Local recognition of graphs, buildings, and related geometries

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A brief overview of results and problems concerning the local recognition of graphs is given, and special attention is drawn toward graphs related to buildings.

1. Introduction

Let Δ be a graph. A graph Γ is said to be *locally* Δ if, for each $\gamma \in \Gamma$, the subgraph (induced on) $\Gamma(\gamma)$ (i.e., the set of vertices adjacent to γ, also called the *neighborhood* or *link* of Γ), is isomorphic to Δ. More generally, when $\mathcal{X}$ is a class of graphs, we say that Γ is *locally* $\mathcal{X}$ if, for each $\gamma \in \Gamma$ (read γ a vertex of Γ) the subgraph $\Gamma(\gamma)$ is isomorphic to a member of $\mathcal{X}$. Of course, by saying that a graph is *locally grid* (a *grid* being a graph that can be written as a product of two cliques), we mean that it is locally $\mathcal{X}$, where $\mathcal{X}$ stands for the class of all grids.

Although it was probably apparent to quite a few finite group theorists that some of the local characterization problems in the classification of finite simple groups can be phrased in terms of graphs, Hall and Shult (to the best of my knowledge) were the first to strip the group theory so as to obtain intriguing problems of merely graph-theoretic nature: given an 'interesting' class $\mathcal{X}$, determine all graphs which are locally $\mathcal{X}$. We shall refer to it as the *local recognition problem (with respect to $\mathcal{X}$)*. The theory of buildings also led to local characterization questions for geometries, which can be reduced to local recognition problems for graphs. These are the two main reasons why I want to survey what is known on the local recognition problem.

Much of the material in this paper can be found scattered in [BCN 1989]. For notions of shadow space, point-line geometry and the like, the reader is referred to [Coh 1986].

The first major paper dealing with the topic known to me is [BuHu 1977] treating graphs that are locally generalized quadrangles. For quite a different angle, see [BHM 1980], where the question is raised whether, for Δ a specified tree, a graph Γ exists that is locally Δ; see also [Hall 1985] for a sequel.

For any graph Γ, denote by $\mathcal{X}(\Gamma)$ the set of all graphs of the form $\Gamma(\gamma)$ for $\gamma \in \Gamma$. Clearly Γ need not be the only graph (up to isomorphism) that is locally $\mathcal{X}(\Gamma)$. For instance, the disjoint union of two copies of Γ has the same property. To remove this obstacle it suffices to demand that Γ be connected. Another instance: any polygon is locally $2K_1$, the disjoint union of two vertices. But then the infinite path is a cover of them all, so constitutes a universal solution. (More generally, if $d(\cdot,\cdot)$ denotes the usual graph-theoretic distance in Γ and A is a subgroup of $Aut\Gamma$ satisfying $d(\gamma,\gamma^a) \geq 4$ for all $\gamma \in \Gamma$ and $a \in A$, then the natural quotient graph Γ/A is locally $\mathcal{X}(\Gamma)$.) A third instance: there are precisely two graphs (up to isomorphism) that are locally a cube (see below); they do not seem to have much more than this property in common. Thus, the best outcome one may expect of local recognition with respect to a class $\mathcal{X}$ is a list of well-described graphs such that any connected graph that is locally $\mathcal{X}$ is a suitable quotient of one of the graphs in the list.

