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Letter Section

On the order of prolongations and restrictions in multigrid procedures

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Abstract: It is well known in the world of multigrid that the order of the prolongation and the order of the restriction in a multigrid method should satisfy certain conditions. A rule of thumb is that the sum of the orders of the prolongation and of the restriction should at least be equal to the order of the differential equation solved. In this note we show the correctness of this rule. We notice that we have to distinguish between low frequency and high frequency orders for the transfer operators. For the restriction, the low frequency order is related with its accuracy, whereas for the interpolation operator both orders are related with the accuracy of the result of the interpolation. If an interpolation rule leaves all polynomials of degree $k-1$ invariant, then both the low and the high frequency order are equal to k. It is the high frequency order that plays a role in the above-mentioned rule of thumb.

Keywords: Multigrid methods, transfer operators.

1. Introduction

The space of discrete l^2-functions on an infinite regular n-dimensional grid is denoted by

$$l_h^2(\mathbb{Z}_h^n) = \left\{ u_h \mid u_h : \mathbb{Z}_h^n \to K;\ \sum_{z \in \mathbb{Z}_h^n} h^n \mid u_h(z) \mid^2 < \infty \right\}, \tag{1}$$

where $K = \mathbb{R}$ or $K = \mathbb{C}$, the field of real or complex numbers, $\mathbb{Z}$ are the integer numbers and

$$\mathbb{Z}_h^n = \{ jh \mid j \in \mathbb{Z}^n \}, \tag{2}$$

where $h = (h_1, h_2, \ldots, h_n)$ with $h_i > 0$, $i = 1, \ldots, n$, and $jh = (j_1 h_1, \ldots, j_n h_n)$. With the obvious norm $\| \cdot \|_{h,2}$ the space $l_h^2(\mathbb{Z}_h^n)$ is a Hilbert space.

The Fourier transform $\mathrm{FT}(u_h) = \hat{u}_h$ of an l_h^2-function u_h is defined by

$$\hat{u}_h(\omega) = \left(\frac{h}{\sqrt{2\pi}} \right)^n \sum_{j \in \mathbb{Z}^n} \mathrm{e}^{-\mathrm{i} jh\omega} u_h(jh), \tag{3}$$

and it is readily seen that

$$\mathrm{FT}: l_h^2(\mathbb{Z}_h^n) \to L^2(T_h^n), \tag{4}$$

with $T_h^n = [-\pi/h, \pi/h]^n$ an n-dimensional torus, is an invertible unitary operator (i.e., $\| u_h \|_{h,2} = \| \hat{u}_h \|_2$).

Convolution or Toeplitz operators $A_h : l_h^2(\mathbb{Z}_h^n) \to l_h^2(\mathbb{Z}_h^n)$ are linear operators

$$(A_h u_h)(jh) = \sum_{k \in \mathbb{Z}^n} a_h(kh) u_h(jh - kh), \tag{5}$$

with $a_h \in l_h^2(\mathbb{Z}_h^n)$. In practice, a_h is mostly associated with a discretisation stencil and A_h is the discrete operator of a linear differential equation with constant coefficients.

It is easily seen [3] that we can introduce the Fourier transform $\mathrm{FT}(A_h) = \hat{A}_h$ of a Toeplitz operator A_h by

$$\widehat{A_h u_h}(\omega) = \hat{A}_h(\omega) \hat{u}_h(\omega). \tag{6}$$

Remark. Let L be a differential operator with constant coefficients and $\hat{L}(\omega)$ its Fourier transform or "symbol". Let us denote by L_h a discretisation of L on a regular rectangular grid, determined by a unique stencil. Then $\hat{L}(\omega)$ is a polynomial of degree M, with M the order of the differential equation, and $\hat{L}_h(\omega)$ is a trigonometric polynomial of the same degree. It is a classical result that

$$| \hat{L}_h(\omega) - \hat{L}(\omega) | = \mathcal{O}(h^{\tilde{p}}) \quad \text{for } h \to 0, \tag{7}$$

with $\tilde{p}$ the order of consistency of the discretisation.

2. Restrictions and prolongations

Although the theory can be made more general [3], for a convenience of notation we restrict ourselves here to grid transfer operators between fine grids $\mathbb{Z}_h^n$ and coarse grids $\mathbb{Z}_{2h}^n$ (also denoted by $\mathbb{Z}_H^n$, $H = 2h$). We first introduce the elementary restriction and prolongation.

