, given by $o_c(x) = \{w \in A^* \mid c(x_w) = 1\}$, that makes the following diagram commute:

$$
\begin{array}{ccc}
& \xrightarrow{\;\;c\;\;} & 2 \\
\nearrow & & \uparrow{\scriptstyle \varepsilon?} \\
(X, \alpha) & \xrightarrow[o_{\mathsf{c}}]{} & (2^{A^*}, \tau)
\end{array}
$$

The function o_c maps a state x to the language $o_c(x)$ accepted by x. Since the language $o_c(x)$ can be viewed as the observable behaviour of x, the function o_c is called the *observability* map for (X, c, α). Summarising, we have set the following scene:

$$
\begin{array}{ccccccc}
1 & \xrightarrow{\;\;x\;\;} & & & \xrightarrow{\;\;c\;\;} & 2 & \\
{\scriptstyle \varepsilon}\downarrow & & \searrow & \nearrow & & \uparrow{\scriptstyle \varepsilon?} & \\
(A^*, \sigma) & \xrightarrow[r_{\mathsf{x}}]{} & (X, \alpha) & & \xrightarrow[o_{\mathsf{c}}]{} & (2^{A^*}, \tau)
\end{array}
\tag{1}
$$

If the reachability map r_x is surjective, then we call (X, x, α) *reachable*. If the observability map o_c is injective, then we call (X, c, α) *observable*. And if r_x is surjective *and* o_c is injective, then we call (X, x, c, α) *minimal*.

Free and cofree automata

Definition 1. A *set of equations* is a bisimulation equivalence relation $E \subseteq A^* \times A^*$ on the automaton (A^*, σ). We define $(X, x, \alpha) \models E$—and say: *the pointed automaton (X, x, α) satisfies E*—by:

$$
(X, x, \alpha) \models E \quad \Leftrightarrow \quad \forall (v, w) \in E, \; x_v = x_w
$$

We define $(X, \alpha) \models E$—and say: *the automaton (X, α) satisfies E*—by:

$$
(X, \alpha) \models E \quad \Leftrightarrow \quad \forall x \colon 1 \to X, \; (X, x, \alpha) \models E
$$

4

We shall sometimes consider also *single* equations $(v, w) \in A^* \times A^*$ and use shorthands such as $(X, \alpha) \models v = w$ to denote $(X, \alpha) \models \mathbf{v=w}$ where $\mathbf{v=w}$ is defined as the smallest bisimulation equivalence on A^* containing (v, w).

Definition 2. A *set of coequations* is a subautomaton $D \subseteq 2^{A^*}$ of the automaton $(2^{A^*}, \tau)$. We say that *the coloured automaton* (X, c, α) *satisfies* D, written $(X, c, \alpha) \models D$, when for all $x \in X$, $o_c(x) \in D$. We say that *the automaton* (X, α) *satisfies* D, written $(X, \alpha) \models D$, if for all $c \colon X \to 2$, $(X, c, \alpha) \models D$.

Let (X, α) be an arbitrary automaton. We show how to construct an automaton that corresponds to the *largest set of equations* satisfied by (X, α). And dually, we construct an automaton that corresponds to the *smallest set of coequations* satisfied by (X, α).

Definition 3. Let $X = \{x_i\}_{i \in I}$ be the set of states of an automaton (X, α). We define a pointed automaton $\mathsf{free}(X, \alpha)$ in two steps as follows:

(i) First, we take the cartesian product of the pointed automata (X, x_i, α) that we obtain by letting the initial element x_i range over X. This yields a pointed automaton $(\Pi X, \overline{x}, \overline{\alpha})$ with
$$\Pi X = \prod_{x \colon 1 \to X} X_x$$
(where $X_x = X$), $\overline{x} = (x_i)_{i \in I}$, and $\overline{\alpha}$ defined component-wise.

(ii) Next we consider the reachability map $r_{\overline{x}} \colon A^* \to \Pi X$ and define:
$$\mathsf{Eq}(X, \alpha) = \mathsf{ker}(r_{\overline{x}}), \qquad \mathsf{free}(X, \alpha) = A^* / \mathsf{Eq}(X, \alpha).$$

This yields the pointed automaton $(\mathsf{free}(X, \alpha), [\varepsilon], [\sigma])$. Note that $\mathsf{free}(X, \alpha)$ is isomorphic to $\mathsf{im}(r_{\overline{x}})$.

Definition 4. Let $X = \{x_i\}_{i \in I}$ be the set of states of an automaton (X, α). We define a coloured automaton $\mathsf{cofree}(X, \alpha)$ in two steps as follows:

(i) First, we take the coproduct of the coloured automata (X, c, α) that we obtain by letting c range over the set of all maps $X \to 2$. This yields a coloured automaton $(\Sigma X, \hat{c}, \hat{\alpha})$ with
$$\Sigma X = \coprod_{c \colon X \to 2} X_c$$
(where $X_c = X$), and $\hat{c}$ and $\hat{\alpha}$ defined component-wise.

(ii) Next we consider the observability map $o_{\hat{c}} \colon \Sigma X \to 2$ and define:
$$\mathsf{coEq}(X, \alpha) = \mathsf{im}(o_{\hat{c}}), \qquad \mathsf{cofree}(X, \alpha) = \mathsf{coEq}(X, \alpha).$$

This yields the coloured automaton $(\mathsf{cofree}(X, \alpha), \varepsilon?, \tau)$. Note that $\mathsf{cofree}(X, \alpha)$ is isomorphic to $\Sigma X / \mathsf{ker}(o_{\hat{c}})$.

The automata $\mathsf{free}(X, \alpha)$ and $\mathsf{cofree}(X, \alpha)$ are *free* and *cofree* on (X, α), respectively, because of the following universal properties:

For every point $x\colon 1 \to X$, there exists a unique homomorphism from $\mathsf{free}(X,\alpha)$ to (X,α) given by the x-th projection. Dually, for every colouring $c\colon X \to 2$, there exists a unique homomorphism from (X,α) to $\mathsf{cofree}(X,\alpha)$, given by the c-th embedding.

Proposition 5. *The set* $\mathsf{Eq}(X,\alpha)$ *is the* largest *set of equations satisfied by* (X,α). *Dually,* $\mathsf{coEq}(X,\alpha)$ *is the* smallest *set of coequations satisfied by* (X,α).