Now, take into consideration a 'natural' class of graphs, such as the dual polar spaces. Their local structure is rather uninformative (a disjoint union of cliques) and does not suffice to pin them down. A partial solution is to extend the notion of local recognition to bigger neighborhoods: for any natural number h, the h-neighborhood of $\gamma \in \Gamma$, notation $\Gamma_{\leq h}(\gamma)$, is the subgraph induced on the set of all vertices at distance at most h to γ. Thus the 1-neighborhood of γ, sometimes denoted by $\gamma^\perp$ rather than $\Gamma_{\leq 1}(\gamma)$, is the complete join of the vertex γ and $\Gamma(\gamma)$. We say that a class Ψ of connected graphs is *locally h-recognizable* if every connected graph Δ with the property that for each $\delta \in \Delta$ there are $\Gamma \in \Psi$ and $\gamma \in \Gamma$ such that $\Delta_{\leq h}(\delta) \cong \Gamma_{\leq h}(\gamma)$ is a suitable quotient of a member of Ψ. In [Coh 1986] the following question was raised: given a 'natural' class of graphs, what is the minimal h such that it is locally h-recognizable? For the class of all dual polar graphs, $1 < h \leq 8$. In general h need not exist. If Ψ consists of a single finite member Γ that is regular (i.e., all vertices have the same finite valency), then h exists.

The minimal h for the point graph Γ of a geometry is obviously related to axiom systems characterizing that geometry: any axiom system should involve a condition that cannot be checked in $\Gamma_{\leq j}(\gamma)$ for $\gamma \in \Gamma$ and $j < h$; in other words, if there is a number j and an axiom system such that the validity of each of its axioms can be verified by mere knowledge of the structure of $\Gamma_{\leq j}(\gamma)$ for $\gamma \in \Gamma$, then $h \leq j$.

2. Some examples and elementary observations

Often, if Δ has diameter at most 2 and there are plenty of edges, the locally Δ graphs can be found. The most extreme case:

Proposition: *If $\Delta = K_n$ (the complete graph, or clique, of size n), then K_{n+1} is the only connected graph (up to isomorphism) that is locally Δ.*

There need not always be a graph that is locally Δ. Take $\Delta = K_{1,2}$, where $K_{m_1,\ldots,m_t}$ is the complete t-partite graph with parts of size $m_1, \ldots, m_t$. More generally,

Proposition (cf. [BCN 1989]): *Let Γ be a connected graph that is locally complete multipartite. Then Γ is either triangle-free or complete multipartite. In particular, if there are $(m_i)_{1 \le i \le t}$ and $\gamma \in \Gamma$ such that $\Gamma(\gamma)$ is locally $K_{m_1,\ldots,m_t}$, then $m_1 = \ldots = m_t$, and Γ is the complete $t+1$-partite graph with parts of size m_1.*

Existence may also depend on finiteness:

Proposition ([BHM 1980]): *Let Δ be the tree on 8 vertices with 3 vertices of valency 3 and 5 of valency 1. There exists an infinite graph that is locally Δ; but no finite one.*

The existence proof is based on a 'free construction', the non-existence proof on a counting argument (the number of edges on a unique triangle can be counted in two ways that lead to distinct answers).

The most 'classical' example of local recognition of graphs inspired by groups is related to $P\Sigma L(2,25)$. This group appears as a sporadic example in the odd transposition papers of [Asch 1973]. The transpositions in question are the 65 outer involutions of a single class D. Let Γ be the graph whose vertex set is D and in which two distinct vertices are adjacent whenever they commute (the so-called 'commuting involutions' graph on D). Then $\Gamma(\gamma)$ can be identified with the commuting involutions graph on the class of transpositions in Sym_5, the symmetric group on 5 letters. In other words, Γ is locally Petersen. The characterization of all groups G having a class of transpositions D such that, for $d \in D$, the set $X = D \cap C_G(d)$ maps to the set of transpositions in Sym_5 under the composition of morphisms $C_G(d) \to C_G(d)/C_G(X) \cong Sym_5$ is a corollary of the determination of all graphs that are locally Petersen:

Proposition ([Hall 1980]): *Apart from the above graph related to $P\Sigma L(2,25)$, there are two connected locally Petersen graphs, viz. the complement of the Johnson graph $J(7,2)$ and a 3-cover of this graph (with automorphism group $3 \cdot Sym_7$).*

The proof of this result is elementary, but requires work. The main step is the determination of the structure induced on the set of points adjacent to two points at distance 2.