The *elementary restriction* $R_{Hh}^0 : l_h^2(\mathbb{Z}_h^n) \to l_H^2(\mathbb{Z}_H^n)$ is defined by

$$(R_{Hh}^0 u_h)(jH) = u_h(j2h), \tag{8}$$

and its Fourier transform is given by

$$\widehat{R_{Hh}^0 u_h}(\omega) = \sum_{p \in \{0,1\}^n} \hat{u}_h\left(\omega + \frac{2\pi p}{H}\right), \tag{9}$$

the sum is taken over all n-tuples with elements taken from $\{0, 1\}$. Any regular restriction $R_{Hh} : l_h^2(\mathbb{Z}_h^n) \to l_H^2(\mathbb{Z}_H^n)$ can be constructed as a combination of the elementary restriction and a convolution operator:

$$R_{Hh} = R_{Hh}^0 A_h. \tag{10}$$

Similar to the elementary restriction, there is an *elementary prolongation* $P_{hH}^0 : l_H^2(\mathbb{Z}_H^n) \to l_h^2(\mathbb{Z}_h^n)$ defined by

$$(P_{hH}^0 u_h)(jh) = \begin{cases} u_h(jh), & \text{if } jh \in \mathbb{Z}_H^n, \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

and any regular prolongation $P_{hH} : l_H^2(\mathbb{Z}_H^n) \to l_h^2(\mathbb{Z}_h^n)$ can be constructed as $P_{hH} = A_h P_{hH}^0$.

To discuss Fourier transforms of prolongations or restrictions, we notice that any element $\omega \in T_h^n$ can be seen as a 2^n-vector $(\omega + \pi p/h)$, $p \in \{0, 1\}^n$ on T_H^n. (The 2^n frequencies $\omega + \pi p/h$ on $\mathbb{Z}_h^n$ are those that correspond with the frequency ω on $\mathbb{Z}_H^n$.) Hence, for $\hat{u}_h \in L^2(T_h^n)$ we also use the notation $\hat{\boldsymbol{u}}_h \in \boldsymbol{L}^2(T_H^n)$, with $\hat{\boldsymbol{u}}_h(\omega)$ the vector with entries $\hat{u}_h(\omega + \pi p/h)$, $p \in \{0, 1\}^n$.

Consistent with this notation, $\hat{A}_h(\omega)$, $\omega \in T_h^n$, may be written as a $2^n \times 2^n$ diagonal matrix

$$\hat{\boldsymbol{A}}_h(\omega), \quad \omega \in T_H^n. \tag{12}$$

The Fourier transform of a restriction $R_{Hh} = R_{Hh}^0 A_h$ is now given by

$$\widehat{R_{Hh}u_h}(\omega) = \sum_p \hat{A}_h\left(\omega + \frac{\pi p}{h}\right)\hat{u}_h\left(\omega + \frac{\pi p}{h}\right), \tag{13}$$

or, in vector notation, with $u_H = R_{Hh}u_h$

$$\hat{u}_H(\omega) = \hat{\boldsymbol{R}}_{Hh}(\omega)\hat{\boldsymbol{u}}_h(\omega), \tag{14}$$

where $\omega \in T_H^n$. Now $\hat{\boldsymbol{R}}_{Hh}(\omega)$ is a 1×2^n matrix with entries $\hat{A}_h(\omega + \pi p/h)$.

Similarly, the Fourier transform of a prolongation $P_{hH} = A_h P_{hH}^0$ with $u_h = P_{hH}u_H$ can be written

$$\hat{\boldsymbol{u}}_h(\omega) = \hat{\boldsymbol{P}}_{hH}(\omega)\hat{u}_H(\omega), \tag{15}$$

with $\omega \in T_H^n$. Here $\hat{\boldsymbol{P}}_{hH}(\omega)$ is a $2^n \times 1$ matrix with entries $\hat{A}_h(\omega + \pi p/h)2^n$.

3. The order of prolongations and restrictions

For the usual prolongations and restrictions, for which the stencils are real and have finite support, the components of their Fourier transforms have the form of a trigonometric polynomial in $\theta = \omega h$. To unify the treatment for both types of transfer operators, we write $\hat{B}(\theta)$ for $\hat{B}_h$ or $2^n\hat{B}_h$ in case of the restriction or prolongation, respectively.

