# BOUND-CONSTRAINED POLYNOMIAL OPTIMIZATION USING ONLY ELEMENTARY CALCULATIONS 

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#### Abstract

We provide a monotone non increasing sequence of upper bounds $f_{k}^{H}(k \geq 1)$ converging to the global minimum of a polynomial $f$ on simple sets like the unit hypercube. The novelty with respect to the converging sequence of upper bounds in [J.B. Lasserre, A new look at nonnegativity on closed sets and polynomial optimization, SIAM J. Optim. 21, pp. 864-885, 2010] is that only elementary computations are required. For optimization over the hypercube, we show that the new bounds $f_{k}^{H}$ have a rate of convergence in $O(1 / \sqrt{k})$. Moreover we show a stronger convergence rate in $O(1 / k)$ for quadratic polynomials and more generally for polynomials having a rational minimizer in the hypercube. In comparison, evaluation of all rational grid points with denominator $k$ produces bounds with a rate of convergence in $O\left(1 / k^{2}\right)$, but at the cost of $O\left(k^{n}\right)$ function evaluations, while the new bound $f_{k}^{H}$ needs only $O\left(n^{k}\right)$ elementary calculations.


## 1. Introduction

Consider the problem of computing the global minimum

$$
\begin{equation*}
f_{\min , \mathbf{K}}=\min \{f(\mathbf{x}): \mathbf{x} \in \mathbf{K}\} \tag{1.1}
\end{equation*}
$$

of a polynomial $f$ on a compact set $\mathbf{K} \subset \mathbb{R}^{n}$. (We will mainly deal with the case where $\mathbf{K}$ is a basic semi-algebraic set.)

A fruitful perspective, introduced by Lasserre [16], is to reformulate problem (1.1) as

$$
f_{\min , \mathbf{K}}=\inf _{\mu} \int_{\mathbf{K}} f d \mu,
$$

where the infimum is taken over all probability measures $\mu$ with support in $\mathbf{K}$. Using this reformulation one may obtain a sequence of lower bounds on $f_{\min , \mathbf{K}}$ that converges to $f_{\min , \mathbf{K}}$, by introducing tractable convex relaxations of the set of probability measures with support in $\mathbf{K}$ (if $\mathbf{K}$ is semi-algebraic). For more details on this approach the interested reader is referred to Lasserre [15, 16, 18, and [20, 17] for a comparison between linear programming (LP) and semidefinite programming (SDP) relaxations.

As an alternative, one may obtain a sequence of upper bounds by optimizing over specific classes of probability distributions. In particular, Lasserre [19] defined the sequence (also called hierarchy) of upper bounds

$$
f_{k}^{s o s}:=\min _{\sigma \in \Sigma_{k}[\mathbf{x}]}\left\{\int_{\mathbf{K}} f(\mathbf{x}) \sigma(\mathbf{x}) d \mathbf{x}: \int_{\mathbf{K}} \sigma(\mathbf{x}) d \mathbf{x}=1\right\}, \quad(k=1,2, \ldots),
$$

where $\Sigma_{k}[\mathbf{x}]$ denotes the cone of sums of squares (SOS) of polynomials of degree at most $2 k$. Thus the optimization is restricted to probability distributions where the probability density function is

[^0]an SOS polynomial of degree at most $2 k$. Lasserre 19 showed that $f_{k}^{s o s} \rightarrow f_{\min , \mathbf{K}}$ as $k \rightarrow \infty$ (see Theorem 2.1 below for a precise statement). In principle this approach works for any compact set $\mathbf{K}$ and any polynomial but for practical implementation it requires knowledge of moments of the measure $\sigma(\mathbf{x}) d \mathbf{x}$. So in practice the approach is limited to simple sets $\mathbf{K}$ like the Euclidean ball, the hypersphere, the simplex, the hypercube and/or their image by a linear transformation.

In fact computing such upper bounds reduces to computing the smallest generalized eigenvalue associated with two real symmetric matrices whose size increases in the hierarchy. For more details the interested reader is referred to Lasserre [19. In a recent paper, De Klerk et al. [6] have provided the first convergence analysis for this hierarchy and shown a bound $f_{k}^{s o s}-f_{\min , \mathbf{K}}=O(1 / \sqrt{k})$ on the rate of convergence. In a related analysis of convergence Romero and Velasco [23] provide a bound on the rate at which one may approximate from outside the cone of nonnegative homogeneous polynomials (of fixed degree) by the hierarchy of spectrahedra defined in [19].

It should be emphasized that it is a difficult challenge in optimization to obtain a sequence of upper bounds converging to the global minimum and having a known estimate on the rate of convergence. So even if the convergence to the global minimum of the hierarchy of upper bounds obtained in 19 is rather slow, and even though it applies to the restricted context of "simple sets", to the best of our knowledge it provides one of the first results of this kind. A notable earlier result was obtained for polynomial optimization over the simplex, where it has been shown that brute force grid search leads to a polynomial time approximation scheme for minimizing polynomials of fixed degree [1, 4]. When minimizing over the set of grid points in the standard simplex with given denominator $k$, the rate of convergence is in $O(1 / k)$ [1, 4] and, for quadratic polynomials (and for general polynomials having a rational minimizer), in $O\left(1 / k^{2}\right)$ [5]. Grid search over the hypercube was also shown to have a rate of convergence in $O(1 / k)[3]$ and, as we will indicate in this paper, a stronger rate of convergence in $O\left(1 / k^{2}\right)$ can be shown. Note however that computing the best grid point in the hypercube $[0,1]^{n}$ with denominator $k$ requires $O\left(k^{n}\right)$ computations, thus exponential in the dimension.

Contribution. As our main contribution we provide a monotone non increasing converging sequence $\left(f_{k}^{H}\right), k \in \mathbb{N}$, of upper bounds $f_{k}^{H} \geq f_{\min , \mathbf{K}}$ such that $f_{k}^{H} \rightarrow f_{\min , \mathbf{K}}$ as $k \rightarrow \infty$. The parameters $f_{k}^{H}$ can be effectively computed when the set $\mathbf{K} \subset[0,1]^{n}$ is a "simple set" like, for example, a Euclidean ball, sphere, simplex, hypercube or any linear transformation of them.

This "hierarchy" of upper bounds is inspired from the one defined by Lasserre in [19, but with the novelty that:

Computing the upper bounds $\left(f_{k}^{H}\right)$ does not require solving an SDP or computing the smallest generalized eigenvalue of some pair of matrices (as is the case in [19]). It only requires elementary calculations (but possibly many of them for good quality bounds).
Indeed, computing the upper bound $f_{k}^{H}$ only requires finding the minimum in a list of $O\left(n^{k}\right)$ scalars $\left(\gamma_{(\eta, \beta)}\right)$, formed from the moments $\gamma$ of the Lebesgue measure on the set $\mathbf{K} \subseteq[0,1]^{n}$ and from the coefficients $\left(f_{\alpha}\right)$ of the polynomial $f$ to minimize. Namely:

$$
\begin{equation*}
f_{k}^{H}:=\min _{(\eta, \beta) \in \mathbb{N}_{k}^{2 n}} \sum_{\alpha \in \mathbb{N}^{n}} f_{\alpha} \frac{\gamma_{(\eta+\alpha, \beta)}}{\gamma_{(\eta, \beta)}} \tag{1.2}
\end{equation*}
$$

where $\mathbb{N}$ denotes the nonnegative integers, $f(\mathbf{x})=\sum_{\alpha \in \mathbb{N}^{n}} f_{\alpha} \mathbf{x}^{\alpha}, \mathbb{N}_{k}^{2 n}=\left\{(\eta, \beta) \in \mathbb{N}^{2 n}:|\eta+\beta|=k\right\}$, and the scalars

$$
\gamma_{(\eta, \beta)}:=\int_{\mathbf{K}} x_{1}^{\eta_{1}} \cdots x_{n}^{\eta_{n}}\left(1-x_{1}\right)^{\beta_{1}} \cdots\left(1-x_{n}\right)^{\beta_{n}} d \mathbf{x}, \quad(\eta, \beta) \in \mathbb{N}^{2 n}
$$

are available in closed-form. (Our informal notion of "simple set" therefore means that the moments $\gamma_{(\eta, \beta)}$ are known a priori.)

The upper bound (1.2) has also a simple interpretation as it reads:

$$
\begin{equation*}
f_{k}^{H}=\min _{(\eta, \beta) \in \mathbb{N}_{k}^{2 n}} \frac{\int_{\mathbf{K}} f(\mathbf{x}) \mathbf{x}^{\eta}(1-\mathbf{x})^{\beta} d \mathbf{x}}{\int_{\mathbf{K}} \mathbf{x}^{\eta}(1-\mathbf{x})^{\beta} d \mathbf{x}}=\min _{\mu}\left\{\int_{\mathbf{K}} f d \mu: \mu \in M(\mathbf{K})_{k}\right\} \tag{1.3}
\end{equation*}
$$

where $M(\mathbf{K})_{k}$ is the set of probability measures on $\mathbf{K}$, absolutely continuous with respect to the Lebesgue measure on $\mathbf{K}$, and whose density is a monomial $\mathbf{x}^{\eta}(1-\mathbf{x})^{\beta}$ with $(\eta, \beta) \in \mathbb{N}_{k}^{2 n}$. (Such measures are in fact products of (univariate) beta distributions, see Section 4.1.) This also proves that at any point $\mathbf{a} \in[0,1]^{n}$ one may approximate the Dirac measure $\delta_{\mathbf{a}}$ with measures of the form $d \mu=\mathbf{x}^{\eta}(1-\mathbf{x})^{\beta} d \mathbf{x}$ (normalized to make then probability measures).

For the case of the hypercube $\mathbf{K}=[0,1]^{n}$, we analyze the rate of convergence of the bounds $f_{k}^{H}$ and show a rate of convergence in $O(1 / \sqrt{k})$ for general polynomials, and in $O(1 / k)$ for quadratic polynomials (and general polynomials having a rational minimizer). As a second minor contribution, we revisit grid search over the rational points with given denominator $k$ in the hypercube and observe that its convergence rate is in $O\left(1 / k^{2}\right)$ (which follows as an easy application of Taylor's theorem). However as observed earlier the computation of the best grid point with denominator $k$ requires $O\left(k^{n}\right)$ function evaluations while the computation of the parameter $f_{k}^{H}$ requires only $O\left(n^{k}\right)$ elementary calculations.

Organization of the paper. We start with some basic facts about the bounds $f_{k}^{H}$ in Section 2 and in Section 3 we show their convergence to the minimum of $f$ over the set $\mathbf{K}$ (see Theorem 3.1).

In Section 4 for the case of the hypercube $\mathbf{K}=[0,1]^{n}$, we analyze the quality of the bounds $f_{k}^{H}$. We show a convergence rate in $O(1 / \sqrt{k})$ for the range $f_{k}^{H}-f_{\text {min }, \mathbf{K}}$ and a stronger convergence rate in $O(1 / k)$ when the polynomial $f$ admits a rational minimizer in $[0,1]^{n}$ (see Theorem 4.8). This stronger convergence rate applies in particular to quadratic polynomials (since they have a rational minimizer) and Example 4.9 shows that this bound is tight. When no rational minimizer exists the weaker rate follows using Diophantine approximations. So again the main message of this paper is that one may obtain non trivial upper bounds with error guarantees (and converging to the global minimum) via elementary calculations and without invoking a sophisticated algorithm.

In Section 5 we revisit the simple technique which consists of evaluating the polynomial $f$ at all rational points in $[0,1]^{n}$ with given denominator $k$. By a simple application of Taylor's theorem we can show a convergence rate in $O\left(1 / k^{2}\right)$. However, in terms of computational complexity, the parameters $f_{k}^{H}$ are easier to compute. Indeed, for fixed $k$, computing $f_{k}^{H}$ requires $O\left(n^{k}\right)$ computations (similar to function evaluations), while computing the minimum of $f$ over all grid points with given denominator $k$ requires an exponential number $k^{n}$ of function evaluations.

In Section 6 we present some additional (simple) techniques to provide a feasible point $\hat{x} \in \mathbf{K}$ with value $f(\hat{x}) \leq f_{k}^{H}$, once the upper bound $f_{k}^{H}$ has been computed, hence also with an error bound guarantee in the case of the box $\mathbf{K}=[0,1]^{n}$. This includes, in the case when $f$ is convex, getting a feasible point using Jensen inequality (Section 6.1) and, in the general case, taking the mode $\hat{x}$ of the optimal density function (i.e., its global maximizer) (see Section 6.2).

In Section 7 we present some numerical experiments, carried out on several test functions on the box $[0,1]^{n}$. In particular, we compare the values of the new bound $f_{k}^{H}$ with the bound $f_{k / 2}^{\text {sos }}$ (whose definition uses a sum of squares density), and we apply the proposed techniques to find a
feasible point in the box. As expected the sos based bound is tighter in most cases but the bound $f_{k}^{H}$ can be computed for much larger values of $k$. Moreover, the feasible points $\hat{x}$ returned by the proposed mode heuristic are often of very good quality for sufficiently large $k$. Finally, in Section 8 we conclude with some remarks on variants of the bound $f_{k}^{H}$ that may offer better results in practice.

## 2. Notation, definitions and preliminary Results

Let $\mathbb{R}[\mathbf{x}]$ denote the ring of polynomials in the variables $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right), \mathbb{R}[\mathbf{x}]_{d}$ is subspace of polynomials of degree at most $d$, and $\Sigma[\mathbf{x}]_{d} \subset \mathbb{R}[\mathbf{x}]_{2 d}$ its subset of sums of squares (SOS) of degree at most $2 d$.

We use the convention that $\mathbb{N}$ denotes the nonnegative integers, and let $\mathbb{N}_{d}^{n}:=\left\{\alpha \in \mathbb{N}^{n}\right.$ : $\left.\sum_{i=1}^{n} \alpha_{i}(=:|\alpha|)=d\right\}$, and similarly $\mathbb{N}_{\leq d}^{n}:=\left\{\alpha \in \mathbb{N}^{n}: \sum_{i=1}^{n} \alpha_{i} \leq d\right\}$. The notation $\mathbf{x}^{\alpha}$ stands for the monomial $x_{1}^{\alpha_{1}} \cdots x_{n}^{\alpha_{n}}$, while $(1-\mathbf{x})^{\alpha}$ stands for $\left(1-x_{1}\right)^{\alpha_{1}} \cdots\left(1-x_{n}\right)^{\alpha_{n}}, \alpha \in \mathbb{N}^{n}$. We will also denote $[n]=\{1,2, \ldots, n\}$.