Example 6. Consider the automaton (Z,γ) below:

The product over all its possible initial states is given by:

Hence $\mathsf{im}(r_{(x,y)})$ is the part reachable from (x,y). We know that $\mathsf{free}(Z,\gamma)$ is isomorphic to $\mathsf{im}(r_{(x,y)})$, which leads to the following isomorphic automaton:
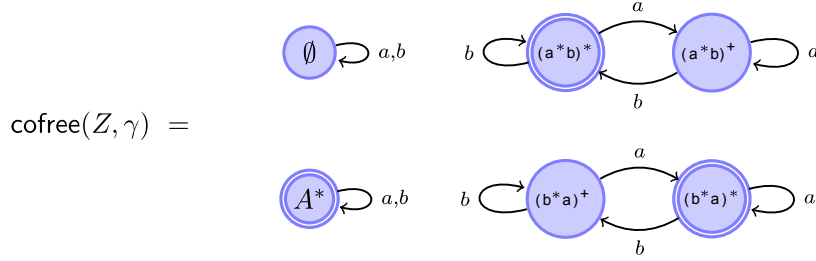

Since $\mathsf{free}(Z,\gamma) = A^*/\mathsf{Eq}(Z,\gamma)$, we can deduce from the above automaton that $\mathsf{Eq}(Z,\gamma)$ consists of $\mathsf{Eq}(Z,\gamma) = \{aa = a,\ bb = b,\ ab = b,\ ba = a\}$. The set $\mathsf{Eq}(Z,\gamma)$ is the largest set of equations satisfied by (Z,γ). Next we turn to coequations. The coproduct of all 4 coloured versions of (Z,γ) is

$$(\Sigma Z, \hat{c}, \hat{\gamma}) \;=\;$$
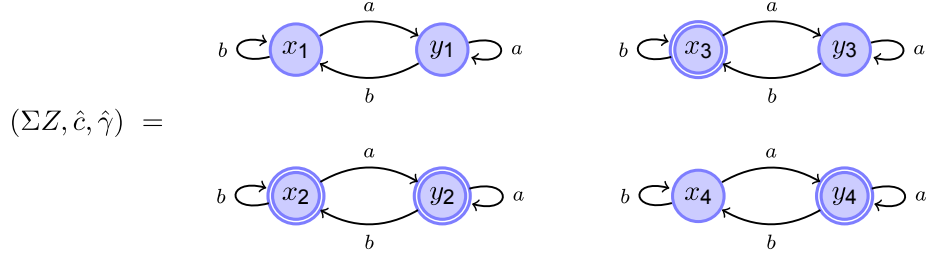
The observability map $o_{\natural} : \Sigma Z \to 2^{A^*}$ is given by

$o_{\natural}(x_1)$	$o_{\natural}(y_1)$	$o_{\natural}(x_2)$	$o_{\natural}(y_2)$	$o_{\natural}(x_3)$	$o_{\natural}(y_3)$	$o_{\natural}(x_4)$	$o_{\natural}(y_4)$
$\emptyset$	$\emptyset$	A^*	A^*	$(a^*b)^*$	$(a^*b)^+$	$(b^*a)^+$	$(b^*a)^*$

Since $\mathsf{cofree}(Z,\gamma) = \mathsf{im}(o_{\natural})$, this yields

$$\mathsf{cofree}(Z,\gamma) \;=\;$$

The set of states of this automaton is $\mathsf{coEq}(Z,\gamma)$, which is the smallest set of coequations satisfied by (Z,γ).

Summarising the present section, we have obtained, for every automaton (X,α), the following refinement of our previous scene (1):
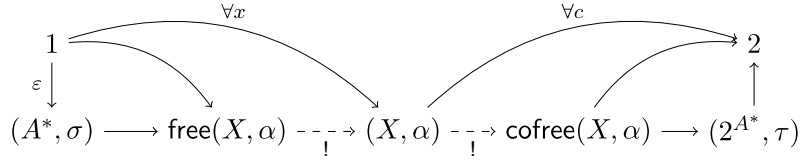
3   A dual equivalence

The purpose of this section is to see how the constructions of free and cofree can be regarded as functors between suitable categories. When we restrict them to certain subcategories, they form a dual equivalence. To this end, we denote:

$\mathcal{A}$: the category of automata (X,α) and automata homomorphisms,

$\mathcal{A}_m$: the category of automata (X,α) and automata monomorphisms,

$\mathcal{A}_e$: the category of automata (X,α) and automata epimorphisms.

If (X, α) and (Y, β) are two objects in $\mathcal{A}_m$ and m is a monomorphism between (X, α) and (Y, β), we have that $\mathsf{Eq}(Y, \beta)$ is contained in $\mathsf{Eq}(X, \alpha)$. This allows us to define a natural epimorphism $\mathsf{free}(m)$ from $\mathsf{free}(Y, \beta)$ to $\mathsf{free}(X, \alpha)$. Therefore $\mathsf{free}: \mathcal{A}_m \to (\mathcal{A}_e)^{\mathsf{op}}$ is a functor.

On the other hand, if e is an epimorphism from (X, α) to (Y, β), we have that $\mathsf{coEq}(Y, \beta) \subseteq \mathsf{coEq}(X, \alpha)$ and therefore the inclusion $\mathsf{cofree}(e)$ is a monomorphism from $\mathsf{cofree}(Y, \beta)$ to $\mathsf{cofree}(X, \alpha)$. Consequently, $\mathsf{cofree}: \mathcal{A}_m \to \mathcal{A}_e{}^{\mathsf{op}}$ is a functor.

Congruence quotients

A *right congruence* is an equivalence relation $E \subseteq A^* \times A^*$ such that, for all $(v, w) \in A^* \times A^*$, if $(v, w) \in E$, then, for all $u \in A^*$, $(vu, wu) \in E$. *Left congruences* are defined analogously. We call E a *congruence* if it is both a right and a left congruence. Note that E is a right congruence if and only if it is a bisimulation equivalence on (A^*, σ). Next we introduce the category $\mathcal{C}$ of *congruence quotients*, which is defined as follows:

$$\mathsf{objects}(\mathcal{C}) = \{(A^*/C, [\sigma]) \mid C \subseteq A^* \times A^* \text{ is a congruence relation}\}$$
$$\mathsf{arrows}(\mathcal{C}) = \{e: A^*/C \to A^*/D \mid e \text{ is an epimorphism of automata}\}$$

Note that $\mathcal{C}$ is a subcategory of $\mathcal{A}_e$.

Theorem 7. $\mathsf{free}(\mathcal{A}_m) = \mathcal{C}^{\mathsf{op}}$.

Preformations of languages

A *preformation of languages* is a set $V \subseteq 2^{A^*}$ such that:

(i) V is a complete atomic Boolean subalgebra of 2^{A^*},

(ii) for all $L \in 2^{A^*}$, if $L \in V$, then, for all $a \in A$, both $L_a \in V$ and $_aL \in V$.