Definition 1. The *low frequency* (LF) *order* of a grid transfer operator B_h is the largest number $m \geqslant 0$ for which

$$\hat{B}(\theta) = 1 + \mathcal{O}(|\theta|^m) \quad \text{for } |\theta| \to 0. \tag{16}$$

Definition 2. The *high frequency* (HF) *order* of a grid transfer operator B_h is the largest number m for which

$$\hat{B}(\theta + p\pi) = \mathcal{O}(|\theta|^m) \quad \text{for } |\theta| \to 0, \tag{17}$$

for all $p \in \{0, 1\}^n$, $p \neq 0^n$.

Remark. We may understand these definitions as: "A high-order transfer operator disturbs low frequencies by a small amount (high LF order), whereas the corresponding high frequencies do not pop up too much (high HF order)".

Examples. Simple computations show for particular transfer operators the following orders.

(1) Injection R_{Hh}^0; stencil $a_h = [0, 1, 0]$, $\hat{A}(\theta) \equiv 1$; the LF order is infinity, the HF order is 0.

(2) The weighted restriction in one dimension, linear interpolation; the stencil is

$$a_h = \tfrac{1}{2}\left[\tfrac{1}{2},\ 1,\ \tfrac{1}{2}\right]; \tag{18}$$

the LF order is 2, the HF order is 2.

(3) Half weighting; the stencil is

$$a_h = \tfrac{1}{2}\begin{bmatrix} 0 & \tfrac{1}{4} & 0 \\ \tfrac{1}{4} & 1 & \tfrac{1}{4} \\ 0 & \tfrac{1}{4} & 0 \end{bmatrix}; \tag{19}$$

the LF order is 2, the HF order is 0.

(4) Full weighting, or bilinear prolongation on rectangles; the stencil is

$$a_h = \tfrac{1}{4}\begin{bmatrix} \tfrac{1}{4} & \tfrac{1}{2} & \tfrac{1}{4} \\ \tfrac{1}{2} & 1 & \tfrac{1}{2} \\ \tfrac{1}{4} & \tfrac{1}{2} & \tfrac{1}{4} \end{bmatrix}; \tag{20}$$

the LF order is 2, the HF order is 2.

(5) Seven point restriction, or linear interpolation on triangles; the stencil is

$$a_h = \tfrac{1}{4}\begin{bmatrix} \tfrac{1}{2} & \tfrac{1}{2} & 0 \\ \tfrac{1}{2} & 1 & \tfrac{1}{2} \\ 0 & \tfrac{1}{2} & \tfrac{1}{2} \end{bmatrix}; \tag{21}$$

the LF order is 2, the HF order is 2.

(6) Cubic interpolation in one dimension; the stencil is

$$a_h = \tfrac{1}{2}\left[-\tfrac{1}{16},\ 0,\ \tfrac{9}{16},\ 1,\ \tfrac{9}{16},\ 0,\ -\tfrac{1}{16}\right]; \tag{22}$$

the LF order is 4, the HF order is 4.

Remark. There is a direct relation between the order of the transfer operator as defined here and the degree of the polynomial that is exactly interpolated by the corresponding interpolation rule. The following statements are easy to verify. (1) If a restriction leaves all polynomials of degree $k-1$ invariant, then the LF order of the operator is k. (2) If a prolongation leaves all polynomials of degree $k-1$ invariant, then both the LF and HF order are at least k.

Proof of (1). If the restriction stencil $[c_j]$ leaves all polynomials P_k of degree $\leqslant k$ invariant, then

$$P_k(x_k) = \sum_j c_j P_k(x_{k-j}), \tag{23}$$

for all P_k, and hence

$$\begin{cases} 1 = \sum_j c_j, \\ 0 = \sum_j c_j(-jh)^{\alpha}, \quad \forall\alpha,\ |\alpha| \leqslant k. \end{cases} \tag{24}$$

Therefore,

$$\hat{C}(\omega)=\sum_j c_j\,\mathrm{e}^{-\mathrm{i}\omega hj}=\sum_j c_j\sum_{|\beta|\geq 0} d_\beta(-\mathrm{i}\omega hj)^\beta$$
$$=\sum_{|\beta|\geq 0}(\mathrm{i}\omega)^\beta d_\beta\sum_j c_j(-hj)^\beta=1+\sum_{|\beta|>k}(\mathrm{i}\omega)^\beta d_\beta\sum_j c_j(-hj)^\beta$$
$$=1+\mathcal{O}\left((\omega h)^{k+1}\right)\quad\text{for } \omega h\to 0.$$

