JOURNAL OF COMBINATORIAL THEORY, Series B **51**, 127–159 (1991)

Disjoint Circuits of Prescribed Homotopies in a Graph on a Compact Surface

A. SCHRIJVER

Mathematical Centre, Kruislaan 413, SJ Amsterdam, The Netherlands, and University of Amsterdam, The Netherlands

Communicated by the Editors

Received June 23, 1988

We give necessary and sufficient conditions for the existence of pairwise vertex-disjoint simple closed curves $\tilde{C}_1, ..., \tilde{C}_k$, homotopic to given closed curves $C_1, ..., C_k$, respectively, in a graph embedded on a compact surface, thus proving a conjecture of L. Lovász and P. D. Seymour. © 1991 Academic Press, Inc.

1. THE THEOREM

We prove the following theorem, conjectured by L. Lovász and P. D. Seymour:

THEOREM. *Let $G = (V, E)$ be a graph, embedded on a compact surface S, and let $C_1, ..., C_k$ be closed curves on S, each not null-homotopic. Then there exist pairwise disjoint simple closed curves $\tilde{C}_1, ..., \tilde{C}_k$ in G such that $\tilde{C}_i$ is homotopic to C_i for $i = 1, ..., k$, if and only if:*

(i) *there exist pairwise disjoint simple closed curves $C'_1, ..., C'_k$ on S such that C'_i is homotopic to C_i for $i = 1, ..., k$;*

(ii) *for each closed curve $D: S_1 \to S$,*

$$\operatorname{cr}(G, D) \geqslant \sum_{i=1}^{k} \operatorname{mincr}(C_i, D);$$

(iii) *for each doubly odd closed curve $D = D_1 \cdot D_2 : S_1 \to S$ with $D_1(1) = D_2(1) \notin G$,*

$$\operatorname{cr}(G, D) > \sum_{i=1}^{k} \operatorname{mincr}(C_i, D). \qquad (1)$$

0095-8956/91 $3.00

Here we use the following conventions, terminology and notation. A graph is said to be *embedded on* S, if it is embedded so that edges intersect each other only at their end points. We identify a graph with its image on S.

A *closed curve* (*on* S) is a continuous function $C: S_1 \to S$, where S_1 denotes the unit circle in the complex plane $\mathbb{C}$. It is *simple* if it is one-to-one. Two closed curves are *disjoint* if their images are disjoint.

Two closed curves C and $\tilde{C}$ are (*freely*) *homotopic* (*on* S), in notation $C \sim \tilde{C}$, if there exists a continuous function $\Phi: S_1 \times [0, 1] \to S$ such that $\Phi(z, 0) = C(z)$ and $\Phi(z, 1) = C(z)$ for all $z \in S_1$. (So we do not fix a base point when speaking of homotopy of closed curves.) Closed curve C is *null-homotopic* if C is homotopic to some constant function.

We assume the reader has some idea of what "crossing" of two curves means. To be precise, let C and D be closed curves on S. A pair $(y, z) \in S_1 \times S_1$ is said to give (or to be) a *crossing* if $C(y) = D(z)$, and $C(y)$ has a neighbourhood $N \simeq \mathbb{C}$ on S such that, in a neighbourhood N_y of $y \in S_1$, C follows the real axis of N, and in a neighbourhood N_z of $z \in S_1$, D follow the imaginary axis of N (real and imaginary axis under the homeomorphism $N \simeq \mathbb{C}$). We will not use this precise definition.

Further we define

$$\begin{aligned} \operatorname{cr}(G, D) &:= |\{z \in S_1 \mid D(z) \in G\}|, \\ \operatorname{cr}(C, D) &:= \text{number of crossings } (y, z) \text{ of } C \text{ and } D, \\ \operatorname{mincr}(C, D) &:= \min\{\operatorname{cr}(\tilde{C}, \tilde{D}) \mid \tilde{C} \sim C, \tilde{D} \sim D\}, \end{aligned} \tag{2}$$

where we restrict $\tilde{C}$ and $\tilde{D}$ to have a finite number of intersections.

If $D_1, D_2: S_1 \to S$ are closed curves with $D_1(1) = D_2(1)$, then $D_1 \cdot D_2$ is the closed curve given by

$$(D_1 \cdot D_2)(z) := \begin{cases} D_1(z^2) & \text{if } \operatorname{Im}(z) \geqslant 0, \\ D_2(z^2) & \text{if } \operatorname{Im}(z) < 0, \end{cases} \tag{3}$$

for $z \in S_1$. We call a closed curve $D: S_1 \to S$ *doubly odd* (with respect to G, $C_1, ..., C_k$) if $D = D_1 \cdot D_2$ for some closed curves D_1, D_2 satisfying

$$\begin{aligned} \operatorname{cr}(G, D_1) &\not\equiv \sum_{i=1}^{k} \operatorname{cr}(C_i, D_1) \pmod 2, \\ \operatorname{cr}(G, D_2) &\not\equiv \sum_{i=1}^{k} \operatorname{cr}(C_i, D_2) \pmod 2. \end{aligned} \tag{4}$$

It is easy to see that the conditions (1) are necessary conditions. The essence of the theorem is sufficiency of (1).

Studying disjoint curves of given homotopies originates from two different sources. One source is the series of papers on Graph Minors by Robertson and Seymour, in which problems on disjoint paths are studied with the help of surfaces and homotopy (cf. [15, 16]). A second source is the layout of VLSI-circuits, in which disjoint connections have to be made between sets of pins in a large planar graph. Again, this problem is handled with the help of homotopy (cf. [3, 8, 11]).

In this paper we first give in Section 2 an auxiliary theorem on integer solutions to certain systems of linear inequalities, and in Section 3 some preliminaries on surfaces. In Section 4 we give a proof of the theorem.

2. An Auxiliary Theorem on Linear Inequalities

A basic ingredient of our proof is a theorem on the existence of an integer solution to a certain system of linear inequalities.

Let W be a finite set, and let M be a set of pairs partitioning W (so M forms a perfect matching on W). Denote $pq := \{p, q\}$ and $pp := \{p\}$. Let

$$E := \{pq \mid p, q \in W\}. \tag{5}$$

So (W, E) is a complete undirected graph, with loops attached at each vertex.

Let

$$\lambda\colon E \to \mathbb{Z} \cup \{\infty\} \tag{6}$$

be a "length" function. We want to know if there exists a function $\psi\colon W \to \mathbb{Z}$ such that

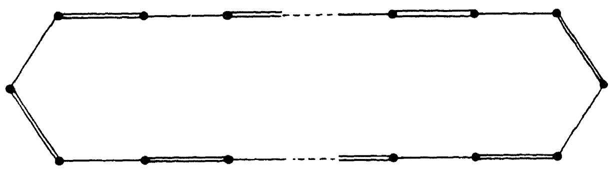
$$\begin{aligned} &\text{(i)} \quad \psi(p) + \psi(q) = 0 \qquad &&\text{if } pq \in M, \\ &\text{(ii)} \quad \psi(p) + \psi(q) \leqslant \lambda(pq) \qquad &&\text{if } pq \in E. \end{aligned} \tag{7}$$

This amounts to solving a certain system of linear inequalities in integers.

To characterize the existence of such a function ψ, call a sequence

$$(p_0, p_1, p_2, ..., p_{2d-1}, p_{2d}) \tag{8}$$

an *alternating cycle* if $p_0 = p_{2d}$ and $p_{2i}p_{2i+1} \in M$ for $i = 0, ..., d-1$. The idea is a cycle of form



(9)

where double lines indicate edges in M, and single lines indicate edges in E. Vertices (and edges) among (8) may coincide, so (9) is not the general picture.

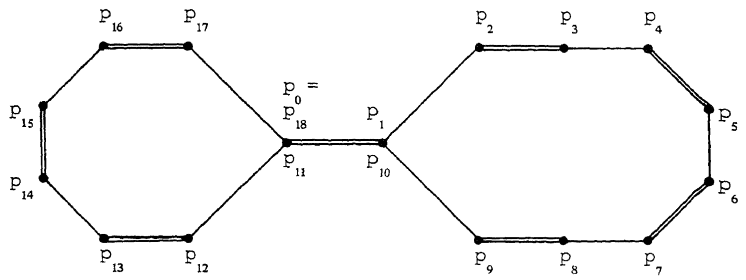
The *length* of (8) is, by definition,

$$\sum_{i=1}^{d} \lambda(p_{2i-1}p_{2i}). \tag{10}$$

We call (8) *doubly odd* if there exists a number t such that

$$\begin{aligned} &\text{(i)} \quad p_0 = p_{2t+1}, \qquad p_1 = p_{2t}, \\ &\text{(ii)} \quad \sum_{i=1}^{t} \lambda(p_{2i-1}p_{2i}) \text{ is odd} \qquad \text{and} \qquad \sum_{i=t+1}^{d} \lambda(p_{2i-1}p_{2i}) \text{ is odd.} \end{aligned} \tag{11}$$

Here "odd" implies finiteness. The idea is a figure of the form



(12)

(here $t=5$, $d=9$; again points may coincide).

Now one has:

AUXILIARY THEOREM. *Given* $\lambda: E \to \mathbb{Z} \cup \{\infty\}$, *there exists a function* $\psi: W \to \mathbb{Z}$ *satisfying* (7), *if and only if*:

(i) *each alternating cycle has nonnegative length*;

(ii) *each doubly odd alternating cycle has positive length.* (13)

Proof. I. *Necessity.* Suppose function $\psi: W \to \mathbb{Z}$ satisfying (7) exists. Then for each alternating cycle (8) one has

$$\sum_{i=1}^{d} \lambda(p_{2i-1}p_{2i}) \geqslant \sum_{i=1}^{d} (\psi(p_{2i-1}) + \psi(p_{2i})) = \sum_{i=1}^{d} (\psi(p_{2i-2}) + \psi(p_{2i-1})) = 0 \tag{14}$$

(using (7) and the fact that $p_0 = p_{2d}$). Moreover, if (8) satisfies (11) then:

$$\sum_{i=1}^{t} \lambda(p_{2i-1}p_{2i}) \geqslant \sum_{i=1}^{t} (\psi(p_{2i-1}) + \psi(p_{2i}))$$

$$= \psi(p_1) + \sum_{i=2}^{t} (\psi(p_{2i-2}) + \psi(p_{2i-1})) + \psi(p_{2t}) = 2\psi(p_1). \tag{15}$$

Hence, by (11)(ii), strict inequality should hold in (15), and therefore also in (14). So (8) has positive length.