We note that, being a subalgebra of 2^{A^*}, a preformation of languages V always contains both $\emptyset$ and A^*. This notion was previously introduced in [BBCLR14] as *varieties of languages*. In order to avoid any possible confusion we have decided to rename it. Next we introduce the category $\mathcal{V}$ of *preformations of languages*, as follows:

$$\mathsf{objects}(\mathcal{V}) = \{(V, \tau) \mid V \subseteq 2^{A^*} \text{ is a preformation of languages}\},$$
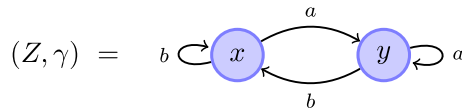$$\mathsf{arrows}(\mathcal{V}) = \{m: V \to W \mid m \text{ is an monomorphism of automata}\}.$$

Note that $\mathcal{V}$ is a subcategory of $\mathcal{A}_m$.
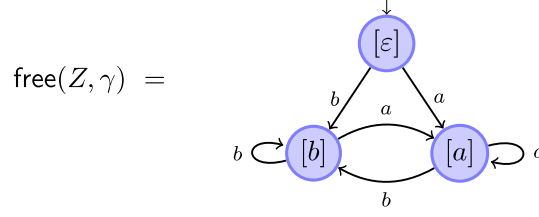
A dual equivalence

The following dual equivalence holds [BBCLR14]:

Theorem 8. $\mathsf{cofree}: \mathcal{C} \cong \mathcal{V}^{\mathsf{op}} : \mathsf{free}$

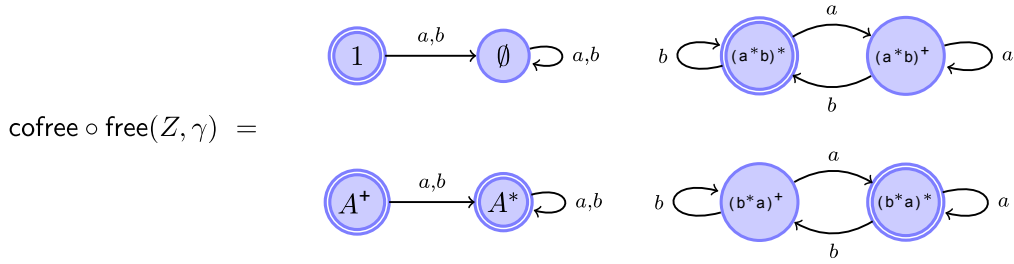**Example 9** (Example 6 continued)**.** Consider our previous example (Z, γ):

$$(Z, \gamma) \quad = $$

8

for which we had computed

$$\mathsf{free}(Z,\gamma) \;=\;$$

By Theorem 7, $\mathsf{free}(Z,\gamma)$ is a congruence quotient over A^*.

By a computation similar to the one done in Example 6, we obtain:

$$\mathsf{cofree} \circ \mathsf{free}(Z,\gamma) \;=\;$$

By Theorem 8, $\mathsf{cofree} \circ \mathsf{free}(Z,\gamma)$ is a preformation of languages. From the dual equivalence between these objects, if we apply free to the last automaton, we will obtain $\mathsf{free}(Z,\gamma)$ again.

In the proof of Theorem 8 one can see that for $w \in A^*$, the equivalence class $[w]$ for a given congruence C can be explicitly computed as the behaviour in $(A^*/C, [\sigma])$ of the initial state $[\varepsilon]$ under a given colouring. It implies that these classes, which are sets of words and, consequently, languages in 2^{A^*}, belong to $\mathsf{cofree}(A^*/C, [\sigma])$. In our running example:

$$[\varepsilon] = 1, \qquad [b] = (a^*b)^+, \qquad [a] = (b^*a)^+ \,.$$

On the converse, preformations of languages (V, τ) are complete atomic Boolean subalgebras of 2^{A^*} having as atoms the corresponding equivalence classes in $\mathsf{free}(V, \tau)$. We can represent such algebras by their Hasse diagrams (indicating language inclusion by edges). In our running example:

$$\mathsf{cofree} \circ \mathsf{free}(Z,\gamma) \;=\;$$
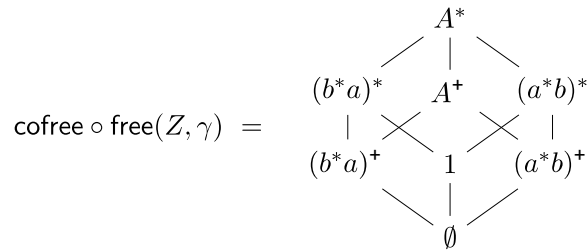
that is, forgetting all the automata structure, in finite objects we recover the classical dual equivalence:

$$\mathsf{powerset} \colon \mathsf{Set} \cong \mathsf{CABA}^{\mathsf{op}} \colon \mathsf{atoms}$$

Finally, the following picture includes an example of an epimorphism e and its image to illustrate the action of free and cofree on arrows:

A^*

$(b^*a)^*$ A^+ $(a^*b)^*$

$(b^*a)^+$ 1 $(a^*b)^+$

$\emptyset$

$[\varepsilon]$

b $[b]$ $[a]$ a

b

a a

b

$e \downarrow$

$\xmapsto{\text{cofree}}$

$\xleftarrow{\text{free}}$

$\uparrow m$

$[\varepsilon]$

a,b

$a,b \;\; [a]$

A^*

1 A^+

$\emptyset$
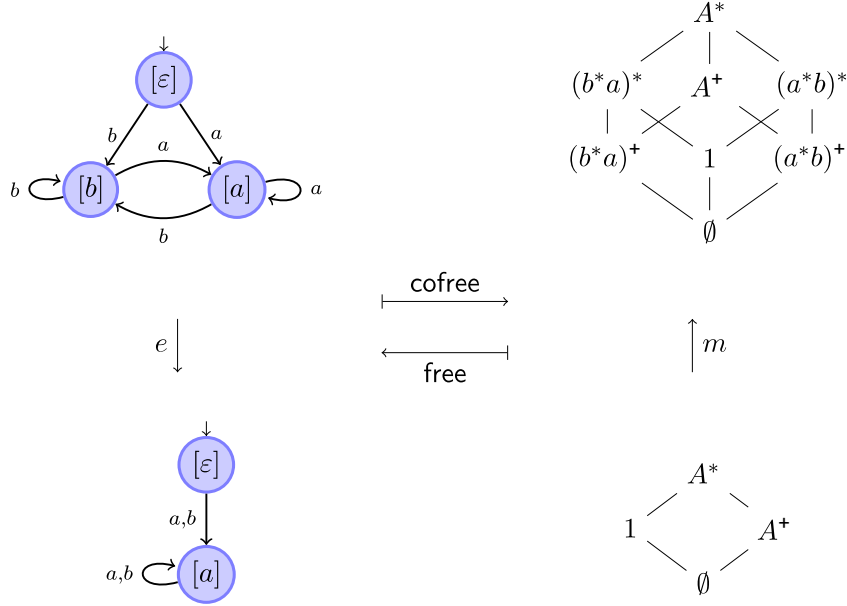
We end this section presenting a useful consequence of the dual equivalence. Here we denote by $\langle L \rangle$ the minimal automaton for a fixed language $L \in 2^{A^*}$.

Proposition 10. *For every congruence C in A^* and every language L in 2^{A^*},*

$$L \in \mathsf{coEq}(A^*/C, [\sigma]) \text{ if and only if } C \subseteq \mathsf{Eq}(\langle L \rangle, \tau).$$

4 Formations

In this section we introduce the notions of formations we will use. For the sake of simplicity, we use the abbreviations (A^*/C) instead of $(A^*/C, [\sigma])$ and $\langle L \rangle$ instead of $(\langle L \rangle, \tau)$.

