TESTING AND ESTIMATING ORDERED PARAMETERS OF PROBABILITY DISTRIBUTIONS

ACADEMISCH PROEFSCHRIFT

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VOORWOORD

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CONTENTS

INTRODUCTION AND SUMMARY

The problem to be considered in the chapters 1-3 and the appendix concerns the maximum likelihood estimation of partially (or completely) ordered parameters of probability distributions. Let $\underline{x}_1,\ldots,\underline{x}_k$ [1] be independent random variables and let, for $i=1,\ldots,k$, $x_{i,1},\ldots,x_{i,n_i}$ be n_i independent observations of $\underline{x}_i$. Assume that the distribution of $\underline{x}_i$ contains one unknown parameter θ_i ($i=1,\ldots,k$). Further assume that information about these parameters $\theta_1,\ldots,\theta_k$ is available in the following two forms. Let, for $i=1,\ldots,k$, $\varphi_i(\theta_i)$ be a given function of θ_i and let I_i be a given closed interval, then it is assumed

1. that $\varphi_1(\theta_1),\ldots,\varphi_k(\theta_k)$ satisfy a number of (non-contradictory) relations of the form $\varphi_i(\theta_i) \leqq \varphi_j(\theta_j)$, i.e. it is assumed that $\varphi_1(\theta_1),\ldots,\varphi_k(\theta_k)$ are partially (or completely) ordered,
2. that, for each $i=1,\ldots,k$, $\varphi_i(\theta_i)$ satisfies the inequality $\varphi_i(\theta_i)\in I_i$.

Then the problem is to determine the maximum likelihood estimates of $\theta_1,\ldots,\theta_k$ given these inequalities.

The problem of estimating $\theta_1,\ldots,\theta_k$ is identical with the problem of maximizing the likelihood function in a subspace of the parameterspace; this subspace is defined by the restrictions (inequalities) imposed on $\theta_1,\ldots,\theta_k$.
In chapter 1 (section 1.2) conditions are given for the existence and uniqueness of this maximum.[2] Further (cf. section 1.3 and 1.4) a recurrent procedure is given by means of which this maximum may be found. Section 1.5 contains an explicit formula for the maximum likelihood estimates and in section 1.6 it is shown that (under certain conditions) the maximum of the likelihood function coincides with the minimum of a sum of squares. In some special cases this implies that the maximum likelihood estimates are identical with the least squares estimates.
In chapter 2 the theorems of chapter 1 are applied to the following special cases:

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1) Random variables will be distinguished from numbers (e.g. from the values they take in an experiment) by underlining their symbols.

2) A part of chapter 1 contains the solution of a more general problem.

1. a binomial distribution with parameters n_i and θ_i,
2. a normal distribution with mean θ_i and known variance,
3. a Poisson distribution with parameter θ_i,
4. an exponential distribution with parameter θ_i,
5. a rectangular distribution between 0 and θ_i,
6. a normal distribution with known mean and variance θ_i.

In some of these cases numerical examples are given.
Chapter 3 contains a proof of the consistency of the maximum likelihood estimates of $\theta_1,\ldots,\theta_k$. In this proof no assumptions are made on the differentiability of the likelihood function.
The appendix contains a description of the solution given by MIRIAM AYER, H.D. BRUNK, et al. (cf. [1] and [2]) for a special case of the abovementioned estimation problem.

In chapter 4 a test for complete ordering of the parameters is given. Let again $\underline{x}_1,\ldots,\underline{x}_k$ be independent random variables and let, for each $i=1,\ldots,k, x_{i,1},\ldots,x_{i,n_i}$ be independent observations of $\underline{x}_i$. Let further, for $i=1,\ldots,k,\theta_i$ denote an unknown parameter of the distribution of $\underline{x}_i$. *Then a class of tests is described for the hypothesis that the parameters $\theta_1,\ldots,\theta_k$ satisfy the inequalities*

$$\theta_1 \leqq \ldots \leqq \theta_k$$

against the alternative hypothesis that at least one value of i *exists with $\theta_i > \theta_{i+1}$.*
These tests can be applied if, for each $i=1,\ldots,k-1$, a test T_i for the hypothesis $\theta_i \leqq \theta_{i+1}$ against the alternative hypothesis $\theta_i > \theta_{i+1}$ is available such that, for each pair of values (i,j) with $i < j$,

$$P[\underline{t}_i \epsilon Z_i \text{ and } \underline{t}_j \epsilon Z_j | \theta_i=\theta_{i+1} \text{ and } \theta_j=\theta_{j+1}] \leqq$$
$$\leqq P[\underline{t}_i \epsilon Z_i | \theta_i=\theta_{i+1}].P[\underline{t}_j \epsilon Z_j | \theta_j=\theta_{j+1}],$$

where $\underline{t}_i$ is the test statistic and Z_i the critical region of T_i. Tests T_i satisfying these conditions are given for the following special cases:

1. a rectangular distribution between 0 and θ_i,
2. an exponential distribution with parameter θ_i,
3. a normal distribution with variance θ_i,
4. a normal distribution with mean θ_i and known variance.

Further an analogous distribution-free test, based on WILCOXON's two sample test, is considered.

CHAPTER 1

THE MAXIMUM OF A FUNCTION IN A CERTAIN CONVEX SET

1.1 *Introduction*

The problem to be treated in this chapter is the following. Let, for $i=1,\dots,k$, I_i be a given closed interval and let G denote the Cartesian product of these intervals

$$(1.1;1) \qquad G \stackrel{\text{def}}{=} \prod_{i=1}^{k} I_i .$$

Let further $H(\eta_1,\dots,\eta_k)$ be a univalued function, subject to certain conditions to be mentioned later and defined for each point $(\eta_1,\dots,\eta_k)\in G$.
Now let D denote the set of all points $(\eta_1,\dots,\eta_k)$ satisfying

$$(1.1;2) \qquad \begin{cases} 1. & \alpha_{i,j}(\eta_i-\eta_j) \leqq 0 \\ 2. & \eta_i \in I_i \end{cases} \qquad (i,j=1,\dots,k),$$

where $\alpha_{i,j}$ $(i,j=1,\dots,k)$ are numbers satisfying

$$(1.1;3) \qquad \begin{cases} 1. & \alpha_{i,j}=-\alpha_{j,i}, \\ 2. & \alpha_{i,j}=0 \text{ if the intersection } I_i \cap I_j \text{ contains at most one point,} \\ 3. & \alpha_{i,j}=0,\ 1 \text{ or } -1 \text{ in all other cases,} \\ 4. & \alpha_{i,j}=1 \text{ if } \alpha_{i,h}=\alpha_{h,j}=1 \text{ for any } h.^{1)} \end{cases}$$

Then (1.1;2.1) represents a partial ordering[2] of the η_i. If (i,j) is a pair of values with $\alpha_{i,j}=0$ then η_i and η_j are not comparable, i.e. the set (1.1;2.1) contains points $(\eta_1,\dots,\eta_k)$ with $\eta_i < \eta_j$, points $(\eta_1,\dots,\eta_k)$ with $\eta_i=\eta_j$ and points $(\eta_1,\dots,\eta_k)$ with $\eta_i > \eta_j$. If $\alpha_{i,j}=1$ then $\eta_i \leqq \eta_j$ for each point $(\eta_1,\dots,\eta_k)$ in the set (1.1;2.1) and this set contains points $(\eta_1,\dots,\eta_k)$ with $\eta_i < \eta_j$ and points $(\eta_1,\dots,\eta_k)$ with $\eta_i=\eta_j$. Further D is a convex subset of G.

The problem to be considered in this chapter is: *to determine the maximum in* D *of the function* $H(\eta_1,\dots,\eta_k)$.

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1) (1.1;3.1) entails that the inequalities (1.1;2.1) for $i < j$ imply those for $i > j$.

2) A complete ordering is considered as a special case of a partial ordering.

Section 1.2 contains conditions for the existence and uniqueness of the maximum of H in D and in section 1.3 a recurrent procedure will be given by means of which this maximum may be found.

In section 1.4-1.6 we restrict ourselves to the special case where $H(\eta_1,\dots,\eta_k)$ may be written in the form

(1.1;4) $$H(\eta_1,\dots,\eta_k)=\sum_{i=1}^{k} H_i(\eta_i)$$

where $H_i(\eta)$ is a univalued function defined for each $\eta\in I_i (i=1,\dots,k)$. For this special case the procedure described in section 1.3 may sometimes be simplified by using one of the theorems of section 1.4. Section 1.5 contains an explicit formula for the maximum of H in D and in section 1.6 the problem will be considered as a minimumproblem for a sum of squares.

Remark 1.1;1

If no confusion is to be expected a point $(y_1,\dots,y_r)$ in an r-dimensional space will be denoted by $\vec{y}$. Then, if $\vec{y}$ and $\vec{y}'$ are two points in this space, $\vec{y}=\vec{y}'$ states that each coordinate of $\vec{y}$ is equal to the corresponding coordinate of $\vec{y}'$; $\vec{y}\neq\vec{y}'$ states that $y_i\neq y_i'$ for at least one value of $i=1,\dots,r$.

1.2 *Conditions*

In the sequel the set $\{1,\dots,k\}$ will be denoted by E. Further for a subset M of E,

(1.2;1) $$I_M \overset{\text{def}}{=} \bigcap_{i\in M} I_i .$$

Now let $M_1,\dots,M_N$ be subsets of E with

(1.2;2) $$\begin{cases} 1. \ \bigcup_{\nu=1}^{N} M_\nu = E, \\ 2. \ M_\nu \cap M_\mu = o^{1)} \quad \text{for each pair of values } (\nu,\mu) \\ \qquad\qquad\qquad\quad \text{with } \nu\neq\mu \ (\nu,\mu=1,\dots,N), \\ 3. \ I_{M_\nu}\neq o \qquad\quad \text{for each } \nu=1,\dots,N. \end{cases}$$

A set of subsets $M_1,\dots,M_N$ satisfying (1.2;2) will be called a *partition* of E and will be denoted by $\mathfrak{p}_N$.

Now consider, for a partition $\mathfrak{p}_N$, the diagonal space consisting of all points $\vec{\eta}$ satisfying, for each $\nu=1,\dots,N$,

(1.2;3) $\eta_i=\eta_j$ for each pair of values $(i,j)\in M_\nu$.

1) o denotes the empty set.

Let further G_N[1] denote the intersection of this diagonal space with G. Then (1.2;2.3) implies that $G_N \neq o$. Further G_N may be written in the form

(1.2;4) $$G_N = \prod_{\nu=1}^{N} I_{M_\nu},$$

i.e. G_N is a parallelepiped. Further the function $H(\vec{\eta})$ reduces, for $\vec{\eta} \in G_N$, to a function of N variables. This function will be denoted by $H^*(\vec{\zeta})$, where $\zeta_1, \ldots, \zeta_N$ denote the coordinates $\eta_1, \ldots, \eta_k$ for $\vec{\eta} \in G_N$. Thus $H^*(\vec{\zeta})$ is the restriction of the function $H(\vec{\eta})$ with respect to the parallelepiped G_N.

We now define

(1.2;5) *Definition: If* $\psi(\vec{y})$ *is a univalued function defined for each point* $\vec{y}$ *in a convex closed set* Γ, *if further the values taken by* $\psi(\vec{y})$ *are real numbers or* $+\infty$, *or* $-\infty$ *then the function* $\psi(\vec{y})$ *will be called strictly unimodal in* Γ *if*

1. $\psi(\vec{y})$ *possesses a unique maximum* $\leqq +\infty$ *in* $\vec{y}^* \in \Gamma$, *such that* $|y_i^*| < \infty$ *for each coordinate* y_i^* *of* $\vec{y}^*$,
2. *each point in* Γ *can be connected with* $\vec{y}^*$ *by a Jordan arc*[2] *in* Γ *such that* $\psi(\vec{y})$ *is strictly increasing along this arc*[3]. *(Such an arc will be called a 'tracer').*

Throughout this chapter it will be supposed that the following condition is satisfied

Condition A: For each partition $\mathfrak{p}_N$ *the function* $H^*(\vec{\zeta})$ *is strictly unimodal in* G_N *and for each point in* G_N *a straight tracer exists.*

1.3 *The maximum of* H *in* D

In the sequel it will be supposed that the variables $\eta_1, \ldots, \eta_k$ are numbered in such a way that

(1.3;1) $\alpha_{i,j} \geqq 0$ for each pair of values $(i,j) \in E$ with $i < j$.

Then we have, for the restrictions $\eta_i \leqq \eta_j$ (i.e. $\alpha_{i,j}=1$),

(1.3;2) $\alpha_{i,h} \cdot \alpha_{h,j} > 0$ for at least one value of h (say h_o) between i and j,

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1) G_N is an abbreviation of $G_{\mathfrak{p}_N}$.

2) A Jordan arc is a biunique continuous image of a line-segment.

3) The fact that $\psi(\vec{y})$ is strictly increasing along this arc implies that $\psi(\vec{y})=-\infty$ in at most one point of the arc. Further the definition implies that $\psi(\vec{y})=+\infty$ in at most one point in Γ.

or

(1.3;3) $\quad \alpha_{i,h}\cdot\alpha_{h,j}=0$ for each h between i and j.

If $\eta_i \leqq \eta_j$ is a restriction satisfying (1.3;2) then $\alpha_{i,h_o}=\alpha_{h_o,j}=1$. Thus in this case the restriction $\eta_i \leqq \eta_j$ follows from the restrictions $\eta_i \leqq \eta_{h_o}$ and $\eta_{h_o} \leqq \eta_j$. Therefore the restrictions $\eta_i \leqq \eta_j$ satisfying (1.3;2) will be called *the non-essential restrictions of* D.

Further (1.3;3) implies that, for each h between i and j, η_h is not comparable with η_i or η_j. Therefore the restrictions $\eta_i \leqq \eta_j$ satisfying (1.3;3) will be called *the essential restrictions defining* D. These essential restrictions will be denoted by $R_1,\dots,R_s$. Then each R_λ corresponds with one pair of values (i,j); this pair will be denoted by (i_λ, j_λ) ($\lambda=1,\dots,s$). Because of the transitivity relations (1.1;3.4) the system $R_1,\dots,R_s$ is equivalent with the inequalities $\alpha_{i,j}(\eta_i-\eta_j) \leqq 0$ $(i,j\in E)$ and uniquely determined by the numbers $\alpha_{i,j}$ $(i,j\in E)$.

Now let, for a partition $\wp_N$,

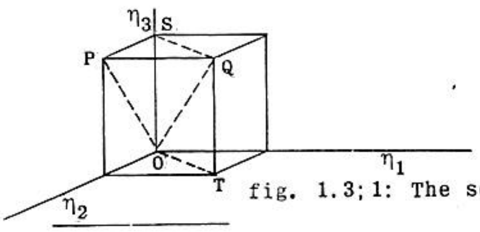
(1.3;4) $$D_{N,s} \overset{\text{def}}{=\!=} D \cap G_N \quad {}^{1)},$$

where s denotes the number of essential restrictions defining D. Then $D_{N,s}$ is a subset of G_N, i.e. $D_{N,s}$ is a subset of the Cartesian product of the N intervals I_{M_ν}. Further $D_{N,s}$ is a subset of D, thus it may be described, in an analogous way as the set D, by means of a set of inequalities

(1.3;5) $$\begin{cases} \alpha'_{\nu,\mu}(\zeta_\nu-\zeta_\mu) \leqq 0 \\ \zeta_\nu \in I_{M_\nu} \end{cases} \qquad (\nu,\mu=1,\dots,N).$$

The numbers $\alpha'_{\nu,\mu}$ then follow from $\wp_N$ and the numbers $\alpha_{i,j}$ $(i,j\in E)$. If e.g. k=3 and if D is the set

(1.3;6) $$\begin{cases} \eta_1 \leqq \eta_2 \leqq \eta_3, \\ 0 \leqq \eta_i \leqq 1 \ (i\in E), \end{cases}$$



D is the tetrahedron OPQS
G_N is the parallelepiped OTQS
$D_{N,s}$ is the triangel OQS

fig. 1.3;1: The sets D, G_N and $D_{N,s}$

1) $D_{N,s}$ is an abbreviation of $D_{\wp_N,s}$

then (cf. fig. (1.3;1)) D is the tetrahedron OPQS. Now let $\mathfrak{p}_N$ be the set of subsets $\{\{1,2\},\{3\}\}$, i.e. let N=2 and

$$\begin{cases} M_1=\{1,2\}, \\ M_2=\{3\}, \end{cases} \tag{1.3;7}$$

then G_N is parallelepiped OTQS and the intersection $D_{N,s}$ of D and G_N is the triangel OQS. Thus $D_{N,s}$ may be described by means of the inequalities

$$\begin{cases} \zeta_1 \leqq \zeta_2, \\ 0 \leqq \zeta_\nu \leqq 1 \quad (\nu=1,2). \end{cases} \tag{1.3;8}$$

Thus, in the same way as for the set D, we can define the essential restrictions defining $D_{N,s}$. If s_N denotes the number of essential restrictions defining $D_{N,s}$ then $s_N \leqq s$. If $s_N=0$ then $D_{N,s}=G_N$. Further if N=k then $I_{M_\nu}=I_\nu$ for each $\nu=1,\ldots,k$, thus $D_{k,s}=D$. If s=0 then D=G, thus $D_{N,0}=G_N$.