One may write every polynomial $f \in \mathbb{R}[\mathbf{x}]_{d}$ in the monomial basis

$$
\mathbf{x} \mapsto f(\mathbf{x})=\sum_{\alpha \in \mathbb{N}_{\leq d}^{n}} f_{\alpha} \mathbf{x}^{\alpha},
$$

with vector of (finitely many) coefficients $\left(f_{\alpha}\right)$.
In [19], Lasserre proved the following.
Theorem 2.1 (Lasserre [19]). Let $\mathbf{K} \subset \mathbb{R}^{n}$ be compact, $f_{\min , \mathbf{K}}$ be as in (1.1), and let

$$
\begin{equation*}
f_{k}^{s o s}:=\inf _{\sigma}\left\{\int_{\mathbf{K}} f(\mathbf{x}) \sigma(\mathbf{x}) d \mathbf{x}: \int_{\mathbf{K}} \sigma(\mathbf{x}) d \mathbf{x}=1, \sigma \in \Sigma[\mathbf{x}]_{k}\right\}, \quad k \in \mathbb{N} \tag{2.1}
\end{equation*}
$$

Then $f_{\min , \mathbf{K}} \leq f_{k}^{\text {sos }} \leq f_{k+1}^{s o s}$ for all $k$ and

$$
\begin{equation*}
f_{\min , \mathbf{K}}=\lim _{k \rightarrow \infty} f_{k}^{s o s} \tag{2.2}
\end{equation*}
$$

We will also use the following important result due to Krivine [13, 14 and Handelman [10].
Theorem 2.2. Let $\mathbf{K}=\left\{\mathbf{x}: g_{j}(\mathbf{x}) \geq 0, j=1, \ldots, m\right\} \subset \mathbb{R}^{n}$ be a polytope with a nonempty interior and where each $g_{j}$ is an affine polynomial, $j=1, \ldots, m$. If $f \in \mathbb{R}[\mathbf{x}]$ is strictly positive on $\mathbf{K}$ then

$$
\begin{equation*}
f(\mathbf{x})=\sum_{\alpha \in \mathbb{N}^{m}} \lambda_{\alpha} g_{1}(\mathbf{x})^{\alpha_{1}} \cdots g_{m}(\mathbf{x})^{\alpha_{m}}, \quad \forall \mathbf{x} \in \mathbb{R}^{n} \tag{2.3}
\end{equation*}
$$

for finitely many positive scalars $\lambda_{\alpha}$.
We will call the expression in (2.3) the Handelman representation of $f$, and call any $f$ that allows a Handelman representation to be of the Handelman type. Throughout we consider the set $\mathcal{H}_{k}$ consisting of the polynomials of the form:

$$
\begin{equation*}
\sum_{(\eta, \beta) \in \mathbb{N}_{k}^{2 n}} \lambda_{\eta, \beta} \mathbf{x}^{\eta}(1-\mathbf{x})^{\beta} \quad \text { where } \lambda_{\eta \beta} \geq 0 \tag{2.4}
\end{equation*}
$$

i.e., all polynomials admitting a Handelman representation of degree at most $k$ in terms of the polynomials $x_{i}, 1-x_{i}$ defining the hypercube $[0,1]^{n}$.

Observe that any term $\mathbf{x}^{\eta}(1-\mathbf{x})^{\beta}$ with degree $|\eta+\beta|<k$ also belongs to the set $\mathcal{H}_{k}$. This follows by iteratively applying the identity: $1=x_{i}+\left(1-x_{i}\right)$, which permits to rewrite $\mathbf{x}^{\eta}(1-\mathbf{x})^{\beta}$
as a conic combination of terms $\mathbf{x}^{\eta^{\prime}}(1-\mathbf{x})^{\beta^{\prime}}$ with degree $\left|\eta^{\prime}+\beta^{\prime}\right|=k$. The next claim follows then as a direct application.

Lemma 2.3. We have the inclusion: $\mathcal{H}_{k} \subseteq \mathcal{H}_{k+1}$ for all $k$.
We may now interpret the new upper bounds $f_{k}^{H}$ in an analogous way as $f_{k}^{s o s}$ (see (2.1)), but where the SOS density function $\sigma \in \Sigma_{k}[\mathbf{x}]$ is replaced by a density $\sigma \in \mathcal{H}_{k}$.

Lemma 2.4. Consider the sequence $\left(f_{k}^{H}\right), k \in \mathbb{N}$, with $f_{k}^{H}$ as in (1.2). Then one has:

$$
f_{k}^{H}=\inf _{\sigma \in \mathcal{H}_{k}}\left\{\int_{\mathbf{K}} f(\mathbf{x}) \sigma(\mathbf{x}) d \mathbf{x}: \int_{\mathbf{K}} \sigma(\mathbf{x}) d \mathbf{x}=1\right\}, \quad k \in \mathbb{N} .
$$

Proof. Note that, for given $k \in \mathbb{N}$,

$$
\begin{aligned}
& \inf _{\sigma}\left\{\int_{\mathbf{K}} f(\mathbf{x}) \sigma(\mathbf{x}) d \mathbf{x}: \int_{\mathbf{K}} \sigma(\mathbf{x}) d \mathbf{x}=1, \sigma \in \mathcal{H}_{k}\right\} \\
= & \inf _{\lambda \geq 0}\{\sum_{\alpha \in \mathbb{N}_{d}^{n}} f_{\alpha}(\sum_{(\eta, \beta) \in \mathbb{N}_{k}^{2 n}} \lambda_{\eta \beta} \underbrace{\int_{\mathbf{K}} \mathbf{x}^{\eta+\alpha}(1-\mathbf{x})^{\beta} d \mathbf{x}}_{\gamma_{(\eta+\alpha, \beta)}}): \sum_{(\eta, \beta) \in \mathbb{N}_{k}^{2 n}} \lambda_{\eta \beta} \int_{\mathbf{K}} \mathbf{x}^{\eta}(1-\mathbf{x})^{\beta} d \mathbf{x}=1\} \\
= & \inf _{\lambda \geq 0}\left\{\sum_{(\eta, \beta) \in \mathbb{N}_{k}^{2 n}} \lambda_{\eta \beta}\left(\sum_{\alpha \in \mathbb{N}_{d}^{n}} f_{\alpha} \gamma_{(\eta+\alpha, \beta)}\right): \sum_{(\eta, \beta) \in \mathbb{N}_{k}^{2 n}} \lambda_{\eta \beta} \gamma_{(\eta, \beta)}=1\right\} \\
= & \min _{(\eta, \beta) \in \mathbb{N}_{k}^{2 n}} \sum_{\alpha \in \mathbb{N}_{d}^{n}} f_{\alpha} \frac{\gamma_{(\eta+\alpha, \beta)}}{\gamma_{(\eta, \beta)}}=f_{k}^{H},
\end{aligned}
$$

where we have used the fact that the penultimate optimization problem is an LP over a simplex that obtains its infimum at one of the vertices.

Example 2.5. We consider the bivariate Styblinski-Tang function

$$
f\left(x_{1}, x_{2}\right)=\sum_{i=1}^{2} \frac{1}{2}\left(10 x_{i}-5\right)^{4}-8\left(10 x_{i}-5\right)^{2}+\frac{5}{2}\left(10 x_{i}-5\right)
$$

over the square $\mathbf{K}=[0,1]^{2}$, with minimum $f_{\min , \mathbf{K}} \approx-78.33198$ and minimizer

$$
\mathbf{x}^{*} \approx(0.20906466,0.20906466)
$$

Here one has $f_{1}^{\text {sos }}=-12.9249$, and the corresponding SOS density of degree 2 is (roughly)

$$
\sigma\left(x_{1}, x_{2}\right)=\left(1.9169-1.005 x_{1}-1.005 x_{2}\right)^{2}
$$

Using a Handelman-type density function, the upper bound of degree 2 is $f_{2}^{H}=-17.3810$, with corresponding density

$$
\sigma\left(x_{1}, x_{2}\right)=6 x_{2}\left(1-x_{2}\right)
$$

On the other hand, if we consider densities of degree 6 then we get $f_{3}^{s o s}=-34.403$ and $f_{6}^{H}=$ -31.429.

Thus there is no general ordering between the bounds $f_{k}^{s o s}$ and $f_{2 k}^{H}$. Having said that, we will show in Section 7 that, for most of the examples we have considered, one has $f_{k}^{\text {sos }} \leq f_{2 k}^{H}$ for all $k$, as one may expect from the relative computational efforts.


Figure 1. Optimal Handelman-type density $\sigma(x)$ of degree 50 on $[0,1]^{2}$ for the bivariate Styblinski-Tang function.

As a final illustration, Figure 1 shows the plot and contour plot of the Handelman-type density corresponding to the bound $f_{50}^{H}=-60.536$ (i.e. degree 50 ).

The figure illustrates the earlier assertion that the optimal density approximates the Dirac delta measure at the minimizer $\mathbf{x}^{*} \approx(0.20906466,0.20906466)$. Indeed, it is clear from the contour plot that the mode of the optimal density is close to $\mathbf{x}^{*}$.

## 3. Convergence proof

Let $\mathbf{K} \subseteq[0,1]^{n}$ be a compact set and for every $(\eta, \beta) \in \mathbb{N}^{2 n}$, let

$$
\begin{equation*}
\gamma_{(\eta, \beta)}:=\int_{\mathbf{K}} \mathbf{x}^{\eta}(1-\mathbf{x})^{\beta} d \mathbf{x} \tag{3.1}
\end{equation*}
$$

Of course when $\mathbf{K}$ is arbitrary one does not know how to compute such generalized moments. But if $\mathbf{K}$ is the unit hypercube $[0,1]^{n}$, the simplex $\Delta:=\left\{\mathbf{x}: \mathbf{x} \geq 0 ; \sum_{i=1}^{n} x_{i} \leq 1\right\}$, a Euclidean ball (or sphere), the hypercube $\{0,1\}^{n}$ and/or their image by a linear mapping, then such moments are available in closed-form. For instance if $\mathbf{K}=[0,1]^{n}$ then

$$
\int_{\mathbf{K}} \mathbf{x}^{\eta}(1-\mathbf{x})^{\beta} d \mathbf{x}=\prod_{i=1}^{n}\left(\int_{0}^{1} x_{i}^{\eta_{i}}\left(1-x_{i}\right)^{\beta_{i}} d x_{i}\right), \quad(\eta, \beta) \in \mathbb{N}^{2 n}
$$

and the univariate integrals may be calculated from

$$
\begin{equation*}
\int_{0}^{1} t^{i}(1-t)^{j} d t=\frac{i!j!}{(i+j+1)!} \quad i, j \in \mathbb{N} \tag{3.2}
\end{equation*}
$$

Theorem 3.1. Let $f \in \mathbb{R}[\mathbf{x}]_{d}$ and let $\gamma_{(\eta, \beta)}$ be as in (3.1). Define as before the parameters

$$
\begin{equation*}
f_{k}^{H}=\min _{(\eta, \beta) \in \mathbb{N}_{k}^{2 n}} \sum_{\alpha \in \mathbb{N}_{\leq d}^{n}} f_{\alpha} \frac{\gamma_{(\eta+\alpha, \beta)}}{\gamma_{(\eta, \beta)}}, \quad \forall k \in \mathbb{N} \tag{3.3}
\end{equation*}
$$

Then the sequence $\left(f_{k}^{H}\right), k \in \mathbb{N}$, is monotone non increasing and $f_{\min , \mathbf{K}}=\lim _{k \rightarrow \infty} f_{k}^{H}$.

Proof. As before, let $f_{k}^{s o s}$ denote the bound obtained by searching over an SOS density $\sigma$ of degree at most $2 k$ :

$$
f_{k}^{s o s}=\min \int_{\mathbf{K}} f(\mathbf{x}) \sigma(\mathbf{x}) d \mathbf{x} \text { such that } \int_{\mathbf{K}} \sigma(\mathbf{x}) d \mathbf{x}=1, \sigma \in \Sigma_{k}
$$

Also recall from Lemma 2.4 that

$$
f_{k}^{H}=\min \int_{\mathbf{K}} f(\mathbf{x}) \sigma(\mathbf{x}) d \mathbf{x} \text { such that } \int_{\mathbf{K}} \sigma(\mathbf{x}) d \mathbf{x}=1, \sigma \in \mathcal{H}_{k}
$$

In view of Lemma 2.3, the sequence $\left(f_{k}^{H}\right)$ is monotone non-increasing. Moreover, $f_{\min , \mathbf{K}} \leq f_{k}^{H}$ for all $k$. Next we show that the sequence $\left(f_{k}^{H}\right)$ converges to $f_{\min , \mathbf{K}}$.