Formations of monoids

Recall that a *variety of monoids* is a class of monoids $\mathbf{V}$ satisfying:

(i) every homomorphic image of a monoid of $\mathbf{V}$ belongs to $\mathbf{V}$,

(ii) every submonoid of a monoid of $\mathbf{V}$ belongs to $\mathbf{V}$,

(iii) the direct product of every family of monoids of $\mathbf{V}$ also belongs to $\mathbf{V}$.

A known theorem of Birkhoff states that varieties of monoids are equationally defined classes of monoids [Neu67].

Following [Gri95, p. 78], we say that a monoid M is a *subdirect product* of the product of a family of monoids $\{M_i \mid i \in I\}$ if M is a submonoid of the direct product $\prod_{i \in I} M_i$ and each induced projection π_i from M onto M_i is surjective. A monoid M which is isomorphic to such a submonoid P is also called a subdirect product of the monoids $\{M_i \mid i \in I\}$. In this case, the projections *separate* the elements of M, in the sense that, if $\pi_i(x) = \pi_i(y)$ for all $i \in I$, then $x = y$. In fact, we have:

Proposition 11 ([Gri95, Proposition 3.1]). *A monoid M is a subdirect product of a family of monoids $(M_i)_{i \in I}$ if and only if there is a family of surjective morphisms $(M \to M_i)_{i \in I}$ which separate the elements of M.*

Subdirect products allow us to introduce the notion of formation of monoids, which is a particular case of the most general notion of formation of algebraic structures, introduced and studied by Shemetkov and Skiba in [SS89]. A *formation of monoids* is a class of monoids **F** satisfying:

(i) every quotient of a monoid of **F** also belongs to **F**,

(ii) the subdirect product of a finite family of monoids of **F** also belongs to **F**.

Every variety of monoids is a formation of monoids, but the converse does not hold, as the class **Z** of monoids with zero shows. If a formation **F** is closed under taking subdirect products of arbitrary families of monoids, then **F** is a variety. This is a theorem of Kogalovskiĭ [Kog65] (see also [Neu67, Grä08]).

Theorem 12. *A class of monoids **F** is a variety if and only if it is closed under taking arbitrary subdirect products and quotients.*

Formations of congruences

Recall that a non-empty subset F of a partially ordered set P is called a *filter* if it satisfies:

(i) if $a, b \in F$, and the infimum $\inf\{a, b\}$ exists, then $\inf\{a, b\} \in F$;

(ii) if $a \in F$ and $a \leq b$, then $b \in F$.

If p is an element in P, the subset $[p) = \{q \in P \mid p \leq q\}$ is always a filter. A filter F is *principal* if it has the form $F = [p)$ for some element $p \in P$.

For a monoid M, the set of all congruences on M shall be denoted by $\mathsf{Con}(M)$. It has a natural order given by inclusion. The set $\mathsf{Con}(M)$ is bounded by the *total relation* on M, denoted by ∇_M, and by the *diagonal relation*, denoted by Δ_M. It is also closed under arbitrary intersections. For arbitrary joins, one takes the transitive closure of the union. It follows that $\mathsf{Con}(M)$ is a complete lattice. For a monoid M, its *residual* with respect to a formation of monoids **F**, written M_{F}, is defined as

$$M_{\mathsf{F}} = \bigcap \{C \in \mathsf{Con}(M) \mid M/C \in \mathbf{F}\}.$$

The above family is not empty as the total relation ∇_M is always included.

Proposition 13. *If **V** is a variety of monoids, for every monoid M, the quotient M/M_{V} is a monoid in **V**.*

Proof. Note that M/M_{V} is the subdirect product of the family of all quotients of M in **V**. Kogalovskiĭ's Theorem 12 guarantees us that this subdirect product is in **V**. $\square$

Remark. Proposition 13 is not true for formations in general. In fact, let **F** be the class of all finite monoids. The set $\mathsf{N} \cup \{0\}$ of all natural numbers and zero with the sum is a monoid whose residual with respect to **F** is the diagonal relation. However, $\mathsf{N} \cup \{0\}$ is not finite.

A *formation of congruences* F is an assignment of a family of congruences to every set A satisfying:

(i) for each set A, the set $\mathsf{F}(A)$ is a filter in $\mathsf{Con}(A^*)$,

(ii) for every two sets A and B, and for every congruence $E \in \mathsf{F}(B)$ with quotient morphism $\eta \colon B^* \to B^*/E$, if there exists a monoid homomorphism $\varphi \colon A^* \to B^*$ such that the composition $\eta \circ \varphi \colon A^* \to B^*/E$ is a surjective monoid homomorphism, then $\mathsf{ker}(\eta \circ \varphi)$ is a congruence in $\mathsf{F}(A)$. It can be depicted as follows:

$$
\begin{array}{ccc}
A^* & \twoheadrightarrow & A^*/\mathsf{ker}(\eta \circ \varphi) \\
\varphi \downarrow & \searrow & \vdots \\
B^* & \xrightarrow{\ \eta\ } & B^*/E
\end{array}
$$

We shall also need an important consequence of the universal property of the free monoid (see [Pin86, p. 10]).

Proposition 14. *Let $\gamma \colon A^* \to Q$ be a morphism and $\eta \colon P \to Q$ be a surjective monoid morphism, then there exists a monoid morphism $\varphi \colon A^* \to P$ with $\eta \circ \varphi = \gamma$.*

$$
\begin{array}{ccc}
A^* & & \\
\varphi \Big\downarrow & \searrow^{\gamma} & \\
P & \xrightarrow{\ \eta\ } & Q
\end{array}
$$

Formations of languages

Finally, a *formation of languages* $\mathcal{F}$ is an assignment to every alphabet A of a family of formal languages satisfying:

(i) for each alphabet A, if L is a language in $\mathcal{F}(A)$, then $\mathsf{coEq}(A^*/\mathsf{Eq}(\langle L \rangle)$ is included in $\mathcal{F}(A)$;

(ii) for each alphabet A, if both $\mathsf{coEq}(A^*/C_1)$, $\mathsf{coEq}(A^*/C_2)$ are included in $\mathcal{F}(A)$, then so is $\mathsf{coEq}(A^*/C_1 \cap C_2)$;

(iii) for every two alphabets A and B, if L is a language in $\mathcal{F}(B)$ and $\eta \colon B^* \to \mathsf{free}(\langle L \rangle)$ denotes the quotient morphism, then for each monoid morphism $\varphi \colon A^* \to B^*$ such that $\eta \circ \varphi$ is surjective, the set $\mathsf{coEq}(A^*/\mathsf{ker}(\eta \circ \varphi))$ belongs to $\mathcal{F}(A)$.