Theorem 1.3;1: For each partition $\mathfrak{p}_N$ *the function* $H^*(\vec{\zeta})$ *is strictly unimodal in* $D_{N,s}$.

Proof:

This theorem will be proved by induction with respect to s (the number of essential restrictions defining D), i.e. supposing the theorem to be proved for each set of numbers $\alpha_{i,j}$ (i, j∈E) satisfying (1.1;3) with s_o (or less) essential restrictions it will be proved that the theorem holds for each set of numbers $\alpha_{i,j}$ (i, j∈E) satisfying (1.1;3) with s_o+1 essential restrictions.

For s=0 we have $D_{N,s}=G_N$, thus in this case the theorem follows from condition A.

Now consider, for a given partition $\mathfrak{p}_N$, a set D_{N,s_o+1}; then the following two cases may be distinguished

I. $D_{N,s_o+1}=G_N$ (i.e. the number of essential restrictions defining D_{N,s_o+1} is zero); then condition A entails that H* is strictly unimodal in D_{N,s_o+1}.

II. $D_{N,s_o+1} \neq G_N$ (this situation is sketched in fig. 1.3;2 for the case that N=2 and $s_o+1=1$); then consider the set D_{N,s_o} obtained by omitting one of the essential restrictions defining D_{N,s_o+1}. Let this be the restriction $\zeta_{\nu_1} \leqq \zeta_{\nu_2}$. Then $D_{N,s_o+1} \subset D_{N,s_o}$.

Now H* has a unique maximum in D_{N,s_o} attained at the point V, say, and the following two cases may be distinguished.

1. V is outside D_{N,s_o+1} (cf. fig. 1.3;2a). Then an arbitrary point P of D_{N,s_o+1} with $\zeta_{\nu_1} < \zeta_{\nu_2}$ can be connected with V by a

tracer within D_{N,s_o} and this tracer must contain at least one borderpoint of D_{N,s_o+1} with $\zeta_{\nu_1}=\zeta_{\nu_2}$. The first of these points when following the tracer from P to V will be denoted by T; then H* assumes a larger value in T than in P. Now T lies in a set $D_{N-1,s_o'}$ with $s_o' \leqq s_o$ and H* is strictly unimodal in $D_{N-1,s_o'}$.

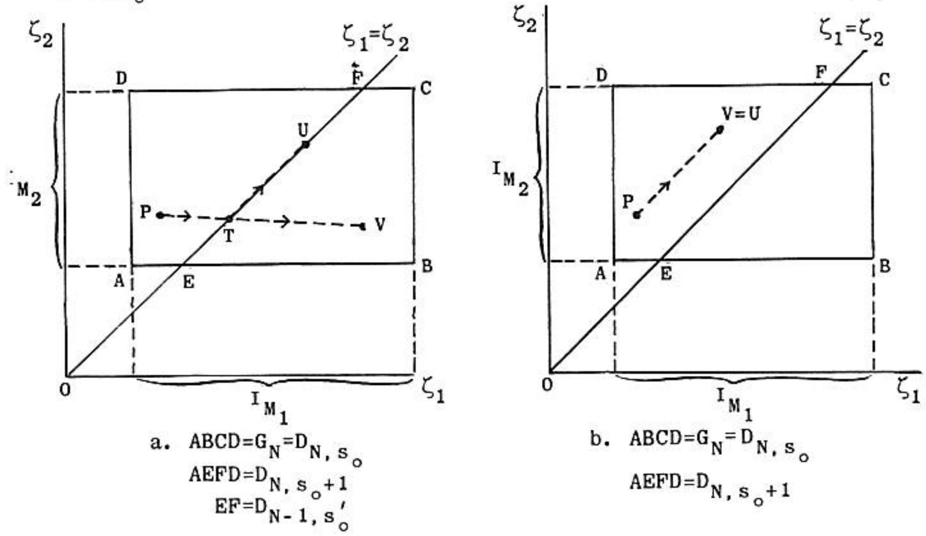


fig. 1.3; 2

A tracer from P to the point U where H* attains its maximum in D_{N,s_o+1} for $s_o+1=1$ and N=2.

Let H* attain its maximum in $D_{N-1,s_o'}$ at the point U, then a tracer in $D_{N-1,s_o'}$ exists from T to U. The tracer from P to T in D_{N,s_o+1} and from T to U in $D_{N-1,s_o'}$ together form a tracer from P to the point U where H* attains its maximum in D_{N,s_o+1}.

2. V is a point of D_{N,s_o+1}. Then H* attains a unique maximum in D_{N,s_o+1} at V and the following two cases may be distinguished

a. H* attains its maximum in G_N at this point V (cf. fig. 1.3; 2b). Then, according to condition A, for each point P in D_{N,s_o+1} a straight tracer (thus a tracer within D_{N,s_o+1}) exists from P to V,

b. V is not the point where H* attains its maximum in G_N; then, if V′ denotes the point where H* attains its maximum in G_N, we have $V' \notin D_{N,s_o+1}$. Further condition A implies that, for each point $P \epsilon D_{N,s_o+1}$, a straight tracer exists from P to V'. This tracer contains a borderpoint of D_{N,s_o+1} between P and V'. Thus in this case V is a borderpoint of D_{N,s_o+1} where at least two ζ_ν from $\zeta_1,\ldots,\zeta_N$ corresponding to an essential restriction for D_{N,s_o+1} are equal. Let this be the restriction $\zeta_{\nu_3} \leqq \zeta_{\nu_4}$, then we consider the set D'_{N,s_o} obtained from D_{N,s_o+1} by omitting the restriction $\zeta_{\nu_3} \leqq \zeta_{\nu_4}$ from the essential restrictions defining D_{N,s_o+1}. The maximum of H^* in D'_{N,s_o} is unique and the point where it is attained is a point with $\zeta_{\nu_3} \geqq \zeta_{\nu_4}$. The rest of the proof is the same as for case II.1.

Taking N=k in theorem 1.3;1 we obtain that $H(\vec{\eta})$ is strictly unimodal in D, i.e. $H(\vec{\eta})$ *possesses a unique maximum in* D. *The point where this maximum is attained will be denoted by* $\vec{u}$. *Further the point where* $H(\vec{\eta})$ *attains its maximum in* G *will be denoted by* $\vec{v}$.

Theorem 1.3;2: If D' *is the set obtained from* D *by omitting the restriction* R_s *and if* $\vec{u}'$ *is the point where* H *attains its maximum in* D' *then*

$$(1.3;9) \qquad \begin{cases} 1. \ \vec{u}=\vec{u}' & \text{if} \quad u'_{i_s} \leqq u'_{j_s}, \\ 2. \ u_{i_s}=u_{j_s} & \text{if} \quad u'_{i_s} > u'_{j_s}. \end{cases}$$

Proof:

If $u'_{i_s} \leqq u'_{j_s}$ then $\vec{u}' \epsilon D$; thus in this case we have $\vec{u}=\vec{u}'$.
If $u'_{i_s} > u'_{j_s}$, then for each point $\vec{\eta} \epsilon D$ a tracer within D' exists from $\vec{\eta}$ to $\vec{u}'$ containing a point $\vec{\eta}'$ with

$$(1.3;10) \qquad \begin{cases} 1. \ \eta'_{i_s}=\eta'_{j_s}, \\ 2. \ H(\vec{\eta}') > H(\vec{\eta}); \end{cases}$$

thus, if $u'_{i_s} > u'_{j_s}$, H attains its maximum in D for $\eta_{i_s}=\eta_{j_s}$; (1.3;9.2) then follows from the uniqueness of the maximum.

By means of theorem 1.3;2 the problem of maximizing H in D is reduced to the following two problems

1. to determine $\vec{u}'$, i.e. the point where H attains its maximum in D',
2. to determine the point, say $\vec{u}''$, where H attains its maximum in the intersection of D and the diagonal $\eta_{i_s}=\eta_{j_s}$.

In order to solve the first problem, i.e. to find $\vec{u}'$, we again apply theorem 1.3;2 to the function H by omitting one of the essential restrictions defining D'. For solving the second problem, i.e. to find $\vec{u}''$, we consider the function H under the restriction $\eta_{i_s}=\eta_{j_s}$ in D, i.e. we consider the function H* in $D_{N,s}$ with N=k-1, where one of the M_ν (cf. (1.2;2)) consists of the numbers i_s and j_s. In order to find $\vec{u}''$ we apply theorem 1.3;2 to the function H* by omitting one of the essential restrictions defining $D_{N,s}$.
Thus by repeated application of theorem 1.3;2 we arrive at the problem of maximizing a function H* in a set D_{N_o,s_o}, where the number of essential restrictions defining D_{N_o,s_o} is zero, i.e. at the problem of maximizing a function H* in a set G_{N_o}.
Consequently the maximum of H in D may be found if the maximum of H* in G_N is known for each partition $\wp_N$.

The procedure described above may sometimes be simplified by using the following theorem.
Theorem 1.3;3: If (i,j) *is a pair of values with*

$$\alpha_{i,j}=0, \tag{1.3;11}$$

if D' *is the subset of* D *where* $\eta_i \leqq \eta_j$ *and if* $\vec{u}'$ *is the point where* H *assumes its maximum in* D' *then*

$$\begin{cases} 1.\ \vec{u}=\vec{u}' & \text{if} \quad u_i' < u_j', \\ 2.\ u_i \geqq u_j & \text{if} \quad u_i' = u_j'. \end{cases} \tag{1.3;12}$$

Proof:
First consider the case that $u_i'=u_j'$, i.e. the case that H attains its maximum in D' for $\eta_i=\eta_j$. Then H attains its maximum in D for $\eta_i \geqq \eta_j$, i.e. $u_i \geqq u_j$. This proves (1.3;12.2). Further if $u_i \geqq u_j$ then it may be proved in the same way as in theorem 1.3;2 that H attains its maximum in D' for $\eta_i=\eta_j$, i.e. $u_i'=u_j'$ if $u_i \geqq u_j$. Consequently $u_i<u_j$ if and only if $u_i' < u_j'$, i.e. $\vec{u}$ coincides with $\vec{u}'$ if $u_i' < u_j'$. This proves (1.3;12.1).
By means of this theorem the problem of maximizing H in D is solved if $u_i' < u_j'$. Further the restriction $\eta_j \leqq \eta_i$ may be added if $u_i'=u_j'$. Thus by repeated application of theorem 1.3;3 the problem is reduced to the case of a complete ordering ($\alpha_{i,j}=1$ for each pair of values $(i,j)\in E$ with $i < j$), which in many cases, e.g. in the special case treated in the following sections, simplifies the procedure.

Finally we have

Theorem 1.3;4: If $\alpha_{i,j}(v_i - v_j) \leqq 0$ *for each pair of values* (i,j) *then*

(1.3;13) $$\vec{u}=\vec{v}.$$

Proof:

This theorem follows from the fact that in this case $\vec{v}\in D$. It also follows from theorem 1.3;2.

1.4 *Some special theorems*

In the sequel of this chapter we restrict ourselves to the special case where the following condition is satisfied.

Condition B: $H(\vec{\eta})$ *may be written in the form*

(1.4;1) $$H(\vec{\eta}) = \sum_{i\in E} H_i(\eta_i),$$

where, for each $i\in E$, $H_i(\eta)$ *is a univalued function defined for each* $\eta\in I_i$, *satisfying*

(1.4;2) $$-\infty \leqq H_i(\eta) < \infty \quad \text{for each } \eta\in I_i.$$

For this special case condition A may be expressed in terms of conditions for the partial sums

(1.4;3) $$H_M(\zeta) \stackrel{\text{def}}{=\!=} \sum_{i\in M} H_i(\zeta) \qquad (\zeta\in I_M),$$

where M is a subset of E with $I_M \neq o$.

We now first prove the following lemma (cf. definition (1.2;5))

Lemma 1.4;1: If Γ *is the Cartesian product of* K *intervals* $\Lambda_\varkappa$ $(\varkappa=1,\ldots,K)$, *if*

(1.4;4) $$\psi(\vec{y}) \stackrel{\text{def}}{=\!=} \sum_{\varkappa=1}^{K} \psi_\varkappa(y_\varkappa),$$

where $\psi_\varkappa(y)$ *is a univalued function defined for each* $y\in\Lambda_\varkappa$ $(\varkappa=1,\ldots,K)$, *if further, for each* $\varkappa=1,\ldots,K$,

(1.4;5) $$-\infty \leqq \psi_\varkappa(y) < \infty \quad \text{for each } y\in\Lambda_\varkappa$$

and if, for each $\varkappa=1,\ldots,K$, $\psi_\varkappa(y)$ *is strictly unimodal in* $\Lambda_\varkappa$, *then* $\psi(\vec{y})$ *is strictly unimodal in* Γ *and for each point in* Γ *a straight tracer exists.*

Proof:

The function $\psi(\vec{y})$ is a univalued function $\geqq -\infty$ defined for each point $\vec{y}\in\Gamma$.

Further the function $\psi_\varkappa(y)$ is strictly unimodal in $\Lambda_\varkappa$, thus $\psi_\varkappa(y)$ possesses a unique maximum in $\Lambda_\varkappa$, attained at, say, the point $y'_\varkappa$ $(\varkappa=1,\ldots,K)$. Then we have, for each $y_\varkappa\in\Lambda_\varkappa$ with $y_\varkappa \neq y'_\varkappa$ (cf. (1.4;5))

(1.4;6) $$-\infty \leqq \psi_\varkappa(y_\varkappa) < \psi_\varkappa(y'_\varkappa) < \infty \quad \text{for each } \varkappa=1,\ldots,K.$$

Thus, $\psi_\varkappa(y'_\varkappa)$ being finite for each $\varkappa=1,\dots,K$, we have, for each point $\vec{y}\in\Gamma$ with $\vec{y}\neq\vec{y}'$,

(1.4;7) $$\psi(\vec{y})=\sum_{\varkappa=1}^{K}\psi_\varkappa(y_\varkappa) < \sum_{\varkappa=1}^{K}\psi_\varkappa(y'_\varkappa)=\psi(\vec{y}')$$

and (1.4;7) entails that $\psi(\vec{y})$ possesses a unique maximum in Γ attained at the point $\vec{y}'$, i.e. $\vec{y}^*=\vec{y}'$.