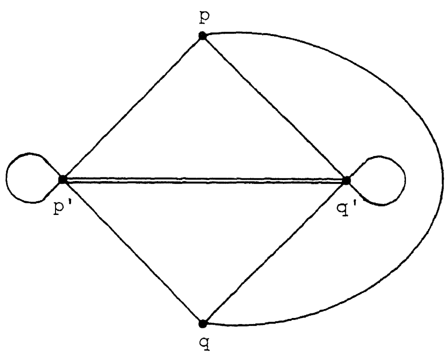
II. *Sufficiency.* The proof is by induction on $|W|$, the case $|W| \leqslant 2$ being trivial. Suppose (13) is satisfied, and $|W| > 2$. Choose $p'q' \in M$. Let

$$W' := W \setminus \{p', q'\}, \qquad E' := \{pq \mid p, q \in W'\}, \qquad M := M \setminus \{p'q'\}. \tag{16}$$

Define $\lambda' \colon E' \to \mathbb{Z} \cup \{\infty\}$ by:

$$\lambda'(pq) := \min\{\lambda(pq), \lambda(pp') + \lambda(q'q), \lambda(pq') + \lambda(p'q),$$
$$\lambda(pp') + \lambda(q'q') + \lambda(p'q), \lambda(pq') + \lambda(p'p') + \lambda(q'q)\}. \tag{17}$$

(The minimum ranges over all 'alternating paths' in



(18)

from p to q. Also the case $p = q$ is included in (17).)

It is straightforward to see that λ' again satisfies (13). Hence, by the induction hypothesis, there exists a function $\psi' \colon W' \to \mathbb{Z}$ so that

$$\begin{aligned} &\text{(i)} \quad \psi'(p) + \psi'(q) = 0 && \text{if } pq \in M', \\ &\text{(ii)} \quad \psi'(p) + \psi'(q) \leqslant \lambda'(pq) && \text{if } pq \in E'. \end{aligned} \tag{19}$$

We define $\psi(p) := \psi'(p)$ for each $p \in W'$. This choice satisfies all inequalities in (7) not containing p' or q'. Next we should choose integer values for $\psi(p')$ and $\psi(q')$ so that the inequalities in (7) containing p' or q' are satisfied. So $\psi(p')$ and $\psi(q')$ should satisfy

$$\begin{aligned} \psi(p') + \psi(q') &= 0, \\ \psi(p') + \psi(p) &\leqslant \lambda(pp'), \qquad \text{for all} \quad p \in W', \\ 2\psi(p') &\leqslant \lambda(p'p'), \\ \psi(q') + \psi(q) &\leqslant \lambda(qq'), \qquad \text{for all} \quad q \in W', \\ 2\psi(q') &\leqslant \lambda(q'q'), \\ \psi(p') + \psi(q') &\leqslant \lambda(p'q'). \end{aligned} \tag{20}$$

We can delete the last inequality, since $\lambda(p'q') \geqslant 0$ (as the alternating cycle (p', q', p') has nonnegative length, by (13)(i)). So (20) is equivalent to

$$\begin{aligned} \max_{q \in W'} (\psi(q) - \lambda(qq')) \leqslant -\psi(q') = \psi(p') &\leqslant \min_{p \in W'} (\lambda(pp') - \psi(p)), \\ -\tfrac{1}{2}\lambda(q'q') \leqslant -\psi(q') = \psi(p') &\leqslant \tfrac{1}{2}\lambda(p'p'). \end{aligned} \tag{21}$$

Now by (17)

$$\begin{aligned} &\max_{q \in W'} (\psi(q) - \lambda(qq')) \leqslant \min_{p \in W'} (\lambda(pp') - \psi(p)), \\ &\max_{q \in W'} (\psi(q) - \lambda(qq')) \leqslant \tfrac{1}{2}\lambda(p'p'), \\ &\qquad -\tfrac{1}{2}\lambda(q'q') \leqslant \min_{p \in W'} (\lambda(pp') - \psi(p)). \end{aligned} \tag{22}$$

Moreover,

$$-\tfrac{1}{2}\lambda(q'q') \leqslant \tfrac{1}{2}\lambda(p'p') \tag{23}$$

by (13)(i) applied to the alternating cycle (p', q', q', p', p'). It implies that we can find integer values for $\psi(p')$ and $\psi(q')$ satisfying (21), except if $-\lambda(q'q') = \lambda(p'p')$ and is odd. But this is excluded by (13)(ii) applied to the doubly odd alternating cycle (p', q', q', p', p'). ∎

3. Preliminaries on Surfaces

We give a brief review of some facts on surfaces, focusing on results used in the proof of our theorem. We refer to Ahlfors and Sario [1], Fenn [5], Massey [9], Moise [10], and Seifert and Threlfall [17] for more extensive treatments.

Surfaces form a class of spaces which are well under control topologically. In this paper we mean by a surface a *triangulable* surface. By a theorem of Radó [13] this is not a restriction for compact surfaces. Beside the compact surfaces, we consider in our proof only three other, noncompact, surfaces, viz.:

— the complex plane;

— the annulus; (24)

— the (open) Möbius strip.

The *annulus* arises from $\mathbb{R} \times [0, 1]$ by identifying $(x, 0)$ and $(x, 1)$, for each $x \in \mathbb{R}$. The *(open) Möbius strip* arises from $\mathbb{R} \times [0, 1]$ by identifying $(x, 0)$ and $(-x, 1)$, for each $x \in \mathbb{R}$.

Dehn and Heegaard [4] classified all compact surfaces as those spaces obtained from the 2-sphere by adding a finite number of handles or a finite number of "cross-caps." By a theorem of Poincaré [12], the 2-sphere and the complex plane are the only surfaces with trivial fundamental group. By a theorem of von Kerékjártó [7], the annulus and the Möbius strip are the only surfaces with infinite cyclic fundamental group (cf. [6, 14]).

The triangulability of the surfaces enables us to apply the theory of "simplicial approximation," by which topological homotopy of curves can be reduced to combinatorial homotopy of curves in a (fine enough) triangular grid on the surface (see Seifert and Threlfall [17:4. Kap.]).

Paths and Closed Curves

A *path* on S is a continuous function P: $[0, 1] \rightarrow S$. It is said to go *from* $P(0)$ *to* $P(1)$. If P and Q are paths on S with $P(1) = Q(0)$, then $P \cdot Q$ is the path defined by

$$(P \cdot Q)(x) := \begin{cases} P(2x) & \text{if } \ 0 \leqslant x \leqslant \frac{1}{2}, \\ Q(2x-1) & \text{if } \ \frac{1}{2} < x \leqslant 1. \end{cases} \tag{25}$$

Similarly one defines $P_1 \cdot \cdots \cdot P_n$ (if $P_{i-1}(1) = P_i(0)$ for $i = 2, ..., n$) and P^n (if $P(1) = P(0)$ and $n \in \mathbb{N}$). Moreover, P^{-1} is the path with $P^{-1}(x) := P(1-x)$ for $x \in [0, 1]$.

As mentioned before, a *closed curve* on S is a continuous function C: $S_1 \rightarrow S$ (where S_1 denotes the unit circle in $\mathbb{C}$). If C and D are closed curves on S with $C(1) = D(1)$, then $C \cdot D$ is the closed curve defined by

$$(C \cdot D)(z) := \begin{cases} C(z^2) & \text{if } \ \mathrm{Im}(z) \geqslant 0, \\ D(z^2) & \text{if } \ \mathrm{Im}(z) < 0, \end{cases} \tag{26}$$

for $z \in S_1$. For $n \in \mathbb{Z}$, closed curve C^n is defined by $C^n(z) := C(z^n)$ for $z \in S_1$.

Homotopy

Two paths $P, Q: [0,1] \to S$ are called *homotopic* (on S), if there exists a continuous function $\Phi: [0,1] \times [0,1] \to S$ such that

$$\begin{aligned} \Phi(x,0) &= P(x), \qquad \Phi(x,1) = Q(x), \\ \Phi(0,x) &= P(0), \qquad \Phi(1,x) = P(1), \end{aligned} \tag{27}$$

for all $x \in [0,1]$. In particular, $P(0) = Q(0)$ and $P(1) = Q(1)$. This defines an equivalence relation on paths, denoted by $\sim$. Path P is called *null-homotopic* if P is homotopic to a constant function (in particular, $P(0) = P(1)$).

Similarly, two closed curves $C, D: S_1 \to S$ are called *freely homotopic*, or just *homotopic* (*on* S), if there exists a continuous function $\Phi: S_1 \times [0,1] \to S$ such that

$$\Phi(z,0) = C(z), \qquad \Phi(z,1) = D(z) \tag{28}$$

for all $z \in S_1$. Again this defines an equivalence relation, denoted by $\sim$. Closed curve C is called *null-homotopic* if C is homotopic to a constant function.

Denote

$$\begin{aligned} \mathrm{hom}(P) &:= \text{homotopy class of } P; \\ \mathrm{Hom}(p,q) &:= \text{set of homotopy classes of paths from } p \text{ to } q. \end{aligned} \tag{29}$$

Now the product $\mathrm{hom}(P) \cdot \mathrm{hom}(Q) := \mathrm{hom}(P \cdot Q)$, for paths P and Q from p to p, is well-defined, and turns $\mathrm{Hom}(p,p)$ into a group, the *fundamental group*. As an abstract group it is independent of the choice of p in S.

A surface is called *simply connected* if its fundamental group consists of one element only. It means that each closed curve on the surface is null-homotopic.

The fundamental groups of the compact surfaces are well-described, and it follows that if S is compact surface, and S is not the projective plane, then the fundamental group of S is torsion-free. In other words:

> if S is a compact surface not equal to the projective plane, and C is a closed curve on S with C^n null-homotopic for some $n \geqslant 2$, then C is null-homotopic. (30)

Covering Surfaces

A *covering surface* of a surface S is a pair S', π, where S' is a surface and $\pi: S' \to S$ maps S' onto S so that each point p of S has a neighbourhood $N \simeq \mathbb{C}$ such that for each component K of $\pi^{-1}[N]$ one has that $\pi | K$ is a homeomorphism from K onto N. The map π is called the *projection function* of the covering surface.

A well-known example of a covering surface of the torus $S_1 \times S_1$ is the pair $\mathbb{R}^2, \pi$, where

$$\pi(x, y) := (\exp(2\pi \mathbf{i} x), \exp(2\pi \mathbf{i} y)), \qquad \text{for} \quad (x, y) \in \mathbb{R}^2. \tag{31}$$

(By $\mathbf{i}$ we denote the imaginary unit.) It corresponds to cutting the torus open, so as to obtain a square, and next sticking infinitely many copies of it together so as to obtain a tesselation of $\mathbb{R}^2$ by squares.

Another covering surface of the torus is the pair $\mathbb{R} \times S_1, \pi$, where

$$\pi(x, z) := (\exp(2\pi \mathbf{i} x), z) \qquad \text{for} \quad (x, z) \in \mathbb{R} \times S_1. \tag{32}$$

It corresponds to cutting the torus open along a circle, so as to obtain a cylinder, and next sticking infinitely many copies of it together, so as to obtain the infinitely long cylinder $\mathbb{R} \times S_1$ ($\simeq$ the annulus).

If we consider the projective plane as obtained from the 2-sphere S_2 by identifying antipodal points, we have that S_2, π is a covering surface of the projective plane (where π is the identification map).

Two covering surfaces S', π and S'', π' of S are called *isomorphic* if there exists a homeomorphism $\phi: S' \to S''$ such that $\pi' \circ \phi = \pi$.

The Universal Covering Surface

If S', π is a covering surface of S and S' is simply connected, then S', π is called a *universal covering surface* of S. In fact, it is unique, up to isomorphism. Therefore, one can speak of *the* universal covering surface of S.