To this end, let $\epsilon>0$. As the sequence $\left(f_{k}^{s o s}\right)$ converges to $f_{\min , \mathbf{K}}$ (Theorem 2.1), there exists an integer $k$ such that

$$
f_{\min , \mathbf{K}} \leq f_{k}^{s o s} \leq f_{\min , \mathbf{K}}+\epsilon
$$

Next, there exists a polynomial $\sigma \in \Sigma_{k}$ such that $\int_{\mathbf{K}} \sigma(\mathbf{x}) d \mathbf{x}=1$ and

$$
f_{k}^{s o s} \leq \int_{\mathbf{K}} f(\mathbf{x}) \sigma(\mathbf{x}) d \mathbf{x} \leq f_{k}^{s o s}+\epsilon
$$

Define now the polynomial $\hat{\sigma}(\mathbf{x})=\sigma(\mathbf{x})+\epsilon$. Then $\hat{\sigma}$ is positive on $[0,1]^{n}$, and thus, by Theorem 2.2. $\hat{\sigma} \in \mathcal{H}_{j_{k}}$ for some integer $j_{k}$. Observe that

$$
\int_{\mathbf{K}} \hat{\sigma}(\mathbf{x}) d \mathbf{x}=\int_{\mathbf{K}}(\sigma(\mathbf{x})+\epsilon) d \mathbf{x} \geq \int_{\mathbf{K}} \sigma(\mathbf{x}) d \mathbf{x}=1
$$

Hence we obtain:
$f_{j_{k}}^{H}-f_{\min , \mathbf{K}} \leq \frac{\int_{\mathbf{K}} f(\mathbf{x}) \hat{\sigma}(\mathbf{x}) d \mathbf{x}}{\int_{\mathbf{K}} \hat{\sigma}(\mathbf{x}) d \mathbf{x}}-f_{\min , \mathbf{K}}=\frac{\int_{\mathbf{K}}\left(f(\mathbf{x})-f_{\min , \mathbf{K}}\right) \hat{\sigma}(\mathbf{x}) d \mathbf{x}}{\int_{\mathbf{K}} \hat{\sigma}(\mathbf{x}) d \mathbf{x}} \leq \int_{\mathbf{K}}\left(f(\mathbf{x})-f_{\min , \mathbf{K}}\right) \hat{\sigma}(\mathbf{x}) d \mathbf{x}$.
The right most term is equal to
$\int_{\mathbf{K}}\left(f(\mathbf{x})-f_{\min , \mathbf{K}}\right) \sigma(\mathbf{x}) d \mathbf{x}+\epsilon \int_{\mathbf{K}}\left(f(\mathbf{x})-f_{\min , \mathbf{K}}\right) d \mathbf{x}=\int_{\mathbf{K}} f(\mathbf{x}) \sigma(\mathbf{x}) d \mathbf{x}-f_{\min , \mathbf{K}}+\epsilon \int_{\mathbf{K}}\left(f(\mathbf{x})-f_{\min , \mathbf{K}}\right) d \mathbf{x}$,
where we used the fact that $\int_{\mathbf{K}} \sigma(\mathbf{x}) d \mathbf{x}=1$. Finally, combining with the fact that $\int_{\mathbf{K}} f(\mathbf{x}) \sigma(\mathbf{x}) d \mathbf{x} \leq$ $f_{k}^{s o s}+\epsilon \leq f_{\min , \mathbf{K}}+2 \epsilon$, we can derive that

$$
f_{j_{k}}^{H}-f_{\min , \mathbf{K}} \leq \epsilon\left(2+\int_{\mathbf{K}}\left(f(\mathbf{x})-f_{\min , \mathbf{K}}\right) d \mathbf{x}\right)=\epsilon C
$$

where $C:=2+\int_{\mathbf{K}}\left(f(\mathbf{x})-f_{\min , \mathbf{K}}\right) d \mathbf{x}$ is a constant. This concludes the proof.
4. Bounding the rate of convergence for the bounds $f_{k}^{H}$ on $\mathbf{K}=[0,1]^{n}$

In this section we analyze the convergence rate of the bounds $f_{k}^{H}$ for the hypercube $\mathbf{K}=[0,1]^{n}$. We prove a convergence rate in $O(1 / \sqrt{k})$ for the range $f_{k}^{H}-f_{\min , \mathbf{K}}$ in general, and a stronger convergence rate in $O(1 / k)$ when $f$ has a rational global minimizer in $[0,1]^{n}$, which is the case, for instance, when $f$ is quadratic.

Our main tool will be exploiting some properties of the moments $\gamma_{(\eta, \beta)}$ which, as we recall below, arise from the moments of the beta distribution.
4.1. The beta distribution. By definition, a random variable $Y \in[0,1]$ has the beta distribution with shape parameters $a>0$ and $b>0$ (denoted by $Y \sim \operatorname{beta}(a, b))$ if its probability density function is given by

$$
y \mapsto \frac{y^{a-1}(1-y)^{b-1}}{\int_{0}^{1} t^{a-1}(1-t)^{b-1} d t}
$$

If $a>1$ and $b>1$, then the (unique) mode of the distribution (i.e. the maximizer of the density function) is

$$
\begin{equation*}
y=(a-1) /(a+b-2) \tag{4.1}
\end{equation*}
$$

Moreover, the $k$-th moment of $Y$ is

$$
\begin{equation*}
\mathbb{E}\left(Y^{k}\right)=\frac{a(a+1) \cdots(a+k-1)}{(a+b)(a+b+1) \cdots(a+b+k-1)}, \quad(k=1,2,3, \ldots) \tag{4.2}
\end{equation*}
$$

(see, e.g, 12, Chapter 24]; this also follows using (3.2)).
4.2. Proof of convergence rate. Given a polynomial $f$, consider a global minimizer $\mathbf{x}^{*}$ of $f$ in $[0,1]^{n}$. In what follows we indicate how to construct a vector of independent random variables $X=\left(X_{1}, \ldots, X_{n}\right)$ so that the $X_{i}$ 's have the beta distribution with suitable shape parameters $\eta_{i}^{*}, \beta_{i}^{*}$, designed to ensure that (roughly) $\mathbb{E}[X]=\mathbf{x}^{*}$.

We will use the following result about Diophantine approximations.
Theorem 4.1 (Dirichlet's theorem). (see e.g. [24, Chapter 6.1]) Consider a real number $x \in \mathbb{R}$ and $0<\epsilon \leq 1$. Then there exist integers $p$ and $q$ satisfying

$$
\begin{equation*}
\left|x-\frac{p}{q}\right|<\frac{\epsilon}{q} \quad \text { and } \quad 1 \leq q \leq \frac{1}{\epsilon} \tag{4.3}
\end{equation*}
$$

If $x \in(0,1)$, then one may moreover assume $0 \leq p \leq q$.
If $x_{i}^{*} \in(0,1)$ is a rational coordinate of $\mathbf{x}^{*}$, then we select integers $p_{i}$ and $q_{i}$ such that $x_{i}^{*}=p_{i} / q_{i}$, so that $1 \leq p_{i}<q_{i}$. When $x_{i}^{*}$ is an irrational coordinate of $\mathbf{x}^{*}$ we use Theorem4.1 to construct a pair of suitable integers $p_{i}, q_{i}$. Namely, we consider an integer $r \geq 1$ and apply Theorem 4.1 with $\epsilon=1 / r$. Then, there exist integers $p_{i}$ and $q_{i}$ satisfying

$$
\begin{equation*}
\left|x_{i}^{*}-\frac{p_{i}}{q_{i}}\right|<\frac{1}{r q_{i}}, \quad 0 \leq p_{i} \leq q_{i} \leq r \text { and } 1 \leq q_{i} \tag{4.4}
\end{equation*}
$$

For convenience, let $I_{0}$ (resp., $I_{1}, I$ ) denote the set of indices $i \in[n]$ for which $x_{i}^{*}$ is irrational and the integers $p_{i}$ and $q_{i}$ in (4.4) satisfy: $p_{i}=0$ (resp., $p_{i}=q_{i}, 1 \leq p_{i}<q_{i}$ ). Moreover, define the set $J$ consisting of all indices $i$ for which $x_{i}^{*} \in(0,1)$ is rational. Then, $x_{i}^{*} \in\{0,1\}$ for all $i \in[n] \backslash\left(I_{0} \cup I_{1} \cup I \cup J\right)$.

We now indicate how to construct the parameters $\eta_{i}^{*}$ and $\beta_{i}^{*}$.
Definition 4.2. Let $r$ be a positive integer. For each coordinate $x_{i}^{*} \in[0,1]$, consider the integers $p_{i}$ and $q_{i}$ defined as above. We define the parameters $\eta_{i}^{*}$ and $\beta_{i}^{*} \in \mathbb{N}$ as follows.
(i) Assume $i \in J \cup I$; that is, either $x_{i}^{*} \in(0,1)$ is rational of the form $x_{i}^{*}=p_{i} / q_{i}$, or $x_{i}^{*}$ is irrational with $1 \leq p_{i}<q_{i}$. Then, we set $\eta_{i}^{*}=r p_{i}$ and $\beta_{i}^{*}=r\left(q_{i}-p_{i}\right)$.
(ii) Assume either $x_{i}^{*}=0$, or $i \in I_{0}$, i.e., $x_{i}^{*}$ is irrational with $p_{i}=0$. Then we set $\eta_{i}^{*}=1$ and $\beta_{i}^{*}=r$.
(iii) Assume either $x_{i}^{*}=1$, or $i \in I_{1}$, i.e., $x_{i}^{*}$ is irrational with $p_{i}=q_{i}$. Then we set $\eta_{i}^{*}=r$ and $\beta_{i}^{*}=1$.

Now we define the vector of independent random variables $X:=\left(X_{1}, \ldots, X_{n}\right)$, where $X_{i} \sim \operatorname{beta}\left(\eta_{i}^{*}, \beta_{i}^{*}\right)$ $(i \in[n])$.

For given $\alpha \in \mathbb{N}^{n}$, we denote $X^{\alpha}=\prod_{i=1}^{n} X_{i}^{\alpha_{i}}$. Since the random variables $X_{i}$ 's are independent we have $\mathbb{E}\left(X^{\alpha}\right)=\prod_{i=1}^{n} \mathbb{E}\left(X_{i}^{\alpha_{i}}\right)$ and the expected value of $f(X)=\sum_{\alpha \in \mathbb{N}^{n}} f_{\alpha} X^{\alpha}$ is given by

$$
\begin{equation*}
\mathbb{E}(f(X))=\sum_{\alpha \in \mathbb{N}^{n}} f_{\alpha} \mathbb{E}\left(X^{\alpha}\right)=\sum_{\alpha \in \mathbb{N}^{n}} f_{\alpha} \prod_{i=1}^{n} \mathbb{E}\left(X_{i}^{\alpha_{i}}\right) \tag{4.5}
\end{equation*}
$$

where $\mathbb{E}\left(X_{i}^{\alpha_{i}}\right)$ can be computed using (4.2). Observe moreover that, by construction,

$$
\begin{equation*}
\mathbb{E}(f(X))=\frac{\int_{[0,1]^{n}} f(\mathbf{x}) \mathbf{x}^{\eta^{*}-\mathbf{1}}(1-\mathbf{x})^{\beta^{*}-\mathbf{1}} d \mathbf{x}}{\int_{[0,1]^{n}} \mathbf{x}^{\eta^{*}-\mathbf{1}}(1-\mathbf{x})^{\beta^{*}-\mathbf{1}} d \mathbf{x}} \geq f_{k_{r}}^{H} \geq f\left(\mathbf{x}^{*}\right) \tag{4.6}
\end{equation*}
$$

where we set

$$
k_{r}:=\sum_{i=1}^{n}\left(\eta_{i}^{*}-1+\beta_{i}^{*}-1\right)
$$

and let $\mathbf{1}$ denote the all-ones vector. We will also use the following estimate on the parameter $k_{r}$.
Lemma 4.3. Consider the parameter $k_{r}=\sum_{i=1}^{n}\left(\eta_{i}^{*}-1+\beta_{i}^{*}-1\right)$. Then the following holds:
(i) If $x^{*} \in \mathbb{Q}$ then $k_{r} \leq$ ar for all $r \geq 1$, where $a>0$ is a constant (not depending on $r$ ).
(ii) If $x^{*} \in \mathbb{R} \backslash \mathbb{Q}$ then $\bar{k}_{r} \leq a^{\prime} r^{2}$ for all $r \geq 1$, where $a^{\prime}>0$ is a constant (not depending on $r$ ).
(iii) For $r=1$, we have that $k_{1}=\sum_{i \in J} q_{i}-2|J|$.

Proof. By construction, $\eta_{i}^{*}+\beta_{i}^{*}-2=r q_{i}-2$ for each $i \in I \cup J$, and $\eta_{i}^{*}+\beta_{i}^{*}-2=r-1$ otherwise. From this one gets $k_{r}=r\left(\sum_{i \in I \cup J} q_{i}+n-|I \cup J|\right)-n-|I \cup J|=$ : ar $-b$, after setting $b:=n+|I \cup J|$ and $a:=\sum_{i \in I \cup J} q_{i}+n-|I \cup J|$, so that $a, b \geq 0$. Thus, $k_{r} \leq a r$ holds.

Next, note that $q_{i} \leq r$ for each $i \in I$, while $q_{i}$ does not depend on $r$ for $i \in J$ (since then $x_{i}^{*}=p_{i} / q_{i}$ ). Hence, in case (i), $I=\emptyset$ and the constant $a$ does not depend on $r$. In case (ii), we obtain: $a \leq r|I|+\sum_{i \in J} q_{i}+n-|I \cup J| \leq a^{\prime} r$, after setting $a^{\prime}:=|I|+\sum_{i \in J} q_{i}+n-|I \cup J|$, which is thus a constant not depending on $r$. Then, $k_{r} \leq a r \leq a^{\prime} r^{2}$.

In the case $r=1$ the set $I$ is empty and thus $k_{1}=\sum_{i \in J} q_{i}-2|J|$, showing (iii).
We can prove the following upper bound for the range $\mathbb{E}(f(X))-f\left(\mathbf{x}^{*}\right)$, which will be crucial for establishing the rate of convergence of the parameters $f_{k}^{H}$.

Theorem 4.4. Given a polynomial $f$, consider a global minimizer $\mathbf{x}^{*}$ of $f$ in $[0,1]^{n}$. Let $r$ be a positive integer. For any $x_{i}^{*} \in[0,1](i \in[n])$, consider the parameters $\eta_{i}^{*}, \beta_{i}^{*}$ and random variables $X_{i}$ in Definition 4.2. Then there exists a constant $C_{f}>0$ (depending only on $f$ ) such that

$$
\mathbb{E}(f(X))-f\left(\mathbf{x}^{*}\right) \leq \frac{C_{f}}{r}
$$

For the proof of Theorem 4.4, we need the following three technical lemmas.
Lemma 4.5. Let $k$ be a positive integer. There exists a constant $C_{k}>0$ (depending only on $k$ ) for which the following relation holds:

$$
\begin{equation*}
\frac{r p(r p+1) \cdots(r p+k-1)}{r q(r q+1) \cdots(r q+k-1)}-\frac{p^{k}}{q^{k}} \leq \frac{C_{k}}{r} \tag{4.7}
\end{equation*}
$$

for all integers $1 \leq p<q$ and $r \geq 1$.