Proof of (2). If the interpolation stencil $[c_j]$ leaves all polynomials P_k of degree $\leqslant k$ invariant, then

$$P_k(x_k)=\sum_j c_{2j-p}P(x_{k+p-2j}),\tag{25}$$

for all multi-integers $p\in\{0,1\}^n$. It follows that, for all p,

$$\begin{cases}1=\sum_j c_{2j-p},\\ 0=\sum_j c_{2j-p}((p-2j)h)^\alpha, & \forall\alpha,\ 0<|\alpha|\leqslant k;\end{cases}\tag{26}$$

and hence

$$\hat{C}(\theta+p\pi)=2^{-n}\sum_j c_j\,\mathrm{e}^{-\mathrm{i}(\omega+p\pi/h)hj}=2^{-n}\sum_{m\in\{0,1\}^n}\sum_j c_{2j-m}\,\mathrm{e}^{-\mathrm{i}(\omega h+p\pi)(2j-m)}$$
$$=2^{-n}\sum_{m\in\{0,1\}^n}\mathrm{e}^{\mathrm{i}p\pi m}\sum_j c_{2j-m}\,\mathrm{e}^{\mathrm{i}\omega h(m-2j)}$$
$$=2^{-n}\sum_{m\in\{0,1\}^n}(-1)^{pm}\sum_j c_{2j-m}\sum_{|\beta|\geqslant 0}d_\beta(\mathrm{i}\omega h(m-2j))^\beta$$
$$=2^{-n}\sum_{m\in\{0,1\}^n}(-1)^{pm}\left\{1+\sum_{|\beta|>k}d_\beta(\mathrm{i}\omega)^\beta\sum_j c_{2j-m}((m-2j)h)^\beta\right\}$$
$$=2^{-n}\left\{\sum_{m\in\{0,1\}^n}(-1)^{pm}\right.$$
$$\left.+\sum_{|\beta|>k}d_\beta(\mathrm{i}\omega)^\beta\sum_{m\in\{0,1\}^n}(-1)^{pm}\sum_j c_{2j-m}((m-2j)h)^\beta\right\}$$
$$=\begin{cases}1+\mathcal{O}\left((\omega h)^{k+1}\right), & \text{if } p=0,\\ \mathcal{O}\left((\omega h)^{k+1}\right), & \text{if } p\neq 0.\end{cases}$$

4. Requirements for transfer operators in multigrid

Let L be a constant-coefficient, linear-differential operator of order M, then

$$\hat{L}(\omega)=\sum_{|m|=0}^{M}c_m\omega^m,\tag{27}$$

where, for the more-dimensional case, we use the multi-integer notation again. Let L_h and L_H, $H = 2h$, be discretisations of L of order $\tilde{p}$. We suppose that L_h and L_H both are determined by the same unique stencil. Then

$$\hat{L}_h(\omega) = \sum_{|m|=0}^{M} c_m \frac{1}{h^m} s_m(\omega h), \tag{28}$$

where $s_m(\omega h)$ is an mth-degree trigonometric polynomial satisfying

$$\begin{aligned} s_m(\omega h) &= \mathcal{O}\left((\omega h)^m\right) && \text{for } \omega h \to 0, \\ s_m(\omega h) &= \mathcal{O}(1) && \text{for } \omega h \in [-\pi, \pi]^n, \end{aligned} \tag{29}$$

and

$$|\hat{L}_h(\omega) - \hat{L}(\omega)| = \mathcal{O}(h^{\tilde{p}}) \quad \text{for } \omega h \to 0. \tag{30}$$

Now we keep ω fixed and let $h \to 0$; hence $\theta = \omega h \to 0$.