Now let $\vec{y}^\circ$ be a fixed point in Γ with $\vec{y}^\circ\neq\vec{y}^*$ and let

(1.4;8) $$\begin{cases}\vec{y}(\beta)\overset{\text{def}}{=\!=}(1-\beta)\vec{y}^\circ+\beta\vec{y}^*,\\ 0\leqq\beta\leqq 1.\end{cases}$$

Then $\vec{y}(\beta)$ is a point in Γ and we have

(1.4;9) $$\psi\{\vec{y}(\beta)\}=\sum_{\varkappa=1}^{K}\psi_\varkappa\{y_\varkappa(\beta)\}.$$

It will be proved that $\psi\{\vec{y}(\beta)\}$ is a strictly increasing function of β in the interval $0\leqq\beta\leqq 1$.
First consider a value of $\varkappa$ with $y^\circ_\varkappa=y^*_\varkappa$; then

(1.4;10) $\qquad y_\varkappa(\beta)=y^*_\varkappa$ for each β with $0\leqq\beta\leqq 1$.

Thus in this case we have

(1.4;11) $-\infty<\psi_\varkappa(y^\circ_\varkappa)=\psi_\varkappa\{y_\varkappa(\beta)\}=\psi_\varkappa(y^*_\varkappa)<\infty$ for each β with $0\leqq\beta\leqq 1$.

Now consider a value of $\varkappa$ with $y^\circ_\varkappa\neq y^*_\varkappa$; then the fact that $\psi_\varkappa(y)$ is strictly unimodal in $\Lambda_\varkappa$ and attains its maximum in $\Lambda_\varkappa$ for $y=y^*_\varkappa$ implies that

(1.4;12) $$\psi_\varkappa(y^\circ_\varkappa)<\psi_\varkappa\{y_\varkappa(\beta_1)\}<\psi_\varkappa\{y_\varkappa(\beta_2)\}<\psi_\varkappa(y^*_\varkappa)$$

for each pair of values (β_1,β_2) with $0<\beta_1<\beta_2<1$.
From (1.4;11), (1.4;12) and the fact that at least one value of $\varkappa$ exists with $y^\circ_\varkappa\neq y^*_\varkappa$ then follows

(1.4;13) $$\psi\{\vec{y}(0)\}<\psi\{\vec{y}(\beta_1)\}<\psi\{\vec{y}(\beta_2)\}<\psi\{\vec{y}(1)\}$$

for each pair of values (β_1,β_2) with $0<\beta_1<\beta_2<1$. Consequently $\psi\{\vec{y}(\beta)\}$ is a strictly increasing function of β in the interval $0\leqq\beta\leqq 1$.

We now apply this lemma to the special case where condition B is satisfied. From (1.4;3) and the definition of $H^*(\vec{\zeta})$ follows

(1.4;14) $$H^*(\vec{\zeta})=\sum_{\nu=1}^{N}H_{M_\nu}(\zeta_\nu).$$

Lemma 1.4;1 then entails that condition A is satisfied if the following condition is satisfied
Condition A'*: For each* $M\subset E$ *with* $I_M\neq o$ *the function* $H_M(\zeta)$ *is strictly unimodal in* I_M.
In the sequel of this chapter we suppose that condition A' is satisfied.

Now let, for a subset M of E with $I_M \neq o$, v_M denote the value of ζ which maximizes $H_M(\zeta)$ in I_M, then (1.4;14) entails that $H^*(\vec{\zeta})$ attains its maximum in G_N at the point $(v_{M_1},\dots,v_{M_N})$. Thus by means of the procedure based on theorem 1.3;2 the coordinates of $\vec{u}$ are expressed in terms of $v_{M_1},\dots,v_{M_N}$.
In this section some theorems will be proved by means of which this procedure may be simplified.

The following theorem will be immediately clear.

Theorem 1.4;1: If M *is a subset of* E *with*

$$\sum_{h \notin M} \alpha^2_{i,h}=0 \quad \text{for each } i \in M, \tag{1.4;15}$$

then the coordinates u_i *of* $\vec{u}$ *for* $i \in M$ *may be found by maximizing* $\sum_{i \in M} H_i(\eta_i)$ *in the set*

$$D_1 \begin{cases} \alpha_{i,j}(\eta_i-\eta_j) \leqq 0 \\ \eta_i \in I_i \end{cases} \qquad (i,j \in M).^{1)} \tag{1.4;16}$$

We now prove the following lemma.

Lemma 1.4;2: If $\vec{\eta}$ *is a point in* D *such that for some pair of values* (i,j)

$$\begin{cases} 1.\ \eta_i < \eta_j, \\ 2.\ v_i \geqq v_j, \end{cases} \tag{1.4;17}$$

then a point $\vec{\eta}'$ *exists satisfying*

$$\begin{cases} 1.\ \eta'_i=\eta'_j \\ \quad \eta'_h=\eta_h \text{ for each } h \in E \text{ with } h \neq i \text{ and } h \neq j, \\ 2.\ H(\vec{\eta}') > H(\vec{\eta}). \end{cases} \tag{1.4;18}$$

If moreover

$$\begin{cases} 1.\ \alpha_{i,h} \leqq \alpha_{j,h} \text{ for each } h > i \text{ with } h \neq j, \\ 2.\ \alpha_{h,i} \geqq \alpha_{h,j} \text{ for each } h < j \text{ with } h \neq i, \end{cases} \tag{1.4;19}$$

then $\vec{\eta}'$ *is a point in* D.

---

1) If we consider, for a subset E′ of E, the set

$$D' \begin{cases} \alpha_{i,j}(\eta_i-\eta_j) \leqq 0 \\ \eta_i \in I_i \end{cases} \qquad (i,j \in E')$$

then the coordinates which are not mentioned may assume any values in their intervals I_i, i.e. D′ is a k-dimensional subset of G and $D \subset D'$.

Proof:

For the proof of (1.4;18) the existence of a number η will be proved satisfying

$$(1.4;20) \qquad \begin{cases} 1. & \eta_i \leqq \eta \leqq \eta_j, \\ 2. & H_i(\eta)+H_j(\eta) > H_i(\eta_i)+H_j(\eta_j). \end{cases}$$

Then the point $\vec{\eta}'$ satisfying

$$(1.4;21) \qquad \begin{cases} 1. & \eta_i'=\eta_j'=\eta, \\ 2. & \eta_h'=\eta_h \quad \text{for each } h\in E \text{ with } h\neq i \text{ and } h\neq j \end{cases}$$

satisfies (1.4;18).

The following cases may be distinguished (cf. fig. 1.4;1)

$$(1.4;22) \qquad \begin{cases} 1. & \eta_i < \eta_j \leqq v_i, \text{ then take } \eta=\eta_j, \\ 2. & v_i \leqq \eta_i < \eta_j, \text{ then take } \eta=\eta_i, \\ 3. & \eta_i < v_i < \eta_j, \text{ then take } \eta=v_i. \end{cases}$$

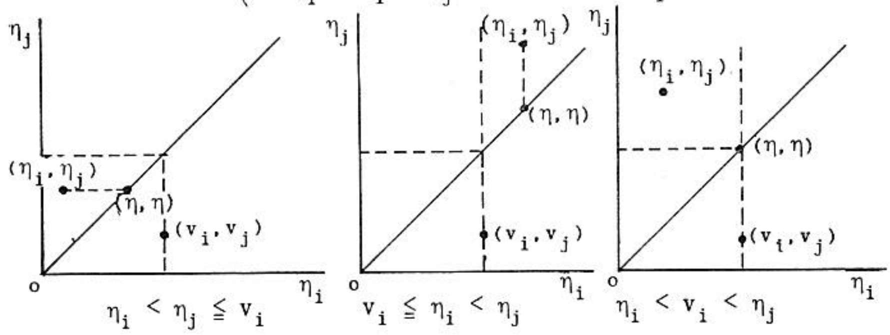


fig. 1.4;1
A number η satisfying (1.4;20)

This number η satisfies (1.4;20.1). We now prove (1.4;20.2). In case (1.4;22.1) we have

$$(1.4;23) \qquad H_j(\eta)=H_j(\eta_j)$$

and

$$(1.4;24) \qquad \eta_i < \eta \leqq v_i.$$

The fact that H_i is strictly unimodal in I_i then entails that

$$(1.4;25) \qquad H_i(\eta) > H_i(\eta_i)$$

and (1.4;20.2) then follows from (1.4;23) and (1.4;25).
For the case (1.4;22.2) we have

$$(1.4;26) \qquad H_i(\eta)=H_i(\eta_i)$$

and (cf. (1.4;17.2))

(1.4;27) $$v_j \leqq v_i \leqq \eta < \eta_j,$$

thus

(1.4;28) $$H_j(\eta) > H_j(\eta_j).$$

Finally for (1.4;22.3)

(1.4;29) $$\begin{cases} 1.\ \eta_i < v_i=\eta, \\ 2.\ v_j \leqq v_i=\eta < \eta_j, \end{cases}$$

thus

(1.4;30) $$\begin{cases} 1.\ H_i(\eta) > H_i(\eta_i), \\ 2.\ H_j(\eta) > H_j(\eta_j). \end{cases}$$

In order to prove that $\vec{\eta}'$ is a point in D it is sufficient to prove that

(1.4;31) $$\begin{cases} 1.\ \alpha_{h,i}(\eta_h-\eta) \leqq 0 & \text{for each } h < i, \\ 2.\ \alpha_{i,h}(\eta-\eta_h) \leqq 0 & \text{for each } h > i, \\ 3.\ \alpha_{h,j}(\eta_h-\eta) \leqq 0 & \text{for each } h < j, \\ 4.\ \alpha_{j,h}(\eta-\eta_h) \leqq 0 & \text{for each } h > j. \end{cases}$$

The fact that η satisfies

(1.4;32) $$\eta_i \leqq \eta \leqq \eta_j$$

and the fact that $\vec{\eta}$ is a point in D imply that

(1.4;33) $$\begin{cases} 1.\ \alpha_{h,i}(\eta_h-\eta) \leqq \alpha_{h,i}(\eta_h-\eta_i) \leqq 0 & \text{for each } h < i, \\ 2.\ \alpha_{j,h}(\eta-\eta_h) \leqq \alpha_{j,h}(\eta_j-\eta_h) \leqq 0 & \text{for each } h > j. \end{cases}$$

This proves (1.4;31.1) and (1.4;31.4).

We now prove (1.4;31.2). For each $h > i$ we have $\alpha_{i,h} \geqq 0$. Now (1.4;31.2) is immediately clear for values of h with $\alpha_{i,h}=0$; thus we only consider values of $h > i$ with $\alpha_{i,h}=1$. Then the following two cases may be distinguished

1. $h=j$, then (cf. (1.4;32))

(1.4;34) $$\alpha_{i,h}(\eta-\eta_h)=\alpha_{i,j}(\eta-\eta_j)=\eta-\eta_j \leqq 0.$$

2. $h \neq j$, then (1.4;19.1) entails that $\alpha_{j,h}=1$, thus $h > j$. Consequently for each $h > i$ with $\alpha_{i,h}=1$ and $h \neq j$ we have

(1.4;35) $$\begin{cases} 1.\ \alpha_{i,h}(\eta-\eta_h)=\alpha_{j,h}(\eta-\eta_h), \\ 2.\ h > j. \end{cases}$$

From (1.4;33.2) and (1.4;35) then follows

(1.4;36) $$\alpha_{i,h}(\eta-\eta_h) \leqq 0 \quad \text{for each } h > i \text{ with } h \neq j \text{ and } \alpha_{i,h}=1.$$

and (1.4;31.2) then follows from (1.4;34) and (1.4;36).
The proof of (1.4;31.3) is analogous.

Theorem 1.4;2: If for some pair of values (i,j)

(1.4;37)
$$\begin{cases} 1.\ \alpha_{i,j}=1 \text{ and } v_i \geqq v_j, & \\ 2.\ \alpha_{i,h}=\alpha_{h,j}=0 & \text{for each } h \text{ between } i \text{ and } j, \\ 3.\ \alpha_{h,i}=\alpha_{h,j} & \text{for each } h < i, \\ 4.\ \alpha_{i,h}=\alpha_{j,h} & \text{for each } h > j, \end{cases}$$

then

(1.4;38)
$$u_i=u_j.$$

Proof:
From (1.4;37) follows

(1.4;39)
$$\begin{cases} 1.\ v_i \geqq v_j, & \\ 2.\ \alpha_{i,h}=\alpha_{j,h} & \text{for each } h > i \text{ with } h\neq j, \\ 3.\ \alpha_{h,i}=\alpha_{h,j} & \text{for each } h < j \text{ with } h\neq i. \end{cases}$$

Now let $\vec{\eta}$ be a point in D with $\eta_i < \eta_j$ then lemma 1.4;2 entails that a point $\vec{\eta}'\epsilon D$ exists with

(1.4;40)
$$\begin{cases} 1.\ \eta_i'=\eta_j', \\ 2.\ H(\vec{\eta}') > H(\vec{\eta}), \end{cases}$$

i.e. H attains its maximum in D for $\eta_i=\eta_j$ and (1.4;38) then follows from the uniqueness of the maximum.

Theorem 1.4;2 is closely related to theorem 1.3;2. Taking for R_s the restriction $\eta_i \leqq \eta_j$ (i.e. $i_s=i, j_s=j$) it follows from (1.4;40) that H attains its maximum in D' for $\eta_i \geqq \eta_j$, thus $u_i' \geqq u_j'$. Theorem 1.3;2 then entails that $u_i=u_j$.
However, the application of theorem 1.4;2 is simpler than the application of theorem 1.3;2; for theorem 1.3;2 we need the point $\vec{u}'$, whereas for theorem 1.4;2 we only need v_i and v_j. But theorem 1.4;2 can only be applied if condition B is satisfied and if moreover (i,j) is a pair of values satisfying (1.4;37); theorem 1.3;2 can always be applied.

By means of theorem 1.4;2 the problem is reduced to the problem of maximizing

(1.4;41)
$$\sum_{h\epsilon E'} H_h(\eta_h)+H_{\{i,j\}}(\eta_i),$$

where

(1.4;42) $E' \stackrel{def}{=\!=} \{1,\ldots,i-1,i+1,\ldots,j-1,j+1,\ldots,k\}$,
under s-1 (or less) restrictions.

Theorem 1.4;3: If $\alpha_{i,j}=1$ *for each pair of values* $(i,j)\in E$ *with* $i < j$ *then*

(1.4;43) $$u_i=u_{i+1} \quad \text{for each } i\in E \text{ with } v_i \geqq v_{i+1}.$$

Proof:

If $\alpha_{i,j}=1$ for each pair of values $(i,j)\in E$ with $i < j$ then each pair of values (i,j) with $v_i \geqq v_j$ and $j=i+1$ satisfies (1.4;37). Thus the theorem follows from theorem 1.4;2.

By means of theorem 1.4;3 a complete ordering with s essential restrictions is reduced to a complete ordering with s-1 essential restrictions. Thus the case of a complete ordering may be solved by means of the theorems 1.4;3 and 1.3;4.

Theorem 1.4;4: If (i,j) *is a pair of values satisfying*

(1.4;44) $$\begin{cases} 1. & v_i \geqq v_j, \\ 2. & \alpha_{i,j}=0, \\ 3. & \alpha_{i,h} \leqq \alpha_{j,h} \quad \text{for each } h > i \text{ with } h\neq j, \\ 4. & \alpha_{h,i} \geqq \alpha_{h,j} \quad \text{for each } h < j \text{ with } h\neq i, \end{cases}$$

then

(1.4;45) $$u_i \geqq u_j.$$

Proof:

If $\vec{\eta}$ is a point in D with $\eta_i < \eta_j$ then it follows from lemma 1.4;2 and (1.4;44) that a point $\vec{\eta}'\in D$ exists with

(1.4;46) $$\begin{cases} 1. & \eta_i'=\eta_j', \\ 2. & H(\vec{\eta}') > H(\vec{\eta}). \end{cases}$$

Now D also contains points $\vec{\eta}$ with $\eta_i > \eta_j$, $\alpha_{i,j}$ being zero. Therefore H attains its maximum in D for $\eta_i \geqq \eta_j$; (1.4;45) then follows from the uniqueness of this maximum.