Local characterizations of Lie graphs (gotten as point graphs of shadow spaces of buildings) are best illustrated by the graphs that are locally polygons. Recall (cf. [BCN 1989]) that the Coxeter graph of type $M_{n,i}$ is the graph obtained from the Coxeter group W of type $M = (m_{j\,k})_{1 \le j,k \le n}$ in the following way: set $r = r_i$; vertices are the cosets with respect to the subgroup $V = \langle r_j \mid j \neq i \rangle$; two distinct vertices xV, yV are adjacent if $y^{-1}x \in VrV$. Let $M = (m_{ij})_{1\leq i,j\leq 3}$, where $m_{12} = 3$, $m_{13} = 2$, and $m_{23} = m$. The Coxeter graph $\Gamma^{(m)} = M_{3,1}$ is a locally m-gon graph. It is finite if and only if $m \leq 5$. [This graph is nothing but the 1-shadow of the thin building (apartment) of type M; edges correspond to objects of type 2, triangles to objects of type 3.]

Proposition: *Let $m \geq 2$. If Γ is a connected graph that is locally the m-gon, then there is a group A of automorphisms of $\Gamma^{(m)}$ such that $\Gamma \cong \Gamma^{(m)}/A$.*

The proof consists of constructing a chamber system of type M from Γ with the above notions for objects, and letting W operate on it as a set of permutations. This gives a morphism $\Gamma^{(m)} \to \Gamma$. The group A consists of all elements in $Aut\ \Gamma^{(m)} \cong W$ fixing the fiber of a given chamber of Γ.

The following elementary and charming argument shows that there is some control over graphs that are locally collinearity graphs of finite polar spaces. Here, we define a *Zara graph* to be a finite non-complete graph with the properties that all maximal cliques have the same size, that the number $e := \mid \gamma^{\perp} \cap M \mid$ is constant for all vertices γ and maximal cliques M not containing γ, and that, for γ and M as before, there exists a maximal clique on γ meeting M in less than e points. (Thus, $e > 0$.)

Lemma ([Pasi 1989]): *Let Γ be a connected locally Zara graph. Then all maximal cliques of Γ have the same size, m say, and the diameter of Γ is bounded from above by $m - 1$.*

Proof: The statement about constant maximal clique size follows directly from the corresponding Zara graph property and connectedness of Γ. Suppose $\gamma \in \Gamma$ and M is a maximal clique. The *distance from γ to M*, denoted by $d(\gamma, M)$, is the minimum over all distances $d(\gamma, \delta)$ for $\delta \in M$. We assert that $d(\gamma, M) = i$ implies $i \leq \mid \Gamma_i(\gamma) \cap M \mid -1$. It clearly proves the lemma.

The assertion follows from induction on i: if $i = 0$, then the inequality reads $0 \leq 0$. So let $i > 0$, and take $\delta \in \Gamma_i(\gamma) \cap M$, $\epsilon \in \Gamma_{i-1}(\gamma) \cap \Gamma(\delta)$. Pick a maximal clique M' on $\{\epsilon, \delta\}$ such that $\mid M \cap M' \mid < \mid \epsilon^{\perp} \cap M \mid$. Then $d(\gamma, M') = i - 1$, so the induction hypothesis yields that $\Gamma_{i-1}(\gamma) \cap M'$ contains at least i vertices. Counting edges between $M \setminus M'$ and $M' \setminus M$ we find that there are at least i vertices in $\bigcup_{\zeta \in \Gamma_{i-1}(\gamma) \cap M'} \zeta^{\perp} \cap M \setminus M'$, whence at least $\mid M \cap M' \mid +i$ vertices in $\Gamma_{\leq i}(\gamma) \cap M$, proving the assertion as $\delta \in M \cap M'$. QED

In many cases, the diameter bound can be considerably improved (although in the locally generalized quadrangle case of the above lemma, it is best possible!); here we have merely derived that there exists a bound, implying that the solutions to the local recognition problem will only involve finite graphs.