Proof. Consider the univariate polynomial $\phi(t)=(t+1) \cdots(t+k-1)=\sum_{i=0}^{k-1} a_{i} t^{i}$, where the scalars $a_{i}>0$ depend only on $k$ and $a_{k-1}=1$. Denote by $\Delta$ the left hand side in (4.7), which can be written as $\Delta=N / D$, where we set

$$
N:=r p q^{k} \phi(r p)-r q p^{k} \phi(r q), \quad D:=r q^{k+1} \phi(r q) .
$$

We first work out the term $N$ :

$$
N=\operatorname{rpq}\left(\sum_{i=0}^{k-2} a_{i} r^{i} p^{i} q^{k-1}-\sum_{i=0}^{k-2} a_{i} r^{i} q^{i} p^{k-1}\right)=r p q \sum_{i=0}^{k-2} a_{i} r^{i} p^{i} q^{i}\left(q^{k-1-i}-p^{k-1-i}\right)
$$

Write: $q^{k-1-i}-p^{k-1-i}=(q-p) \sum_{j=0}^{k-2-i} q^{j} p^{k-2-i-j} \leq(q-p) q^{k-2-i}(k-1-i)$, where we use the fact that $p<q$. This implies:

$$
N \leq r p q(q-p) \sum_{i=0}^{k-2} a_{i} r^{i} p^{i} q^{k-2}(k-1-i)=r p q^{k-1}(q-p) \sum_{i=0}^{k-2} a_{i}(k-1-i) r^{i} p^{i}=: N^{\prime}
$$

Thus we get:

$$
\Delta \leq \frac{N^{\prime}}{D}=\frac{p(q-p)}{q^{2}} \cdot \frac{\sum_{i=0}^{k-2} a_{i}(k-1-i) r^{i} p^{i}}{\phi(r q)}
$$

The first factor is at most 1 , since one has: $p(q-p) \leq q^{2}$, as $q^{2}-p(q-p)=(q-p)^{2}+p q$. Second, we bound the sum $\sum_{i=0}^{k-2} a_{i}(k-1-i) r^{i} p^{i}$ in terms of $\phi(r q)=\sum_{j=0}^{k-1} a_{j} r^{j} q^{j}$. Namely, define the constant

$$
C_{k}:=\max _{0 \leq i \leq k-2} \frac{a_{i}(k-1-i)}{a_{i+1}},
$$

which depends only on $k$. We show that

$$
a_{i}(k-1-i) r^{i} p^{i} \leq \frac{C_{k}}{r}
$$

Indeed, for each $0 \leq i \leq k-2$, using $p^{i} \leq q^{i+1}$ and the definition of $C_{k}$, we get:

$$
r \cdot a_{i}(k-1-i) r^{i} p^{i} \leq a_{i}(k-1-i) r^{i+1} q^{i+1} \leq C_{k} a_{i+1} r^{i+1} q^{i+1}
$$

Summing over $i=0,1, \ldots, k-2$ gives:

$$
r \sum_{i=0}^{k-2} a_{i}(k-1-i) r^{i} p^{i} \leq C_{k} \sum_{i=0}^{k-2} a_{i+1} r^{i+1} q^{i+1} \leq C_{k} \phi(r q)
$$

and thus

$$
\Delta \leq \frac{N^{\prime}}{D} \leq \frac{C_{k}}{r}
$$

as desired.
Lemma 4.6. Let $r$ be a positive integer. For any $x_{i}^{*} \in[0,1]$, we consider the parameters $\eta_{i}^{*}, \beta_{i}^{*}$ and random variables $X_{i}$ in Definition 4.2. For any integer $k \geq 1$, there exists a constant $C_{k}^{\prime}>0$ (depending only on $k$ ) for which the following holds:

$$
\left|\mathbb{E}\left(X_{i}^{k}\right)-\left(x_{i}^{*}\right)^{k}\right| \leq \frac{C_{k}^{\prime}}{r}
$$

Proof. (i) We consider first the case when $x_{i}^{*} \in(0,1)$ with $i \in J \cup I$. Then, by Definition 4.2 (i), one has $\eta_{i}^{*}=r p_{i}$ and $\beta_{i}^{*}=r\left(q_{i}-p_{i}\right)$, where the integers $p_{i}, q_{i}$ satisfy $1 \leq p_{i}<q_{i}$, and either $x_{i}^{*}=p_{i} / q_{i}$ if $x_{i}^{*}$ is rational, or $\left|x_{i}^{*}-p_{i} / q_{i}\right| \leq 1 /\left(q_{i} r\right) \leq 1 / r$ if $x_{i}^{*}$ is irrational. Then, by (4.2), the $k$-th moment of $X_{i}$ is

$$
\mathbb{E}\left(X_{i}^{k}\right)=\frac{\eta_{i}^{*}\left(\eta_{i}^{*}+1\right) \cdots\left(\eta_{i}^{*}+k-1\right)}{\left(\eta_{i}^{*}+\beta_{i}^{*}\right)\left(\eta_{i}^{*}+\beta_{i}^{*}+1\right) \cdots\left(\eta_{i}^{*}+\beta_{i}^{*}+k-1\right)}=\frac{r p_{i}\left(r p_{i}+1\right) \cdots\left(r p_{i}+k-1\right)}{r q_{i}\left(r q_{i}+1\right) \cdots\left(r q_{i}+k-1\right)}
$$

and we obtain:

$$
\left|\mathbb{E}\left(X_{i}^{k}\right)-\left(x_{i}^{*}\right)^{k}\right| \leq \underbrace{\left|\frac{r p_{i}\left(r p_{i}+1\right) \cdots\left(r p_{i}+k-1\right)}{r q_{i}\left(r q_{i}+1\right) \cdots\left(r q_{i}+k-1\right)}-\frac{p_{i}^{k}}{q_{i}^{k}}\right|}_{=: T_{1}}+\underbrace{\left|\frac{p_{i}^{k}}{q_{i}^{k}}-\left(x_{i}^{*}\right)^{k}\right|}_{=: T_{2}}
$$

For the term $T_{1}$, Lemma 4.5 implies:

$$
T_{1} \leq \frac{C_{k}}{r}
$$

For the term $T_{2}$, we have:

$$
T_{2}=\left|\frac{p_{i}^{k}}{q_{i}^{k}}-\left(x_{i}^{*}\right)^{k}\right|=\left|\frac{p_{i}}{q_{i}}-x_{i}^{*}\right| \cdot\left(\sum_{h=0}^{k-1}\left(\frac{p_{i}}{q_{i}}\right)^{h}\left(x_{i}^{*}\right)^{k-h-1}\right) \leq \frac{k}{r}
$$

since the first factor is at most $1 / r$ and, in the second factor, each term in the summation is bounded by 1. Summarizing, we obtain: $\left|\mathbb{E}\left(X_{i}^{k}\right)-\left(x_{i}^{*}\right)^{k}\right| \leq T_{1}+T_{2} \leq\left(C_{k}+k\right) / r$.
(ii) When $x_{i}^{*}=0$, by Definition 4.2 (ii), one has $\eta_{i}^{*}=1$ and $\beta_{i}^{*}=r$. Thus we have:

$$
\mathbb{E}\left(X_{i}^{k}\right)-\left(x_{i}^{*}\right)^{k}=E\left(X_{i}^{k}\right)=\frac{k!}{(r+1)(r+2) \cdots(r+k)} \leq \frac{k!}{r}
$$

(iii) When $i \in I_{0}$, then $x_{i}^{*} \leq 1 /\left(q_{i} r\right) \leq 1 / r$ and, using the above inequality in (ii), we get:

$$
\left|\mathbb{E}\left(X_{i}^{k}\right)-\left(x_{i}^{*}\right)^{k}\right| \leq E\left(X_{i}^{k}\right)+\left(x_{i}^{*}\right)^{k} \leq \frac{k!}{r}+\frac{1}{r^{k}} \leq \frac{k!+1}{r}
$$

(iv) When $x_{i}^{*}=1$, by Definition 4.2 (iii), one has $\eta_{i}^{*}=r$ and $\beta_{i}^{*}=1$. Thus we have:

$$
\left|\mathbb{E}\left(X_{i}^{k}\right)-\left(x_{i}^{*}\right)^{k}\right|=\left|\mathbb{E}\left(X_{i}^{k}\right)-1\right|=\frac{k}{r+k} \leq \frac{k}{r}
$$

(v) Finally, if $i \in I_{1}$, then $1-x_{i}^{*} \leq 1 /\left(q_{i} r\right) \leq 1 / r$ and, using the above inequality in (iv), we get:

$$
\left|\mathbb{E}\left(X_{i}^{k}\right)-\left(x_{i}^{*}\right)^{k}\right| \leq\left|\mathbb{E}\left(X_{i}^{k}\right)-1\right|+\left(1-\left(x_{i}^{*}\right)^{k}\right) \leq \frac{k}{r}+\frac{k}{r} \leq \frac{2 k}{r}
$$

where we have used $1-\left(x_{i}^{*}\right)^{k}=\left(1-x_{i}^{*}\right) \sum_{h=0}^{k-1}\left(x_{i}^{*}\right)^{k-h-1} \leq k\left(1-x^{*}\right) \leq k / r$.
In all cases (i)-(v), we found $\left|\mathbb{E}\left(X_{i}^{k}\right)-\left(x_{i}^{*}\right)^{k}\right| \leq C_{k}^{\prime} / r$, after setting $C_{k}^{\prime}=\max \left\{C_{k}+k, k!+1,2 k\right\}$.
Lemma 4.7. For any $x, y \in \mathbb{R}^{n}$, one has the following equality:

$$
\prod_{i=1}^{n} x_{i}-\prod_{i=1}^{n} y_{i}=\sum_{i=1}^{n}\left[\left(x_{i}-y_{i}\right) \prod_{j=1}^{i-1} y_{j} \prod_{j=i+1}^{n} x_{j}\right]
$$

Proof. Proof by direct verification.

Now we can prove Theorem 4.4.
Proof. (of Theorem 4.4) We write $f(\mathbf{x})=\sum_{\alpha \in \mathbb{N}^{n}} f_{\alpha} \mathbf{x}^{\alpha}$. From the definition (4.5), we have

$$
\mathbb{E}(f(X))-f\left(\mathbf{x}^{*}\right)=\sum_{\alpha \in \mathbb{N}^{n}} f_{\alpha} \underbrace{\left(\prod_{i=1}^{n} \mathbb{E}\left(X_{i}^{\alpha_{i}}\right)-\prod_{i=1}^{n}\left(x_{i}^{*}\right)^{\alpha_{i}}\right)}_{=: A} .
$$

By Lemma 4.7, one has

$$
A=\sum_{i=1}^{n}(\left(\mathbb{E}\left(X_{i}^{\alpha_{i}}\right)-\left(x_{i}^{*}\right)^{\alpha_{i}}\right) \underbrace{\prod_{j=1}^{i-1}\left(x_{j}^{*}\right)^{\alpha_{j}}}_{=: B} \underbrace{\prod_{j=i+1}^{n} \mathbb{E}\left(X_{j}^{\alpha_{j}}\right)}_{=: C}) .
$$

Since $x_{i}^{*} \in[0,1]$ for any $i \in[n]$, then $0 \leq B \leq 1$. Moreover, by Definition 4.2 and (4.2), one has that $\mathbb{E}\left(X_{i}^{\alpha_{i}}\right) \in[0,1]$ for any $i \in[n]$, and thus $0 \leq C \leq 1$. Combining with Lemma 4.6, we can conclude: $|A| \leq\left(\sum_{i=1}^{n} C_{\alpha_{i}}^{\prime}\right) / r$. Therefore, we obtain that

$$
\mathbb{E}(f(X))-f\left(\mathbf{x}^{*}\right) \leq \sum_{\alpha \in \mathbb{N}^{n}}\left|f_{\alpha}\right| \sum_{i=1}^{n}\left|\mathbb{E}\left(X_{i}^{\alpha_{i}}\right)-\left(x_{i}^{*}\right)^{\alpha_{i}}\right|
$$

where the right hand side is at most $C_{f} / r$, after setting $C_{f}:=\sum_{\alpha \in \mathbb{N} n}\left|f_{\alpha}\right| \sum_{i=1}^{n} C_{\alpha_{i}}^{\prime}$. This concludes the proof.

We can now show the following results for the rate of convergence of the sequence $f_{k}^{H}$.
Theorem 4.8. Given a polynomial $f$, let $x^{*}$ be a global minimizer of $f$ in $[0,1]^{n}$ and consider as before the parameters

$$
f_{k}^{H}=\min _{(\eta, \beta) \in \mathbb{N}_{k}^{2 n}} \frac{\int_{[0,1]^{n}} f(\mathbf{x}) \mathbf{x}^{\eta}(1-\mathbf{x})^{\beta} d \mathbf{x}}{\int_{[0,1]^{n}} \mathbf{x}^{\eta}(1-\mathbf{x})^{\beta} d \mathbf{x}} \quad(k=1,2, \ldots)
$$

There exists a constant $M_{f}$ (depending only on $f$ ) such that

$$
\begin{equation*}
f_{k}^{H}-f\left(\mathbf{x}^{*}\right) \leq \frac{M_{f}}{\sqrt{k}} \quad \text { for all } k \geq k_{1} \tag{4.8}
\end{equation*}
$$

where $k_{1}=\sum_{i \in J} q_{i}-2|J|$ (as in Lemma 4.3 (iii)). Moreover, if $f$ has a rational global minimizer $x^{*}$, then there exists a constant $M_{f}^{\prime}$ (depending only on $f$ ) such that

$$
\begin{equation*}
f_{k}^{H}-f\left(\mathbf{x}^{*}\right) \leq \frac{M_{f}^{\prime}}{k} \quad \text { for all } k \geq k_{1} \tag{4.9}
\end{equation*}
$$

In particular, the convergence rate is in $O(1 / k)$ when $f$ is a quadratic polynomial.
Proof. Consider an integer $r \geq 1$ and, as in Definition 4.2, the parameters $\eta_{i}^{*}, \beta_{i}^{*}$ and the random variables $X_{i} \sim \operatorname{beta}\left(\eta_{i}^{*}, \beta_{i}^{*}\right)$ for $i \in[n]$. Recall also the parameter $k_{r}$ from Lemma 4.3. Then, as observed earlier in (4.6), by the definition of the parameter $f_{k_{r}}^{H}$, we have that $\mathbb{E}(f(X)) \geq f_{k_{r}}^{H}$. Moreover, by Theorem 4.4, $\mathbb{E}(f(X))-f\left(\mathbf{x}^{*}\right) \leq C_{f} / r$ for some constant $C_{f}$ (depending only on $f$ ). Therefore, we have the following inequality:

$$
\begin{equation*}
f_{k_{r}}^{H}-f\left(\mathrm{x}^{*}\right) \leq \frac{C_{f}}{r} \tag{4.10}
\end{equation*}
$$

We now consider an arbitrary integer $k \geq k_{1}$. Let $r \geq 1$ be the largest integer for which $k \geq k_{r}$. Then we have $k_{r} \leq k<k_{r+1}$. As $k_{r} \leq k$, we have the inequality $f_{k}^{H}-f\left(\mathbf{x}^{*}\right) \leq f_{k_{r}}^{H}-f\left(\mathbf{x}^{*}\right)$ and thus, using (4.10), $f_{k}^{H}-f\left(\mathbf{x}^{*}\right) \leq \frac{C_{f}}{r}$. We now bound $1 / r$ in terms of $k$.