In [BBPSE12], the authors gave a different definition of formation of languages. We reproduce their definition to see that, for regular languages, both definitions coincide. In order to avoid confusion, we will rename that concept. Hence, a *r-formation of languages* $\mathcal{R}$ is an assignment to every alphabet A of a family of languages over A satisfying:

(F1) for each alphabet A, $\mathcal{R}(A^*)$ is closed under Boolean operations and derivatives;

(F2) for every two alphabets A and B, if L is a language in $\mathcal{R}(B)$ and $\eta \colon B^* \to \mathsf{free}(\langle L \rangle)$ denotes the quotient morphism, then for each monoid morphism $\varphi \colon A^* \to B^*$ such that $\eta \circ \varphi$ is surjective, the language $\varphi^{-1}(L)$ belongs to $\mathcal{R}(A)$.

Proposition 15. *If $\mathcal{F}$ is a formation of languages, then $\mathcal{F}$ is an r-formation of languages.*

Proof. Let L be a language in $\mathcal{F}(A)$, then $\mathsf{coEq}(A^*/\mathsf{Eq}(\langle L \rangle))$ is included in $\mathcal{F}(A)$. As $\mathsf{coEq}(A^*/\mathsf{Eq}(\langle L \rangle))$ is a preformation of languages containing L, then the complement and every derivative of L belong to it. Let L_1 and L_2 be two languages in in $\mathcal{F}(A)$, then $\mathsf{coEq}(A^*/\mathsf{Eq}(\langle L_1 \rangle))$ and $\mathsf{coEq}(A^*/\mathsf{Eq}(\langle L_2 \rangle))$ are included in $\mathcal{F}(A)$. It follows that

$$D = \mathsf{coEq}(A^*/[\mathsf{Eq}(\langle L_1 \rangle) \cap \mathsf{Eq}(\langle L_2 \rangle)])$$

is also included in $\mathcal{F}(A)$. By Proposition 10, we have that L_1 and L_2 are languages in D, which is a preformation of languages, then $L_1 \cap L_2$ and $L_1 \cup L_2$ are languages in D.

Now, consider two alphabets A and B, and let L be a language in $\mathcal{F}(B)$. Let η denote the quotient morphism $\eta \colon B^* \to \mathsf{free}(\langle L \rangle, \tau)$ and let $\varphi \colon A^* \to B^*$ be a monoid morphism such that $\eta \circ \varphi$ is surjective. Then $\mathsf{coEq}(A^*/\mathsf{ker}(\eta \circ \varphi))$ is included in $\mathcal{F}(A)$. Let L' be a language in $\langle \varphi^{-1}(L) \rangle$, then there exists some word $u \in A^*$ with $L' = [\varphi^{-1}(L)]_u$. Let (v, w) be a pair in $\mathsf{ker}(\eta \circ \varphi)$, then

$$
\begin{aligned}
L'_v = [\varphi^{-1}(L)]_{uv} &= \{x \in A^* \mid uvx \in \varphi^{-1}(L)\} = \{x \in A^* \mid \varphi(uvx) \in L\} \\
&= \{x \in A^* \mid \varphi(u)\varphi(v)\varphi(x) \in L\} = \{x \in A^* \mid \varphi(x) \in L_{\varphi(u)\varphi(v)}\}.
\end{aligned}
$$

Recall that $L_{\varphi(u)}$ is a language in $\langle L \rangle$, and $(\varphi(v), \varphi(w))$ is a pair in $\mathsf{Eq}(\langle L \rangle)$, therefore:

$$L'_v = \{x \in A^* \mid \varphi(x) \in L_{\varphi(u)\varphi(v)}\} = \{x \in A^* \mid \varphi(x) \in L_{\varphi(u)\varphi(w)}\} = L'_w.$$

It follows that $\mathsf{ker}(\eta \circ \varphi) \subseteq \mathsf{Eq}(\langle \varphi^{-1}(L) \rangle, \tau)$. Again by Proposition 10, we have $\varphi^{-1}(L)$ is a language in $\mathsf{coEq}(A^*/\mathsf{ker}(\eta \circ \varphi))$. $\qquad\square$

It follows that the concept of formation of languages presented here is stronger than the concept of r-formation of languages. However, if all the alphabets considered are finite, and the assignment maps each alphabet to a regular language, both definitions coincide. We shall firstly prove an easy lemma.

Lemma 16. *Let $\mathcal{R}$ be an r-formation of languages. For an alphabet A, if C is a congruence on A^* with finite quotient A^*/C, then*

$$\mathsf{coEq}(A^*/C) \subseteq \mathcal{R}(A) \quad \textit{if and only if} \quad [w]_C \in \mathcal{R}(A) \textit{ for all } w \in A^*.$$

Proof. Let w be a word in A^*, for the coloration $c_w \colon A^*/C \to 2$, given by $c_w([u]_C) = 1$ if and only if $[u]_C = [w]_C$, we have that $[w]_C = o_{c_w}([\varepsilon])$ is a language in $\mathsf{coEq}(A^*/C)$ which is included in $\mathcal{R}(A)$. On the converse, let L be language in $\mathsf{coEq}(A^*/C)$, then $L = \bigcup \{[w]_C \mid w \in L\}$. As A^*/C is finite, L is a finite union of languages in $\mathcal{R}(A)$. Hence, L belongs to $\mathcal{R}(A)$. $\qquad\square$

Proposition 17. *If $\mathcal{R}$ is an r-formation of languages which assigns to each finite alphabet A a class of regular languages over A, then $\mathcal{R}$ is a formation of languages.*

13

Proof. Let L be a language in $\mathcal{R}(A)$. It is well known that a language L over an alphabet A is regular if and only if the set $\{{_vL_w} \mid v, w \in A^*\}$ is finite. Let $[u]$ be an element in $A^*/\mathsf{Eq}(\langle L \rangle)$, then it holds by [BBPSE12, Proposition 2.14] that

$$[u] = \bigcap \{{_vL_w} \mid u \in {_vL_w}\} \setminus \bigcup \{{_vL_w} \mid u \notin {_vL_w}\}.$$

That is, every atom in $\mathsf{coEq}(A^*/\mathsf{Eq}(\langle L \rangle))$ belongs to the Boolean algebra generated by the derivatives of L. It follows from Lemma 16 that $\mathsf{coEq}(A^*/\mathsf{Eq}(\langle L \rangle))$ is included in $\mathcal{R}(A)$.

Now, assume that $\mathsf{coEq}(A^*/C_1)$ and $\mathsf{coEq}(A^*/C_2)$ are both included in $\mathcal{R}(A)$. Let $[w]_{C_1 \cap C_2}$ be an atom in $\mathsf{coEq}(A^*/C_1 \cap C_2)$. The equation $[w]_{C_1 \cap C_2} = [w]_{C_1} \cap [w]_{C_2}$ trivially holds. Note that $[w]_{C_i}$ is a language in $\mathsf{coEq}(A^*/C_i)$ for $i = 1, 2$, and hence, included in $\mathcal{R}(A)$. We conclude that $[w]_{C_1 \cap C_2}$ is a language in $\mathcal{R}(A)$. By Lemma 16, $\mathsf{coEq}(A^*/C_1 \cap C_2)$ is included in $\mathcal{R}(A)$.