Theorem 1.4;4 is related to theorem 1.3;3. If D' is the subset of D where $\eta_i \leqq \eta_j$ then it follows from (1.4;46) that H attains its maximum in D' for $\eta_i=\eta_j$; thus $u_i'=u_j'$. Theorem 1.3;3 then entails $u_i \geqq u_j$.

However, the application of theorem 1.3;3 is more complicated, because the point $\vec{u}'$ must be known. For theorem 1.4;4 we only need v_i and v_j but this theorem cannot always be applied.

1.5 *An explicit formula*

Let, for each $i \in E$,

$$(1.5;1) \qquad \begin{cases} 1. & S_i \stackrel{def}{=\!=} i \cup \text{Ens}\{j \mid \alpha_{j,i}=1\}, \\ 2. & T_i \stackrel{def}{=\!=} i \cup \text{Ens}\{j \mid \alpha_{i,j}=1\} \end{cases}$$

and let further, for a subset M of E,

$$(1.5;2) \qquad \begin{cases} 1. & S_M \stackrel{def}{=\!=} \bigcup_{i \in M} S_i, \\ 2. & T_M \stackrel{def}{=\!=} \bigcup_{i \in M} T_i. \end{cases}$$

Further the complement of a subset M of E will be denoted by $\bar{M}$, i.e.

$$(1.5;3) \qquad \begin{cases} M \cap \bar{M} = o, \\ M \cup \bar{M} = E. \end{cases}$$

In H.D.BRUNK [2] S_M is called a lower layer and T_M an upper layer. From the definition of S_M and T_M it follows at once that the complement $\bar{S}_M$ of a lower layer S_M is an upper layer with respect to another M and vice versa.

In the sequel of this section the following theorem will be proved.

Theorem 1.5;1

$$(1.5;4) \qquad u_i = \max_M . \min_{M'} \{ v_{T_M \cap S_{M'}} \mid i \in T_M \cap S_{M'} \} \qquad (i \in E),$$

where (cf. section 1.4) v_M is the value of ζ which maximizes $H_M(\zeta)$ in I_M.

For the proof of this theorem we need the following lemmas.

Lemma 1.5;1: If for any pair of values (i,j)

$$(1.5;5) \qquad \begin{cases} 1. & \alpha_{i,j}=0, \\ 2. & u_i \leqq u_j \end{cases}$$

and if $\vec{u}'$ *is the point where* H *attains its maximum in the subset* D' *of* D *where* $\eta_i \leqq \eta_j$, *then* $\vec{u} = \vec{u}'$.

Proof:

This lemma follows immediately from the fact that $\vec{u} \in D' \subset D$ and the uniqueness of the maximum.

Lemma 1.5;2: If for any value of λ

$$(1.5;6) \qquad u_{i_\lambda} < u_{j_\lambda},$$

if D' *is the set obtained from* D *by omitting the restriction* R_λ *and if* $\vec{u}'$ *is the point where* H *attains its maximum in* D' *then* $\vec{u} = \vec{u}'$.

Proof:

Theorem 1.3;2 entails that $u_{i_\lambda}=u_{j_\lambda}$ if and only if $u'_{i_\lambda} \geqq u'_{j_\lambda}$. From (1.5;6) then follows $u'_{i_\lambda} < u'_{j_\lambda}$, thus (cf. theorem 1.3;2) $\vec{u}=\vec{u}'$.

Lemma 1.5;3: If $\alpha_{i,j}=1$ *for each pair of values* (i,j) *with* $i < j$, *if a value* $i \geqq 0$ *and a value* $h \geqq 2$ *exist with*

(1.5;7) $$u_{i+1}=\dots=u_{i+h},$$

if $l_1,\dots,l_h$ *is a permutation of the numbers* $i+1,\dots,i+h$ *with*

(1.5;8) $$v_{l_1} \geqq \dots \geqq v_{l_h}$$

and if $\vec{u}'$ *is the point where* H *attains its maximum under the restrictions*

(1.5;9) $$\begin{cases}\eta_1 \leqq \dots \leqq \eta_i \leqq \eta_{l_1} \leqq \dots \leqq \eta_{l_h} \leqq \eta_{i+h+1} \leqq \dots \leqq \eta_k, \\ \eta_i \in I_i\ (i \in E),\end{cases}$$

then $\vec{u}=\vec{u}'$.

Proof:

If D′ is the subset of D where $\eta_{i+1}=\dots=\eta_{i+h}$ then (1.5;7) entails that $\vec{u}\in D'$. Further theorem 1.4;3 entails that $\vec{u}'\in D'$. The lemma then follows from the uniqueness of the maximum.

Lemma 1.5;4: If

(1.5;10) $$u_1=\dots=u_k$$

then

(1.5;11) $$v_{S_M} \geqq v_{\bar{S}_M} \quad \text{for each } M\subset E.$$

Proof:

We first consider the case that $\alpha_{i,j}=1$ for each pair of values (i,j) with $i < j$. Then, if M is a subset of E and if

(1.5;12) $$r \stackrel{\text{def}}{=} \max_{i\in M} i,$$

S_M consists of the numbers $1,\dots,r$ and thus $\bar{S}_M$ consists of the numbers $r+1,\dots,k$. Now let $l_1,\dots,l_r$ be a permutation of the numbers $1,\dots,r$ with $v_{l_1} \geqq \dots \geqq v_{l_r}$ and let $l_{r+1},\dots,l_k$ be a permutation of the numbers $r+1,\dots,k$ with $v_{l_{r+1}} \geqq \dots \geqq v_{l_k}$. Let further $\vec{u}'$ denote the point where H assumes its maximum in the set

(1.5;13) $$D' \begin{cases}\eta_{l_1} \leqq \dots \leqq \eta_{l_k}, \\ \eta_i \in I_i\ (i\in E),\end{cases}$$

then lemma 1.5;3 implies that

$$\vec{u}=\vec{u}'. \tag{1.5;14}$$

Now let D″ denote the set obtained from D′ by omitting the restriction $\eta_{1_r} \leqq \eta_{1_{r+1}}$ and let $\vec{u}''$ denote the point where H attains its maximum in D″. Then theorem 1.3;2 implies that $u'_{1_r}=u'_{1_{r+1}}$ if and only if $u''_{1_r} \geqq u''_{1_{r+1}}$. From (1.5;10) and (1.5;14) then follows $u''_{1_r} \geqq u''_{1_{r+1}}$ and the lemma then follows from (cf. theorem 1.4;3)

$$u''_{1_r}=v_{S_M},\, u''_{1_{r+1}}=v_{\bar{S}_M}. \tag{1.5;15}$$

Now consider the case that at least one pair of values (i,j) exists with $\alpha_{i,j}=0$. From lemma 1.5;1 and (1.5;10) then follows that for each pair of values (i,j) with $\alpha_{i,j}=0$ the restriction $\eta_i \leqq \eta_j$ or the restriction $\eta_i \geqq \eta_j$ may be added. Such a restriction is added for each pair of values $(i,j)\epsilon S_M$ with $\alpha_{i,j}=0$ and for each pair of values $(i,j)\epsilon \bar{S}_M$ with $\alpha_{i,j}=0$ in such a way that within S_M and within $\bar{S}_M$ a complete ordering is obtained. This new ordering is denoted by $\alpha'_{i,j}(\eta_i-\eta_j) \leqq 0(i,j\epsilon E)$. Then a value $h_1\epsilon S_M$ exists with $\alpha'_{j,h_1}=1$ for each $j\epsilon S_M$ and a value $h_2\epsilon\bar{S}_M$ with $\alpha'_{h_2,j}=1$ for each $j\epsilon\bar{S}_M$. Further it follows from the definition of S_M that $\alpha_{i,j} \geqq 0$ for each pair of values (i,j) with $i\epsilon S_M$ and $j\epsilon\bar{S}_M$, thus $\alpha'_{h_1,h_2} \geqq \alpha_{h_1,h_2} \geqq 0$. If $\alpha'_{h_1,h_2}=0$ we add (cf. lemma 1.5;1) the restriction $\eta_{h_1} \leqq \eta_{h_2}$. In this way we obtain a complete ordering and the lemma then follows from the first part of the proof.

Remark:1.5;1

The lemma holds analogously for any T_M and $\bar{T}_M$.

$$v_{T_M} \leqq v_{\bar{T}_M}.$$

Lemma 1.5;5: If M_1 *and* M_2 *are two subsets of* E *with*

$$\begin{cases} 1.\ M_1 \cap M_2 = o, \\ 2.\ v_{M_1} \leqq v_{M_2}, \end{cases} \tag{1.5;16}$$

then

$$v_{M_1} \leqq v_{M_1 \cup M_2} \leqq v_{M_2}. \tag{1.5;17}$$

Proof:

This lemma follows easily from condition A′.

Lemma 1.5;6: If

$$u_1=\dots=u_k(=u), \tag{1.5;18}$$

then

$$u=\max_M v_{T_M}=\min_M v_{S_M}. \tag{1.5;19}$$

Proof:

From lemma 1.5;4 and lemma 1.5;5 follows

$$v_{S_M} \geqq v_{S_M \cup \bar{S}_M} \geqq v_{\bar{S}_M} \tag{1.5;20}$$

and from (1.5;18) follows

$$v_{S_M \cup \bar{S}_M}=v_E=u. \tag{1.5;21}$$

Thus

$$v_{S_M} \geqq u \geqq v_{\bar{S}_M}. \tag{1.5;22}$$

From (1.5;21) and the first inequality of (1.5;22) then follows

$$u=\min_M v_{S_M}. \tag{1.5;23}$$

The other part of (1.5;19) follows analogously.

Now let $E_\nu(\nu=1,\dots,K)$ be subsets of E with

$$\begin{cases} 1. & \bigcup_{\nu=1}^{K} E_\nu=E, \\ 2. & u_i < u_j \quad \text{for each pair of values } (i,j) \text{ with } i\in E_\nu, j\in E_\mu (\nu < \mu;\ \nu,\mu=1,\dots,K), \\ 3. & u_i=u_j \quad \text{for each pair of values } (i,j)\in E_\nu(\nu=1,\dots,K) \end{cases} \tag{1.5;24}$$

and let, for $\nu=1,\dots,K$, D_ν denote the set

$$\begin{cases} \alpha_{i,j}(\eta_i-\eta_j) \leqq 0 \\ \eta_i \in I_i \end{cases} \qquad (i,j\in E_\nu). \tag{1.5;25}$$

Then we have

Lemma 1.5;7: The coordinates u_i *of* $\vec{u}$ *for* $i\in E_\nu$ *may also be found by maximizing* $\sum_{i\in E_\nu} H_i(\eta_i)$ *in* $D_\nu(\nu=1,\dots,K)$.

Proof:

Let $\vec{u}'$ denote the point where H attains its maximum in the set

$$B \overset{\text{def}}{=} \bigcap_{\nu=1}^{K} D_\nu. \tag{1.5;26}$$

Then lemma 1.5;2 implies that $\vec{u}=\vec{u}'$. Further theorem 1.4;1 implies that the coordinates u'_i of $\vec{u}'$ for $i \in E_\nu$ may be found by maximizing $\sum_{i \in E_\nu} H_i(\eta_i)$ in D_ν ($\nu=1,\ldots,K$).

Proof of theorem 1.5;1

We first consider the case that $u_1=\ldots=u_k$. Then, according to lemma 1.5;6 it is sufficient to prove that, for each $i \in E$,

$$(1.5;27) \qquad \max_M \min_{M'}\{v_{T_M \wedge S_{M'}} \mid i \in T_M \wedge S_{M'}\} = \min_M v_{S_M} = \max_M v_{T_M}.$$

The following relation always holds

$$(1.5;28) \quad \max_M \min_{M'}\{v_{T_M \wedge S_{M'}} \mid i \in T_M \wedge S_{M'}\} \geqq \min_{M'}\{v_{T_{M_o} \wedge S_{M'}} \mid i \in T_{M_o} \wedge S_{M'}\}$$

for any M_o.

Thus, taking $T_{M_o}=E$, we have, for each $i \in E$,

$$(1.5;29) \quad \max_M \min_{M'}\{v_{T_M \wedge S_{M'}} \mid i \in T_M \wedge S_{M'}\} \geqq \min_{M'}\{v_{S_{M'}} \mid i \in S_{M'}\} \geqq \min_M v_{S_M}.$$

In an analogous way it may be proved, that, for each $i \in E$,

$$(1.5;30) \qquad \max_M \min_{M'}\{v_{T_M \wedge S_{M'}} \mid i \in T_M \wedge S_{M'}\} \leqq \max_M v_{T_M}$$

and (1.5;27) then follows from (1.5;29), (1.5;30) and lemma 1.5;6.

Now consider the case that at least one pair of values $(i,j) \in E$ exists with $u_i \neq u_j$ and let E_ν ($\nu=1,\ldots,K$) be subsets of E satisfying (1.5;24). Then, if u'_ν denotes the value of u_i for $i \in E_\nu$ ($\nu=1,\ldots,K$), lemma 1.5;7 implies that

$$(1.5;31) \qquad u'_\nu = \min_M v_{S_M \wedge E_\nu} = \max_M v_{T_M \wedge E_\nu} \qquad (\nu=1,\ldots,K)$$

and from (1.5;31) follows

$$(1.5;32) \qquad v_{S_M \wedge E_\nu} \geqq u'_\nu \quad \text{for each } M \subset E \; (\nu=1,\ldots,K).$$

Thus if $E'_\nu \overset{\text{def}}{=} \bigcup_{\mu=\nu}^{K} E_\mu$ we have, for each $M \subset E$, (cf. lemma 1.5;5)

$$(1.5;33) \quad v_{S_M \wedge E'_\nu} = v_{\bigcup_{\mu=\nu}^{K}(E_\mu \wedge S_M)} \geqq \min_{\nu \leqq \mu \leqq K} v_{E_\mu \wedge S_M} \geqq \min_{\nu \leqq \mu \leqq K} u'_\mu = u'_\nu \; (\nu=1,\ldots,K).$$

Further, according to (1.5;28) with $T_{M_o}=E'_\nu$, for $i \in E_\nu$,

$$(1.5;34) \quad \max_M \min_{M'}\{v_{T_M \wedge S_{M'}} \mid i \in T_M \wedge S_{M'}\} \geqq \min_{M'}\{v_{S_{M'} \wedge E'_\nu} \mid i \in S_{M'} \wedge E'_\nu\} \geqq$$

$$\geqq u'_\nu = \min_M v_{S_M \wedge E_\nu} \; (\nu=1,\ldots,K).$$

In an analogous way it may be proved that, for each $i \in E_\nu$,

$$(1.5;35) \quad \max_M \min_{M'}\{v_{T_M \wedge S_{M'}} \mid i \in T_M \wedge S_{M'}\} \leqq \max_M v_{T_M \wedge E_\nu} \; (\nu=1,\ldots,K).$$

Theorem 1.5;1 then follows from (1.5;31), (1.5;34) and (1.5;35).