For example, $\mathbb{R}^2, \pi$ (with π as in (31)) is the universal covering surface of the torus. Similarly, the 2-sphere, with π as above, is the universal covering surface of the projective plane. Moreover, S_2, id is the universal covering surface of S_2 (where id denotes the identity map).

In fact, one has the following helpful result:

> if S is a compact surface, not equal to the 2-sphere or the projective plane, and S', π is the universal covering surface of S, then S' is homeomorphic to the complex plane $\mathbb{C}$. (33)

This was shown by Schwarz and by Poincaré [12]. It means that copies of the "fundamental polygon" of any compact surface (except the 2-sphere and the projective plane) can be stuck together so as to form a space homeomorphic to the complex plane $\mathbb{C}$.

Lifting of Paths and Closed Curves

An important property of covering surfaces is the capability of "lifting" paths and closed curves. Let S', π be a covering surface of the surface S.

Then

> if $P: [0, 1] \to S$ is a path on S and $p \in \pi^{-1}(P(0))$, then there exists a unique path $P': [0, 1] \to S'$ with $P'(0) = p$ and $\pi \circ P' = P$. (34)

Path P' is called a *lifting* of P to S'.

Similarly one has

> if $C: S_1 \to S$ is a closed curve on S and $p \in \pi^{-1}(C(1))$, then there exists a unique map $C': \mathbb{R} \to S'$ with $C'(0) = p$ and $\pi \circ C'(x) = C(\exp(2\pi\mathbf{i}x))$ for all $x \in \mathbb{R}$. (35)

Again, C' is called a *lifting* of C to S'.

Crossings of Closed Curves

We finally study the number of crossings of closed curves on a compact surface S. Let C and D be closed curves on S, having a finite number of intersections. Let S', π be the universal covering surface of S. Then:

PROPOSITION. *If each lifting of C to S' crosses each lifting of D to S' at most once, then* $\operatorname{cr}(C, D) = \operatorname{mincr}(C, D)$.

Proof. By the theory of simplicial approximation there exist triangulations Γ and Δ of S such that Γ and Δ have only a finite number of intersections, such that C is part of Γ and D is part of Δ, and such that $\operatorname{mincr}(C, D)$ is attained by closed curves $\tilde{C}$ in Γ and $\tilde{D}$ in Δ. Moreover, $\tilde{C}$ and $\tilde{D}$ arise from C and D by shifting over triangles of Γ and Δ, respectively.

Define

$$X(C, D) := \text{set of crossings of } C \text{ and } D. \tag{36}$$

For any closed curve $B: S_1 \to S$ and $y, y' \in S_1$, call a path $P: [0, 1] \to S$ a *$y - y'$ walk along B* if there exist $x, x' \in \mathbb{R}$ such that

$$\begin{aligned} &\text{(i)} \quad y = \exp(2\pi\mathbf{i}x), \qquad y' = \exp(2\pi\mathbf{i}x'), \\ &\text{(ii)} \quad P(\lambda) = B(\exp(2\pi\mathbf{i}((1-\lambda)x + \lambda x')), \qquad \text{for} \quad \lambda \in [0, 1]. \end{aligned} \tag{37}$$

Define an equivalence relation on $X(C, D)$ by

$$(y, z) \approx (y', z') \Leftrightarrow \text{some } y - y' \text{ walk along } C \text{ is homotopic to some } z - z' \text{ walk along } D. \tag{38}$$

Define

$$\text{odd}(C, D) := \text{number of equivalence classes of} \approx \text{with an odd number of elements.} \tag{39}$$

Now odd(C, D) is invariant under shifts of C and D over triangles of Γ and Δ, respectively, as one easily checks. Hence if $\tilde{C}$ and $\tilde{D}$ attain mincr(C, D), then

$$\text{mincr}(C, D) = \text{cr}(\tilde{C}, \tilde{D}) \geqslant \text{odd}(\tilde{C}, \tilde{D}) = \text{odd}(C, D). \tag{40}$$

However, if each lifting of C to S' crosses each lifting of D to S' at most once, then odd$(C, D) = $ cr(C, D), since each equivalence class of $\approx$ then contains exactly one element. Hence we have that cr$(C, D) =$ mincr(C, D). ∎

In fact, using a theorem of Baer [2] it can be shown that the condition given in the Proposition is near to a necessary condition for the property cr$(C, D) = $ mincr(C, D).

4. Proof of the Theorem

I. *Necessity*

Let $\tilde{C}_1, ..., \tilde{C}_k$ be pairwise disjoint simple closed curves in G so that $\tilde{C}_i$ is homotopic to C_i, for $i = 1, ..., k$. Clearly, this implies (1)(i). Condition (1)(ii) is also direct:

$$\text{cr}(G, D) \geqslant \sum_{i=1}^{k} \text{cr}(\tilde{C}_i, D) \geqslant \sum_{i=1}^{k} \text{mincr}(C_i, D). \tag{41}$$

To see condition (1)(iii), let $D = D_1 \cdot D_2$ be a doubly odd closed curve with $D_1(1) = D_2(1) \notin G$. So D_1 and D_2 satisfy (4). Now for each $i = 1, ..., k$,

$$\text{cr}(\tilde{C}_i, D_1) \equiv \text{cr}(C_i, D_1) \pmod 2, \tag{42}$$

since the parity of the number of crossings of two curves is invariant under homotopic transformations. Hence by (4)(i),

$$\text{cr}(G, D_1) \not\equiv \sum_{i=1}^{k} \text{cr}(\tilde{C}_i, D_1) \pmod 2. \tag{43}$$

Since $\tilde{C}_1, ..., \tilde{C}_k$ are pairwise disjoint we know that

$$\text{cr}(G, D_1) \geqslant \sum_{i=1}^{k} \text{cr}(\tilde{C}_i, D_1), \tag{44}$$

and hence, by (43), we should have strict inequality here. Therefore

$$\mathrm{cr}(G, D) = \mathrm{cr}(G, D_1) + \mathrm{cr}(G, D_2) > \sum_{i=1}^{k} \mathrm{cr}(\tilde{C}_i, D_1) + \sum_{i=1}^{k} \mathrm{cr}(\tilde{C}_i, D_2)$$

$$= \sum_{i=1}^{k} \mathrm{cr}(\tilde{C}_i, D_1 \cdot D_2) = \sum_{i=1}^{k} \mathrm{cr}(\tilde{C}_i, D) \geqslant \sum_{i=1}^{k} \mathrm{mincr}(C_i, D). \quad (45)$$

II. *Sufficiency*

Let (1) be satisfied. We first show:

CLAIM 1. *We may assume that each face of G ($=$ component of $S \setminus G$) is simply connected, i.e., homeomorphic to $\mathbb{C}$.*

Proof. By the triangulability of S, we can extend G to a graph G' embedded on S so that each face of G' is homeomorphic to $\mathbb{C}$. Let us choose such a G' containing G, so that G' has a minimum number of edges. One easily checks that G' satisfies (1) again. Moreover

> for each edge e of G' not in G there is a closed curve D on S crossing e once, and not having any other intersection with G'. (46)

To see this, let F and F' be the faces of G'' incident to e. If $F = F'$, then clearly a closed curve D as required exists. If $F \neq F'$, deleting e from G' would join F and F' to a face homeomorphic to $\mathbb{C}$, contradicting the minimality of G'.

Now assuming the theorem to hold for graphs with all faces simply connected, we know that G' contains pairwise disjoint simple closed curves $\tilde{C}_1, ..., \tilde{C}_k$ so that $\tilde{C}_i$ is homotopic to C_i for $i = 1, ..., k$. We show that $\tilde{C}_1, ..., \tilde{C}_k$ in fact are in G.

Suppose to the contrary that $\tilde{C}_i$ uses some edge e of G' not in G. Let D be as in (46). Then $\mathrm{cr}(\tilde{C}_i, D) = 1$, and hence $\mathrm{mincr}(C_i, D)$ is odd, and hence is also equal to 1. This gives

$$\mathrm{cr}(G, D) = 0 < \sum_{i=1}^{k} \mathrm{mincr}(C_i, D), \quad (47)$$

contradicting (1)(ii). ∎

So we assume that each face of G is simply connected. Next we observe that the theorem is trivial if S is the 2-sphere. Moreover, the theorem is easy if S is the projective plane: in that case, $k \leqslant 1$, since any two non-null-homotopic closed curves cross. Since each face of G is simply connected, it follows that G contains a simple non-null-homotopic closed curve, which is $\tilde{C}_1$.

So from now on we assume that S is not the 2-sphere or the projective plane. (This allows us to use below the facts that the universal covering surface of S is homeomorphic to $\mathbb{C}$, and that for each closed curve C on S, if C^n is null-homotopic for some $n \geqslant 2$, then C itself is null-homotopic.)

As each face of G is simply connected, we can pass to the dual graph G^* of G. So G^* is obtained from G by putting a vertex F^* in each face F of G, connecting two of these vertices F^*, H^* by an edge e^* crossing edge e of G, if F and H both are incident to e. (It might create a loop if $F = H$ is incident to e at both sides of e.)

Now the existence of pairwise disjoint simple closed curves $\tilde{C}_1, ..., \tilde{C}_k$ in G such that $\tilde{C}_i \sim C_i$ for $i = 1, ..., k$ is equivalent to

> there exist pairwise disjoint simple closed curves $\tilde{C}_1, ..., \tilde{C}_k$ on S, not intersecting $V(G^*)$, such that $\tilde{C}_i \sim C_i$ for $i = 1, ..., k$, and such that each face of G^* is passed through at most once by $\tilde{C}_1, ..., \tilde{C}_k$ (48)

(i.e., so that for each face F of G^* the set $F \cap \bigcup_{i=1}^{k} \tilde{C}_i[S_1]$ has at most one component). Here $V(G^*)$ denotes the vertex set of G^*.

Moreover, condition (1) for G is equvalent to almost the same condition for G^*:

(i) there exist pairwise disjoint simple closed curves $\tilde{C}_1, ..., \tilde{C}_k$ on S such that $\tilde{C}_i$ is homotopic to C_i for $i = 1, ..., k$;

(ii) for each closed curve $D: S_1 \to S$,

$$\mathrm{cr}(G^*, D) \geqslant \sum_{i=1}^{k} \mathrm{mincr}(C_i, D);$$

(iii) for each doubly odd closed curve $D = D_1 \cdot D_2$: $S_1 \to S$ with $D_1(1) = D_2(1) \in V(G^*)$,

$$\mathrm{cr}(G^*, D) > \sum_{i=1}^{k} \mathrm{mincr}(C_i, D). \tag{49}$$

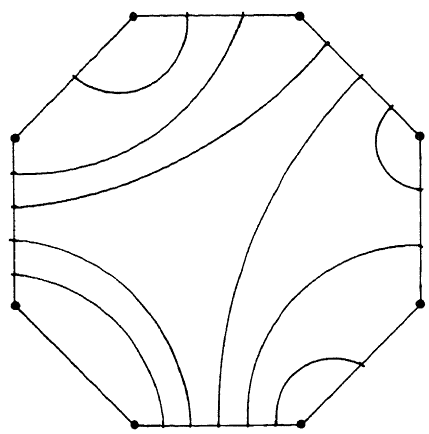
This follows from the fact that each closed curve D in (1) and (49) may be assumed not to intersect any edge of G of G^*. This implies $\mathrm{cr}(G, D) = \mathrm{cr}(G^*, D)$.