Finally, consider two alphabets A and B, and let L be a language in $\mathcal{R}(B)$. Let η denote the quotient morphism $\eta \colon B^* \to \mathsf{free}(\langle L \rangle, \tau)$ and let $\varphi \colon A^* \to B^*$ be a monoid morphism such that $\eta \circ \varphi$ is surjective. We have that $\varphi^{-1}(\langle L \rangle)$ is a language in $\mathcal{R}(A)$, hence, by the first item of this proof, we have that $\mathsf{coEq}(A^*/\mathsf{Eq}(\langle \varphi^{-1}(L) \rangle))$ is included in $\mathcal{R}(A)$. Let us check $\ker(\eta \circ \varphi) = \mathsf{Eq}(\langle \varphi^{-1}(L) \rangle)$. We will only check that $\mathsf{Eq}(\langle \varphi^{-1}(L) \rangle)$ is included in $\ker(\eta \circ \varphi)$; for the other inclusion, see the proof of Proposition 15. Let (v, w) be a pair in $\mathsf{Eq}(\langle \varphi^{-1}(L) \rangle)$, we claim that $(\varphi(v), \varphi(w))$ is a pair in $\mathsf{Eq}(\langle L \rangle)$. As $\eta \circ \varphi$ is surjective, for every word $u \in B^*$, there exists some word $u' \in A^*$, with $(u, \varphi(u')) \in \mathsf{Eq}(\langle L \rangle)$. Let L_u be a language in $\langle L \rangle$.

$$
\begin{aligned}
L_{u\varphi(v)} &= L_{\varphi(u)\varphi(v)} && (\eta \circ \varphi \text{ surjective}) \\
&= L_{\varphi(uv)} && (\text{monoid homomorphism}) \\
&= \{x \in B^* \mid \varphi(u'v)x \in L\} \\
&= \{x \in B^* \mid \varepsilon \in L_{\varphi(u'v)x}\} \\
&= \{x \in B^* \mid \varepsilon \in L_{\varphi(u'v)\varphi(x')}\} && (\eta \circ \varphi \text{ surjective}) \\
&= \{x \in B^* \mid \varphi(u'vx') \in L\} && (\text{monoid homomorphism}) \\
&= \{x \in B^* \mid u'vx' \in \varphi^{-1}(L)\} \\
&= \{x \in B^* \mid u'wx' \in \varphi^{-1}(L)\} && ((v, w) \in \mathsf{Eq}(\langle \varphi^{-1}(L) \rangle)) \\
&= \cdots \\
&= L_{u\varphi(w)}
\end{aligned}
$$

Finally, $\ker(\eta \circ \varphi) = \mathsf{Eq}(\langle \varphi^{-1}(L) \rangle)$ and $\mathcal{R}$ is a formation of languages. $\qquad\square$

We do not know whether or not r-formations of languages are in general formations of languages.

5 Formation theorems

Monoids vs congruences

Proposition 18. *Every formation of monoids* $\mathbf{F}$ *induces a formation of congruences* F.

Proof. Consider the assignment:

$$\mathsf{F} \colon A \longmapsto \{C \in \mathsf{Con}(A^*) \mid A^*/C \in \mathbf{F}\}.$$

Let C_1 and C_2 be congruences in $\mathsf{F}(A)$, then A^*/C_1 and A^*/C_2 are monoids in $\mathbf{F}$. Note that $C_1 \cap C_2$ is included in C_i for $i = 1, 2$. If we consider the corresponding quotient homomorphisms $\pi_i \colon A^*/C_1 \cap C_2 \to A^*/C_i$ for $i = 1, 2$, we have that $\{\pi_1, \pi_2\}$ is a family of surjective morphisms separating the elements of $A^*/C_1 \cap C_2$. It follows from Proposition 11 that $A^*/C_1 \cap C_2$ is a subdirect product of two monoids in $\mathbf{F}$. Therefore $C_1 \cap C_2$ is a congruence in $\mathsf{F}(A)$. Now, if C is a congruence in $\mathsf{F}(A)$ and D is a congruence on A^* with $C \subseteq D$, we have that A^*/D is a quotient of A^*/C. It follows that A^*/D is a monoid in $\mathbf{F}$, and we conclude that D is a congruence in $\mathsf{F}(A)$. Therefore, $\mathsf{F}(A)$ is a filter in $\mathsf{Con}(A^*)$.

Let A and B be two sets, and let E be a congruence in $\mathsf{F}(B)$ with quotient morphism $\eta \colon B^* \to B^*/E$. Let $\varphi \colon A^* \to B^*$ be a monoid homomorphism such that the composition $\eta \circ \varphi \colon A^* \to B^*/E$ is a surjective monoid homomorphism. Hence, $A^*/\ker(\eta \circ \varphi)$ is isomorphic to B^*/E, which is a monoid in $\mathbf{F}$. It follows that $A^*/\ker(\eta \circ \varphi)$ is in $\mathbf{F}$ and $\ker(\eta \circ \varphi)$ is a congruence in $\mathsf{F}(A)$. $\qquad\square$

Remark. If $\mathbf{V}$ is a variety of monoids, the assignment of Proposition 18 maps each set A to the principal filter given by the residual of A^* over $\mathbf{V}$

$$\mathsf{V} \colon A \longmapsto \{C \in \mathsf{Con}(A^*) \mid A^*/C \in \mathbf{V}\} = [A^*_{\mathsf{V}})$$

Proposition 19. *Every formation of congruences F induces a formation of monoids $\mathbf{F}$.*

Proof. We take $\mathbf{F}$ to be the class of all monoids M for which there exists a set A and a congruence $C \in \mathsf{F}(A)$ satisfying $M \cong A^*/C$. We claim that this class is a formation of monoids.

Let $f \colon M \to N$ be the surjective monoid homomorphism defined on a monoid M in $\mathbf{F}$. Then there exists a set A and a congruence $C \in \mathsf{F}(A)$ satisfying $M \cong A^*/C$. Let $\gamma \colon A^* \to M$ be a monoid homomorphism with kernel C. Then $f \circ \gamma \colon A^*/C \to N$ is a surjective monoid homomorphism. Moreover, $C \subseteq \ker(f \circ \gamma)$, which implies that $\ker(f \circ \gamma)$ is a congruence in $\mathsf{F}(A)$. Finally, $A^*/\ker(f \circ \gamma)$ is isomorphic to N, and so N belongs to $\mathbf{F}$.