1.6 *The problem as a minimumproblem for a sum of squares*

Let, for each $i\epsilon E$, (cf. condition B) $H_i'(\eta)$ be a univalued function defined for each $\eta\epsilon I_i$, satisfying

$$(1.6;1) \qquad -\infty \leqq H_i'(\eta) < \infty \quad \text{for each } \eta\epsilon I_i.$$

Let further

$$(1.6;2) \qquad \begin{cases} H'(\vec{\eta}) \overset{def}{=} \sum_{i\epsilon E} H_i'(\eta_i) & (\vec{\eta}\epsilon G) \\ H_M'(\zeta) \overset{def}{=} \sum_{i\epsilon M} H_i'(\zeta) & (\zeta\epsilon I_M) \end{cases}$$

and let (cf. condition A'), for each $M\subset E$ with $I_M \neq o$, $H_M'(\zeta)$ be strictly unimodal in I_M. Then $H'(\vec{\eta})$ possesses a unique maximum in D. Let $\vec{u}'$ denote the point where $H'(\vec{\eta})$ attains its maximum in D and let v_M' denote the value of ζ maximizing $H_M'(\zeta)$ in I_M.
In this section conditions will be given for the coincidence of $\vec{u}$ and $\vec{u}'$.
We first prove the following lemma
Lemma 1.6;1: If

$$(1.6;3) \qquad u_1=\ldots=u_k(=u)$$

and

$$(1.6;4) \qquad \max_M v'_{T_M} = \min_M v'_{S_M} = u$$

then $\vec{u}=\vec{u}'$.
Proof:

From (1.5;29) and (1.5;30) it follows that

$$(1.6;5) \qquad \min_M v'_{S_M} \leqq u_i' \leqq \max_M v'_{T_M} \quad \text{for each } i\epsilon E.$$

From (1.6;4) and (1.6;5) then follows

$$(1.6;6) \qquad u_i'=u \quad \text{for each } i\epsilon E.$$

Now let $E_\nu(\nu=1,\ldots,K)$ be subsets of E satisfying (1.5;24) and let $D_\nu(\nu=1,\ldots,K)$ denote the set (1.5;25). Then
Theorem 1.6;1: If

$$(1.6;7) \qquad \max_M v'_{T_M\wedge E_\nu} = \min_M v'_{S_M\wedge E_\nu} = v_{E_\nu} \quad \text{for each } \nu=1,\ldots,K,$$

then $\vec{u}=\vec{u}'$
Proof:

Lemma 1.6;1 and (1.6;7) imply that, for each $\nu=1,\ldots,K$, the point where $\sum_{i\epsilon E_\nu} H_i(\eta_i)$ attains its maximum in D_ν coincides with the point where $\sum_{i\epsilon E_\nu} H_i'(\eta_i)$ attains its maximum in D_ν. Thus (cf.

theorem 1.4:1 and the proof of lemma 1.5;7) the point where H attains its maximum in B coincides with the point where H' attains its maximum in B. The theorem then follows the fact that D is a subset of B containing the point $\vec{u}$.

Remark 1.6;1

Condition (1.6;7) is e.g. satisfied if, for each $M \subset E_\nu$ and each $\nu=1,\dots,K$,

(1.6;8) $$v_M = v'_M$$

and (1.6;8) is e.g. satisfied if

(1.6;9) $$v_M = v'_M \quad \text{for each } M \subset E.$$

Now let J_i be a closed interval such that $I_i \subset J_i$ for each $i \in E$. Let further, for a subset M of E,

(1.6;10) $$J_M \overset{\text{def}}{=} \bigcap_{i \in M} J_i$$

and let the following conditions be satisfied

Condition B′: $H(\vec{\eta})$ *may for* $\vec{\eta} \in \prod_{i \in E} J_i$ *be written in the form*

(1.6;11) $$H(\vec{\eta}) = \sum_{i \in E} H_i(\eta_i),$$

where, for each $i \in E$, $H_i(\eta)$ *is a univalued function defined for each* $\eta \in J_i$ satisfying

(1.6;12) $$-\infty \leqq H_i(\eta) < \infty \quad \text{for each } \eta \in J_i$$

and

Condition A″: *For each* $M \subset E$ *with* $J_M \neq o$ *the function* $H_M(\zeta)$ *is strictly unimodal in* J_M.

Then, I_M being a closed subinterval of J_M, the conditions B and A′ are satisfied.

Now let, for each $i \in E$, $[c_i, d_i]$ denote the interval I_i and let, for $M \subset E$,

(1.6;13) $$\begin{cases} c_M \overset{\text{def}}{=} \max_{i \in M} c_i, \\ d_M \overset{\text{def}}{=} \min_{i \in M} d_i, \end{cases}$$

then, if $I_M \neq o$, I_M is the interval $[c_M, d_M]$. Further let w_M denote the value of ζ which maximizes $H_M(\zeta)$ in J_M, then

(1.6;14) $$v_M = \begin{cases} w_M & \text{if } c_M < w_M < d_M, \\ c_M & \text{if } w_M \leqq c_M, \\ d_M & \text{if } w_M \geqq d_M. \end{cases}$$

We now take for H' the function $-\sum_{i\in E} q_i(\eta_i - w_i)^2$, where $q_1, \dots, q_k$ are positive numbers; then $-\sum_{i\in M} q_i(\zeta - w_i)^2$ is strictly unimodal in the interval $(-\infty, \infty)$. Thus, J_M being a closed interval, $-\sum_{i\in M} q_i(\zeta - w_i)^2$ is strictly unimodal in J_M. Further if w'_M denotes the value of ζ which maximizes $-\sum_{i\in M} q_i(\zeta - w_i)^2$ in the interval $(-\infty, \infty)$, then

(1.6;15) $$w'_M = \sum_{i\in M} q_i w_i \Big/ \sum_{i\in M} q_i .$$

From theorem 1.6;1 then follows

Theorem 1.6;2: If $q_1, \dots, q_k$ *are chosen in such a way that*

(1.6;16) $$\begin{cases} 1.\ \ (1.6;7) \text{ is satisfied,} \\ 2.\ \ q_i > 0 \quad \text{for each } i\in E, \end{cases}$$

then the point where H *attains its maximum in* B *coincides with the point where*

(1.6;17) $$Q = Q(\vec{\eta}) \overset{\text{def}}{=\!=} \sum_{i\in E} q_i(\eta_i - w_i)^2$$

attains its minimum in B.

Remark 1.6;2

From (1.6;8) it follows (cf. also (1.6;14)) that (1.6;16) is satisfied e.g. if, for each $M \subset E_\nu$ and each $\nu = 1, \dots, K$,

(1.6;18) $$\begin{cases} w'_M = w_M \text{ if } c_M < w_M < d_M, \\ w'_M \leqq c_M \text{ if } w_M \leqq c_M, \\ w'_M \geqq d_M \text{ if } w_M \geqq d_M \end{cases}$$

and (1.6;18) is satisfied e.g. if, for each $M \subset E_\nu$ and each $\nu = 1, \dots, K$,

(1.6;19) $$w'_M = w_M .$$

Theorem 1.6;3: If $q_1, \dots, q_k$ *are chosen in such a way that (1.6;16) is satisfied and if* $\vec{u} \neq \vec{w}$ *then the ellipsoid*

(1.6;20) $$\sum_{i\in E} q_i \left(\eta_i - \frac{u_i + w_i}{2} \right)^2 = \sum_{i\in E} q_i \left(\frac{u_i - w_i}{2} \right)^2$$

touches the set B *(and the set* D*) on the outside in the point* $\vec{u}$.

Proof:

If $\vec{u} \neq \vec{w}$ then $\vec{w} \notin B$.

Further $\vec{u}$ is a borderpoint of B and if $0 < \beta < 1$ then $\beta\vec{w} + (1-\beta)\vec{u} \notin B$. This may be seen as follows. Lemma 1.4;1 entails that $H\{\beta\vec{w} + (1-\beta)\vec{u}\}$

is a strictly increasing function of β in the interval $0 \leqq \beta \leqq 1$. Thus if $0 < \beta_o < 1$ then $H\{\beta_o\vec{w}+(1-\beta_o)\vec{u}\} > H(\vec{u})$.
The fact that H attains its maximum in B at the point $\vec{u}$ then entails that $\beta_o\vec{w}+(1-\beta_o)\vec{u}\notin B$ and that $\vec{u}$ is a borderpoint of B.
Now let

$$(1.6;21)\qquad \begin{cases} \eta_i' \stackrel{def}{=} \sqrt{q_i}\,(\eta_i - \frac{u_i+w_i}{2}), \\ u_i'' \stackrel{def}{=} \sqrt{q_i}\,(u_i - \frac{u_i+w_i}{2}) = \sqrt{q_i}\,\frac{u_i-w_i}{2}, \\ w_i'' \stackrel{def}{=} \sqrt{q_i}\,(w_i - \frac{u_i+w_i}{2}) = -\sqrt{q_i}\,\frac{u_i-w_i}{2}, \end{cases}$$

then (1.6;20) reduces to

$$(1.6;22)\qquad \sum_{i\in E} \eta_i'^2 = \sum_{i\in E} u_i''^2 \;(= \sum_{i\in E} w_i''^2)$$

and B reduces to a convex set B'. Further $\vec{u}''$ is a borderpoint of B', $\vec{w}''\notin B'$ and if $0 < \beta < 1$ then $\beta\vec{w}''+(1-\beta)\vec{u}''\notin B'$.
From (1.6;21) follows

$$(1.6;23)\qquad \sum_{i\in E} q_i(\eta_i - w_i)^2 = \sum_{i\in E} (\eta_i' - w_i'')^2$$

and theorem 1.6;2 then entails that $\sum_{i\in E} (\eta_i'-w_i'')^2$ attains its minimum in B' at the point $\vec{u}''$, i.e. the sphere (1.6;22) touches B' on the outside in the point $\vec{u}''$; thus the ellipsoid (1.6;20) touches B on the outside in the point $\vec{u}$.

We now prove the following lemma
Lemma 1.6;2: Let C *be a convex set and* S *a point on its boundary. Let* K_S *be an ellipsoid touching* C *on the outside in* S *and let the diameter of* K_S *passing through* S *intersect* K_S *in a point* U($\neq$S). *Let further* Y *be a point inside* C *or on its boundary and* K_Y *an ellipsoid with diameter* YU, *with axes parallel to those of* K_S *and with the length of the axes proportional to those of* K_S. *Then* S *lies inside or on* K_Y.
Proof:
We apply a linear transformation such that K_S reduces to a sphere K_S'; then (cf. fig. 1.6;1) K_Y reduces to a sphere K_Y', C to a convex set C', S to a point S' on the boundary of C' and Y to a point Y' inside C' or on its boundary. Further the sphere K_S' touches C' in S'.
Let (cf. fig. 1.6;1) P denote the centre of the sphere K_Y' then it is sufficient to prove that $S'P \leqq U'P$. Now the fact that Y' lies inside C' or on its boundary implies that $PQ \leqq PY'=U'P$; thus $PQ \leqq S'P \leqq U'P$.

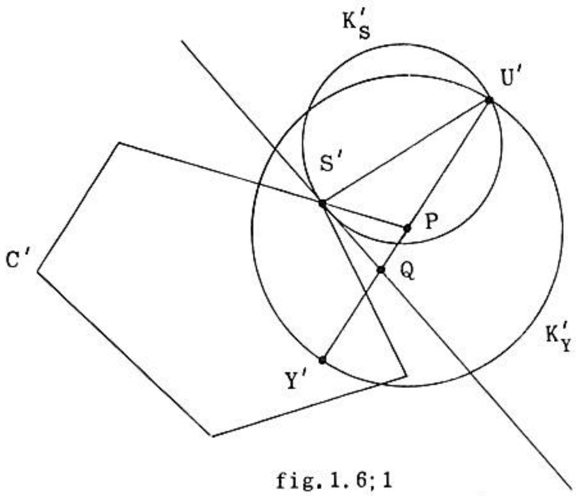


fig. 1.6;1

S' lies inside or on K'_Y

Theorem 1.6;4: If $q_1,\ldots,q_k$ *are chosen in such a way that (1.6;16) is satisfied then*

$$\sum_{i\in E} q_i(u_i-w_i)(u_i-\eta_i^\circ) \leqq 0 \quad \text{for each } \vec{\eta}^\circ\in B. \tag{1.6;24}$$

Proof:

If $\vec{u}=\vec{w}$ then (1.6;16) reduces to

$$q_i > 0 \quad \text{for each } i\in E. \tag{1.6;25}$$

Then the theorem is immediately clear. If $\vec{u}\neq\vec{w}$ then the ellipsoid (1.6;20) touches B on the outside in $\vec{u}$. Thus (cf. lemma 1.6;2) if $\vec{\eta}^\circ\in B$ then $\vec{u}$ lies inside or on the ellipsoid

$$\sum_{i\in E} q_i\left(\eta_i-\frac{\eta_i^\circ+w_i}{2}\right)^2 = \sum_{i\in E} q_i\left(\frac{\eta_i^\circ-w_i}{2}\right)^2, \tag{1.6;26}$$

i.e. $\vec{u}$ satisfies

$$\sum_{i\in E} q_i\left(u_i-\frac{\eta_i^\circ+w_i}{2}\right)^2 \leqq \sum_{i\in E} q_i\left(\frac{\eta_i^\circ-w_i}{2}\right)^2 \tag{1.6;27}$$

and (1.6;27) is identical with (1.6;24)

Further the foregoing implies that the following theorem holds.

Theorem 1.6;5: If $q_1,\ldots,q_k$ *are chosen in such a way that (1.6;16) is satisfied then exactly one point* $\vec{\eta}\in B$ *exists satisfying the inequalities*

(1.6;28) $$\sum_{i\in E} q_i(\eta_i-\eta_i^\circ)(\eta_i-w_i) \leqq 0,$$

where the point $\vec{\eta}^\circ$ *runs through the set* B.

In some cases (cf. chapter 2) numbers $q_1,\ldots,q_k$ independent of $\vec{u}$ may be found such that (1.6;16) is satisfied. In these cases the point $\vec{u}$ may also be found by minimizing Q in D or by solving the inequalities (1.6;28) for the point $\vec{\eta}$, where $\vec{\eta}^\circ$ runs through the set D.

CHAPTER 2

THE MAXIMUM LIKELIHOOD ESTIMATES OF ORDERED PARAMETERS OF PROBABILITY DISTRIBUTIONS

2.1 *Introduction*

Consider k independent random variables $\underline{x}_1,\dots,\underline{x}_k$ and n_i independent observations $x_{i,1},\dots,x_{i,n_i}$ of $\underline{x}_i$ ($i\in E\overset{def}{=}\{1,\dots,k\}$). Assume that the distribution of $\underline{x}_i$ contains one unknown parameter θ_i ($i\in E$) and that its distribution function is

(2.1;1) $$F_i(x|\theta_i)\overset{def}{=}P[\underline{x}_i \leqq x|\theta_i] \quad (i\in E).$$

Two types of restrictions are imposed on the parameters $\theta_1,\dots,\theta_k$. Let I_i' be a subset of the set of all values of τ for which $F_i(x|\tau)$ is a distribution function, let $\varphi_i(\tau)$ be a given function defined for each $\tau\in I_i'$ and let I_i be a given closed *interval* such that

(2.1;2) $$\begin{cases} 1.\ \varphi_i(\tau)\in I_i \text{ if and only if } \tau\in I_i', \\ 2.\ \text{if } \tau\in I_i' \text{ and } \tau'\in I_i' \text{ then } \varphi_i(\tau)\neq\varphi_i(\tau') \\ \qquad \text{if and only if } \tau\neq\tau'. \end{cases}$$

Let further, for $\eta\in I_i$, $\Phi_i(\eta)$ be the inverse of the function $\varphi_i(\tau)$ ($\tau\in I_i'$), then $F_i(x|\Phi_i(\eta))$ is a distribution function for each $\eta\in I_i$ ($i\in E$), but I_i is *not* necessarily the set of *all* values of η for which $F_i(x|\Phi_i(\eta))$ is a distribution function.
Let further $\alpha_{i,j}$ ($i,j\in E$) be given numbers satisfying (1.1;3), then the restrictions imposed on $\theta_1,\dots,\theta_k$ are

(2.1;3) $$\begin{cases} \alpha_{i,j}\{\varphi_i(\theta_i)-\varphi_j(\theta_j)\} \leqq 0 \\ \varphi_i(\theta_i)\in I_i \end{cases} \quad (i,j\in E)$$

and *the problem to be considered in this chapter is the maximum likelihood estimation of the parameters* $\theta_1,\dots,\theta_k$ *under the restrictions (2.1;3).*
The parameter θ_i is e.g. a probability or the mean of a normal distribution or the parameter of an exponential distribution. Further in most practical cases I_i' will be an interval and $\varphi_i(\tau)$ will be a continuous function of τ for each $i\in E$.