So it suffices to show that (49) implies (48), and the remainder of this paper is devoted to this. To simplify notation:

from now on we write G for G^*. (50)

By (49)(i) we may assume that $C_1, ..., C_k$ themselves are pairwise disjoint and simple. We can also assume that $C_1, ..., C_k$ do not intersect any vertex of G. Moreover, we can assume that each face of G is passed only a finite number of times—that is, $\text{cr}(G, C_i)$ is finite for $i = 1, \ldots, k$. This follows by the theory of simplicial approximation, since we can take $G, C_1, \ldots, C_k$ in one common triangulation of S.

Consider now some face F of G, together with all curves C_i traversing F. For example:



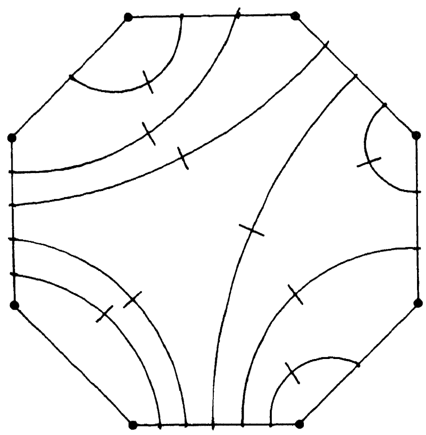
(51)

Consider the components of

$$F \cap \bigcup_{i=1}^{k} C_i[S_1], \tag{52}$$

which we call the *curve parts* on F. For each curve part P let l_P be a "short" line segment in F crossing P, and not intersecting any other curve part. We do this in such a way that if $P \neq P'$ then l_P and $l_{P'}$ are disjoint.

E.g., (51) becomes



(53)

We do this for each face of G. Define

$$\begin{aligned} \mathscr{L} &:= \text{collection of all line segments chosen;} \\ W &:= \text{collection of end points of line segments in } \mathscr{L}. \end{aligned} \tag{54}$$

So $|W| = 2|\mathscr{L}|$. We call two points in W *mates* if they are end points of one common line segment in $\mathscr{L}$. Define

$$M := \{pq \mid p \text{ and } q \text{ are mates}\}. \tag{55}$$

(As in Section 2, we denote $\{p, q\}$ by pq.)

We next show that there exists a function $\psi: W \to \mathbb{Z}$ satisfying

$$\begin{aligned} &\text{(i)} \quad \psi(p) + \psi(q) = 0 && \text{if } p \text{ and } q \text{ are mates;} \\ &\text{(ii)} \quad \psi(p) + \psi(q) \leqslant \lambda(pq) && \text{for all} \quad p, q \in W, \end{aligned} \tag{56}$$

where

$$\lambda(pq) := \min_P (\mathrm{cr}(G, P) - 1), \tag{57}$$

the minimum ranging over

all paths P connecting p and q, which are

(i) homotopic to some path in $S \setminus \bigcup_{i=1}^k C_i[S_1]$, but

(ii) not homotopic to any path in $\bigcup_{i=1}^k C_i[S_1] \cup \bigcup_{l \in \mathscr{L}} l$.

(58)

We take $\lambda(pq) = \infty$ if no such path P exists. This occurs when p and q belong to different components of $S \setminus \bigcup_{i=1}^k C_i[S_1]$ and also when each path connecting p and q is homotopic to some path in $\bigcup_{i=1}^k C_i[S_1] \cup \bigcup_{l \in \mathscr{L}} l$. If p and q are end points of $l, l' \in \mathscr{L}$ intersecting *different* closed curves C_i and $C_{i'}$, then condition (58)(ii) is automatically satisfied.

The integers $\psi(p)$ form an indication of how far we must shift the corresponding closed curves to obtain the closed curves required by (48).

Before showing that integers $\psi(p)$ satisfying (56) exist (with the help of the auxiliary theorem in Section 2), we interpret (56) in terms of the universal covering surface S', π of S. Since $C_1, \ldots, C_k$ are pairwise disjoint, simple, and non-null-homotopic, and since S is not the 2-sphere and not the projective plane we have:

CLAIM 2. (i) *Each lifting of each C_i is a one-to-one function.*

(ii) *If $i \neq i'$ then any lifting of C_i is disjoint from any lifting of $C_{i'}$.*

(iii) *If C_i' and C_i'' are liftings of C_i, then either they are disjoint, or there exists an $n \in \mathbb{Z}$ so that for all $x \in \mathbb{R}$, $C_i'(x+n) = C_i''(x)$.*

Proof. (i) follows from the fact that each C_i is non-null-homotopic, and hence C_i^n is non-null-homotopic, for each $n \geqslant 2$.

(ii) follows from the fact that $C_1, ..., C_k$ are pairwise disjoint.

(iii) follows from the fact that each C_i is simple. For suppose that say $C_i'(y) = C_i''(z)$ for some $y, z \in \mathbb{R}$. Then

$$C_i(\exp(2\pi \mathbf{i} y)) = (\pi \circ C_i')(y) = (\pi \circ C_i'')(z) = C_i(\exp(2\pi \mathbf{i} z)). \quad (59)$$

Hence $\exp(2\pi \mathbf{i} y) = \exp(2\pi \mathbf{i} z)$ (as C_i is simple), and therefore $y = z + n$ for some $n \in \mathbb{Z}$. We show that $C_i'(x+n) = C_i''(x)$ for all $x \in \mathbb{R}$.

Let $D: S_1 \to S$ be defined by $D(w) := C_i(w \cdot \exp(2\pi \mathbf{i} y))$ for $w \in S_1$, $D': \mathbb{R} \to S'$ by $D'(x) := C_i'(x+y)$ for $x \in \mathbb{R}$, and $D'': \mathbb{R} \to S'$ by $D''(x) := C_i''(x+z)$ for $x \in \mathbb{R}$. Then D' and D'' are liftings of D to S' with $D'(0) = D''(0)$. By the uniqueness of liftings it follows that $D' = D''$, implying that $C_i'(x+n) = C_i''(x)$ for all $x \in \mathbb{R}$. ∎

Let $\mathcal{M}$ denote the collection of images of all liftings of all C_i to S', that is,

$$\mathcal{M} := \{ C_i'[\mathbb{R}] \mid C_i' \text{ is a lifting of } C_i \text{ to } S', \text{ for some } i \in \{1, ..., k\} \}. \quad (60)$$

For simplicity, we also call the sets in $\mathcal{M}$ *liftings*.

Define

$$\begin{aligned} G' &:= \pi^{-1}[G], \\ V' &:= \pi^{-1}[V]. \end{aligned} \quad (61)$$

Then G' is an infinite graph, with vertex set V', embedded on S' so that V' is discrete.

If line segment $l \in \mathcal{L}$ crosses C_i, then each component of $\pi^{-1}[l]$ is a line segment crossing some lifting of C_i. Define

$$\begin{aligned} \mathcal{L}' &:= \bigcup_{l \in \mathcal{L}} \text{set of components of } \pi^{-1}[l], \\ W' &:= \pi^{-1}[W]. \end{aligned} \quad (62)$$

For each line segment $l \in \mathcal{L}'$, there exists a unique lifting $L \in \mathcal{M}$ crossing l. If p and q are the end points of $l \in \mathcal{L}'$, then $\pi(p)$ and $\pi(q)$ are the end points of $\pi[l]$. Again, we call p and q *mates*.

Now, for $p, q \in W$, we can describe $\lambda(pq)$ in terms of S', π, with

$$\lambda(pq) = \min_{P'} (\operatorname{cr}(G', P') - 1), \quad (63)$$

where P' ranges over all paths in S' connecting points $p', q' \in W'$ such that

(i) $\pi(p') = p, \quad \pi(q') = q$;
(ii) p' and q' belong to the same component of $S' \setminus \bigcup_{L \in \mathcal{M}} L$;
(iii) p' and q' are end points of line segments in $\mathcal{L}'$ crossing different liftings in $\mathcal{M}$. (64)

($\mathrm{cr}(G', P')$ denotes the number of times P' intersects G'.)

This follows from the facts that for each P' in the range of (63) the path $P := \pi \circ P'$ is in the range of (57), and that conversely for each path P in the range of (57) any lifting P' of P to S' is in the range of (63). Moreover, $\mathrm{cr}(G', P') = \mathrm{cr}(G, P)$.

Having given this interpretation of $\lambda(pq)$, we show that our auxiliary theorem in Section 2 applies to (56).

CLAIM 3. *The function λ satisfies* (13).

Proof. Let

$$(p_0, p_1, p_2, ..., p_{2d-1}, p_{2d}) \tag{65}$$

be an alternating cycle with respect to M. In order to check (13), we may assume that $\lambda(p_{j-1} p_j)$ is finite for each even j.

For j odd, let P_j be the path following the line segment in $\mathcal{L}$ from p_{j-1} to p_j. For j even, let Q_j be a path in $S \setminus \bigcup_{i=1}^{k} C_i[S_1]$ from p_{j-1} to p_j not homotopic to any path in $\bigcup_{i=1}^{k} C_i[S_1] \cup \bigcup_{l \in \mathcal{L}} l$. Let P_j be a path homotopic to Q_j with the property that

$$\lambda(p_{j-1} p_j) = \mathrm{cr}(G, P_j) - 1. \tag{66}$$

Such Q_j and P_j exist by the definition of $\lambda(p_{j-1} p_j)$.

Let D and B be the closed curves following

$$P_1 \cdot P_2 \cdot P_3 \cdot P_4 \cdot \cdots \cdot P_{2d-1} \cdot P_{2d} \quad \text{and} \quad P_1 \cdot Q_2 \cdot P_3 \cdot Q_4 \cdot \cdots \cdot P_{2d-1} \cdot Q_{2d}, \tag{67}$$

respectively. So D and B are homotopic. We now first show:

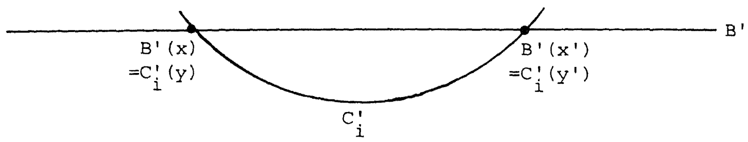
SUBCLAIM 3a. *For* $i = 1, ..., k$: $\mathrm{cr}(C_i, B) = \mathrm{mincr}(C_i, B)$.

Proof. If $\sum_{i=1}^{k} \mathrm{cr}(C_i, B) \leqslant 1$, the subclaim is trivial. So we may assume

$$\sum_{i=1}^{k} \mathrm{cr}(C_i, B) \geqslant 2. \tag{68}$$

By the proposition in Section 3 it suffices to show that any lifting C_i' of C_i to the universal covering surface S' intersects any lifting B' of B to S' at most once.