If $x^{*} \in \mathbb{Q}$ then, by Lemma 4.3 (i), $k_{r+1} \leq a(r+1) \leq 2 a r$, which implies $k \leq k_{r+1} \leq 2 a r$, where the constant $a$ does not depend on $r$. Thus, $f_{k}^{H}-f\left(\mathrm{x}^{*}\right) \leq \frac{C_{f}}{r} \leq \frac{2 a C_{f}}{k}=\frac{M_{f}}{k}$, where the constant $M_{f}=2 a C_{f}$ depends only on $f$. This shows (4.9).

If $x^{*} \notin \mathbb{Q}$ then, by Lemma 4.3 (ii), $k_{r+1} \leq a^{\prime}(r+1)^{2} \leq 4 a^{\prime} r^{2}$, which implies $k \leq k_{r+1} \leq 4 a^{\prime} r^{2}$ and thus $\frac{1}{r} \leq \frac{2 \sqrt{a^{\prime}}}{\sqrt{k}}$, where the constant $a^{\prime}$ does not depend on $r$. Therefore, $f_{k}^{H}-f\left(\mathbf{x}^{*}\right) \leq \frac{C_{f}}{r} \leq \frac{2 \sqrt{a^{\prime}} C_{f}}{\sqrt{k}}$, which shows that $f_{k}^{H}-f\left(\mathbf{x}^{*}\right) \leq \frac{M_{f}^{\prime}}{\sqrt{k}}$ and thus (4.8), after setting $M_{f}^{\prime}=2 \sqrt{a^{\prime}} C_{f}$.

Finally, if $f$ is quadratic then, by a result of Vavasis [25], $f$ has a rational minimizer over the hypercube and thus the rate of convergence is $O(1 / k)$.

Note that the inequalities (4.8) and (4.9) hold for all $k \geq k_{1}$, where $k_{1}$ depends only on the rational components in $(0,1)$ of the minimizer $x^{*}$. The constant $k_{1}$ can be in $O(1)$, e.g., when all but $O(1)$ of these rational components have a small denominator (say, equal to 2 ). Thus we can, for some problem classes, get a bound with an error estimate in polynomial time.

Example 4.9. Consider the polynomial $f=\sum_{i=1}^{n} x_{i}$ and the set $\mathbf{K}=[0,1]^{n}$. Then $f_{\text {min }, \mathbf{K}}=0$ is attained at the zero vector. Using the relations (3.1), (3.2) and (3.3) it follows that $f_{k}^{H}=$ $\min _{(\eta, \beta) \in \mathbb{N}_{k}^{2 n}} \sum_{i=1}^{n} \frac{\eta_{i}+1}{\eta_{i}+\beta_{i}+2}$. Since $\eta_{i}+\beta_{i} \leq k$ and $\eta_{i} \geq 0$ (for any $i \in[n]$ ), we have $f_{k}^{H} \geq \frac{n}{k+2}$.
By this example, there does not exist any $\delta>0$ such that, for any $f, f_{k}^{H}-f_{\min , \mathbf{K}}=O\left(1 / k^{1+\delta}\right)$. Therefore, when a rational minimizer exists, the convergence rate from Theorem 4.8 in $O(1 / k)$ for $f_{k}^{H}$ is tight.

## 5. Bounding the rate of convergence for grid search over $\mathbf{K}=[0,1]^{n}$

As an alternative to computing $f_{k}^{H}$ on $\mathbf{K}=Q:=[0,1]^{n}$, one may minimize $f$ over the regular grid:

$$
Q(k):=\left\{\mathbf{x} \in Q=[0,1]^{n} \mid k \mathbf{x} \in \mathbb{N}^{n}\right\}
$$

i.e., the set of rational points in $[0,1]^{n}$ with denominator $k$. Thus we get the upper bound

$$
f_{\min , Q(k)}:=\min _{\mathbf{x} \in Q(k)} f(x) \geq f_{\min , Q} \quad k=1,2, \ldots
$$

De Klerk and Laurent [3] showed a rate of convergence in $O(1 / k)$ for this sequence of upper bounds:

$$
\begin{equation*}
f_{\min , Q(k)}-f_{\min , Q} \leq \frac{L(f)}{k}\binom{d+1}{3} n^{d} \text { for any } k \geq d \tag{5.1}
\end{equation*}
$$

where $d$ is the degree of $f$ and $L(f)$ is the constant

$$
L(f)=\max _{\alpha}\left|f_{\alpha}\right| \frac{\prod_{i=1}^{n} \alpha_{i}!}{|\alpha|!}
$$

We can in fact show a stronger convergence rate in $O\left(1 / k^{2}\right)$.
Theorem 5.1. Let $f$ be a polynomial and let $\mathbf{x}^{*}$ be a global minimizer of $f$ in $[0,1]^{n}$. Then there exists a constant $C_{f}$ (depending on $f$ ) such that

$$
f_{\min , Q(k)}-f\left(\mathrm{x}^{*}\right) \leq \frac{C_{f}}{k^{2}} \quad \text { for all } k \geq 1
$$

Proof. Fix $k \geq 1$. By looking at the grid point in $Q(k)$ closest to $\mathbf{x}^{*}$, there exists $\mathbf{h} \in[0,1]^{n}$ such that $\mathbf{x}^{*}+\mathbf{h} \in Q(k)$ and $\|\mathbf{h}\| \leq \frac{\sqrt{n}}{k}$. Then, by Taylor's theorem, we have that

$$
\begin{equation*}
f\left(\mathbf{x}^{*}+\mathbf{h}\right)=f\left(\mathbf{x}^{*}\right)+\mathbf{h}^{T} \nabla f\left(\mathbf{x}^{*}\right)+\frac{1}{2} \mathbf{h}^{T} \nabla^{2} f(\zeta) \mathbf{h} \tag{5.2}
\end{equation*}
$$

for some point $\zeta$ lying in the segment $\left[\mathbf{x}^{*}, \mathbf{x}^{*}+\mathbf{h}\right] \subseteq[0,1]^{n}$.
Assume first that the global minimizer $\mathbf{x}^{*}$ lies in the interior of $[0,1]^{n}$. Then $\nabla f\left(\mathbf{x}^{*}\right)=0$ and thus

$$
f_{\min , Q(k)}-f\left(\mathbf{x}^{*}\right) \leq f\left(\mathbf{x}^{*}+\mathbf{h}\right)-f\left(\mathbf{x}^{*}\right) \leq C\|\mathbf{h}\|^{2} \leq \frac{n C}{k^{2}}
$$

after setting $C:=\max _{\zeta \in[0,1]^{n}}\left\|\nabla^{2} f(\zeta)\right\| / 2$.
Assume now that $\mathbf{x}^{*}$ lies on the boundary of $[0,1]^{n}$ and let $I_{0}$ (resp., $I_{1}, I$ ) denote the set of indices $i \in[n]$ for which $x_{i}^{*}=0$ (resp., $x_{i}^{*}=1, x_{i}^{*} \in(0,1)$ ). Define the polynomial $g(y)=$ $f(y, 0, \ldots, 0,1, \ldots, 1)$ (with 0 at the positions $i \in I_{0}$ and 1 at the positions $i \in I_{1}$ ) in the variable $y \in \mathbb{R}^{|I|}$. Then $\mathbf{x}_{I}^{*}=\left(x_{i}^{*}\right)_{i \in I}$ is a global minimizer of $g$ over $[0,1]^{|I|}$ which lies in the interior. So we may apply the preceding reasoning to the polynomial $g$ and conclude that $g_{\min , Q(k)}-g\left(\mathbf{x}_{I}^{*}\right) \leq \frac{C^{\prime}}{k^{2}}$ for some constant $C^{\prime}$ (depending on $g$ and thus on $f$ ). As $f_{\min , Q(k)} \leq g_{\min , Q(k)}$ and $f\left(\mathbf{x}^{*}\right)=g\left(\mathbf{x}_{I}^{*}\right)$ the result follows.

Therefore the bounds $f_{\min , Q(k)}$ obtained through grid search have a faster convergence rate than the bounds $f_{k}^{H}$. However, for any fixed value of $k$, for the bound $f_{k}^{H}$ one needs a polynomial number $O\left(n^{k}\right)$ of computations (similar to function evaluations), while computing the bound $f_{\min , Q(k)}$ requires an exponential number $k^{n}$ of function evaluations. Hence the 'measure-based' guided search producing the bounds $f_{k}^{H}$ is superior to the brute force grid search technique in terms of complexity.

## 6. Obtaining feasible points $\mathbf{x}$ with $f(\mathbf{x}) \leq f_{k}^{H}$

In this section we describe how to generate a point $\mathbf{x} \in \mathbf{K} \subseteq[0,1]^{n}$ such that $f(\mathbf{x}) \leq f_{k}^{H}$ (or that $f(\mathbf{x}) \leq f_{k}^{H}+\epsilon$ for some small $\left.\epsilon>0\right)$.

We will discuss in turn:

- the convex case (and related cases), and
- the general case.
6.1. The convex case (and related cases): using the Jensen inequality. Our main tool for treating the convex case (and related cases) will be the Jensen inequality.

Lemma 6.1 (Jensen inequality). If $\mathcal{C} \subseteq \mathbb{R}^{n}$ is convex, $\phi: \mathcal{C} \rightarrow \mathbb{R}$ is a convex function, and $X \in \mathcal{C}$ a random variable, then

$$
\phi(\mathbb{E}(X)) \leq \mathbb{E}(\phi(X))
$$

Theorem 6.2. Assume that $\mathbf{K} \subseteq[0,1]^{n}$ is closed and convex, and $(\eta, \beta) \in \mathbb{N}_{k}^{2 n}$ is such that

$$
f_{k}^{H}=\frac{\int_{\mathbf{K}} f(\mathbf{x}) \mathbf{x}^{\eta}(1-\mathbf{x})^{\beta} d \mathbf{x}}{\int_{\mathbf{K}} \mathbf{x}^{\eta}(1-\mathbf{x})^{\beta} d \mathbf{x}}
$$

Let $X=\left(X_{1}, \ldots, X_{n}\right)$ be a vector of random variables with $X_{i} \sim \operatorname{beta}\left(\eta_{i}+1, \beta_{i}+1\right)(i \in[n])$.
Then one has $f(\mathbb{E}(X)) \leq f_{k}^{H}$ in the following cases:
(1) $f$ is convex;
(2) $f$ has only nonnegative coefficients;
(3) $f$ is square-free, i.e., $f(\mathbf{x})=\sum_{\alpha \in\{0,1\}^{n}} f_{\alpha} \mathbf{x}^{\alpha}$.

Proof. The proof uses the fact that, by construction,

$$
f_{k}^{H}=\mathbb{E}(f(X))
$$

Thus the first item follows immediately from Jensen's inequality. For the proof of the second item, recall that

$$
f_{k}^{H}=\mathbb{E}(f(X))=\sum_{\alpha \in \mathbb{N}^{n}} f_{\alpha} \prod_{i=1}^{n} \mathbb{E}\left(X_{i}^{\alpha_{i}}\right)
$$

where we now assume $f_{\alpha} \geq 0$ for all $\alpha$. Since $\phi\left(X_{i}\right)=X_{i}^{\alpha_{i}}$ is convex on $[0,1](i \in[n])$, Jensen's inequality yields $\mathbb{E}\left(X_{i}^{\alpha_{i}}\right) \geq\left[\mathbb{E}\left(X_{i}\right)\right]^{\alpha_{i}}$. Thus

$$
f_{k}^{H} \geq \sum_{\alpha \in \mathbb{N}^{n}} f_{\alpha} \mathbb{E}(X)^{\alpha}
$$

as required. For the third item, where $f$ is assumed square-free, one has

$$
f_{k}^{H}=\mathbb{E}(f(X))=\sum_{\alpha \in \mathbb{N}^{n}} f_{\alpha} \prod_{i=1}^{n} \mathbb{E}\left(X_{i}^{\alpha_{i}}\right)
$$

where all $\alpha \in\{0,1\}^{n}$ so that $\mathbb{E}\left(X_{i}^{\alpha_{i}}\right)=\left[\mathbb{E}\left(X_{i}\right)\right]^{\alpha_{i}}$, and consequently

$$
f_{k}^{H}=\sum_{\alpha \in \mathbb{N} n} f_{\alpha} \mathbb{E}(X)^{\alpha}
$$

This completes the proof.

### 6.2. The general case.

Sampling. One may generate random samples $\mathbf{x} \in \mathbf{K}$ from the density $\sigma$ on $\mathbf{K}$ using the well-known method of conditional distributions (see e.g., [21, Section 8.5.1]). For $\mathbf{K}=[0,1]^{n}$, the procedure is described in detail in [6, Section 3]. In this way one may obtain, with high probability, a point $\mathbf{x} \in \mathbf{K}$ with $f(\mathbf{x}) \leq f_{k}^{H}+\epsilon$, for any given $\epsilon>0$. (The size of the sample depends on $\epsilon$.) Here we only mention that this procedure may be done in time polynomial in $n$ and $1 / \epsilon$; for details the reader is referred to [6, Section 3].

A heuristic based on the mode. As an alternative, one may consider the heuristic that returns the mode (i.e. maximizer) of the density $\sigma$ as a candidate solution; $c f$. Example 2.5. The mode may be calculated one variable at a time using (4.1).