The special case that $\varphi_i(\tau)\equiv\tau$ for each $i\in E$ has been treated by the present author in [8], [9] and [10]. Further H.D.BRUNK [2] considers the case that, for each $i \in E, \varphi_i(\tau) \equiv \tau$ and moreover I_i is the set of *all* values of η for which $F_i(x|\eta)$ is a distribution function.

2.2 *The maximum likelihood estimates of* $\theta_1, \ldots, \theta_k$

In the following it will be supposed that the parameters $\theta_1, \ldots, \theta_k$ are numbered in such a way that $\alpha_{i,j} \geqq 0$ for each pair of values $(i,j) \in E$ with $i < j$. No other restrictions than (2.1;3) are imposed on $\theta_1, \ldots, \theta_k$. Consequently all points $\vec{\tau}$ satisfying

$$(2.2;1) \qquad \begin{cases} \alpha_{i,j}\{\varphi_i(\tau_i)-\varphi_j(\tau_j)\} \leqq 0 \\ \varphi_i(\tau_i) \in I_i \end{cases} \qquad (i,j \in E)$$

belong to the parameterspace.

Now let

$$(2.2;2) \qquad f_i(x|\theta_i) \stackrel{\text{def}}{=} \begin{cases} \text{the density function of } \underline{x}_i \text{ if } \underline{x}_i \text{ possesses a continuous probability distribution,} \\ P[\underline{x}_i = x|\theta_i] \text{ if } \underline{x}_i \text{ possesses a discrete probability distribution,} \end{cases}$$

and

$$(2.2;3) \qquad \begin{cases} L_i = L_i(\tau) \stackrel{\text{def}}{=} \sum_{\gamma=1}^{n_i} \ln f_i(x_{i,\gamma}|\tau) & (\tau \in I_i'; i \in E), \\ L = L(\vec{\tau}) \stackrel{\text{def}}{=} \sum_{i \in E} L_i(\tau_i) & (\vec{\tau} \in \prod_{i \in E} I_i'). \end{cases}$$

Then the maximum likelihood estimates of $\theta_1, \ldots, \theta_k$ under the restrictions (2.1;3) are the values of $\tau_1, \ldots, \tau_k$ which maximize L in the set (2.2;1).

Now let

$$(2.2;4) \qquad \begin{cases} H_i = H_i(\eta) \stackrel{\text{def}}{=} L_i\{\Phi_i(\eta)\} & (\eta \in I_i; i \in E), \\ H = H(\vec{\eta}) \stackrel{\text{def}}{=} \sum_{i \in E} H_i(\eta_i) & (\vec{\eta} \in G). \end{cases}$$

Let further, for $I_M \neq o$,

$$(2.2;5) \qquad H_M(\zeta) \stackrel{\text{def}}{=} \sum_{i \in M} H_i(\zeta) \qquad (\zeta \in I_M)$$

and let condition A′ and B (cf. p. 21, 22) be satisfied.

The value of ζ which maximizes $H_M(\zeta)$ in I_M will be denoted by v_M. Then $v_1, \ldots, v_k$ are the maximum likelihood estimates of $\varphi_1(\theta_1), \ldots, \varphi_k(\theta_k)$ under the restrictions $\varphi_i(\theta_i) \in I_i (i \in E)$.
From theorem 1.3;1 it follows that H possesses a unique maximum in the set D and this maximum may be found by means of the procedure described in chapter 1. Let this maximum be attained at the point $\vec{u}$, then $u_1, \ldots, u_k$ are the maximum likelihood estimates of

$\varphi_1(\theta_1),\ldots,\varphi_k(\theta_k)$ under the restrictions (2.1;3). It then follows from (2.1;2) that L possesses a unique maximum in the set (2.2;1), i.e. the maximum likelihood estimates of $\theta_1,\ldots,\theta_k$ under the restrictions (2.1;3) exist. These estimates will be denoted by $t_1,\ldots,t_k$ and we have $t_i=\Phi_i(u_i)(i\epsilon E)$.

In the following sections some examples will be given. In all examples we take

(2.2;6) $$\varphi_i(\tau)=\alpha_i+\beta_i\tau \qquad (i\epsilon E),$$

where $\beta_i\neq 0$ and α_i are given numbers ($i\epsilon E$). Further we take for J_i (cf. section 1.6) the set of *all* values of η for which $F_i(x|\Phi_i(\eta))$ is a distribution function; then in all examples J_i is an interval for each $i\epsilon E$. Finally we only consider examples where the conditions A″ and B′ (cf. p. 34) are satisfied. Then the results of section 1.6 may be applied.

The value of ζ which maximizes $H_M(\zeta)$ in J_M will be denoted by w_M. Then $w_1,\ldots,w_k$ are the 'ordinary' maximum likelihood estimates of $\varphi_1(\theta_1),\ldots,\varphi_k(\theta_k)$, i.e. the maximum likelihood estimates of $\varphi_1(\theta_1),\ldots,\varphi_k(\theta_k)$ without any restrictions.

2.3 *The binomial case*

Let, for each $i\epsilon E$,

(2.3;1) $$\begin{cases} P[\underline{x}_i=1]=\theta_i, P[\underline{x}_i=0]=1-\theta_i & (0<\theta_i<1), \\ a_i \overset{def}{=\!=} \sum_{\gamma=1}^{n_i} x_{i,\gamma}, b_i \overset{def}{=\!=} n_i-a_i \end{cases}$$

An example of ordered probabilities may be found in bio-assay. E.g. let, for each $i\epsilon E$, n_i animals be injected with a certain drug at a dose-level l_i, such that $l_1<\ldots<l_k$. Further let the response of the animals consist of a plus-or-minus-response and let, for each $i\epsilon E$, θ_i denote the probability of a plus-response at dose level l_i.

Now let the probability of a plus-response be a non-decreasing function of the dose; then we have $\theta_1\leqq\ldots\leqq\theta_k$, i.e. the probabilities $\theta_1,\ldots,\theta_k$ are completely ordered.

An incomplete ordering of $\theta_1,\ldots,\theta_k$ may e.g. be obtained if two different drugs are used each at one or more dose-levels. E.g. consider three samples of animals, where the animals of sample 1 and sample 3 are injected with a drug D_1 at dose-levels l_1 and l_2 respectively. Then we have $\theta_1\leqq\theta_3$. The animals of sample 2 are injected with a drug D_2 at dose-level l_2. Now let it be known that, for each dose-level, the probability of a plus-response for drug D_2 is not larger than the probability of a plus-response for drug D_1. Then we have: $\theta_2 \leqq \theta_3$; thus in this case θ_1, θ_2 and θ_3 are partially ordered with $\alpha_{1,3}=1$, $\alpha_{2,3}=1$, $\alpha_{1,2}=0$.

In this case we have (cf. (2.2;2))

$$(2.3;2) \quad f_i(x|\theta_i)= \begin{cases} \theta_i^x(1-\theta_i)^{1-x} & x=0 \text{ and } x=1 \\ 0 & x\neq 0 \text{ and } \neq 1 \end{cases} \quad (i\epsilon E),$$

thus (cf. (2.2;3))

$$(2.3;3) \quad L_i(\tau)=a_i \ln \tau+b_i \ln(1-\tau) \quad (0 \leqq \tau \leqq 1; i\epsilon E).$$

From (2.2;6) then follows

$$(2.3;4) \quad H_i(\eta)=L_i\{\Phi_i(\eta)\}=a_i \ln\frac{\eta-\alpha_i}{\beta_i}+b_i \ln(1-\frac{\eta-\alpha_i}{\beta_i}) \; (\eta\epsilon J_i; i\epsilon E),$$

where J_i is the set of all values of η for which $F_i(x|\frac{\eta-\alpha_i}{\beta_i})$ is a distribution function; thus for all $i\epsilon E$

$$(2.3;5) \quad J_i=[\min(\alpha_i,\alpha_i+\beta_i), \max(\alpha_i,\alpha_i+\beta_i)].$$

Further

$$(2.3;6) \quad H_M(\zeta)=\sum_{i\epsilon M}\{a_i \ln\frac{\zeta-\alpha_i}{\beta_i}+b_i \ln(1-\frac{\zeta-\alpha_i}{\beta_i})\} \quad (\zeta\epsilon J_M),$$

thus, for ζ in the open interval J_M

$$(2.3;7) \quad \begin{cases} 1. \; \dfrac{dH_M(\zeta)}{d\zeta}=\sum\limits_{i\epsilon M}\dfrac{a_i\beta_i-n_i(\zeta-\alpha_i)}{(\zeta-\alpha_i)(\alpha_i+\beta_i-\zeta)}, \\ 2. \; \dfrac{d^2H_M(\zeta)}{d\zeta^2}=\sum\limits_{i\epsilon M}\dfrac{-n_i(\zeta-\alpha_i)^2+2a_i\beta_i(\zeta-\alpha_i)-a_i\beta_i^2}{(\zeta-\alpha_i)^2(\alpha_i+\beta_i-\zeta)^2}. \end{cases}$$

Further, for each $i\epsilon E$,

$$(2.3;8) \quad -n_i(\zeta-\alpha_i)^2+2a_i\beta_i(\zeta-\alpha_i)-a_i\beta_i^2 \begin{cases} < 0 \text{ for each } \zeta\epsilon J_M & \text{if } 0 < a_i < n_i, \\ =-n_i(\zeta-\alpha_i)^2 & \text{if } a_i=0, \\ =-n_i(\alpha_i+\beta_i-\zeta)^2 & \text{if } a_i=n_i, \end{cases}$$

thus, for each ζ in the open interval J_M, $\frac{d^2H_M(\zeta)}{d\zeta^2} < 0$ and this implies that at most one value of ζ in the open interval J_M exists with $\frac{dH_M(\zeta)}{d\zeta}=0$. Consequently $H_M(\zeta)$ is strictly unimodal in J_M. Further (cf. (2.3;4)) $H_i(\eta) \leqq 0$ for each $\eta\epsilon J_i$ and each $i\epsilon E$; thus the conditions A″ and B′ (cf. p. 34) are satisfied.

The function $\frac{dH_M(\zeta)}{d\zeta}$ is sketched in fig. 2.3;1 for the case that $M=\{1,2,3\}$, $\alpha_3+\beta_3 < \alpha_1 < \alpha_2 < \alpha_1+\beta_1 < \alpha_3 < \alpha_2+\beta_2$ and $0 < a_i < n_i$ for each i=1,2,3.

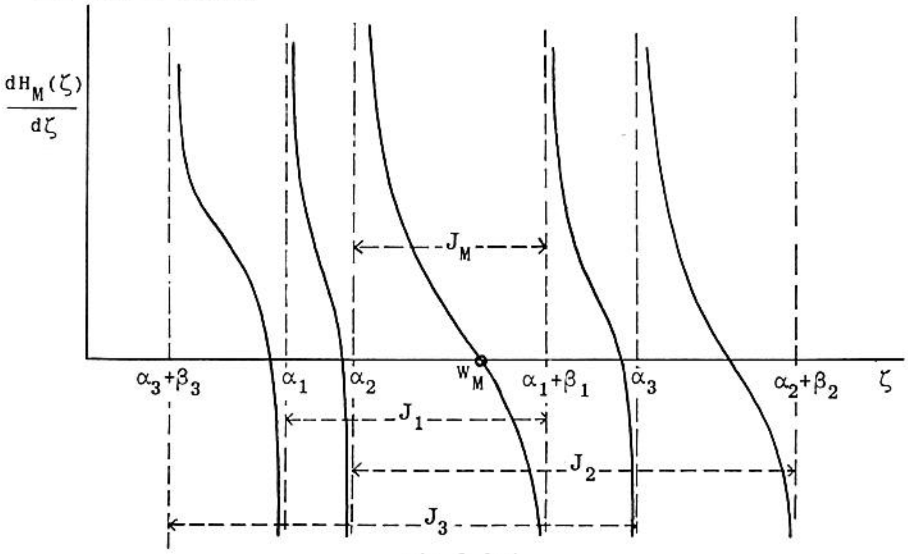


Fig.2.3;1

The function $\frac{dH_M(\zeta)}{d\zeta}$ for $M=\{1,2,3\}$ and $0 < a_i < n_i$ for each i=1,2,3.

In this special case w_M is the value of ζ satisfying

(2.3;9) $$\begin{cases} \frac{dH_M(\zeta)}{d\zeta}=0, \\ \zeta\in J_M. \end{cases}$$

If however $\frac{dH_M(\zeta)}{d\zeta} < 0$ for each $\zeta\in J_M$ (in the special case of fig. 2.3;1 this occurs e.g. if $a_1=a_2=b_3=0$) then $H_M(\zeta)$ is strictly decreasing in J_M, thus in this case we have (cf. (2.3;5))

(2.3;10) $$w_M=\max_{i\in M}\{\min(\alpha_i\alpha_i+\beta_i)\}.$$

In the same way, if $\frac{dH_M(\zeta)}{d\zeta} > 0$ for each ζ in J_M then

(2.3;11) $$w_M=\min_{i\in M}\{\max(\alpha_i,\alpha_i+\beta_i)\}.$$

Further (cf. (2.3;7)) the value of η which maximizes $H_i(\eta)$ in J_i (i.e. w_i) satisfies

(2.3;12) $$w_i=\alpha_i+\beta_i\frac{a_i}{n_i} \quad (i\in E).$$

The procedure will now be illustrated by means of the following example.

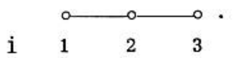
Example 2.3;1

Let $k=3$ and

(2.3;13) $$\alpha_{1,2}=\alpha_{2,3}=1.$$

This (complete) ordering may be denoted by

(2.3;14)



where a line connecting the points i and j (with i on the left hand side of j) denotes that $\alpha_{i,j}=1$.

Let further

(2.3;15)

i	1	2	3
a_i	3	15	6
n_i	25	33	25
α_i	0	0	1
β_i	3	1	-3,

then (cf. (2.3;5) and (2.3;12))

(2.3;16)

i	1	2	3
J_i	[0;3]	[0;1]	[-2;1]
w_i	0,36	0,45	0,28.[1]

Finally let

(2.3;17)

i	1	2	3
I_i	[0;3]	[0;0,4]	[-2;1]

then the problem is the maximum likelihood estimation of the parameters θ_1, θ_2, θ_3 under the restrictions (cf. (2.3;13) and (2.3;17))

(2.3;18) $$\begin{cases}\varphi_1(\theta_1) \leqq \varphi_2(\theta_2) \leqq \varphi_3(\theta_3)\\ \varphi_i(\theta_i)\in I_i\,(i=1,2,3),\end{cases}$$

where $\varphi_i(\tau)=\alpha_i+\beta_i\tau\,(i=1,2,3)$.