Now, let M be a monoid that can be expressed as the subdirect product of a finite family $\{M_i \mid 1 \leq i \leq n\}$ of monoids in $\mathbf{F}$. Therefore, for each index $i \in \{1, \ldots, n\}$ there exists a set A_i and a congruence $C_i \in \mathsf{Con}(A_i)$ satisfying $M_i \cong A_i^*/C_i$. Let us denote the corresponding quotient homomorphisms as $\eta_i \colon A_i^* \to A_i^*/C_i$. Consider the set $B = \bigsqcup_{i=1}^n A_i$. By the universal property of the free monoid, we can construct a monoid epimorphism $\varphi_i \colon B^* \to A_i^*$ for all $i \in \{1, \ldots, n\}$. Thus, $\eta_i \circ \varphi_i \colon B^* \to A_i^*/C_i$ is a surjective monoid homomorphism for all $i \in \{1, \ldots, n\}$. As F is a formation of congruences, the congruence $D_i = \ker(\eta_i \circ \varphi_i)$ belongs to $\mathsf{F}(B)$ for all $i \in \{1, \ldots, n\}$. Note that M can be expressed as the subdirect product of the finite family of monoids $\{B^*/D_i \mid 1 \leq i \leq n\}$. Since B generates each monoid in the family, M is generated by B. It follows that $M \cong B^*/F$ for some congruence F on B^*. Since M is a subdirect product of the monoids B^*/D_i, we have that $\bigcap_{i=1}^n D_i \subseteq F$. Note that $\bigcap_{i=1}^n D_i$ is a congruence in $\mathsf{F}(B)$ as it is a finite intersection of congruences in $\mathsf{F}(B)$. Finally, F is a congruence in $\mathsf{F}(B)$ and M belongs to $\mathbf{F}$. $\qquad\square$

Theorem 20. *The assignments $\mathsf{F} \mapsto \mathbf{F}$ and $\mathbf{F} \mapsto \mathsf{F}$ define mutually inverse correspondences between formations of congruences and formations of monoids.*

Proof. Consider a formation of monoids $\mathbf{F}$. The first correspondence gives us the formation of congruences F that assigns to each set A the set $\{C \in \mathsf{Con}(A^*) \mid A^*/C \in \mathbf{F}\}$ of all congruences whose quotient belongs to $\mathbf{F}$. Let $\mathbf{H}$ be the class of all monoids M for which there exists

15

a set A and a congruence $C \in \mathsf{F}(A)$ satisfying $M \cong A^*/C$. It immediately follows that $\mathbf{H}$ is included in $\mathbf{F}$. The other inclusion follows easily since every monoid can be written as a quotient of a free monoid.

Now, let F be a formation of congruences. The first correspondence gives us $\mathbf{F}$, which is equal to the class of all monoids M for which there exists a set A and a congruence $C \in \mathsf{F}(A)$ satisfying $M \cong A^*/C$. Let H denote the formation of congruence quotients that assigns to each set A the set $\{C \in \mathsf{Con}(A^*) \mid A^*/C \in \mathbf{F}\}$. For a fixed set A, if C is a congruence in $\mathsf{F}(A)$, then A^*/C is a monoid in $\mathbf{F}$ and C belongs to $\mathsf{H}(A)$. Let C be a congruence in $\mathsf{H}(A)$, then A^*/C is a monoid in $\mathbf{F}$. Therefore, there exists a set B and a congruence $D \in \mathsf{F}(B)$ satisfying $A^*/C \cong B^*/D$. Let $\eta \colon B^* \to B^*/D$ and $\delta \colon A^* \to A^*/C$ be the corresponding quotient homomorphisms. Let $\rho \colon A^*/C \to B^*/D$ be the corresponding monoid isomorphism. It follows that $\gamma = \rho \circ \delta$ is a monoid epimorphism from A^* onto B^*/D. By Proposition 14, there exists a monoid homomorphism $\varphi \colon A^* \to B^*$ with $\eta \circ \varphi = \gamma$. Summarising,

$$
\begin{array}{ccc}
A^* & \xrightarrow{\quad\delta\quad} & A^*/C \\
{\scriptstyle\varphi}\Big\downarrow & {\scriptstyle\gamma}\searrow & \Big\downarrow{\scriptstyle\rho} \\
B^* & \xrightarrow[\quad\eta\quad]{} & B^*/D
\end{array}
$$

As F is a formation of congruences, $\ker(\eta \circ \varphi)$ belongs to $\mathsf{F}(A)$. Finally, C is in $\mathsf{F}(A)$ as $\ker(\eta \circ \varphi) = \ker(\gamma) = \ker(\rho \circ \delta) = C$. $\qquad\square$

Remark. In the case of varieties, Theorem 20 gives a correspondence between varieties $\mathbf{V}$ and formations of congruences V satisfying that for all A, the set $\mathsf{V}(A)$ is a principal filter in $\mathsf{Con}(A^*)$.

Congruences vs languages

Proposition 21. *Every formation of congruences F induces a formation of languages $\mathcal{F}$.*

Proof. Consider the assignment:

$$
\mathcal{F} \colon A \longmapsto \bigcup \{\mathsf{coEq}(A^*/C) \mid C \in \mathsf{F}(A)\}.
$$

Let L be a language in $\mathcal{F}(A)$, then there exists a congruence C in $\mathsf{F}(A)$ for which L is a language in $\mathsf{coEq}(A^*/C)$. By Proposition 10, we have that $C \subseteq \mathsf{Eq}(\langle L \rangle)$. Thus, $\mathsf{Eq}(\langle L \rangle)$ is a congruence in $\mathsf{F}(A)$. Hence, $\mathsf{coEq}(A^*/\mathsf{Eq}(\langle L \rangle))$ is included in $\mathcal{F}(A)$. Now, if $\mathsf{coEq}(A^*/C_1)$ and $\mathsf{coEq}(A^*/C_2)$ are included in $\mathcal{F}(A)$, then the congruences C_1, C_2 are in $\mathsf{F}(A)$. By assumption, the congruence $C_1 \cap C_2$ also belongs to $\mathsf{F}(A)$. Thus, $\mathsf{coEq}(A^*/C_1 \cap C_2)$ is in $\mathcal{F}(A)$. Let L be a language of $\mathcal{F}(B)$ with quotient morphism $\eta \colon B^* \to \mathsf{free}(\langle L \rangle)$. Let $\varphi \colon A^* \to B^*$ such that $\eta \circ \varphi$ is surjective, then $\ker(\eta \circ \varphi)$ is a congruence in $\mathsf{F}(A)$. Thus, $\mathsf{coEq}(A^*/\ker(\eta \circ \varphi))$ belongs to $\mathcal{F}(A)$. Hence, $\mathcal{F}$ is a formation of languages. $\qquad\square$

Proposition 22. *Every formation of languages $\mathcal{F}$ induces a formation of congruences F.*

Proof. Consider the assignment:

$$
\mathsf{F} \colon A \longmapsto \{C \in \mathsf{Con}(A^*) \mid \mathsf{coEq}(A^*/C) \subseteq \mathcal{F}(A)\}.
$$

16

Let C be a congruence in $\mathsf{F}(A)$. If D is a congruence on A^* with $C \subseteq D$, then, by Theorem 8, $\mathsf{coEq}(A^*/D)$ is included in $\mathsf{coEq}(A^*/C)$, which is included in $\mathcal{F}(A)$ by assumption. Now, let C_1 and C_2 be two congruences in $\mathsf{F}(A)$, then $\mathsf{coEq}(A^*/C_1)$ and $\mathsf{coEq}(A^*/C_2)$ belong to $\mathcal{F}(A)$. As $\mathcal{F}$ is a formation of languages, then $\mathsf{coEq}(A^*/C_1 \cap C_2)$ is included in $\mathcal{F}(A)$. Hence, $C_1 \cap C_2$ is a congruence in $\mathsf{F}(A)$. Hence, F maps each alphabet A to a filter in $\mathsf{Con}(A^*)$.