3. Covers and imprimitivity

In group theory, the notion of imprimitivity helps to reduce arbitrary permutation representations to primitive ones. For association schemes, there is a satisfactory counterpart, cf. [BCN 1989]. Graph-theoretically, there is no unified approach. In [Asch 1976], a partial interpretation has been given in terms of *contractions*, i.e., surjective morphisms of graphs $\Gamma \to \Delta$ such that any two distinct vertices δ and δ' are adjacent if and only if every vertex in the fiber of δ is adjacent to every vertex in the fiber of δ'. One reason why it is partial is that contraction could also be considered for the graph $\Gamma_{\mathcal{I}}$, whose vertex set coincides with that of Γ and in which two vertices are adjacent whenever they are at distance i in Γ for some $i \in \mathcal{I}$.

For example, by the contraction criterion, the disjoint union of cliques is imprimitive. If Γ is the 3-cover in the locally Petersen proposition, the graph Γ_3 is a disjoint union of cliques; the corresponding primitive quotient – obtained by calling two cliques X, Y of Γ_3 adjacent if there is an edge of Γ joining a vertex from X with one from Y – is the Johnson graph $J(7,2)$. The surjective morphism defined on Γ thus obtained is an isomorphism when restricted to $\Gamma_{\leq 1}(\gamma)$ for each $\gamma \in \Gamma$.

We give two examples regarding the question of finding local conditions that are sufficient for Γ to be imprimitive.

Proposition ([JoSh 1988]): *Let Γ be a graph. Call γ and δ equivalent whenever $\gamma^\perp = \delta^\perp$. Then $\Gamma \to \Gamma/\equiv$ is a contraction. Moreover, $\Gamma/\equiv$ is reduced (i.e., $\gamma^\perp = \delta^\perp$ implies $\gamma = \delta$). If Γ is finite, regular, and reduced, then Γ is locally nondegenerate (a graph is called* nondegenerate *if no vertex is adjacent to all others).*

Proposition (cf. [BCN 1989]): *Let Δ be a finite graph whose complement is not connected, and suppose Γ is a connected graph that is locally Δ. If $\Gamma(\gamma) \cap \Gamma(\delta)$ has the same size for all $\gamma, \delta \in \Gamma$ at distance 2, then Γ is a complete multipartite graph.*

4. Coxeter graphs

Since the Coxeter graphs $M_{n,i}$ are shadow spaces of apartments in buildings, local recognition of these graphs serves as a prelude to local recognition of the shadow spaces of buildings themselves. If M is irreducible and spherical, we shall adopt Bourbaki's node labeling of the diagram for M (cf. [Bourb 1968]). The graph $A_{n,i}$ is the Johnson graph $J(n+1, i)$. It is locally an $i \times (n-i)$ grid. The following result is known in various guises and special cases (see [BCN 1989] for further details and references).

Theorem: *Let Γ be a connected locally grid graph. Suppose that no connected component of the common neighbor subgraph $\Gamma(\gamma)\cap\Gamma(\delta)$ of two vertices at distance 2 has strictly more than 4 points. Then Γ is a Johnson graph or the quotient of a Johnson graph $J(2n,n)$ by a suitable involutory automorphism.*

Connected locally $2\times n$ grid graphs necessarily satisfy the supposition. (Locally $J(k,2)$ graphs have been studied in [Neum 1985].)

There are precisely two locally 3×3 grid graphs, namely the Johnson graph $J(6,3)$ and the complement of the 4×4 grid. More generally, [Hall 1989] has identified these graphs with line graphs of certain Fischer spaces. As a consequence all locally $3\times n$ grid graphs are known. [BlBr 1984] proved that, besides $J(8,4)$ and the (unique) quotient by the automorphism 'taking complements', there are precisely two more connected locally 4×4 grid graphs, each having 40 vertices and diameter 3.