First we determine v_1, v_2 and v_3, i.e. the maximum likelihood estimates of $\varphi_1(\theta_1),\varphi_2(\theta_2)$ and $\varphi_3(\theta_3)$ under the restrictions $\varphi_i(\theta_i)\in I_i\,(i=1,2,3)$. According to (1.6;14) we have

(2.3;19) $$v_i=\begin{cases}w_i \text{ if } c_i < w_i < d_i,\\ c_i \text{ if } w_i \leqq c_i,\\ d_i \text{ if } w_i \geqq d_i,\end{cases}$$

---

1) The comma denotes the decimal sign, thus e.g. $0,36=\frac{9}{25}$.

where I_i is the interval $[c_i, d_i]$ (i=1,2,3). Thus (cf. (2.3;16) and (2.3;17))

$$\begin{cases} i & 1 & 2 & 3 \\ v_i & 0{,}36 & 0{,}4 & 0{,}28 \end{cases} \tag{2.3;20}$$

and these estimates do not satisfy the inequalities $v_1 \leqq v_2 \leqq v_3$, i.e. $\vec{v} \notin D$.
The estimates u_1, u_2 and u_3 of $\varphi_1(\theta_1), \varphi_2(\theta_2)$ and $\varphi_3(\theta_3)$ may now be found by means of theorem 1.4;3. We have $v_2 > v_3$ and theorem 1.4;3 then entails that H attains its maximum in D for $\eta_2 = \eta_3$. This reduces the problem to maximizing the function

$$H^* = H^*(\zeta_1, \zeta_2) \stackrel{\text{def}}{=} H_1(\zeta_1) + H_{\{2,3\}}(\zeta_2) \tag{2.3;21}$$

in the set (cf. (1.3;4))

$$D_{2,2} \quad \begin{cases} \zeta_1 \leqq \zeta_2, \\ \zeta_1 \in I_1, \zeta_2 \in I_{\{2,3\}}, \end{cases} \tag{2.3;22}$$

where $I_{\{2,3\}} = I_2 \cap I_3 = [0; 0{,}4]$.
Further (2.3;7.1) and (2.3;15) entail that $\frac{dH_{\{2,3\}}(\zeta)}{d\zeta}$ is positive for $\zeta = 0{,}4$; thus, $\frac{d^2 H_{\{2,3\}}(\zeta)}{d\zeta^2}$ being negative for each ζ with $0 < \zeta < 1$, we have

$$\frac{dH_{\{2,3\}}(\zeta)}{d\zeta} > 0 \quad \text{for each } \zeta \in I_{\{2,3\}}. \tag{2.3;23}$$

Consequently

$$v_{\{2,3\}} = 0{,}4, \tag{2.3;24}$$

i.e. H* attains its maximum in $G_2 \stackrel{\text{def}}{=} I_1 \times I_{\{2,3\}}$ at the point (0,36; 0,4). This is a point in $D_{2,2}$ thus (cf. theorem 1.3;4)

$$u_1 = 0{,}36; u_2 = u_3 = 0{,}4 \tag{2.3;25}$$

and from

$$t_i = \frac{u_i - \alpha_i}{\beta_i} \quad (i \in E) \tag{2.3;26}$$

then follows

$$t_1 = 0{,}12; t_2 = 0{,}4; t_3 = 0{,}2. \tag{2.3;27}$$

The estimates u_1, u_2 and u_3 of $\varphi_1(\theta_1), \varphi_2(\theta_2)$ and $\varphi_3(\theta_3)$ under the restrictions (2.3;18) may also be found by means of the formula (1.5;4). Thus e.g.

$$u_2 = \max_{M} \min_{M'} \{ v_{T_M \cap S_{M'}} \mid 2 \in T_M \cap S_{M'} \}. \tag{2.3;28}$$

Now we have

$$(2.3;29)\quad v_{T_M\cap S_{M'}}=\begin{cases} v_2 & \text{if } T_M=\{2,3\}, S_{M'}=\{1,2\},\\ v_{\{2,3\}} & \text{if } T_M=\{2,3\}, S_{M'}=\{1,2,3\},\\ v_{\{1,2\}} & \text{if } T_M=\{1,2,3\}, S_{M'}=\{1,2\},\\ v_{\{1,2,3\}} & \text{if } T_M=\{1,2,3\}, S_{M'}=\{1,2,3\},\end{cases}$$

thus

$$(2.3;30)\quad u_2=\max\{\min(v_2, v_{\{2,3\}}), \min(v_{\{1,2\}}, v_{\{1,2,3\}})\}.$$

Further (cf. (2.3;17), (2.3;20) and (2.3;24))

$$(2.3;31)\quad \begin{cases} 1. & I_{\{1,2\}}=I_{\{1,2,3\}}=[0;0,4],\\ 2. & v_2=v_{\{2,3\}}=0,4\end{cases}$$

and (2.3;31.1) entails

$$(2.3;32)\quad v_{\{1,2\}} \leqq 0,4; v_{\{1,2,3\}} \leqq 0,4.$$

Consequently

$$(2.3;33)\quad \begin{cases} \min(v_2, v_{\{2,3\}})=0,4,\\ \min(v_{\{1,2\}}, v_{\{1,2,3\}}) \leqq 0,4\end{cases}$$

and from (2.3;30) and (2.3;33) then follows $u_2=0,4$ (cf. (2.3;25)). In a similar way u_1 and u_3 may be found by means of formula (1.5;4).

We now consider the function Q (cf. section 1.6). In the special case of this section we have (cf. (2.3;12))

$$(2.3;34)\quad Q(\vec{\eta})=\sum_{i\in E} q_i(\eta_i-\alpha_i-\beta_i\frac{a_i}{n_i})^2$$

and if (cf. (1.6;15)) w'_M is the value of ζ which minimizes $Q_M(\zeta)\overset{\text{def}}{=}\sum_{i\in M} q_i(\zeta-\alpha_i-\beta_i\frac{a_i}{n_i})^2$ in the interval $(-\infty,\infty)$ then

$$(2.3;35)\quad w'_M=\frac{\sum_{i\in M} q_i(\alpha_i+\beta_i\frac{a_i}{n_i})^2}{\sum_{i\in M} q_i}.$$

Further if v'_M is the value of ζ which minimizes $Q_M(\zeta)$ in the interval I_M then (cf. (1.6;14))

$$(2.3;36)\quad v'_M=\begin{cases} w'_M & \text{if } c_M < w'_M < d_M,\\ c_M & \text{if } w'_M \leqq c_M,\\ d_M & \text{if } w'_M \geqq d_M,\end{cases}$$

Now let (cf. (1.5;24)) $E_\nu(\nu=1,\dots,K)$ be subsets of E with

$$(2.3;37)\begin{cases}1.\ \bigcup_{\nu=1}^{K} E_\nu=E, & \\ 2.\ u_i < u_j & \text{for each pair of values } (i,j) \text{ with } i\in E_\nu, j\in E_\mu (\nu<\mu; \nu,\mu=1,\dots,K), \\ 3.\ u_i=u_j & \text{for each pair of values } (i,j)\in E_\nu\ (\nu=1,\dots,K)\end{cases}$$

and let (cf. (1.5;25)), for $\nu=1,\dots,K, D_\nu$ denote the set

$$(2.3;38)\qquad \begin{cases}\alpha_{i,j}(\eta_i-\eta_j) \leqq 0 \\ \eta_i \in I_i\end{cases} \qquad (i,j\in E_\nu).$$

Finally let (cf. (1.5;26))

$$(2.3;39)\qquad B \overset{def}{=\!=} \bigcap_{\nu=1}^{K} D_\nu,$$

then, according to theorem 1.6;2, the point where H attains its maximum in B coincides with the point where Q attains its minimum in B if $q_1,\dots,q_k$ are chosen in such a way that

$$(2.3;40)\qquad \begin{cases}1.\ \max_M v'_{T_M \wedge E_\nu} = \min_{M} v'_{S_M\wedge E_\nu} = v_{E_\nu} & \text{for each } \nu=1,\dots,K, \\ 2.\ q_i > 0 \text{ for each } i\in E. & \end{cases}$$

According to (1.6;19) a sufficient condition for (2.3;40) is

$$(2.3;41)\qquad \begin{cases}1.\ w_M=w'_M & \text{for each } M\subset E_\nu (\nu=1,\dots,K), \\ 2.\ q_i > 0 & \text{for each } i\in E.\end{cases}$$

The question now arises if numbers $q_1,\dots,q_k$ exist satisfying (2.3;40). In the general case we are unable to answer this question. Therefore we consider, in the sequel of this section, the special case that $\varphi_i(\tau)\equiv\tau$ (i.e. $\alpha_i=0$ and $\beta_i=1$) for each $i\in E$. Then numbers $q_1,\dots,q_k$ (independent of $a_1,\dots,a_k$ and $I_1,\dots,I_k$) exist satisfying (2.3;40). Moreover considerable simplifications occur in the procedure by means of which the estimates $t_1,\dots,t_k$ may be found.

If $\alpha_i=0$ and $\beta_i=1$ for each $i\in E$ then $L_i\equiv H_i$, $u_i=t_i$ and J_i is the interval $[0;1]$ for each $i\in E$. Consequently the maximum likelihood estimates $t_1,\dots,t_k$ are the values of $\eta_1,\dots,\eta_k$ which maximize

$$(2.3;42)\qquad H=H(\vec{\eta})=\sum_{i\in E}\{a_i \ln \eta_i + b_i \ln(1-\eta_i)\}$$

in the set

$$(2.3;43)\qquad D\ \begin{cases}\alpha_{i,j}(\eta_i-\eta_j) \leqq 0 \\ \eta_i\in I_i\end{cases} \qquad (i,j\in E).$$

The special case that I_i is the interval $[0;1]$ for each $i \in E$ has been treated by the present author in [7], whereas MIRIAM AYER, H.D.BRUNK, G.M.EWING, W.T.REID and EDWARD SILVERMAN in [1] consider the case that $I_i=[0;1]$ for each $i \in E$ and that moreover $\alpha_{i,j}=1$ for each pair of values (i,j) with $i < j$.

Now let

(2.3;44)
$$\begin{cases} a_M \overset{\text{def}}{=} \sum_{i \in M} a_i, \; b_M \overset{\text{def}}{=} \sum_{i \in M} b_i, \\ n_M \overset{\text{def}}{=} a_M + b_M = \sum_{i \in M} n_i, \end{cases}$$

then (cf. (2.3;42))

(2.3;45)
$$H_M(\zeta)=a_M \ln \zeta + b_M \ln(1-\zeta).$$

Consequently $H_M(\zeta)$ attains its maximum in J_M for $\zeta=\frac{a_M}{n_M}$, i.e.

(2.3;46)
$$w_M=\frac{a_M}{n_M}.$$

Further if H attains its maximum in D for $\eta_i=\eta_j$ then the problem of maximizing H reduces to the problem of maximizing

(2.3;47)
$$\sum_{h \notin M} H_h(\eta_h)+H_M(\eta_i) \quad \text{with } M=\{i,j\},$$

where (cf. (2.3;45)) $H_M(\zeta)$ is the likelihood function for the pooled samples of $\underline{x}_i$ and $\underline{x}_j$.

We now consider an example with $\varphi_i(\tau)\equiv\tau$ for each $i \in E$, $k=4$ and an incomplete ordering of $\theta_1,\dots,\theta_4$.

Example 2.3;2

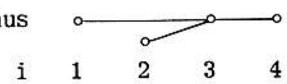
Let $k=4$,

(2.3;48)

i	1	2	3	4
a_i	13	7	15	2
n_i	20	10	30	5
$w_i=\frac{a_i}{n_i}$	0,65	0,7	0,5	0,4
I_i	[0;0,5]	[0;1]	[0;1]	[0,6;0,8]

and

(2.3;49)
$$\begin{cases} \alpha_{1,3}=\alpha_{2,3}=\alpha_{3,4}=1 \\ \alpha_{1,2}=0 \end{cases}, \text{ thus}$$



Then (cf. (1.1;3.4))

(2.3;50)
$$\alpha_{1,4}=\alpha_{2,4}=1.$$

Thus the problem is the maximum likelihood estimation of the parameters $\theta_1,\dots,\theta_4$ under the restrictions

$$(2.3;51) \qquad \begin{cases} \theta_1 \leqq \theta_3 \leqq \theta_4, \theta_2 \leqq \theta_3, \\ \theta_i \in I_i \, (i=1,\dots,4). \end{cases}$$

First we determine $v_1,\dots,v_4$, i.e. the maximum likelihood estimates of $\theta_1,\dots,\theta_4$ under the restrictions $\theta_i \in I_i \, (i=1,\dots,4)$.
From (1.6;14) and (2.3;48) follows

$$(2.3;52) \qquad \begin{cases} i & 1 & 2 & 3 & 4 \\ v_i & 0,5 & 0,7 & 0,5 & 0,6, \end{cases}$$

thus $\vec{v} \notin D$.
In order to find $u_1,\dots,u_4$, i.e. the estimates of $\theta_1,\dots,\theta_4$ under the restrictions (2.3;51), we apply theorem 1.4;4 with i=2, j=1; we have (cf. (1.4;44))

$$(2.3;53) \qquad \begin{cases} 1. \; v_1 < v_2, \\ 2. \; \alpha_{2,1}=0, \\ 3. \; \alpha_{1,3}=\alpha_{2,3}, \alpha_{1,4}=\alpha_{2,4}, \end{cases}$$

thus H attains its maximum in D for $\eta_1 \leqq \eta_2$, i.e. we may add the restriction $\eta_1 \leqq \eta_2$. The problem then reduces to maximizing H in the set

$$(2.3;54) \qquad D' \; \begin{cases} \eta_1 \leqq \dots \leqq \eta_4, \\ \eta_i \in I_i \, (i \in E) \end{cases}$$

and this problem may be solved by means of the theorems 1.4;3 and 1.3;4.
From (2.3;52) and theorem 1.4;3 it follows that H attains its maximum in D′ for $\eta_2=\eta_3$. Substituting $\eta_2=\eta_3$ into H (i.e. pooling the sample of $\underline{x}_2$ and $\underline{x}_3$) the problem reduces to the case of k-1=3 samples with

$$(2.3;55) \qquad \begin{cases} M & \{1\} & \{2,3\} & \{4\} \\ a_M & 13 & 22 & 2 \\ n_M & 20 & 40 & 5 \\ w_M=\frac{a_M}{n_M} & 0,65 & 0.55 & 0,4 \\ I_M & [0;0,5] & [0;1] & [0,6;0,8] \\ v_M & 0,5 & 0,55 & 0,6 \end{cases}$$

and from (2.3;55) and theorem 1.3;4 then follows

$$(2.3;56) \qquad t_1=0,5; t_2=t_3=0,55; t_4=0,6.$$

Further the function Q (cf. (2.3;34)) reduces to

(2.3;57) $$Q(\vec{\eta}) = \sum_{i\in E} q_i(\eta_i - \frac{a_i}{n_i})^2$$

with (cf. (2.3;35))

(2.3;58) $$w'_M = \frac{\sum_{i\in M} q_i \frac{a_i}{n_i}}{\sum_{i\in M} q_i}.$$

Thus if we take

(2.3;59) $$q_i = g_\nu n_i \qquad (i\in E_\nu;\ \nu=1,\ldots,K),$$

where $g_\nu(\nu=1,\ldots,K)$ are positive numbers then

(2.3;60) $$w'_M = \frac{\sum_{i\in M} a_i}{\sum_{i\in M} n_i} = \frac{a_M}{n_M} = w_M \quad \text{for each } M\subset E_\nu(\nu=1,\ldots,K)$$

and (2.3;41) then entails that (2.3;40) is satisfied. Consequently the point where H attains its maximum in B coincides with the point where

(2.3;61) $$\sum_{\nu=1}^{K} g_\nu \sum_{i\in E_\nu} n_i(\eta_i - \frac{a_i}{n_i})^2$$

attains its minimum in B. Further the inequality (1.6;24) reduces to

(2.3;62) $$\sum_{\nu=1}^{K} g_\nu \sum_{i\in E_\nu} n_i(t_i - \eta_i^\circ)(t_i - \frac{a_i}{n_i}) \leqq 0 \quad \text{for each point } \vec{\eta}^\circ\in B.$$

This is a generalization of the inequality (obtained by taking $g_\nu=1$ for each $\nu=1,\ldots,K$ and $\vec{\eta}^\circ\in D$)

(2.3;63) $$\sum_{i\in E} n_i(\eta_i^\circ - \frac{a_i}{n_i})^2 \geqq \sum_{i\in E} n_i\left\{(t_i - \eta_i^\circ)^2 + (t_i - \frac{a_i}{n_i})^2\right\} \text{ for each point } \vec{\eta}^\circ\in D,$$

which is mentioned in [1] (p. 644).