Suppose C_i' intersects B' more than once, say $C_i'(y) = B'(x)$ and $C_i'(y') = B'(x')$, where $y \neq y' \in \mathbb{R}$ and $x < x' \in \mathbb{R}$:



(69)

We may assume that we have chosen i, x, x', y, y' so that $x' - x$ is as small as possible. Then

> no lifting $C_{i'}^-$ of any $C_{i'}$ to S' has a crossing with B' on the interval (x, x'). (70)

Indeed, suppose $C_{i'}^-$ crosses B' at $B'(x'')$ with $x'' \in (x, x')$. Then $C_{i'}^-[\mathbb{R}] \neq C_i'[\mathbb{R}]$, since otherwise we could decrease $x' - x$ to $x'' - x$. So by Claim 2, $C_{i'}^-$ and C_i' are disjoint. Since part $[x, x']$ of B' forms a closed curve on S' with part $[y, y']$ of C_i', $C_{i'}^-$ should have a second crossing with B', say at $B'(x''')$, with $x''' \in (x, x')$. However, $|x''' - x''| < x' - x$, contradicting the minimality of $x' - x$. This shows (70).

In particular we have $x' - x < 1$ (since B' also has a crossing with some lifting of C_i at $x + 1$, and since (68) holds). So

$$(\pi \circ B') \,|\, [x, x'] \tag{71}$$

is the part of B in between two consecutive crossings with C_i, having no other crossings with any C_i on this part. So this part contains a Q_j (with j even), connecting two line segments l', l'' crossing C_i. Since $B' \,|\, [x, x']$ is homotopic on S' to $C_i' \,|\, [y, y']$, it follows that Q_j is homotopic to some path contained in $C_i[S_1] \cup l' \cup l''$—a contradiction. ∎

Next one has:

Subclaim 3b. $\sum_{j=1}^{d} \lambda(p_{2j-1} p_{2j}) = \mathrm{cr}(G, D) - \sum_{i=1}^{k} \mathrm{mincr}(C_i, D)$.

Proof. First note that

$$\sum_{i=1}^{k} \mathrm{cr}(C_i, B) = d, \tag{72}$$

as for odd j, P_j follows some $l \in \mathscr{L}$, and hence crosses the C_i once, and for even j, Q_j is disjoint from every C_i.

This implies

$$\sum_{j=1}^{d} \lambda(p_{2j-1}p_{2j}) = \sum_{j=1}^{d} (\mathrm{cr}(G, P_{2j}) - 1) = \left[\sum_{j=1}^{d} \mathrm{cr}(G, P_{2j})\right] - d$$
$$= \mathrm{cr}(G, D) - d = \mathrm{cr}(G, D) - \sum_{i=1}^{k} \mathrm{cr}(C_i, B)$$
$$= \mathrm{cr}(G, D) - \sum_{i=1}^{k} \mathrm{mincr}(C_i, B)$$
$$= \mathrm{cr}(G, D) - \sum_{i=1}^{k} \mathrm{mincr}(C_i, D). \quad \blacksquare \tag{73}$$

Subclaim 3b shows that condition (13)(i) directly follows from condition (49)(ii). To see (13)(ii), let (65) be doubly odd. We have to show it has positive length. Since (65) is doubly odd, there exists a t such that

$$\text{(i)}\quad p_0 = p_{2t+1}, \qquad p_1 = p_{2t};$$
$$\text{(ii)}\quad \sum_{j=1}^{t} (\mathrm{cr}(G, P_{2j}) - 1) \text{ is odd} \quad \text{and} \quad \sum_{j=t+1}^{d} (\mathrm{cr}(G, P_{2j}) - 1) \text{ is odd.} \tag{74}$$

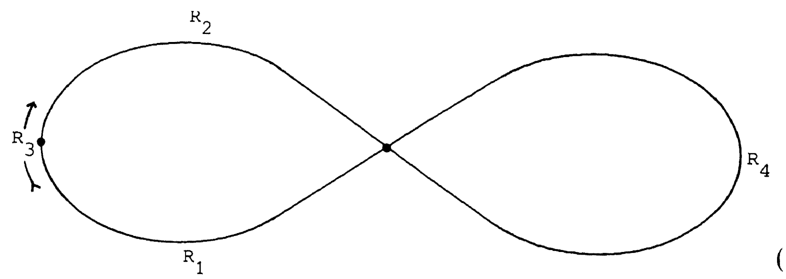
Let D_3 and D_4 be the closed curves on S following the 'closed' paths

$$R_3 := P_1 \cdot P_2 \cdot P_3 \cdot \cdots \cdot P_{2t} \cdot P_{2t+1} \quad \text{and}$$
$$R_4 := P_{2t+2} \cdot P_{2t+3} \cdot P_{2t+4} \cdot \cdots \cdot P_{2d-1} \cdot P_{2d}, \tag{75}$$

respectively. So $D_3(1) = D_4(1)$. We can identify D with $D_3 \cdot D_4$. However, $D_3(1) = D_4(1)$ is not a vertex of G, so we cannot appeal directly to condition (49)(iii). Therefore, we consider the closed curve

$$D_3 \cdot D_4 \cdot D_3^{-1} \cdot D_4. \tag{76}$$

We can decompose R_3 as $R_1 \cdot R_2$ so that R_1 and R_2 are paths with $R_1(1) = R_2(0)$ being a vertex of G (assuming without loss of generality that R_3 intersects G only in vertices of G):



(77)

Let D_1 and D_2 be the closed curves following the "closed" paths

$$R_2 \cdot R_4 \cdot R_2^{-1} \quad \text{and} \quad R_1^{-1} \cdot R_4 \cdot R_1, \tag{78}$$

respectively. Now $D_1(1) = R_2(0) = R_1(1) = D_2(1)$ is a vertex of G. Furthermore:

SUBLAIM 3c. *$D_1 \cdot D_2$ is doubly odd.*

Proof. By (74)(ii) we have

$$\begin{aligned}\text{cr}(G, D_1) &= 2\,\text{cr}(G, R_2) + \text{cr}(G, R_4) - 3 \not\equiv \text{cr}(G, R_4)\\ &= \sum_{j=t+1}^{d} \text{cr}(G, P_{2j}) \not\equiv d - t \pmod 2.\end{aligned} \tag{79}$$

Moreover,

$$\begin{aligned}\sum_{i=1}^{k} \text{cr}(C_i, D_1) &\equiv \sum_{i=1}^{k} \text{cr}(C_i, D_4) = \sum_{j=2t+2}^{2d} \sum_{i=1}^{k} \text{cr}(C_i, P_j)\\ &\equiv \sum_{j=t+2}^{d} \sum_{i=1}^{k} \text{cr}(C_i, P_{2j-1}) = d - t - 1 \pmod 2.\end{aligned} \tag{80}$$

This follows from the fact that D_1 and D_4 are homotopic, from (75), and from the fact that $\sum_{i=1}^{k} \text{cr}(C_i, P_{2j-1}) = 1$ or all j, while $\sum_{i=1}^{k} \text{cr}(C_i, P_{2j}) \equiv \sum_{i=1}^{k} \text{cr}(C_i, Q_{2j}) = 0 \pmod 2$ for all j.

Combining (79) and (80) gives

$$\text{cr}(G, D_1) \not\equiv \sum_{i=1}^{k} \text{cr}(C_i, D_1) \pmod 2. \tag{81}$$

Similarly one has

$$\text{cr}(G, D_2) \not\equiv \sum_{i=1}^{k} \text{cr}(C_i, D_2) \pmod 2. \tag{82}$$

So $D_1 \cdot D_2$ is doubly odd. ∎

Applying Subclaim 3b again we now have

$$\begin{aligned}2 \cdot \sum_{j=1}^{d} \lambda(p_{2j-1} p_{2j}) &= \text{cr}(G, D_3 \cdot D_4 \cdot D_3^{-1} \cdot D_4)\\ &\quad - \sum_{i=1}^{k} \text{mincr}(C_i, D_3 \cdot D_4 \cdot D_3^{-1} \cdot D_4)\\ &= \text{cr}(G, D_1 \cdot D_2) - \sum_{i=1}^{k} \text{mincr}(C_i, D_1 \cdot D_2) > 0,\end{aligned} \tag{83}$$

by (49)(iii). This shows condition (13)(ii). ∎

Hence, by the auxiliary theorem, we know that there exist integers $\psi(p)$ satisfying (56). We assume we have taken such $\psi(p)$ with

$$\sum_{p \in W} |\psi(p)| \tag{84}$$

as small as possible. We next show

CLAIM 4. *Let P be a path on S connecting $p, q \in W$ which is both homotopic to a path in $S \setminus \bigcup_{i=1}^{k} C_i[S_1]$ and homotopic to a path in $\bigcup_{i=1}^{k} C_i[S_1] \cup \bigcup_{l \in \mathscr{L}} l$. Then*

$$\psi(p) - \psi(q) \leqslant \mathrm{cr}(G, P). \tag{85}$$

Proof. Suppose (85) does not hold. Let $\bar{p}$ and $\bar{q}$ denote the mates of p and q, respectively. So $\psi(\bar{p}) = -\psi(p)$ and $\psi(\bar{q}) = -\psi(q)$. Let $\bar{P}$ be the path obtained from P by extending P over the line segments $p\bar{p}$ and $q\bar{q}$ in $\mathscr{L}$. Then

$$\begin{aligned} &\psi(p) - \psi(q) > \mathrm{cr}(G, P), \\ &\psi(\bar{q}) - \psi(\bar{p}) > \mathrm{cr}(G, P) = \mathrm{cr}(G, \bar{P}). \end{aligned} \tag{86}$$

Hence, by symmetry, we may assume that $|\psi(q)| = |\psi(\bar{q})| \geqslant |\psi(p)| = |\psi(\bar{p})|$. Since $\psi(\bar{q}) - \psi(\bar{p}) > 0$, it follows that $\psi(\bar{q}) > 0$. Hence $\psi(q) < 0$. Now let $\tilde{\psi}: W \to \mathbb{Z}$ be defined by

$$\begin{aligned} &\tilde{\psi}(\bar{q}) := \psi(\bar{q}) - 1, \\ &\tilde{\psi}(q) := \psi(q) + 1, \\ &\tilde{\psi}(r) := \psi(r) \qquad \text{for all} \quad r \in W \setminus \{q, \bar{q}\}. \end{aligned} \tag{87}$$

We show that $\tilde{\psi}$ again is a solution of (56), contradicting the minimality of (84).