[Hall 1987] studied local recognition of the maximal distance graphs $J(n,i)_i$ of $J(n,i)$. He showed that if $n\geq 3i+1$ any connected graph that is locally $J(n,i)_i$ must be isomorphic to $J(n+i,i)_i$. The bound is sharp in the sense that for $(n,i)=(6,2)$ (cf. [BuHu 1977]), there are two more connected graphs that are locally $J(6,2)_2$, viz. the complements of an elliptic and of a hyperbolic quadric in $Sp(4,2)$.

The graphs $D_{n,1}$ and $B_{n,1}=C_{n,1}$ are all isomorphic to the n-partite graph with parts of size 2, so are dealt with above. At the other end node of the Coxeter diagram we have the n-cube $B_{n,n}$ and its halved graph $D_{n,n}$. Now the hypercubes are clearly not locally 1-recognizable. On the other hand, they are 3-recognizable as Brouwer worked out, cf. [BCN 1989]. Finally, the halved n-cubes are locally $J(n,2)$. Such graphs are treated by [Neum 1985].

The graph $E_{6,1}$, known as the *Schläfli graph*, is the complement of the collinearity graph of the unique generalized quadrangle of order $(2,4)$; it is the unique connected graph that is locally $D_{5,5}$. The graph $E_{7,7}$, known as the *Gosset graph*, is the unique connected locally Schläfli graph. The graph $E_{8,8}$ is the unique connected graph that is locally $E_{7,7}$.

The graph $F_{4,1}$ is locally a cube. The only other connected locally cube graph is the complement of the 4×5 grid (see [Bus 1983]).

Concerning $H_{n,1}$, the locally icosahedral graphs (so $n=4$) can be determined in the same way as the locally pentagon graphs (so $n=3$), see [BBBC 1985]. (Two references on the nonspherical case of locally dodecahedron graphs are: [Cox 1954] and [VC 1985].)

5. Point-line geometries of spherical buildings

During the last few years, the thick analogues of the graphs $M_{n,i}$ for M of spherical type, i.e., the collinearity graphs of shadow spaces of spherical buildings – also called *Lie graphs* – have received some more attention beyond what has been described in [Coh 1986]. Here we content ourselves with an update of those notes. See [BCN 1989] for several properties of Lie graphs. Using (a slight variation of) the concept of parapolar space of polar rank 3, [HaTh 1988] have given an axiom system characterizing the finite Lie graphs of type $A_{n,i}$ $(2 \leq i \leq n-1)$, $C_{n,n-2}$ $(n \geq 3)$, $D_{n,n-1}$ $(n \geq 4)$, $E_{7,4}$, $E_{8,5}$, and $F_{4,1}$. The axioms beyond those for a parapolar space only concern configurations of symplecta intersecting in a line (which in fact follow from 1-local knowledge), so the result provides a local 2-recognition of these graphs. See [Shult 1989] for another local characterization of some of these graphs. The Grassmannians of polar spaces have also been characterized: Starting with a parapolar space that is locally a graph whose reduced graph is a Lie graph of type $A_m \cup B_n$ $(m \geq 1$ and $n \geq 3)$ [Hanss 1987] constructed globally defined subspaces of type $A_{n,2}$; from this he derived (with existing methods) that such a parapolar space is a suitable quotient of a Lie graph of type $B_{m+n+1,m+1}$. Thus, the Grassmannians of polar spaces are locally 2-recognizable. [Shult 1988] has shown that 4-shadow spaces of buildings with affine diagram F_4 (so the diagram has rank 5 diagram and maximal singular subspaces have rank 2) are locally 2-recognizable.

6. Beyond buildings

Now that the shadow spaces of spherical buildings are reasonably well understood, it is time to explore related geometries and graphs that are no longer of Lie type.