To find the conditions for the grid transfer operators to satisfy in a multigrid algorithm, we first consider

$$\left|\frac{\hat{L}_h(\omega)}{\hat{L}_{2h}(\omega)}\right| \approx \left|\frac{\hat{L}(\omega) + Ch^{\tilde{p}} + \cdots}{\hat{L}(\omega) + C(2h)^{\tilde{p}} + \cdots}\right| = 1 + \mathcal{O}(\theta^{\tilde{p}}) \quad \text{for } \theta \to 0, \tag{31}$$

and for $p \in \{0, 1\}^n$, $p \neq 0^n$ and $\theta \to 0$,

$$\left|\frac{\hat{L}_h(\omega + \pi p/h)}{\hat{L}_{2h}(\omega + \pi p/2h)}\right| = \left|\frac{\hat{L}_h(\omega + \pi p/h)}{\hat{L}_{2h}(\omega)}\right| \approx \left|\frac{\hat{L}_h(\omega + \pi p/h)}{\sum_{|m|=0}^{M} c_m (1/2h)^m (\omega 2h)^m}\right| = \mathcal{O}\left((\omega h)^{-M}\right). \tag{32}$$

The amplification operator of the two-grid coarse-grid-correction operator is [2]

$$M_h^{\text{CGC}} = I_h - P_{hH} L_H^{-1} R_{Hh} L_h, \tag{33}$$

and hence

$$\hat{\boldsymbol{M}}_h^{\text{CGC}}(\theta) = \hat{\boldsymbol{I}}_h - \hat{\boldsymbol{P}}_{hH} \hat{L}_H^{-1} \hat{\boldsymbol{R}}_{Hh} \hat{\boldsymbol{L}}_h, \tag{34}$$

or

$$\hat{\boldsymbol{M}}_h^{\text{CGC}}(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \hat{L}_H^{-1} \begin{pmatrix} \hat{P}_{hH}(\omega) \\ \hat{P}_{hH}(\omega + \pi p/h) \end{pmatrix} \left(\hat{R}_{Hh}(\omega), \hat{R}_{Hh}(\omega + \pi p/h)\right) \times \begin{pmatrix} \hat{L}_h(\omega) & 0 \\ 0 & \hat{L}_h(\omega + \pi p/h) \end{pmatrix}.$$

Now we denote the LF and HF order of the prolongation (restriction) by m_1 and m_2

(respectively by n_1 and n_2), to obtain for $\theta \to 0$

$$\hat{M}_h^{\mathrm{CGC}}(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1+\mathcal{O}(\theta^{m_1}) \\ \mathcal{O}(\theta^{m_2}) \end{pmatrix} (1+\mathcal{O}(\theta^{n_1}),\ \mathcal{O}(\theta^{n_2})) \begin{pmatrix} 1+\mathcal{O}(\theta^{\bar{p}}) & 0 \\ 0 & \mathcal{O}(\theta^{-M}) \end{pmatrix}$$
$$= \begin{pmatrix} \mathcal{O}(\theta^{m_1})+\mathcal{O}(\theta^{n_1})+\mathcal{O}(\theta^{\bar{p}}) & \mathcal{O}(\theta^{n_2-M}) \\ \mathcal{O}(\theta^{m_2}) & 1+\mathcal{O}(\theta^{m_2+n_2-M}) \end{pmatrix}.$$

For $\theta \to 0$ the eigenvalues of $\hat{M}(\theta)$ approach

$$\lambda_1 \approx 1 \quad \text{and} \quad \lambda_2 \approx \mathcal{O}(\theta^{n_2+m_2-M}). \tag{35}$$

Hence, a necessary condition for nonincreasing high frequencies arising from a coarse grid correction is $n_2 + m_2 \geqslant M$. This corresponds with the well-known rule of thumb, used in the multigrid community [1,2,4]. However, we have to be aware that it is the HF and not the LF order that shows up in this rule.

In [4] it is shown by an example that the rule gives a necessary condition indeed. There it is seen that a set of transfer operators that satisfies the conditions for a convection operator fails as soon as an additional diffusion operator becomes significant.

In order that the *norm* of the amplification operator of the error is also bounded, one should require $n_2 \geqslant M$ and $m_2 \geqslant 0$.

Similarly, for *the residual amplification operator*, i.e., $\overline{M}_h^{\mathrm{CGC}} = I_h - L_h P_{hH} L_H^{-1} R_{Hh}$, we find

$$\hat{\overline{M}}_h^{\mathrm{CGC}}(\theta) = \begin{pmatrix} \mathcal{O}(\theta^{m_1})+\mathcal{O}(\theta^{n_1})+\mathcal{O}(\theta^{\bar{p}}) & \mathcal{O}(\theta^{n_2}) \\ \mathcal{O}(\theta^{m_2-M}) & 1+\mathcal{O}(\theta^{m_2+n_2-M}) \end{pmatrix}. \tag{36}$$

If we want its norm to be bounded, then we find the conditions $m_2 \geqslant M$ and $n_2 \geqslant 0$.

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