We only have to check those inequalities among (56)(ii) containing $\tilde{\psi}(q)$. Let $r \in W$, and let Q be a path on S connecting q and r, so that Q is homotopic to some path in $S \setminus \bigcup_{i=1}^{k} C_i[S_1]$, but not homotopic to any path in $\bigcup_{i=1}^{k} C_i[S_1] \cup \bigcup_{l \in \mathscr{L}} l$. We must show that

$$\tilde{\psi}(q) + \tilde{\psi}(r) \leqslant \mathrm{cr}(G, Q) - 1. \tag{88}$$

If $r \neq q$ then

$$\tilde{\psi}(q)+\tilde{\psi}(r)=\psi(q)+1+\psi(r)=(\psi(q)-\psi(p))+(\psi(p)+\psi(r))+1$$
$$\leqslant(-\mathrm{cr}(G,P)-1)+(\mathrm{cr}(G,P^{-1}\cdot Q)-1)+1$$
$$=\mathrm{cr}(G,Q)-1. \tag{89}$$

If $r=q$ then

$$\tilde{\psi}(q)+\tilde{\psi}(r)=\psi(q)+\psi(q)+2=2(\psi(q)-\psi(p))+(\psi(p)+\psi(p))+2$$
$$\leqslant 2(-\mathrm{cr}(G,P)-1)+(\mathrm{cr}(G,P^{-1}\cdot Q\cdot P)-1)+2$$
$$=\mathrm{cr}(G,Q)-1. \quad \blacksquare \tag{90}$$

We now want to shift the C_i over distances given by ψ. For any lifting $L\in\mathscr{M}$ and any face F of G' define

$$\Pi_L(F):=\min_P\,(\mathrm{cr}(G',P)-\psi(\pi(P(0)))), \tag{91}$$

where P ranges over all paths P: $[0,1]\to S'$ satisfying

(i) $P(0)$ is an end point of a line segment in $\mathscr{L}'$ crossing L;
(ii) P crosses L an even number of times;
(iii) $P(1)\in F$. (92)

Since ψ takes only a finite number of values, (91) is well-defined. Let

$$U_L:=\text{collection of faces } F \text{ of } G' \text{ with } \Pi_L(F)=0. \tag{93}$$

We next show:

CLAIM 5. *If $L', L''\in\mathscr{M}$ with $L'\neq L''$, then $U_{L'}\cap U_{L''}=\varnothing$.*

Proof. Suppose $F\in U_{L'}\cap U_{L''}$. So $\Pi_{L'}(F)=\Pi_{L''}(F)=0$. Let P' and P'' attain the minimum in (91) with respect to L' and L'', respectively. So

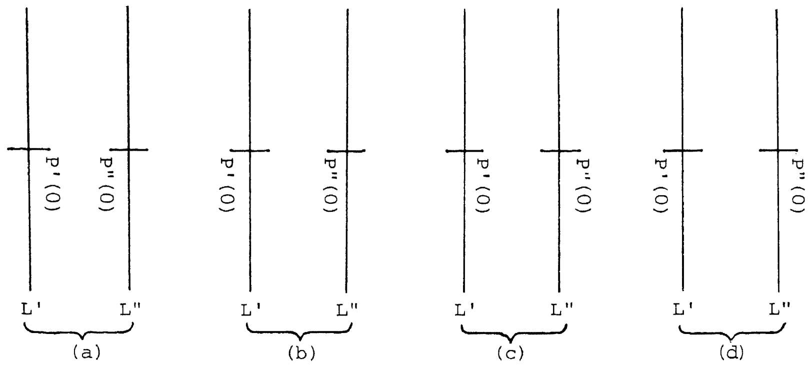
$$\mathrm{cr}(G',P')=\psi(\pi(P'(0))),$$
$$\mathrm{cr}(G',P'')=\psi(\pi(P''(0))). \tag{94}$$

Let l' and l'' be the line segments in $\mathscr{L}'$ so that $P'(0)$ and $P''(0)$ are end points of l' and l'', respectively. Let Q be a path in F from $P'(1)$ to $P''(1)$. So $\mathrm{cr}(G',Q)=0$. Consider the path

$$P'\cdot Q\cdot(P'')^{-1}. \tag{95}$$

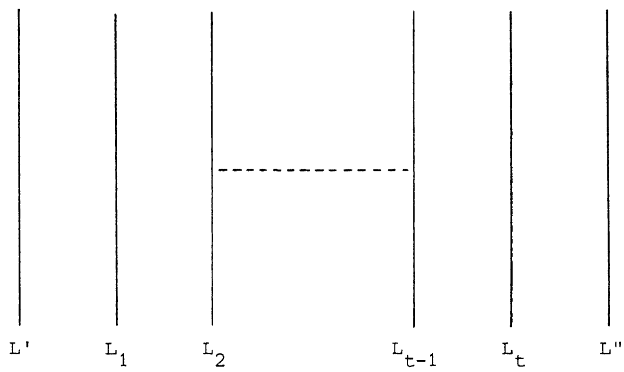
We may assume that this path, when it crosses any lifting $L \in \mathscr{M}$, then it crosses L over one of the line segments in $\mathscr{L}'$.

The liftings L' and L'' divide S' (which is homeomorphic to $\mathbb{C}$) into three parts. Hence there are four possible types of positions of $P'(0)$ and $P''(0)$ relative to L' and L'':



(96)

Denote by $L_1, ..., L_t$ the liftings in $\mathscr{M}$ "separating" L' and L'':



(97)

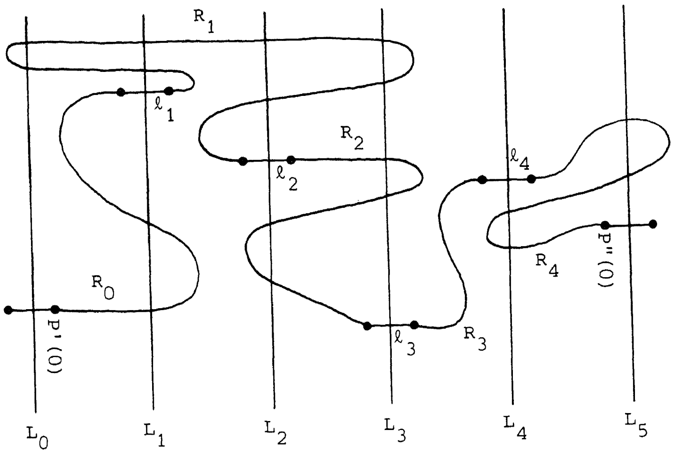
Define $L_0 := L'$ and $L_{t+1} := L''$.

We consider the four cases given by (96).

Case A. Configuration (96)(a) applies. We can split path $P' \cdot Q \cdot (P'')^{-1}$ as

$$R_0 \cdot l_1 \cdot R_1 \cdot l_2 \cdot \cdots \cdot R_{t-1} \cdot l_t \cdot R_t, \tag{98}$$

where l_j is a path passing a line segment crossing L_j from left to right in (96), for $j=1, ..., t$. E.g., for $t=4$,



(99)

Now, by inequality (58)(ii), using (63),

$$\psi(\pi(R_j(0))) + \psi(\pi(R_j(1))) \leqslant \text{cr}(G', R_j) - 1, \qquad \text{for} \quad j=0, ..., t. \quad (100)$$

Hence

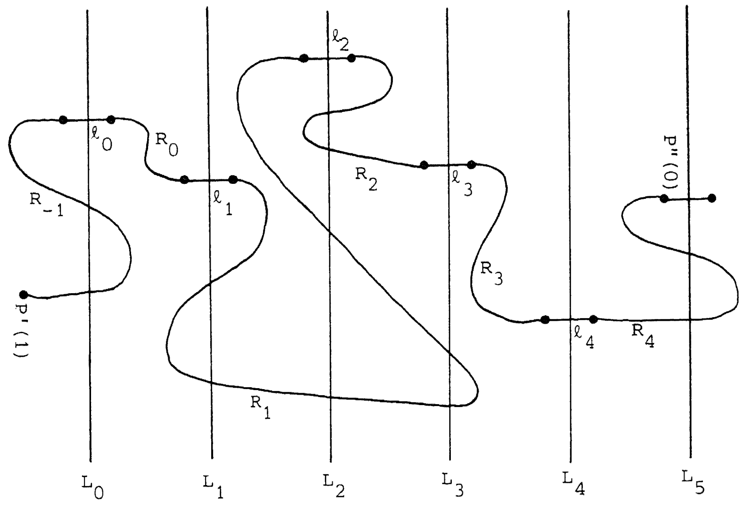
$$\begin{aligned}\text{cr}(G', P') + \text{cr}(G', P'') &= \text{cr}(G', P' \cdot Q \cdot (P'')^{-1}) \\ &= \sum_{j=0}^{t} \text{cr}(G', R_j) \geqslant \sum_{j=0}^{t} (\psi(\pi(R_j(0))) + \psi(\pi(R_j(1))) + 1) \\ &= t + 1 + \psi(\pi(R_0(0))) + \psi(\pi(R_t(1))) \\ &= t + 1 + \psi(\pi(P'(0))) + \psi(\pi(P''(0))) \\ &> \psi(\pi(P'(0))) + \psi(\pi(P''(0))), \end{aligned} \qquad (101)$$

where we use the fact that $\psi(\pi(R_j(1))) = -\psi(\pi(R_{j+1}(0)))$ for $j=0, ..., t-1$. This contradicts (94).

Case B. Configuration (96)(b) applies. By (92)(ii) applied to P' and L' we know that $P'(1)$ is at the left side of L'. So we can split $Q \cdot (P'')^{-1}$ as

$$R_{-1} \cdot l_0 \cdot R_0 \cdot l_1 \cdot R_1 \cdot l_2 \cdot \cdots \cdot R_{t-1} \cdot l_t \cdot R_t, \qquad (102)$$

where l_j is path passing a line segment crossing L_j from left to right in (96), for $j=0, ..., t$. E.g., for $t=4$,



(103)

Since $\Pi_{L'}(F) \geqslant 0$ we know that

$$\mathrm{cr}(G', R_{-1}) - \psi(\pi(R_{-1}(1))) \geqslant 0. \tag{104}$$

Moreover, again (100) holds. Hence

$$\begin{aligned}\mathrm{cr}(G', P'') &= \mathrm{cr}(G', Q \cdot (P'')^{-1}) = \sum_{j=-1}^{t} \mathrm{cr}(G', R_j) \\ &\geqslant \psi(\pi(R_{-1}(1))) + \sum_{j=0}^{t} (\psi(\pi(R_j(0))) + \psi(\pi(R_j(1))) + 1) \\ &= t + 1 + \psi(\pi(R_t(1))) \\ &= t + 1 + \psi(\pi(P''(0))) > \psi(\pi(P''(0))).\end{aligned} \tag{105}$$

This contradicts (94).

Case C. Configuration (96)(c) applies. This case is similar to Case B.

Case D. Configuration (96)(d) applies. We can split Q as

$$R_{-1} \cdot l_0 \cdot R_0 \cdot l_1 \cdot R_1 \cdot l_2 \cdot \cdots \cdot R_{t-1} \cdot l_t \cdot R_t \cdot l_{t+1} \cdot R_{t+1}, \tag{106}$$

where, again l_j is a path passing a line segment crossing L_j from left to right in (96), for $j=0, ..., t+1$.