Let A and B be two sets and let C be a congruence in $\mathsf{F}(B)$. Consider the corresponding quotient homomorphism $\eta \colon B^* \to B^*/C$. Let $\varphi \colon A^* \to B^*$ be a monoid homomorphism such that the composition $\eta \circ \varphi \colon A^* \to B^*/C$ is a surjective monoid homomorphism. Since $\mathcal{F}$ is a formation of languages, $\mathsf{coEq}(A^*/\mathsf{ker}(\eta \circ \varphi))$ is included in $\mathcal{F}(A)$. It follows that $\mathsf{ker}(\eta \circ \varphi))$ is a congruence in $\mathsf{F}(A)$. $\qquad\square$

Theorem 23. *The assignments $\mathsf{F} \mapsto \mathcal{F}$ and $\mathcal{F} \mapsto \mathsf{F}$ define mutually inverse correspondences between formations of congruences and formations of languages.*

Proof. It immediately follows from the assignments we have chosen. $\qquad\square$

Languages vs monoids

Proposition 24. *Every formation of languages $\mathcal{F}$ induces a formation of monoids $\mathbf{F}$.*

Proof. Just consider the composition of the correspondences given by Propositions 22 and 19. Hence, we take $\mathbf{F}$ to be the class of all monoids M that are isomorphic to A^*/C for some congruence C on A^* satisfying that $\mathsf{coEq}(A^*/C) \subseteq \mathcal{F}(A)$. $\qquad\square$

Proposition 25. *Every formation of monoids $\mathbf{F}$ induces a formation of languages $\mathcal{F}$.*

Proof. Just consider the composition of the correspondences given by Propositions 18 and 21. Hence, for a set A we take the set of languages $\mathcal{F}(A) = {}^{\mathsf{S}}\{\mathsf{coEq}(A^*/C) \mid A^*/C \in \mathbf{F}\}$. $\qquad\square$

Theorem 26. *The assignments $\mathbf{F} \mapsto \mathcal{F}$ and $\mathcal{F} \mapsto \mathbf{F}$ define mutually inverse correspondences between formations of monoids and formations of languages.*

Proof. It immediately follows from Theorems 20 and 23. $\qquad\square$

6 Relatively regular monoids and disjunctive languages

We present an example of an application of Theorem 26.

A monoid M is called *relatively regular* (*r-regular* for short) (see [LSG08]) if it contains a finite ideal. We shall denote by $\mathbf{R}$ the class of all r-regular monoids. A monoid with zero is an example of a r-regular monoid. For each finite set Ω, the monoid of its transformations $\mathsf{T} = \{f \mid f \colon \Omega \to \Omega\}$ has a finite ideal given by all its constant mappings.

Proposition 27. *The class $\mathbf{R}$ is a formation of monoids which is not a variety.*

Proof. The set of all integers Z with the usual multiplication is r-regular as it is a monoid with zero. The set Z^* of nonzero integers is a submonoid of Z without finite ideals, therefore $\mathbf{R}$ is not a variety. We need to check that $\mathbf{R}$ is closed under quotients and subdirect products. For a monoid M with finite ideal I, and for a congruence $\equiv$ on M, the subset $[I]_{\equiv} = \{[x]_{\equiv} \mid x \in I\}$ is a finite ideal of the quotient monoid $M/\equiv$. Let M be a subdirect product of a family $\{M_1, \dots, M_n\}$ of monoids in $\mathbf{R}$. Let us denote the finite ideal of the monoid M_i by I_i, for

$i \in \{1, \ldots, n\}$. Consider the set $I = \{x \in M \mid \gamma_i(x) \in I_i \text{ for all } 1 \leq i \leq n\}$. For $p_i \in I_i$, there exists an element $x_i \in M$ satisfying $\gamma_i(x_i) = p_i$. Consider the element $x = x_1 \cdots x_n$. It satisfies:

$$\gamma_i(x) = \gamma_i(x_1)\gamma_i(x_2) \cdots p_i \cdots \gamma_i(x_n) \in I_i.$$

It follows that I is not empty, and it is finite as there are at most $\prod_{i=1}^n |I_i|$ elements in I. Finally, I is an ideal: Let $m \in M$ and $x \in I$, we have that $\gamma_i(mx) = \gamma_i(m)\gamma_i(x) \in I_i$ for $1 \leq i \leq n$. We can apply an analogous argument on the right. Thus, M has a finite ideal. $\qquad \square$

The following results can be found in [ZGS14]. All the notation has been translated to our notation. We denote the *cardinality* of a language L over A by $|L|$. We call a language L over A *disjunctive* if $\mathsf{Eq}(\langle L \rangle)$ is the diagonal relation on A^*. We call a language L over A *dense* if $A^* w A^* \cap L \neq \emptyset$ for every $w \in A^*$; otherwise, the language L is said to be *thin*. According to Reis and Shyr, a language L is dense if and only if L contains a disjunctive language [RS78]. We shall call a language over A *relatively f-disjunctive* [*relatively disjunctive*] (*rf-disjunctive* [r-disjunctive] for short) if there exists a dense language D over A such that for all $w \in A^*$

$$|[w]_{\mathsf{Eq}(\langle L \rangle)} \cap D| < \infty \qquad [|[w]_{\mathsf{Eq}(\langle L \rangle)} \cap D| \leq 1].$$

It has been shown in [GRT88] that L is rf-disjunctive if and only if L is r-disjunctive, if and only if either A^* has no dense $\mathsf{Eq}(\langle L \rangle)$-classes or has infinitely many dense $\mathsf{Eq}(\langle L \rangle)$-classes.

We shall denote by $\mathcal{F}(A)$ the set of all non-r-disjunctive languages over A. The next result can be found in [LSG08]. It relates r-disjunctive languages and r-regular monoids.

Theorem 28. *A language L over A is r-disjunctive if and only if $\mathsf{free}(\langle L \rangle)$ is not r-regular.*

As a consequence of our previous section, we have a result on disjunctive languages that does not follow immediately from the definition.

Corollary 29. *The assignment $\mathcal{F} \colon A \to \mathcal{F}(A)$ is a formation of languages. In particular, $\mathcal{F}(A)$ is closed under Boolean operations and derivatives.*

Proof. It follows from Theorem 26, Proposition 27 and the contrapositive version of Theorem 28. $\qquad \square$

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