In Section 7 below, we will illustrate the performance of all the strategies described in this section on numerical examples.

## 7. Numerical examples

In this section we will present numerical examples to illustrate the behaviour of the sequences of upper bounds, and of the techniques to obtain feasible points.
7.1. The complexity of computing $f_{k}^{H}$ and $f_{k}^{s o s}$. We let $N_{f}$ denote the set of indices $\alpha \in \mathbb{N}^{n}$ for which $f_{\alpha} \neq 0$; note that $\left|N_{f}\right| \leq\binom{ n+d}{d}$ if $d$ is the total degree of $f$. The computation of $f_{k}^{H}$ is done by computing

$$
\sum_{\alpha \in N_{f}} f_{\alpha} \frac{\gamma_{(\eta+\alpha, \beta)}}{\gamma_{(\eta, \beta)}}
$$

for all $(\eta, \beta) \in \mathbb{N}_{k}^{2 n}$, and taking the minimum. (We assume that the values $\gamma_{(\eta, \beta)}$ are pre-computed for all $(\eta, \beta) \in \mathbb{N}_{k+d}^{2 n}$.)

Thus, for fixed $(\eta, \beta) \in \mathbb{N}_{k}^{2 n}$, one may first compute the inner product of the vectors with components $f_{\alpha}$ and $\gamma_{(\eta+\alpha, \beta)}$ (indexed by $\alpha$ ). Note that these vectors are of size $\left|N_{f}\right|$. Since there are $\binom{2 n+k-1}{k}$ pairs $(\eta, \beta) \in \mathbb{N}_{k}^{2 n}$, the entire computation requires $\left(2\left|N_{f}\right|+1\right)\binom{2 n+k-1}{k}$ flops $\mathbb{\square}$.

As mentioned before, the computation of the upper bounds $f_{k}^{s o s}$ may be done by finding the smallest generalized eigenvalue $\lambda$ of the system:

$$
A x=\lambda B x \quad(x \neq 0)
$$

for suitable symmetric matrices $A$ and $B$ of order $\binom{n+k}{k}$. In particular, the rows and columns of the two matrices are indexed by $\mathbb{N}_{\leq k}^{n}$, and

$$
A_{\alpha, \beta}=\sum_{\delta \in N_{f}} f_{\delta} \int_{\mathbf{K}} \mathbf{x}^{\alpha+\beta+\delta} d \mathbf{x}, \quad B_{\alpha, \beta}=\int_{\mathbf{K}} \mathbf{x}^{\alpha+\beta} d \mathbf{x} \quad \alpha, \beta \in \mathbb{N}_{\leq k}^{n}
$$

Note that the matrices $A$ and $B$ depend on the moments of the Lebesgue measure on $\mathbf{K}=[0,1]^{n}$, and that these moments may be computed beforehand, by assumption. One may compute $A_{\alpha, \beta}$ by taking the inner product of $\left(f_{\delta}\right)_{\delta \in N_{f}}$ with the vector of moments $\left(\int_{\mathbf{K}} \mathbf{x}^{\alpha+\beta+\delta} d \mathbf{x}\right)_{\delta \in N_{f}}$. Thus computation of the elements of $A$ require a total of $\left|N_{f}\right|\left(\binom{n+k}{k}+1\right)^{2}$ flops.

Also note that the matrix $B$ is a positive definite (Gram) matrix. Thus one has to solve a socalled symmetric-definite generalized eigenvalue problem, and this may be done in $14\binom{n+k}{k}^{3}$ flops; see e.g. [9, Section 8.7.2]. Thus one may compute $f_{k}^{s o s}$ in at most $14\binom{n+k}{k}^{3}+\left|N_{f}\right|\left(\binom{n+k}{k}+1\right)^{2}$ flops.
7.2. Test functions and results. We consider several well-known polynomial test functions from global optimization (also used in [6]), that are listed in Table 1. Note that the Booth and Matyas functions are convex. Note also that the functions have a rational minimizer in the hypercube (except the Styblinski-Tang function).

We start by listing the upper bounds $f_{k}^{H}$ for these test functions in Table 2 for densities with degree up to $k=50$.

One notices that the observed convergence rate is more-or-less in line with the $O(1 / k)$ bound.
In a next experiment, we compare the Handelman-type densities ( $f_{k}^{H}$ bounds) to SOS densities ( $f_{k / 2}^{s o s}$ bounds); see Tables 3 and 4 .

[^1]Table 1. Test functions

| Name | Formula | Minimum $\left(f_{\min , \mathbf{K})}\right.$ <br> main (K) |  |
| :--- | :--- | :--- | :--- |
| Booth Function | $f=\left(20 x_{1}+40 x_{2}-37\right)^{2}+\left(40 x_{1}+20 x_{2}-35\right)^{2}$ | $f(0.55,0.65)=0$ | $[0,1]^{2}$ |
| Matyas Function | $f=0.26\left[\left(20 x_{1}-10\right)^{2}+\left(20 x_{2}-10\right)^{2}\right]-$ <br> $0.48\left(20 x_{1}-10\right)\left(20 x_{2}-10\right)$ | $f(0.5,0.5)=0$ |  |
| Motzkin Polynomial | $f=\left(4 x_{1}-2\right)^{4}\left(4 x_{2}-2\right)^{2}+\left(4 x_{1}-2\right)^{2}\left(4 x_{2}-\right.$ <br> $2)^{4}-3\left(4 x_{1}-2\right)^{2}\left(4 x_{2}-2\right)^{2}+1$ | $f\left(\frac{1}{4}, \frac{1}{4}\right)=f\left(\frac{1}{4}, \frac{3}{4}\right)=f\left(\frac{3}{4}, \frac{1}{4}\right)=f\left(\frac{3}{4}, \frac{3}{4}\right)=0$ |  |
| Three-Hump Camel <br> Function | $f=2\left(10 x_{1}-5\right)^{2}-1.05\left(10 x_{1}-5\right)^{4}+\frac{1}{6}\left(10 x_{1}-\right.$ <br> $5)^{6}+\left(10 x_{1}-5\right)\left(10 x_{2}-5\right)+\left(10 x_{2}-5\right)^{2}$ | $f(0.5,0.5)=0$ |  |
| Styblinski-Tang Func- <br> tion | $f=\sum_{i=1}^{n} \frac{1}{2}\left(10 x_{i}-5\right)^{4}-8\left(10 x_{i}-5\right)^{2}+$ <br> $\frac{5}{2}\left(10 x_{i}-5\right)$ | $f(0.20906466, \ldots, 0.20906466)=-39.16599 n$ |  |
| Rosenbrock Function | $f=\sum_{i=1}^{n-1} 100\left(4.096 x_{i+1}-2.048-\right.$ <br> $\left.\left(4.096 x_{i}-2.048\right)^{2}\right)^{2}+\left(4.096 x_{i}-3.048\right)^{2}$ | $f\left(\frac{3048}{4096}, \ldots, \frac{3048}{4096}\right)=0$ | $[0,1]^{n}$ |

Table 2. $f_{k}^{H}$ for Booth, Matyas, Motzkin, Three-Hump Camel, Styblinski-Tang and Rosenbrock Functions.

| $k$ | Booth | Matyas | Motzkin | T-H. Camel | $\begin{aligned} & \text { St.-Tang } \\ & (n=2) \\ & \hline \end{aligned}$ | Rosen. ( $n=$ 2) | $\text { Rosen. }(n=$ 3) | Rosen. ( $n=$ 4) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 280.667 | 17.3333 | 4.2000 | 265.77 | -12.5 | 303.16 | 794.818 | 1289.9 |
| 2 | 250.667 | 12.0000 | 2.1886 | 86.091 | -17.381 | 235.68 | 603.931 | 1097.7 |
| 3 | 214.0 | 11.0667 | 2.1886 | 86.091 | -21.548 | 177.91 | 536.449 | 906.76 |
| 4 | 184.0 | 8.8000 | 1.2743 | 40.593 | -26.429 | 148.6 | 478.673 | 839.28 |
| 5 | 172.0 | 8.1333 | 1.2743 | 40.593 | -28.929 | 142.2 | 411.191 | 781.51 |
| 6 | 151.333 | 6.9867 | 1.0218 | 24.354 | -31.429 | 130.43 | 343.863 | 714.02 |
| 7 | 143.905 | 6.5524 | 1.0218 | 24.354 | $-32.778$ | 120.17 | 314.559 | 646.68 |
| 8 | 130.762 | 5.9048 | 0.8912 | 17.322 | -34.127 | 103.43 | 296.24 | 579.2 |
| 9 | 125.429 | 5.6190 | 0.8912 | 17.322 | -34.921 | 100.03 | 266.936 | 511.86 |
| 10 | 117.571 | 5.2245 | 0.8538 | 13.867 | -35.714 | 91.011 | 252.003 | 482.56 |
| 11 | 109.556 | 5.0317 | 0.8538 | 13.867 | -36.956 | 87.425 | 239.06 | 460.14 |
| 12 | 106.222 | 4.7778 | 0.8384 | 10.534 | -38.305 | 76.959 | 225.146 | 430.83 |
| 13 | 99.4545 | 4.6444 | 0.8384 | 10.534 | -39.516 | 75.033 | 212.057 | 406.9 |
| 14 | 94.7407 | 4.4741 | 0.8366 | 8.6752 | -40.31 | 69.148 | 203.723 | 377.6 |
| 15 | 90.6667 | 4.3798 | 0.8339 | 8.6752 | -41.003 | 66.266 | 189.252 | 362.66 |
| 16 | 85.6364 | 4.2618 | 0.8336 | 7.2466 | -42.483 | 60.434 | 179.188 | 349.718 |
| 17 | 83.0909 | 4.1939 | 0.8242 | 7.2466 | -43.694 | 59.243 | 169.714 | 334.462 |
| 18 | 78.6434 | 4.1102 | 0.8139 | 6.1763 | -44.905 | 55.276 | 163.392 | 321.52 |
| 19 | 75.8648 | 4.0606 | 0.8062 | 6.1763 | -45.598 | 52.947 | 155.662 | 309.927 |
| 20 | 73.5152 | 4.0000 | 0.8025 | 5.3826 | -46.291 | 49.381 | 150.066 | 294.517 |
| 25 | 61.6535 | 3.4324 | 0.7762 | 4.2267 | -49.633 | 40.704 | 121.272 | 242.747 |
| 30 | 53.1228 | 2.8927 | 0.7474 | 3.1892 | -52.976 | 33.338 | 101.914 | 205.889 |
| 35 | 46.5982 | 2.5989 | 0.7067 | 2.7367 | -55.193 | 28.72 | 86.9293 | 177.821 |
| 40 | 41.6416 | 2.2609 | 0.6625 | 2.2626 | -57.411 | 24.883 | 75.5008 | 155.681 |
| 45 | 37.4988 | 2.0800 | 0.6254 | 2.0337 | -58.998 | 21.984 | 67.1078 | 138.990 |
| 50 | 34.0573 | 1.8595 | 0.5914 | 1.7768 | -60.536 | 19.739 | 59.6395 | 124.115 |

As described in Example 2.5, there is no ordering possible in general between $f_{k / 2}^{s o s}$ and $f_{k}^{H}$, but one observes that, in most cases, $f_{k / 2}^{s o s} \leq f_{k}^{H}$, i.e. the SOS densities usually give better bounds for a given degree, but at a higher computational cost.

Next we consider the strategies for generating feasible points corresponding to the bounds $f_{k}^{H}$, as described in Section 6, see Table 5

In Table 5, the columns marked $f(\mathbb{E}(X))$ refer to the convex case in Theorem 6.2 The columns marked $f(\hat{\mathbf{x}})$ correspond to the mode $\hat{\mathbf{x}}$ of the optimal density; an entry '-' in these columns means that the mode of the optimal density was not unique.

For the convex Booth and Matyas functions $f(\mathbb{E}(X))$ gives the best upper bound. For sufficiently large $k$ the mode $\hat{\mathbf{x}}$ gives a better bounds than $f_{k}^{H}$, indicating that this heuristic is useful in the non-convex case.

Table 3. Comparison of two upper bounds for Booth, Matyas, Three-Hump Camel and Motzkin Functions

| degree $k$ | Booth |  | Matyas |  | Three-Hump Camel |  | Motzkin |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $f_{k / 2}^{\text {sos }}$ | $f_{k}^{H}$ | $f_{k / 2}^{\text {sos }}$ | $f_{k}^{H}$ | $f_{k / 2}^{s o s}$ | $f_{k}^{H}$ | $f_{k / 2}^{\text {sos }}$ | $f_{k}^{H}$ |
| 1 |  | 280.667 |  | 17.3333 |  | 265.77 |  | 4.2 |
| 2 | 244.680 | 250.667 | 8.26667 | 12.0 | 265.774 | 86.091 | 4.2 | 2.1886 |
| 3 |  | 214.0 |  | 11.0667 |  | 86.091 |  | 2.1886 |
| 4 | 162.486 | 184.0 | 5.32223 | 8.8000 | 29.0005 | 40.593 | 1.06147 | 1.2743 |
| 5 |  | 172.0 |  | 8.1333 |  | 40.593 |  | 1.2743 |
| 6 | 118.383 | 151.333 | 4.28172 | 6.9867 | 29.0005 | 24.354 | 1.06147 | 1.0218 |
| 7 |  | 143.905 |  | 6.5524 |  | 24.354 |  | 1.0218 |
| 8 | 97.6473 | 130.762 | 3.89427 | 5.9048 | 9.58064 | 17.322 | 0.829415 | 0.8912 |
| 9 |  | 125.429 |  | 5.6190 |  | 17.322 |  | 0.8912 |
| 10 | 69.8174 | 117.571 | 3.68942 | 5.2245 | 9.58064 | 13.867 | 0.801069 | 0.8538 |
| 11 |  | 109.556 |  | 5.0317 |  | 13.867 |  | 0.8538 |
| 12 | 63.5454 | 106.222 | 2.99563 | 4.7778 | 4.43983 | 10.534 | 0.801069 | 0.8384 |
| 13 |  | 99.4545 |  | 4.6444 |  | 10.534 |  | 0.8384 |
| 14 | 47.0467 | 94.7407 | 2.54698 | 4.4741 | 4.43983 | 8.6752 | 0.708889 | 0.8366 |
| 15 |  | 90.6667 |  | 4.3798 |  | 8.6752 |  | 0.8339 |
| 16 | 41.6727 | 85.6364 | 2.04307 | 4.2618 | 2.55032 | 7.2466 | 0.565553 | 0.8336 |
| 17 |  | 83.909 |  | 4.1939 |  | 7.2466 |  | 0.8242 |
| 18 | 34.2140 | 78.6434 | 1.83356 | 4.1102 | 2.55032 | 6.1763 | 0.565553 | 0.8139 |
| 19 |  | 75.8648 |  | 4.0606 |  | 6.1763 |  | 0.8062 |
| 20 | 28.7248 | 73.5152 | 1.47840 | 4.0000 | 1.71275 | 5.3826 | 0.507829 | 0.8025 |