Since $\Pi_{L'}(F)=\Pi_{L''}(F)=0$ we know

$$\begin{aligned}\operatorname{cr}(G', R_{-1})-\psi(\pi(R_{-1}(1)))\geqslant 0,\\ \operatorname{cr}(G', R_{t+1})-\psi(\pi(R_{t+1}(0)))\geqslant 0.\end{aligned} \tag{107}$$

Moreover, again (100) holds. Hence,

$$\begin{aligned}0=\operatorname{cr}(G', Q)&=\sum_{j=-1}^{t+1}\operatorname{cr}(G', R_j)\\ &\geqslant\psi(\pi(R_{-1}(1)))+\left(\sum_{j=1}^{t}(\psi(\pi(R_j(0)))+\psi(\pi(R_j(1)))+1\right)\\ &\quad+\psi(\pi(R_{t+1}(0)))=t+1>0,\end{aligned} \tag{108}$$

a contradiction. ∎

We next describe how to shift C_i. We fix some $i\in\{1, ..., k\}$ (this i will be fixed until Claim 9 below). We construct a covering surface of S which, roughly speaking, arises from S' by "rolling up" S' along some lifting of C_i to S'.

More precisely, let C_i' be some lifting of C_i to S'. Let $p':=C_i'(0)$. For $n\in\mathbb{Z}$, let $\varphi_n\colon S'\to S'$ be the unique homeomorphism of S' satisfying

$$\varphi_n(p')=C_i'(n) \qquad\text{and}\qquad \pi\circ\varphi_n=\pi. \tag{109}$$

Note that

$$\{\varphi_n \mid n\in\mathbb{Z}\} \tag{110}$$

forms a group of automorphism which is infinitely cyclic. Define an equivalence relation $\leftrightarrow$ on S' by $q\leftrightarrow r \Leftrightarrow \exists n\in\mathbb{Z}: \varphi_n(q)=r$. In particular, if $q\leftrightarrow r$ then $\pi(q)=\pi(r)$.

Now let S'' be the quotient space obtained from S' by identifying equivalent points. In particular, $C_i'(x)$ and $C_i'(x+n)$ are identified, for each $x\in\mathbb{R}$ and $n\in\mathbb{Z}$.

The space S'' is again a covering surface of S, with projection function $\pi'\colon S''\to S$ given by: $\pi'(\langle q\rangle):=\pi(q)$ (where $\langle ..\rangle$ denotes the class of ..).

Moreover, S' is a covering surface of S'', with projection function $\pi''\colon S'\to S''$ given by $\pi''(q):=\langle q\rangle$. One has $\pi'\circ\pi''=\pi$, or in diagram

$$\begin{array}{ccc} S' & \xrightarrow{\quad\pi\quad} & S\\ & {\scriptstyle\pi''}\searrow \quad \nearrow{\scriptstyle\pi'} & \\ & S'' & \end{array} \tag{111}$$

Now S'' has fundamental group isomorphic to (110) and hence to the infinite cyclic group (cf. Massey [9, Chap. 5, Theorem 10.2]). Hence, topologically, S'' is homeomorphic to the annulus or to the (open) Möbius strip (by von Kerékjártó's classification theorem [7]).

To fix a representation, let the *annulus* be obtained from $\mathbb{R} \times [0, 1]$ by identifying $(x, 0)$ and $(x, 1)$, for each $x \in \mathbb{R}$. Similarly, let the *Möbius* strip be obtained by identifying $(x, 0)$ and $(-x, 1)$, for each $x \in \mathbb{R}$.

Let P: $[0, 1] \to S''$ be defined by

$$P(x) := \langle C_i'(x) \rangle \qquad (\text{for } x \in [0, 1]). \tag{112}$$

Then hom(P) generates the fundamental group of S''. Hence we can assume that P forms the mid circle of the annulus or the Möbius strip. In the representation above, the *mid circle* is the image of $\{0\} \times [0, 1]$ after identification.

Since $P(0) = P(1)$, we can consider P as a closed curve $C_i'': S_1 \to S''$, i.e.,

$$C_i''(\exp(2\pi \mathbf{i} x)) := P(x) \qquad (\text{for } x \in [0, 1]). \tag{113}$$

So $C_i = \pi' \circ C_i''$. Hence C_i'' is a simple closed curve on S'' (as C_i is simple).

Moreover, C_i' is a lifting of C_i'' to the universal covering surface S', π'' of S'', since $\pi'' \circ C_i'(x) = C_i''(\exp(2\pi \mathbf{i} x))$ for all $x \in \mathbb{R}$. In fact, C_i' is the only lifting of C_i'' to S', up to translations of $\mathbb{R}$ over an integer distance: if $C': \mathbb{R} \to S'$ is a lifting of C_i'' to S', then there exists an $n \in \mathbb{Z}$ with $C'(x) = C_i'(x + n)$ for each $x \in \mathbb{R}$. (*Proof*: Since C' and C_i' are liftings of C_i, by (iii) of Claim 2 it suffices to show that C' and C_i' are not disjoint. Now C' being a lifting of C_i'' to S' means that $\pi'' \circ C'(x) = C_i''(\exp(2\pi \mathbf{i} x))$ for each $x \in \mathbb{R}$. Hence $\langle C'(0) \rangle = C_i''(1) = \langle C_i'(0) \rangle$, implying that $C'(0) = C_i'(n)$ for some $n \in \mathbb{Z}$.)

Define

$$\begin{aligned} G'' &:= (\pi')^{-1} [G], \\ V'' &:= (\pi')^{-1} [V]. \end{aligned} \tag{114}$$

The G'' is an infinite graph, with vertex set V''.

Let $L := C_i'[\mathbb{R}]$. We next show:

CLAIM 6. *S'' contains a closed curve $\tilde{C}_i''$ homotopic (on S'') to C_i'', such that*:

(i) *$\tilde{C}_i''$ does not pass through any vertex of G'', and does not pass through any face of G'' more than once*;

(ii) *if F is a face of G'' so that $\tilde{C}_i''$ passes through $\pi''[F]$, then $F \in U_L$.*

Proof. Denote

$$\mathcal{L}'' := \text{set of components of } (\pi')^{-1}\left[\bigcup_{l \in \mathcal{L}'} l\right], \qquad W'' := (\pi')^{-1}[W]. \tag{115}$$

So $\mathcal{L}''$ is a (generally infinite) collection of line segments, and W'' is the collection of end points of line segments in $\mathcal{L}''$.

Let V_0 be the collection of vertices v of G'' with the property that there exists a path P: $[0, 1] \to S''$ satisfying

(i) $P(0)$ is an end point of one of the line segments in $\mathcal{L}''$ crossing C''_i;
(ii) P crosses C''_i an even number of times; (116)
(iii) $P(1) = v$;
(iv) $\text{cr}(G'', P) \leqslant \psi(\pi'(P(0)))$.

(Note that (iv) implies $\psi(\pi'(P(0))) > 0$.) The set V_0 is finite, since there are only a finite number of line segments in $\mathcal{L}''$ crossing C''_i and since each face of G'' is incident with only a finite number of faces.

Next let:

$E_1 :=$ set of edges of G'' crossed an odd number of times by C''_i;

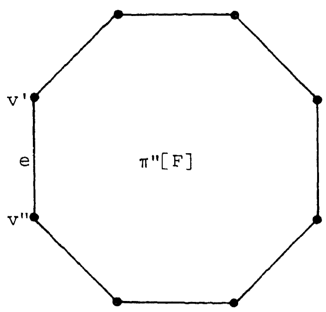
$E_v :=$ set of edges incident with v (excluding loops); (117)

$$E_0 := E_1 \triangle \left(\bigtriangleup_{v \in V_0} E_v\right),$$

where $\triangle$ and $\bigtriangleup$ denote symetric difference.

SUBCLAIM 6a. *Let F be a face of G' with $\pi''[F]$ incident to some $e \in E_0$. Then $F \in U_L$.*

Proof. We must show that $\Pi_L(F) = 0$. Consider $\pi''[F]$:



(118)

I. To show that $\Pi_L(F) \geqslant 0$, suppose to the contrary that

$$\mathrm{cr}(G', P) < \psi(\pi(P(0))) \tag{119}$$

for some path P: $[0, 1] \to S'$ satisfying (92). Cosider $P' := \pi'' \circ P$. Then P' satisfies:

$$\begin{aligned}
&\text{(i)} \quad P'(0) \text{ is end point of a line segment in } \mathscr{L}'' \text{ crossing } C_i'';\\
&\text{(ii)} \quad P' \text{ crosses } C_i'' \text{ an even number of times;}\\
&\text{(iii)} \quad P'(1) \in \pi''[F];\\
&\text{(iv)} \quad \mathrm{cr}(G'', P') < \psi(\pi'(P(0))).
\end{aligned} \tag{120}$$

Let Q_1: $[0, 1] \to S''$ be a path starting in $P'(1)$ so that $Q_1([0, 1)) \subseteq \pi''[F]$ and $Q_1(1) = v'$. Denote by α_1 the number of times Q_1 crosses C_i''. We show that

$$\alpha_1 \text{ is even } \Leftrightarrow v' \in V_0. \tag{121}$$

If α_1 is even, then $P' \cdot Q_1$ satisfies (116) for $v := v'$, and hence $v' \in V_0$. Suppose next α_1 is odd and $v' \in V_0$. Let P'' be a path satisfying (116) for $v := v'$. Let Q_3: $[0, 1] \to S''$ be the path following the line segment $l \in \mathscr{L}''$ containing $P'(0)$, so that $Q_3(1) = P'(0)$ and $Q_3(0)$ are the end points of l. As α_1 is odd, $Q_3 \cdot P' \cdot Q_1$ crosses C_i'' an even number of times. This implies that also $Q_3 \cdot P' \cdot Q_1 \cdot (P'')^{-1}$ crosses C_i'' an even number of times. So by Claim 4,

$$\mathrm{cr}(G'', Q_3 \cdot P' \cdot Q_1 \cdot (P'')^{-1}) \geqslant \psi(\pi'(P''(0))) - \psi(\pi'(Q_3(0))). \tag{122}$$

However,

$$\mathrm{cr}(G'', Q_3 \cdot P' \cdot Q_1 \cdot (P'')^{-1}) = \mathrm{cr}(G'', P') + \mathrm{cr}(G'', P''), \tag{123}$$

and by (120)(iv) and (119),

$$\begin{aligned}
&\psi(\pi'(P''(0))) \geqslant \mathrm{cr}(G'', P''),\\
&-\psi(\pi'(Q_3(0)) = \psi(\pi'(P'(0))) > \mathrm{cr}(G'', P').
\end{aligned} \tag{124}$$

Now (122), (123), and (124) form a contradiction. So $v' \notin V_0$, and we have shown (121).

Similarly, let Q_2: $[0, 1] \to S''$ be a path in $P'(1)$ so that $Q_2([0, 1)) \subseteq \pi''[F]$ and $Q_2(1) = v''$. Denote by α_2 the number of times Q_2 crosses C_i''. Then one has

$$\alpha_2 \text{ is even} \Leftrightarrow v'' \in V_0. \tag{125}$$

However, $\alpha_1+\alpha_2$ has the same parity as the number of times C_i'' crosses edge e. But this implies $e \notin E_0$, contradicting the assumption.

II. To show that $\Pi_L(F) \leqslant 0$, note that by definition of E_0 at least one of the following should hold: (i) $v' \in V_0$, (ii) $v'' \in V_0$, (iii) e is crossed by C_i''.