Removing a geometric hyperplane H (i.e., a subset such that each line either has a single point or all of its points inside of it) from a shadow space S of a spherical building leads to a partial linear space on $S \setminus H$ with the same local structure as S (that is, for $x \in S \setminus H$, the lines and planes of $S \setminus H$ on x form a space isomorphic to the space of lines and planes of S on x). We call such a space an *affine space* because if S is a projective space, the usual affine space appears, and (hence) if S is arbitrary, every (singular) plane of S not contained in H becomes a 'classical' affine plane in the space $S \setminus H$. In [CoSh 1987], a local recognition theorem is given for affine polar spaces of rank ≥ 3. Thus the classification of affine polar spaces amounts to that of classifying geometric hyperplanes of polar spaces. If the shadow space S is embedded in a projective space, geometric hyperplanes can be obtained as intersections of projective hyperplanes of the ambient projective space not containing all points of the embedded geometry. In [*loc. cit.*] it is also shown that, in case S is a polar space of rank ≥ 3, every hyperplane can be obtained by this construction if a projective embedding of S exists, and is of the form $x^{\perp}$ for some point x of S otherwise. In general, that is, for an arbitrary shadow space S, the local recognition of corresponding affine spaces is complicated by the fact that the geometric hyperplanes come in many kinds (there are at least as many orbits under the corresponding group G of Lie type as the number of projective hyperplane orbits of a projective space in which S is embedded, which equals the number of point orbits of the corresponding G-module).

The two sources of inspiration for local recognition mentioned at the outset both yield classes of geometries that merit further study. First, as pointed out by [Lyons 1988], the classification of finite simple groups uses 'component type' theorems which are related to commuting tori graphs (i.e., graphs whose vertex sets are conjugacy classes of 1-dimensional tori or semi-simple elements of a group of Lie type, and in which two vertices are adjacent when they commute). Second, according to a result of [Seitz 1974] the primitive parabolic permutation representations (i.e., those on the point sets of our Lie graphs) are the only primitive permutation representations of the corresponding groups of Lie type whose permutation ranks do not depend on the order of the underlying field. One step beyond these are the transitive permutation representations of groups of a given Lie type that have a permutation rank that is linear as a function of the order of the underlying field. In various cases, an association scheme can be found whose number of classes is independent of the order of the field.

An example of what I have in mind is the permutation representation of $A_n(q) \cong PSL(n+1, q)$ on the set P of antiflags (= nonincident point hyperplane pairs) in the associated projective space. The permutation rank of the full group $Aut A_n(q)$ (graph automorphisms included) on P is $q+3$ (cf. [Dar 1986]). Nevertheless, the projective geometry naturally produces an association scheme of 6 classes (one class is empty if the field has order 2). Taking two antiflags to be adjacent when the point of each of them lies in the hyperplane of the other, we turn P into the commuting tori graph described above with respect to the class of tori whose centralizer has 'simple component' of type A_{n-1}. (It is not always true that small permutation rank, say linear in the order q of the underlying field, implies that the point stabilizer normalizes a torus, the case of ${}^2B_2(q)$ in $Sp(4, q)$ being a counterexample).

Other examples come from association schemes on the nonisotropic points in an orthogonal or unitary space. The beautiful result [HaSh 1985] on locally cotriangular graphs is the first local recognition theorem for graphs related to those schemes (only very small fields appear in the conclusion of the theorem). The characterization of affine polar spaces in [CoSh 1987] uses the same ideas of proof. This is not so surprising as it may seem at first sight because certain spaces, whose points are the nonisotropic points, occur as quotients of suitable affine polar spaces. For example, consider the unitary polar space S embedded in a projective space over $\mathbb{F}_{q^2}$, and the geometric hyperplane $H = x^\perp \cap S$ of all points of S perpendicular (with respect to the unitary form associated with S) to some nonisotropic point x. The relation 'having distance 3 in the collinearity graph' is an equivalence relation on $S \setminus H$. The quotient space of $S \setminus H$ by the equivalence relation is the space N induced on the set of all nonisotropic points of $x^\perp$ whose lines are the tangents to H. The planes are affine again, but given a point x and a line l not on x, there are either 0, q, or q^2 points of l collinear with x. Recently, H. Cuypers has further extended the methods of [CoSh 1987] and [Pasi 1988] to obtain a characterization of the space N, valid for all finite fields.

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