Table 4. Comparison of two upper bounds for Styblinski-Tang and Rosenbrock Functions

| degree $k$ | Sty.-Tang $(n=2)$ |  | Rosenb. $(n=2)$ |  | Rosenb. $(n=3)$ |  | Rosenb. $(n=4)$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $f_{k / 2}^{\text {sos }}$ | $f_{k}^{H}$ | $f_{k / 2}^{\text {sos }}$ | $f_{k}^{H}$ | $f_{k / 2}^{s o s}$ | $f_{k}^{H}$ | $f_{k / 2}^{s o s}$ | $f_{k}^{H}$ |
| 1 |  | -12.5 |  | 303.16 |  | 794.818 |  | 1289.9 |
| 2 | -12.9249 | -17.381 | 214.648 | 235.68 | 629.086 | 603.931 | 1048.19 | 1097.7 |
| 3 |  | -21.548 |  | 177.91 |  | 536.449 |  | 906.76 |
| 4 | -25.7727 | -26.429 | 152.310 | 148.6 | 394.187 | 478.673 | 690.332 | 839.28 |
| 5 |  | -28.929 |  | 142.2 |  | 411.191 |  | 781.51 |
| 6 | -34.4030 | -31.429 | 104.889 | 130.43 | 295.811 | 343.863 | 536.367 | 714.02 |
| 7 |  | -32.778 |  | 120.17 |  | 314.559 |  | 646.68 |
| 8 | -41.4436 | -34.127 | 75.6010 | 103.43 | 206.903 | 296.24 | 382.729 | 579.2 |
| 9 |  | -34.921 |  | 100.03 |  | 266.936 |  | 511.86 |
| 10 | -45.1032 | -35.714 | 51.5037 | 91.011 | 168.135 | 252.003 | 314.758 | 482.56 |
| 11 |  | -36.956 |  | 87.425 |  | 239.06 |  | 460.14 |
| 12 | -51.0509 | -38.305 | 41.7878 | 76.959 | 121.558 | 225.146 | 236.709 | 430.83 |
| 13 |  | -39.516 |  | 75.033 |  | 212.057 |  | 406.9 |
| 14 | -56.4050 | -40.31 | 30.1392 | 69.148 | 101.953 | 203.723 | 202.674 | 377.6 |
| 15 |  | -41.003 |  | 66.266 |  | 189.252 |  | 362.66 |
| 16 | -58.6004 | -42.483 | 25.8329 | 60.434 | 77.4797 | 179.188 | 156.295 | 349.718 |
| 17 |  | -43.694 |  | 59.243 |  | 169.714 |  | 334.462 |
| 18 | -60.7908 | -44.905 | 19.4972 | 55.276 | 66.6954 | 163.392 | 137.015 | 321.52 |

As a final comparison, we also look at the general sampling technique via the method of conditional distributions; see Tables 6and 7. We present results for the Motzkin polynomial and the Three hump camel function.

For each degree $k$, we use the sample sizes 10 and 100. In Tables 6 and 7 we record the mean, variance and the minimum value of these samples. (Recall that the expected value of the sample mean equals $f_{k}^{H}$.) We also generate samples uniformly from $[0,1]^{n}$, for comparison.

The mean of the sample function values approximates $f_{k}^{H}$ reasonably well for sample size 100 , but less so for sample size 10 . Moreover, the mean sample function value for uniform sampling from $[0,1]^{n}$ is much higher than $f_{k}^{H}$. Also, the minimum function value for sampling is significantly lower than the minimum function value obtained by uniform sampling for most values of $k$.

Table 5. Comparing strategies for generating feasible points for Booth, Matyas, Motzkin, and Three-Hump Camel Functions. Here, $\hat{\mathbf{x}}$ denotes the mode of the optimal density.

| $k$ | Booth |  |  | Matyas |  |  | Motzkin |  | Three-H. Camel |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $f_{k}^{H}$ | $f(\hat{\mathbf{x}})$ | $f(\mathbb{E}(X))$ | $f_{k}^{H}$ | $f(\hat{\mathbf{x}})$ | $f(\mathbb{E}(X))$ | $f_{k}^{H}$ * | $f(\hat{\mathbf{x}})$ | $f_{k}^{H}$ | $f(\hat{\mathbf{x}})$ |
| 1 | 280.667 | - | 2.8889 | 17.3333 | - | 0 | 4.2000 | - | 265.77 | - |
| 2 | 250.667 | - | 9.0 | 12.0000 | 4.0 | 0.4444 | 2.1886 | - | 86.091 | - |
| 3 | 214.0 | 194.0 | 2.8889 | 11.0667 | 4.0 | 1.3889 | 2.1886 | - | 86.091 | - |
| 4 | 184.0 | 194.0 | 9.0 | 8.8000 | 4.0 | 1.0 | 1.2743 | 1.0 | 40.593 | - |
| 5 | 172.0 | 96.222 | 17.0 | 8.1333 | 4.0 | 1.460 | 1.2743 | 1.0 | 40.593 | - |
| 6 | 151.333 | 96.222 | 18.0 | 6.9867 | 4.0 | 1.440 | 1.0218 | 1.0 | 24.354 | - |
| 7 | 143.905 | 96.222 | 24.222 | 6.5524 | 4.0 | 1.7156 | 1.0218 | 1.0 | 24.354 | - |
| 8 | 130.762 | 122.0 | 16.204 | 5.9048 | 4.0 | 1.7778 | 0.8912 | 1.0 | 17.322 | - |
| 9 | 125.429 | 26.0 | 2.9796 | 5.6190 | 4.0 | 1.9637 | 0.8912 | 1.0 | 17.322 | 25.0 |
| 10 | 117.571 | 96.222 | 25.806 | 5.2245 | 4.0 | 2.0408 | 0.8538 | 1.0 | 13.867 | - |
| 11 | 109.556 | 26.0 | 2.9796 | 5.0317 | 4.0 | 2.1760 | 0.8538 | 1.0 | 13.867 | 25.0 |
| 12 | 106.222 | 42.889 | 9.0 | 4.7778 | 4.0 | 2.2500 | 0.8384 | 1.0 | 10.534 | 0 |
| 13 | 99.4545 | 26.0 | 2.9796 | 4.6444 | 4.0 | 2.3534 | 0.8384 | 1.0 | 10.534 | 0 |
| 14 | 94.7407 | 13.592 | 0.91358 | 4.4741 | 4.0 | 2.4198 | 0.8366 | 1.0 | 8.6752 | 0 |
| 15 | 90.6667 | 27.580 | 7.6777 | 4.3798 | 4.0 | 2.5017 | 0.8339 | 1.0 | 8.6752 | 0.273 |
| 16 | 85.6364 | 9.0 | 2.0 | 4.2618 | 4.0 | 2.5600 | 0.8336 | 1.0 | 7.2466 | 0 |
| 17 | 83.0909 | 17.210 | 4.5785 | 4.1939 | 4.0 | 2.6268 | 0.8242 | 1.0 | 7.2466 | 0 |
| 18 | 78.6434 | 9.0 | 2.0 | 4.1102 | 4.0 | 2.6777 | 0.8139 | 1.0 | 6.1763 | 0 |
| 19 | 75.8648 | 5.951 | 0.35445 | 4.0606 | 4.0 | 2.7332 | 0.8062 | 1.0 | 6.1763 | 0.209 |
| 20 | 73.5152 | 9.0 | 2.0 | 4.0000 | 0.16 | 0.1111 | 0.8025 | 1.0 | 5.3826 | 0 |
| 25 | 61.6535 | 4.5785 | 1.8107 | 3.4324 | 0.3161 | 0.2404 | 0.7762 | 1.0 | 4.2267 | 0.1653 |
| 30 | 53.1228 | 1.6403 | 0.41428 | 2.8927 | 0.0178 | 0.0138 | 0.7474 | 1.0 | 3.1892 | 0 |
| 35 | 46.5982 | 1.0923 | 0.53061 | 2.5989 | 0.1071 | 0.0897 | 0.7067 | 0.4214 | 2.7367 | 0.110 |
| 40 | 41.6416 | 0.8454 | 0.64566 | 2.2609 | 0 | 0 | 0.6625 | 0.2955 | 2.2626 | 0 |
| 45 | 37.4988 | 2.0 | 0.80157 | 2.0800 | 0 | 0 | 0.6254 | 0.1985 | 2.0337 | 0.0783 |
| 50 | 34.0573 | 0.9784 | 0.22222 | 1.8595 | 0 | 0 | 0.5914 | 0.1297 | 1.7768 | 0 |

TABLE 6. Sampling results for Motzkin Polynomial

|  |  | Sample size 10 |  |  | Sample size 100 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k$ | $f_{k}^{H}$ | Mean | Variance | Minimum | Mean | Variance | Minimum |
| 1 | 4.2000 | 6.2601 | 66.2605 | 0.6183 | 6.5027 | 188.1445 | 0.0060 |
| 2 | 2.1886 | 1.4972 | 1.6084 | 0.9158 | 1.8377 | 12.5387 | 0.0657 |
| 3 | 2.1886 | 1.9658 | 5.0427 | 0.0644 | 2.8413 | 68.2093 | 0.0036 |
| 4 | 1.2743 | 1.1776 | 1.8501 | 0.0421 | 0.8571 | 0.6764 | 0.0042 |
| 5 | 1.2743 | 0.8330 | 0.0466 | 0.2790 | 1.1590 | 4.2023 | 0.0525 |
| 6 | 1.0218 | 1.7002 | 6.2647 | 0.3196 | 0.9336 | 0.8998 | 0.0002 |
| 7 | 1.0218 | 0.8350 | 0.1672 | 0.2416 | 0.9863 | 1.3777 | 0.0070 |
| 8 | 0.8912 | 0.6108 | 0.1451 | 0.0218 | 0.8431 | 1.4834 | 0.0070 |
| 9 | 0.8912 | 0.7545 | 0.0679 | 0.1656 | 0.8879 | 0.2752 | 0.0175 |
| 10 | 0.8538 | 0.7005 | 0.0800 | 0.1862 | 0.8435 | 0.1448 | 0.1149 |
| 11 | 0.8538 | 0.8244 | 0.0779 | 0.1123 | 0.8673 | 0.2565 | 0.1100 |
| 12 | 0.8384 | 0.8912 | 0.0213 | 0.5919 | 0.7835 | 0.2554 | 0.0188 |
| 13 | 0.8384 | 0.8286 | 0.0412 | 0.3205 | 0.7664 | 0.0714 | 0.0112 |
| 14 | 0.8366 | 0.7698 | 0.0781 | 0.2083 | 0.9574 | 1.2157 | 0.0778 |
| 15 | 0.8339 | 0.9063 | 0.0153 | 0.6069 | 0.8465 | 0.0932 | 0.0593 |
| 16 | 0.8336 | 0.7482 | 0.0750 | 0.1759 | 0.7209 | 0.0875 | 0.0648 |
| 17 | 0.8242 | 0.7430 | 0.0706 | 0.1500 | 0.8051 | 0.0718 | 0.0984 |
| 18 | 0.8139 | 0.8546 | 0.0493 | 0.4460 | 0.7749 | 0.0785 | 0.0038 |
| 19 | 0.8062 | 0.6621 | 0.0892 | 0.1836 | 0.7850 | 0.1273 | 0.0408 |
| 20 | 0.8025 | 0.7704 | 0.0336 | 0.3826 | 0.9326 | 1.6454 | 0.0040 |
| 25 | 0.7762 | 0.7995 | 0.1014 | 0.2433 | 0.7493 | 0.0717 | 0.0722 |
| 30 | 0.7474 | 1.0104 | 1.2852 | 0.1091 | 0.8290 | 0.8620 | 0.0522 |
| 35 | 0.7067 | 0.5930 | 0.0981 | 0.1940 | 0.7647 | 1.3012 | 0.0016 |
| 40 | 0.6625 | 0.6967 | 0.0497 | 0.2867 | 0.6028 | 0.1371 | 0.0021 |
| 45 | 0.6254 | 0.6258 | 0.0500 | 0.3548 | 0.7007 | 0.2242 | 0.0090 |
| 50 | 0.5914 | 0.6244 | 0.0718 | 0.3000 | 0.5782 | 0.1406 | 0.0154 |
| Uniform Sample | 4.2888 | 37.4427 | 0.5290 | 3.7397 | 53.8833 | 0.0492 |  |

## 8. Concluding remarks

One may consider several strategies to improve the upper bounds $f_{k}^{H}$, and we list some in turn.