If $v' \in V_0$, let P satisfy (116) for $v := v'$. Extending P to $\pi''[F]$ yields a curve P' satisfying

$$\mathrm{cr}(G'', P') = \mathrm{cr}(G'', P) \leqslant \psi(\pi'(P(0))) = \psi(\pi'(P'(0))). \tag{126}$$

Hence $\Pi_L(F) \leqslant 0$. Similarly, if $v'' \in V_0$ then $\Pi_L(F) \leqslant 0$.

If e is crossed by C_i'', some line segment $l \in \mathscr{L}''$ is contained in $\pi''[F]$. Then one of the end points p of l satisfies $\psi(\pi'(p)) \geqslant 0$. Hence the path P: $[0, 1] \to S''$ defined by $P(x) = p$ for all $x \in [0, 1]$ satisfies (92). Moreover

$$\mathrm{cr}(G'', P) = 0 \leqslant \psi(\pi'(p)). \tag{127}$$

So $\Pi_L(F) \leqslant 0$. ∎

Now consider the dual graph $(G'')^*$ of G'' on S''. Let E_0^*, E_1^*, E_v^* denote the sets of edges of $(G'')^*$ dual to E_0, E_1, E_v, respectively.

By (117), each vertex of $(G'')^*$ is incident with an even number of edges in E_0^*. Moreover, E_0^* is finite. Hence there exist simple closed curves $D_1, ..., D_t$ in $(G'')^*$ not passing any edge not in E_0^* and passing each edge in E_0^* exactly once. We finally show that

$$\text{at least one of } D_1, ..., D_t \text{ is homotopic on } S'' \text{ to } C_i''. \tag{128}$$

By Subclaim 6a, this gives the $\tilde{C}_i''$ as required.

To see (128), we use that S'' can be identified with the annulus or the Möbius strip, in such a way that C_i'' follows the mid circle. Using the representation described above, we consider the "cut" $\mathbb{R} \times \{\frac{1}{2}\}$. We may assume that $\mathbb{R} \times \{\frac{1}{2}\}$ does not intersect any vertex of $(G'')^*$.

Now $\mathbb{R} \times \{\frac{1}{2}\}$ crosses the mid circle an odd number of times (in fact, exactly once). So $\mathbb{R} \times \{\frac{1}{2}\}$ has an odd number of crossings with the edges in E_1^*. Moreover, for each $v \in V_0$, $\mathbb{R} \times \{\frac{1}{2}\}$ has an even number of crossings with the edges in E_v^*. Hence $\mathbb{R} \times \{\frac{1}{2}\}$ has an odd number of crossings with the edges in E_0^*. Therefore, at least one of $D_1, ..., D_t$ crosses $\mathbb{R} \times \{\frac{1}{2}\}$ an odd number of times; say D_1. As D_1 is simple, it is homotopic to the mid circle, i.e., to C_i''. This shows (128). ∎

Having $\tilde{C}_i''$, we define

$$\tilde{C}_i := \pi' \circ \tilde{C}_i''. \tag{129}$$

Since $\tilde{C}_i''$ is homotopic to C_i'' on S'', it follows that $\pi' \circ \tilde{C}_i'' = \tilde{C}_i$ is homotopic to $\pi' \circ C_i'' = C_i$ on S. Moreover, $\tilde{C}_i$ does not intersect V, since $\tilde{C}_i''$ does not intersect V'' and since $\pi'[V''] = V$.

We next show that shifting C_i to $\tilde{C}_i$ corresponds to shifting a lifting L of C_i to U_L. More precisely, let θ: $[0, 1] \to S''$ be a path on S'' such that

$$\theta(0) = C_i''(1) \quad \text{and} \quad \theta(1) = \tilde{C}_i''(1). \tag{130}$$

Let θ' be the lifting of θ to S' with $\theta'(0) = C_i'(0)$. So $\pi \circ \theta'(1) = \pi' \circ \tilde{C}_i''(1) = \tilde{C}_i(1)$. As a direct consequence of Claim 6 and the construction of C_i'' we have

Claim 7. *Let* $L := C_i'[\mathbb{R}]$, *and let* $\tilde{C}_i'$ *be the lifting of* $\tilde{C}_i$ *with* $\tilde{C}_i'(0) = \theta'(1)$. *Then each face passed by* $\tilde{C}_i'$ *belongs to* U_L.

Proof. Note that $\tilde{C}_i$is a lifting of $\tilde{C}_i''$ to S', since

(i) $\pi'' \circ \tilde{C}_i'(0) = \pi'' \circ \theta'(1) = \theta(1) = \tilde{C}_i''(1)$;

(ii) $\pi' \circ (\pi'' \circ \tilde{C}_i')(x) = \pi \circ \tilde{C}_i'(x) = \tilde{C}_i \exp(2\pi \mathbf{i} x)) = \pi' \circ \tilde{C}_i''(\exp(2\pi \mathbf{i} x))$, for all $x \in \mathbb{R}$. (131)

Hence, by the uniqueness of liftings,

$$\pi'' \circ \tilde{C}_i'(x) = \tilde{C}_i''(\exp(2\pi \mathbf{i} x)) \qquad \text{for all} \quad x \in \mathbb{R}. \tag{132}$$

Let F be a face of G' passed by $\tilde{C}_i'$. Then $\pi''[F]$ is passed through by $\pi''[C_i']$, and hence by C_i''. Therefore, by Claim 6, $F \in U_L$. ▮

This implies

Claim 8. $\tilde{C}_i$ *does not pass any face of G more than once.*

Proof. Suppose $\tilde{C}_i$ passes some face more than once. Since $\tilde{C}_i = \pi' \circ \tilde{C}_i''$ and since $\tilde{C}_i''$ does not pass through any face of G'' more than once, there exist faces F'' and H'' of G'' passed through by $\tilde{C}_i''$ so that $F'' \neq H''$ and $\pi'[F''] = \pi'[H'']$. Let $\tilde{C}_i'$ again be a lifting of $\tilde{C}_i''$ to S'. So the faces passed through by $\tilde{C}_i'$ are contained in U_L, where $L = C_i'[\mathbb{R}]$ for some lifting C_i' of C_i. Hence U_L contains faces F' and H' such that $\pi''[F'] \neq \pi''[H']$ and $\pi[F'] = \pi[H']$. Let φ: $S' \to S'$ be the homeomorphism satisfying $\pi \circ \varphi = \varphi$ and $\varphi[F'] = H'$. Then $\varphi[L]$ is again a lifting of C_i. Since, by symmetry, $\varphi[U_L] = U_{\varphi[L]}$ and since $H' \in U_L$ and $H' = \varphi[F'] \in \varphi[U_L] = U_{\varphi[L]}$, it follows from Claim 5 that $\varphi[L] = L$. So $\varphi \circ C_i'$ is a lifting of C_i intersecting C_i'. Hence by (iii) of Claim 2 there exists an $n \in \mathbb{Z}$ so that $\varphi \circ C_i'(x) = C_i'(x+n)$ for all $x \in \mathbb{R}$. Therefore, $\varphi = \varphi_n$. So $H' = \varphi_n[F']$, and hence $\pi''[H'] = \pi''[F']$, a contradiction. ▮

Doing this for each $i = 1, ..., k$, we obtain $\tilde{C}_1, ..., \tilde{C}_k$. These are closed curves as required, since

CLAIM 9. *$\tilde{C}_1, ..., \tilde{C}_k$ satisfy* (48).

Proof. By Claim 8 it suffices to show that no two of $\tilde{C}_1, ..., \tilde{C}_k$ pass through the same face of G. Suppose that, say, $\tilde{C}_1$ and $\tilde{C}_2$ pass through a common face. Then, by the symmetry of the universal covering surface S', there exist liftings $\tilde{C}'_1$ of $\tilde{C}_1$ and $\tilde{C}'_2$ of $\tilde{C}_2$ to S' passing a common face of G'. However, by Claim 7, all faces passed by $\tilde{C}'_1$ are contained in U_L for some lifting L of C_1, while all faces passed by $\tilde{C}'_2$ are contained in $U_{L'}$ for some lifting L' of C_2. Since $L \neq L'$, it follows by Claim 5 that $U_L \cap U_{L'} = \emptyset$, contradicting the assumption that $\tilde{C}'_1$ and $\tilde{C}'_2$ have a face in common. ▮

This finishes the proof of the theorem. ▮

ACKNOWLEDGMENTS

I thank András Frank, Laci Lovász, and Paul Seymour for very stimulating discussions on the problem discussed in this paper. Moreover, I thank Bill Pulleyblank, Paul Seymour, and Bruce Shepherd for carefully reading the paper, leading to several improvements in the presentation of the results.

REFERENCES

1. L. V. AHLFORS AND L. SARIO, "Riemann Surfaces," Princeton Univ. Press, Princeton, NJ, 1960.
2. R. BAER, Kurventypen auf Flächen, *J. Reine Angew. Math.* **156** (1927), 231–246.
3. R. COLE AND A. SIEGEL, River routing every which way, but loose, *in* "Proc. 25th Ann. Symp. Foundations of Computer Sci.," pp. 65–73, 1984.
4. M. DEHN AND P. HEEGAARD, Analysis situs, *in* "Encyclopädie der mathematischen Wissenschaften, III. Band, I. Teil, I. Hälfte," pp. 153–220, Teubner, Leipzig, 1907–1910.
5. R. A. FENN, "Techniques of Geometric Topology," Cambridge Univ. Press, Cambridge, UK, 1983.
6. M. E. GOLDMAN, An algebraic classification of noncompact 2-manifolds, *Trans. Amer. Math. Soc.* **156** (1971), 241–258.
7. B. VON KERÉKJÁRTÓ, "Vorlesungen über Topologie I: Flächentopologie," Springer-Verlag, Berlin, 1923.
8. C. E. LEISERSON AND F. M. MALEY, Algorithms for routing and testing routability of planar VLSI-layouts, *in* "Proc. 17th Ann. ACM Symp. Theory of Computing," pp. 69–78, 1985.
9. W. S. MASSEY, "Algebraic Topology: An Introduction," Springer-Verlag, New York, 1967.
10. E. E. MOISE, "Geometric Topology in Dimensions 2 and 3," Springer-Verlag, New York, 1977.
11. R. Y. PINTER, River routing: methodology and analysis, *in* "Third CalTech Conf. on Very Large Scale Integration," pp. 141–163, Springer-Verlag, Berlin, 1983.

12. H. Poincaré, 5e complément à l'Analysis Situs, *Rend. Circ. Mat. Palermo* **18** (1904), 45–110.
13. T. Radó, Über den Begriff der Riemannschen Fläche, *Acta Litt. Sci. Szeged* **2** (1925), 101–121.
14. I. Richards, On the classification of noncompact surfaces, *Trans. Amer. Math. Soc.* **106** (1963), 259–269.
15. N. Robertson and P. D. Seymour, Graph minors. VI. Disjoint paths across a disc, *J. Combin. Theory Ser. B* **41** (1986), 115–138.
16. N. Robertson and P. D. Seymour, Graph minors. VII. Disjoint paths on a surface, *J. Combin. Theory Ser. B* **45** (1988), 212–254.
17. H. Seifert and W. Threlfall, "Lehrbuch der Topologie," Teubner, Leipzig, 1934 [English translation: "A Textbook of Topology," Academic Press, New York, 1980].