Table 7. Sampling results for Three-Hump Camel function

|  |  | Sample size 10 |  |  | Sample size 100 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k$ | $f_{k}^{H}$ | Mean | Variance | Minimum | Mean | Variance | Minimum |
| 1 | 265.77 | 359.98 | 274477.0 | 2.4493 | 300.34 | 245144.0 | 0.011095 |
| 2 | 86.091 | 88.717 | 24117.0 | 1.1729 | 122.12 | 76646.0 | 0.082513 |
| 3 | 86.091 | 14.712 | 186.23 | 2.219 | 58.186 | 15987.0 | 0.492 |
| 4 | 40.593 | 55.091 | 19297.0 | 0.10296 | 44.844 | 21297.0 | 0.19439 |
| 5 | 40.593 | 91.872 | 27065.0 | 0.90053 | 53.656 | 14575.0 | 0.58086 |
| 6 | 24.354 | 12.961 | 77.377 | 0.8186 | 34.115 | 7862.5 | 0.019021 |
| 7 | 24.354 | 33.96 | 1745.4 | 0.65266 | 27.072 | 10632.0 | 0.33813 |
| 8 | 17.322 | 10.029 | 60.746 | 1.0931 | 12.307 | 314.46 | 0.074663 |
| 9 | 17.322 | 9.4932 | 100.22 | 0.0027565 | 20.185 | 7279.8 | 0.11239 |
| 10 | 13.867 | 11.312 | 45.784 | 0.8916 | 14.273 | 382.98 | 0.018985 |
| 11 | 13.867 | 8.3991 | 87.108 | 0.0031527 | 11.928 | 357.45 | 0.01384 |
| 12 | 10.534 | 5.013 | 52.681 | 0.30303 | 12.377 | 547.42 | 0.25952 |
| 13 | 10.534 | 14.281 | 401.82 | 0.52373 | 7.8673 | 253.02 | 0.11989 |
| 14 | 8.6752 | 5.2897 | 43.81 | 0.3909 | 9.4462 | 362.49 | 0.051331 |
| 15 | 8.6752 | 5.6281 | 31.311 | 0.21853 | 10.373 | 778.32 | 0.022282 |
| 16 | 7.2466 | 9.5801 | 95.901 | 1.7112 | 6.465 | 122.72 | 0.013084 |
| 17 | 7.2466 | 5.2511 | 23.863 | 2.0409 | 6.0633 | 56.495 | 0.18354 |
| 18 | 6.1763 | 6.0327 | 34.298 | 0.85182 | 5.2985 | 35.953 | 0.071544 |
| 19 | 6.1763 | 5.3006 | 52.994 | 0.6699 | 5.0383 | 41.619 | 0.040785 |
| 20 | 5.3826 | 3.5174 | 16.053 | 0.43269 | 9.4178 | 653.27 | 0.041752 |
| 25 | 4.2267 | 10.741 | 776.55 | 0.59616 | 5.0642 | 112.61 | 0.039463 |
| 30 | 3.1892 | 2.2515 | 8.6915 | 0.063265 | 2.2096 | 6.2611 | 0.040845 |
| 35 | 2.7367 | 1.5032 | 1.4626 | 0.0085016 | 3.0679 | 16.47 | 0.24175 |
| 40 | 2.2626 | 1.3941 | 1.1995 | 0.21653 | 2.3431 | 17.735 | 0.069473 |
| 45 | 2.0337 | 2.3904 | 10.934 | 0.57818 | 1.8928 | 3.6581 | 0.050042 |
| 50 | 1.7768 | 1.664 | 3.3983 | 0.061995 | 1.6301 | 1.6966 | 0.048476 |
| Uniform Sample | 306.96 | 275366.0 | 0.15602 | 368.28 | 296055.0 | 0.59281 |  |

- A natural idea is to use density functions that are convex combinations of SOS and Handelman-type densities, i.e., that belong to $\mathcal{H}_{k}+\Sigma[x]_{r}$ for some nonnegative integers $k, r$. Unfortunately one may show that this does not yield a better upper bound than $\min \left\{f_{r}^{s o s}, f_{k}^{H}\right\}$, namely

$$
\min \left\{f_{r}^{s o s}, f_{k}^{H}\right\}=\inf _{\sigma \in \mathcal{H}_{k}+\Sigma[x]_{r}}\left\{\int_{\mathbf{K}} f(\mathbf{x}) \sigma(\mathbf{x}) d \mathbf{x}: \int_{\mathbf{K}} \sigma(\mathbf{x}) d \mathbf{x}=1\right\}, \quad k, r \in \mathbb{N} .
$$

(We omit the proof since it is straightforward, and of limited interest.)

- For optimization over the hypercube, a second idea is to replace the integer exponents in Handelman representations of the density by more general positive real exponents. (This is amenable to analysis since the beta distribution is defined for arbitrary positive shape parameters and with its moments available via relation (4.2).) If we drop the integrality requirement for $(\eta, \beta)$ in the definition of $f_{k}^{H}$ (see (1.2)), we obtain the bound:

$$
f_{k}^{H} \geq f_{k}^{b e t a}:=\min _{(\eta, \beta) \in \Delta_{k}^{2 n}} \sum_{\alpha \in \mathbb{N}_{\leq d}^{n}} f_{\alpha} \frac{\gamma_{(\eta+\alpha, \beta)}}{\gamma_{(\eta, \beta)}}, \quad k \in \mathbb{N},
$$

where $\Delta_{k}^{2 n}$ is the simplex $\Delta_{k}^{2 n}:=\left\{(\eta, \beta) \in \mathbb{R}_{+}^{2 n}: \sum_{i=1}^{n}\left(\eta_{i}+\beta_{i}\right)=k\right\}$.
As with $f_{k}^{H}$, when $(\eta, \beta)$ is such that $f_{k}^{b e t a}=\sum_{\alpha \in \mathbb{N}_{\leq d}^{n}} f_{\alpha} \frac{\gamma_{(\eta+\alpha, \beta)}}{\gamma_{(\eta, \beta)}}$, one has that $f_{k}^{b e t a}=$ $\mathbb{E}(f(X))$ where $X=\left(X_{1}, \ldots, X_{n}\right)$ and $X_{i} \sim \operatorname{beta}\left(\eta_{i}+1, \bar{\beta}_{i}+1\right)(i \in[n])$. Using the moments of the beta distribution in (4.2), we obtain

$$
f_{k}^{b e t a}=\min _{(\eta, \beta) \in \Delta_{k}^{2 n}} \sum_{\alpha \in \mathbb{N}_{d}^{n}} f_{\alpha} \prod_{i=1}^{n} \frac{\left(\eta_{i}+1\right) \cdots\left(\eta_{i}+\alpha_{i}\right)}{\left(\eta_{i}+\beta_{i}+2\right) \cdots\left(\eta_{i}+\beta_{i}+\alpha_{i}+1\right)}, \quad k \in \mathbb{N} .
$$

Thus one may obtain the bounds $f_{k}^{b e t a}$ by minimizing a rational function over a simplex. A question for future research is whether one may approximate $f_{k}^{b e t a}$ to any fixed accuracy in time polynomial in $k$ and $n$. (This may be possible, since the minimization of fixed-degree polynomials over a simplex allows a PTAS [4, and the relevant algorithmic techniques have been extended to rational objective functions [11].)

One may also use the value of $(\eta, \beta) \in \Delta_{k}^{2 n}$ that gives $f_{k}^{H}$ as a starting point in the minimization problem (8.1), and employ any iterative method to obtain a better upper bound heuristically. Subsequently, one may use the resulting density function to obtain 'good' feasible points as described in Section 6. Of course, one may also use the feasible points (generated by sampling) as starting points for iterative methods. Suitable iterative methods for bound-constrained optimization are described in the books [2, 7, 8, and the latest algorithmic developments for bound constrained global optimization are surveyed in the recent thesis 22 .

- Perhaps the most promising practical variant of the $f_{k}^{H}$ bound is the following parameter:

$$
\begin{aligned}
f_{r, k}^{H} & =\min _{(\eta, \beta) \in \mathbb{N}_{k}^{2 n}} \frac{\int_{\mathbf{K}} f(\mathbf{x})\left(\mathbf{x}^{\eta}(1-\mathbf{x})^{\beta}\right)^{r} d \mathbf{x}}{\int_{\mathbf{K}}\left(\mathbf{x}^{\eta}(1-\mathbf{x})^{\beta}\right)^{r} d \mathbf{x}} \\
& =\min _{(\eta, \beta) \in \mathbb{N}_{k}^{2 n}} \sum_{\alpha \in \mathbb{N}^{n}} f_{\alpha} \frac{\gamma_{(r \eta+\alpha, r \beta)}}{\gamma_{(r \eta, r \beta)}} \quad \text { for } r, k \in \mathbb{N} .
\end{aligned}
$$

Thus, the idea is to replace the density $\sigma(x)=\mathbf{x}^{\eta}(1-\mathbf{x})^{\beta} / \int_{\mathbf{K}} \mathbf{x}^{\eta}(1-\mathbf{x})^{\beta} d \mathbf{x}$ by the density $\sigma(x)^{r} / \int_{\mathbf{K}} \sigma(r)^{r} d \mathbf{x}$ for some power $r \in \mathbb{N}$. Hence, for $r=1, f_{1, k}^{H}=f_{k}^{H}$. Note that the calculation of $f_{r, k}^{H}$ requires exactly the same number of elementary operations as the calculation of $f_{k}^{H}$, provided all the required moments are available. (Also note that, for $K=[0,1]^{n}$, one could allow an arbitrary $r>0$ since the moments are still available as pointed out above.)

In Tables 8, 9, and 10, we show some numerical values for the parameter $f_{r, k}^{H}$.

Table 8. $\quad f_{r, k}^{H}$ for the Styblinski-Tang function $(n=2)$

| $k$ | $r=1$ | $r=2$ | $r=3$ | $r=4$ | $r=5$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | -12.5 | -10.06 | -8.3333 | -8.3333 | -8.3333 |
| 2 | -17.381 | -17.857 | -16.919 | -15.793 | -14.744 |
| 3 | -21.548 | -21.686 | -22.582 | -23.179 | -23.62 |
| 4 | -26.429 | -27.381 | -28.256 | -30.263 | -31.736 |
| 5 | -28.929 | -31.209 | -31.167 | -32.872 | -34.435 |
| 6 | -31.429 | -35.038 | -36.842 | -38.025 | -38.906 |
| 7 | -32.778 | -38.76 | -42.505 | -45.109 | -47.022 |
| 8 | -34.127 | -42.483 | -48.179 | -52.193 | -55.138 |
| 9 | -34.921 | -44.387 | -50.577 | -54.802 | -57.837 |
| 10 | -35.714 | -46.291 | -52.976 | -57.411 | -60.536 |

A first important observation is that, for fixed $k$, the values of $f_{r, k}^{H}$ are not monotonically decreasing in $r$; see e.g. the row $k=2$ in Table 8, Likewise, the sequence $f_{r, k}^{H}$ is not monotonically decreasing in $k$ for fixed $r$; see, e.g., the column $r=5$ in Table 9 ,

Table 9. $\quad f_{r, k}^{H}$ for the Rosenbrock function $(n=3)$

| $k$ | $r=1$ | $r=2$ | $r=3$ | $r=4$ | $r=5$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 794.818 | 727.337 | 698.032 | 683.822 | 676.526 |
| 2 | 603.931 | 512.228 | 473.974 | 454.193 | 443.769 |
| 3 | 536.449 | 449.625 | 398.566 | 367.869 | 350.671 |
| 4 | 478.673 | 368.873 | 294.499 | 253.135 | 227.526 |
| 5 | 411.191 | 274.121 | 235.89 | 228.906 | 232.996 |
| 6 | 343.863 | 225.146 | 191.935 | 151.455 | 119.98 |
| 7 | 314.559 | 225.768 | 166.179 | 128.62 | 106.417 |
| 8 | 296.24 | 198.861 | 144.94 | 111.721 | 88.0661 |
| 9 | 266.936 | 185.145 | 133.379 | 103.162 | 84.3506 |
| 10 | 252.003 | 158.448 | 111.33 | 87.8805 | 70.0394 |

Table 10. $f_{r, k}^{H}$ for the Rosenbrock function $(n=4)$

| $k$ | $r=1$ | $r=2$ | $r=3$ | $r=4$ | $r=5$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1289.9 | 1223.8 | 1194.5 | 1180.3 | 1173.0 |
| 2 | 1097.7 | 1006.9 | 968.53 | 948.71 | 938.29 |
| 3 | 906.76 | 790.03 | 742.57 | 717.15 | 703.61 |
| 4 | 839.28 | 727.43 | 669.06 | 632.76 | 612.44 |
| 5 | 781.51 | 606.15 | 502.34 | 446.68 | 413.72 |
| 6 | 714.02 | 515.76 | 397.34 | 330.93 | 289.5 |
| 7 | 646.68 | 421.01 | 338.74 | 306.71 | 294.97 |
| 8 | 579.2 | 371.11 | 293.83 | 229.25 | 181.95 |
| 9 | 511.86 | 331.44 | 269.02 | 206.42 | 168.39 |
| 10 | 482.56 | 323.69 | 246.84 | 189.36 | 149.9 |

On the other hand, it is clear from Tables 8, 9, and 10 that $f_{r, k}^{H}$ can provide a much better bound than $f_{k}^{H}$ for $r>1$.

Since $f_{r, k}^{H}$ is not monotonically decreasing in $r$ (for fixed $k$ ), or in $k$ (for fixed $r$ ), one has to consider the convergence question. An easy case is when $\mathbf{K}=[0,1]^{n}$ and the global minimizer $\mathbf{x}^{*}$ is rational. Say $x_{i}^{*}=\frac{p_{i}}{q_{i}}(i \in[n])$, setting $q_{i}=1$ and $p_{i}=x_{i}^{*}$ when $x_{i}^{*} \in\{0,1\}$. Consider the following variation of the parameters $\eta_{i}^{*}, \beta_{i}^{*}$ from Definition 4.2 $\eta_{i}^{*}=r p_{i}+1$ and $\beta_{i}^{*}=r\left(q_{i}-p_{i}\right)+1$ for $i \in[n]$, so that $\sum_{i=1}^{n} \eta_{i}^{*}+\beta_{i}^{*}-2=r\left(\sum_{i=1}^{n} q_{i}\right)$. Combining relation (4.6) and Theorem 4.4, we can conclude that the following inequality holds:

$$
f_{r, k}^{H}-f\left(\mathbf{x}^{*}\right) \leq \frac{C_{f}}{r} \quad \text { for all } k \geq \sum_{i=1}^{n} q_{i} \text { and } r \geq 1
$$

where $C_{f}$ is a constant that depends on $f$ only.
For more general sets $\mathbf{K}$, one may ensure convergence by considering instead the following parameter (for fixed $R \in \mathbb{N}$ ):

$$
\min _{r \in[R]} f_{k, r}^{H} \leq f_{k}^{H} \quad(k \in \mathbb{N})
$$

Then convergence follows from the convergence results for $f_{k, r}^{H}$. Moreover, this last parameter may be computed in polynomial time if $k$ is fixed, and $R$ is bounded by a polynomial in $n$.

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[^1]:    ${ }^{1}$ We define floating point operations (flops) as in [9, p. 18]; in particular, by this definition the inner product of two $n$-vectors requires $2 n$ flops.