Further if $g_\nu=1$ for each $\nu=1,\ldots,K$ then (2.3;61) reduces to

(2.3;64) $$\sum_{i\in E} n_i(\eta_i - \frac{a_i}{n_i})^2,$$

i.e. the maximum likelihood estimates of $\theta_1,\ldots,\theta_k$ in D are identical with the least squares estimates in D.

Finally if $I_i=J_i$ for each $i\in E$ then $v_M=w_M$ for each $M\subset E$. Thus in this case formula (1.5;4) reduces to

(2.3;65) $$t_i = u_i = \max_M \min_{M'}\left\{ \sum_{j\in T_M\cap S_{M'}} a_j \Big/ \sum_{j\in T_M\cap S_{M'}} n_j \,\Big|\, i\in T_M\cap S_{M'}\right\} \qquad (i\in E)$$

and if $\alpha_{i,j}=1$ for each pair of values (i,j) with $i < j$ then (2.3;65) reduces to (cf. also [1], p. 644)

$$t_i = \max_{1 \leq r \leq i} \min_{i \leq s \leq k} \frac{a_r + \ldots + a_s}{n_r + \ldots + n_s} \quad (i \in E). \tag{2.3;66}$$

2.4 *A normal distribution with unknown mean*

Let, for each $i \in E$, $\underline{x}_i$ possess a normal distribution with mean θ_i and known variance $\sigma_i^2 > 0$.

A practical example of ordered means of normal distributions may be found in bio-assay (cf. the example of section 2.3), if quantitative measurements of the response are available. These measurements may e.g. consist of changes in weight of the animals. In such cases the mean response will be a non-decreasing function of the dose.
Thus if the response at doselevel l_i is normally distributed with mean θ_i ($i \in E$) an example of ordered means of normal distribution is obtained.

In this case we have

$$f_i(x|\theta_i) = \frac{1}{\sigma_i\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\theta_i)^2}{\sigma_i^2}} \quad (-\infty < x < \infty;\, i \in E), \tag{2.4;1}$$

thus

$$L_i(\tau) = -\tfrac{1}{2} n_i \ln 2\pi\sigma_i^2 - \tfrac{1}{2}\frac{\sum_{\gamma=1}^{n_i}(x_{i,\gamma}-\tau)^2}{\sigma_i^2} \quad (-\infty < \tau < \infty;\, i \in E). \tag{2.4;2}$$

Further J_i is the interval $(-\infty,\infty)$ $(i \in E)$ and if

$$x'_{i,\gamma} \stackrel{\text{def}}{=} \alpha_i + \beta_i x_{i,\gamma} \quad (\gamma = 1, \ldots, n_i;\, i \in E), \tag{2.4;3}$$

then

$$H_M(\zeta) = -\tfrac{1}{2}\sum_{i \in M} n_i \ln 2\pi\sigma_i^2 - \tfrac{1}{2}\sum_{i \in M}\frac{\sum_{\gamma=1}^{n_i}(x'_{i,\gamma}-\zeta)^2}{(\beta_i\sigma_i)^2} \quad (\zeta \in J_M) \tag{2.4;4}$$

and

$$\frac{dH_M(\zeta)}{d\zeta} = \sum_{i \in M}\frac{\sum_{\gamma=1}^{n_i} x'_{i,\gamma}}{(\beta_i\sigma_i)^2} - \zeta\sum_{i \in M}\frac{n_i}{(\beta_i\sigma_i)^2} \quad (\zeta \in J_M). \tag{2.4;5}$$

Thus $H_M(\zeta)$ is strictly unimodal in J_M. Further $H_i(\eta) < \infty$ for each $\eta \in J_i$ and each $i \in E$; thus the conditions A″ and B′ (cf. p. 34) are satisfied.

From (2.4;5) follows

(2.4;6)
$$\begin{cases} w_M = \sum_{i \in M} \frac{\sum_{\gamma=1}^{n_i} x'_{i,\gamma}}{(\beta_i \sigma_i)^2} \Big/ \sum_{i \in M} \frac{n_i}{(\beta_i \sigma_i)^2}, \\ w_i = \frac{1}{n_i} \sum_{\gamma=1}^{n_i} x'_{i,\gamma}. \end{cases}$$

Further let

(2.4;7)
$$\underline{x}'_i \overset{\text{def}}{=\!=} \alpha_i + \beta_i \underline{x}_i \qquad (i \in E),$$

then $x'_{i,\gamma}(\gamma=1,\dots,n_i)$ are n_i independent observations of $\underline{x}'_i (i \in E)$. Then, if (i,j) is a pair of values with $\beta_i \sigma_i = \beta_j \sigma_j$, we have, for $M = \{i,j\}$,

(2.4;8)
$$H_M(\zeta) = -\tfrac{1}{2} n_j \ln \frac{\beta_i^2}{\beta_j^2} - \tfrac{1}{2} n_M \ln 2\pi\sigma_i^2 + -\tfrac{1}{2} \frac{\sum_{\gamma=1}^{n_M} (x'_{M,\gamma} - \zeta)^2}{(\beta_i \sigma_i)^2},$$

where $n_M \overset{\text{def}}{=\!=} n_i + n_j$ and where $x'_{M,\gamma}(\gamma=1,\dots,n_M)$ denotes the pooled samples of $\underline{x}'_i$ and $\underline{x}'_j$.
Thus if (i,j) is a pair of values with $\beta_i \sigma_i = \beta_j \sigma_j$ and if H attains its maximum in D for $\eta_i = \eta_j$ then the problem reduces to the case of k-1 samples, i.e. the sample of $\underline{x}'_1, \dots, \underline{x}'_{i-1}, \underline{x}'_{i+1}, \dots, \underline{x}'_{j-1}, \underline{x}'_{j+1}, \dots, \underline{x}'_k$ and the pooled samples of $\underline{x}'_i$ and $\underline{x}'_j$.

Remark 2.4;1

From (2.4;6) it follows that the procedure described in chapter 1 may also be applied if the σ_i^2 are unknown and σ_i^2/σ_j^2 is known for each pair of values (i,j). Then if

(2.4;9)
$$K_i \overset{\text{def}}{=\!=} \frac{\sigma_i^2}{\sigma_1^2} \qquad (i \in E),$$

we have

(2.4;10)
$$w_M = \sum_{i \in M} \frac{\sum_{\gamma=1}^{n_i} x'_{i,\gamma}}{\beta_i^2 \, K_i} \Big/ \sum_{i \in M} \frac{n_i}{\beta_i^2 \, K_i}$$

and the maximum likelihood estimate s_i^2 of σ_i^2 is

$$s_i^2=\frac{K_i}{n}\sum_{j\in E}\frac{\sum_{\gamma=1}^{n_j}(x'_{j,\gamma}-u_j)^2}{\beta_j^2\,K_j}=\frac{K_i}{n}\sum_{j\in E}\frac{\sum_{\gamma=1}^{n_j}(x_{j,\gamma}-t_j)^2}{K_j}\quad (i\in E), \tag{2.4;11}$$

where

$$n\overset{def}{=\!=}\sum_{i\in E} n_i. \tag{2.4;12}$$

The procedure will now be illustrated by means of the following example.

Example 2.4;1

Let k=4,

$$\begin{cases}\alpha_{1,2}=\alpha_{1,4}=\alpha_{3,4}=1\\ \alpha_{1,3}=\alpha_{2,3}=\alpha_{2,4}=0\end{cases}, \text{ thus} \tag{2.4;13}$$

i 1 2 3 4

and

(2.4;14)

i	1	2	3	4
$x_{i,\gamma}$	2,63 -0,75 0,14 1,53 1,69 1,06	-1,27 0,64 0,81	0,58 2,01 0,22 -1,22 -0,10	0,67 -2,41 0,99 -1,07
$\frac{1}{n_i}\sum_{\gamma=1}^{n_i} x_{i,\gamma}$	1,14	0,06	2,98	-0,79
α_i	4	0	2	-1
β_i	3	3	-2	5
I_i	$(-\infty;5]$	$(-\infty;\infty)$	$[0;\infty)$	$(-\infty;\infty)$
K_i	1	1	5	3

From (2.4;6) follows

$$w_i=\alpha_i+\beta_i\frac{1}{n_i}\sum_{\gamma=1}^{n_i} x_{i,\gamma}\quad (i\in E), \tag{2.4;15}$$

thus

$$\begin{cases} i & 1 & 2 & 3 & 4\\ w_i & 7{,}39 & 0{,}18 & 1{,}40 & -4{,}95.\end{cases} \tag{2.4;16}$$

From $I_1, \dots, I_4$ and (2.4;16) then follows

(2.4;17) $$\begin{cases} i & 1 \quad 2 \quad 3 \quad 4 \\ v_i & 5 \quad 0{,}18 \quad 1{,}40 \quad -4{,}95. \end{cases}$$

In this case theorem 1.4;2 cannot be applied; for the pair i=1, j=2 e.g. we have

(2.4;18) $$\alpha_{1,2}=1, v_1 > v_2,$$

but $\alpha_{1,4} \neq \alpha_{2,4}$. In the same way it may be seen that the pairs i=1, j=4 and i=3, j=4 do not satisfy (1.4;37).

Also theorem 1.4;4 cannot be applied; for the pair i=3, j=2 e.g. we have

(2.4;19) $$v_2 < v_3, \alpha_{3,2}=0,$$

but $\alpha_{1,3} < \alpha_{1,2}$. In the same way it may be seen that the pairs i=1, j=3 and i=2, j=4 do not satisfy (1.4;44).
Thus in this case theorem 1.3;2 or theorem 1.3;3 must be applied. We use theorem 1.3;2 with $i_s=1, j_s=4$, i.e. we omit the restriction $\eta_1 \leqq \eta_4$. Then $\vec{u}'$ is the point where H attains its maximum in the set

(2.4;20) $$D' \quad \begin{cases} \eta_1 \leqq \eta_2, \eta_3 \leqq \eta_4, \\ \eta_i \in I_i \quad (i \in E) \end{cases}$$

and according to theorem 1.4;1 the maximum of H in D' may be found by maximizing $\sum_{i=1}^{2} H_i(\eta_i)$ in the set

(2.4;21) $$\begin{cases} \eta_1 \leqq \eta_2, \\ \eta_i \in I_i (i=1,2) \end{cases}$$

and maximizing $\sum_{i=3}^{4} H_i(\eta_i)$ in the set

(2.4;22) $$\begin{cases} \eta_3 \leqq \eta_4, \\ \eta_i \in I_i (i=3,4). \end{cases}$$

From theorem 1.4;3 and (2.4;17) it follows that $\sum_{i=1}^{2} H_i(\eta_i)$ attains its maximum in the set (2.4;21) for $\eta_1=\eta_2$, i.e.

(2.4;23) $$u_1'=u_2'=v_{\{1,2\}}.$$

Now we have (cf. (2.4;10))

(2.4;24) $$w_{\{1,2\}}=\sum_{i=1}^{2} \frac{\sum_{\gamma=1}^{n_i} x'_{i,\gamma}}{\beta_i^2 \kappa_i} \Bigg/ \sum_{i=1}^{2} \frac{n_i}{\beta_i^2 \kappa_i} = 0{,}846.$$

Further $I_{\{1,2\}}$ is the interval $(-\infty;5]$, thus

(2.4;25) $$u_1'=u_2'=v_{\{1,2\}}=0,846.$$

Analogously we have

(2.4;26) $$u_3'=u_4'=v_{\{3,4\}}.$$

From (2.4;10) follows $w_{\{3,4\}}=0,0138$; thus $I_{\{3,4\}}$ being the interval $[0,\infty)$ we have

(2.4;27) $$u_3'=u_4'=0,0138.$$

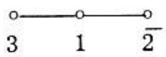
Thus

(2.4;28) $$u'_{i_s}=u_1' > u_4'=u'_{j_s}$$

and from theorem 1.3;2 then follows that H attains its maximum in D for $\eta_1=\eta_4$. This reduces the problem to maximizing the function

(2.4;29) $$H^*=H^*(\zeta_1,\zeta_2,\zeta_3)\overset{def}{=\!=}H_{\{1,4\}}(\zeta_1)+\sum_{i=2}^{3}H_i(\zeta_i)$$

in the set

(2.4;30) $$D_{3,3}\ \begin{cases}\zeta_3 \leqq \zeta_1 \leqq \zeta_2\\ \zeta_1\in I_{\{1,4\}},\ \zeta_i\in I_i\ (i=2,3)\end{cases},$$ thus 

.

The function H* attains its maximum in the set $G_3\overset{def}{=\!=}I_{\{1,4\}}\times I_2\times I_3$ at the point $(v_{\{1,4\}},v_2,v_3)$. Now it follows from (2.4;10) that $w_{\{1,4\}}=1,05$; thus, $I_{\{1,4\}}$ being the interval $(-\infty;5]$, we have $v_{\{1,4\}}=1,05$. Consequently

(2.4;31) $$v_3 > v_{\{1,4\}} > v_2$$

and from (2.4;30) and (2.4;31) then follows (cf. theorem 1.4;3) that H* attains its maximum in $D_{3,3}$ for $\zeta_1=\zeta_2=\zeta_3$; i.e.

(2.4;32) $$u_1=u_2=u_3=u_4=v_{\{1,2,3,4\}}.$$

Now it follows from (2.4;10) that $w_{\{1,2,3,4\}}=0,7535$; thus, $I_{\{1,2,3,4\}}$ being the interval $[0;5]$, we have

(2.4;33) $$u_1=u_2=u_3=u_4=0,7535.$$

From

(2.4;34) $$t_i=\frac{u_i-\alpha_i}{\beta_i}\qquad (i\in E)$$

then follows

(2.4;35) $$t_1=-1,08;\ t_2=0,25;\ t_3=0,62;\ t_4=0,38.$$

(2.7;9) $$w_i=X_i\qquad (i\in E).$$

From (2.4;11) follows

(2.4;36) $$s_1^2=s_2^2=2,38;\quad s_3^2=11,92;\quad s_4^2=7,15.$$

Further if I_i is the interval $(-\infty,\infty)$ for each $i\in E$ then $v_M=w_M$ for each $M\subset E$. Thus in this case (1.5;4) reduces to

(2.4;37) $$\alpha_i+\beta_i t_i=u_i=\max_M \min_{M'}\left\{\sum_{j\in T_M\cap S_{M'}} \frac{\sum_{\gamma=1}^{n_j} x'_{j,\gamma}}{\beta_j^2 K_j} \Big/ \sum_{j\in T_M\cap S_{M'}} \frac{n_j}{\beta_j^2 K_j} \,\Big|\, i\in T_M\cap S_{M'}\right\}.$$

Further the function Q (cf. section 1.6) reduces to

(2.4;38) $$Q(\vec{\eta})=\sum_{i\in E} q_i\Big(\eta_i-\frac{1}{n_i}\sum_{\gamma=1}^{n_i} x'_{i,\gamma}\Big)^2$$

with

(2.4;39) $$w'_M=\sum_{i\in M}\frac{q_i}{n_i}\sum_{\gamma=1}^{n_i} x'_{i,\gamma}\Big/\sum_{i\in M} q_i.$$

Thus if we take

(2.4;40) $$q_i=g_\nu\frac{n_i}{\beta_i^2 K_i}\qquad (i\in E_\nu;\ \nu=1,\dots,K),$$

where $g_\nu(\nu=1,\dots,K)$ are positive numbers, then $w'_M=w_M$ for each $M\subset E_\nu(\nu=1,\dots,K)$ and (1.6;19) then entails that (1.6;16) is satisfied. Thus the point where H attains its maximum in B coincides with the point where

(2.4;41) $$\sum_{\nu=1}^{K} g_\nu \sum_{i\in E_\nu}\frac{n_i}{\beta_i^2 K_i}\Big(\eta_i-\frac{1}{n_i}\sum_{\gamma=1}^{n_i} x'_{i,\gamma}\Big)^2$$

attains its minimum in B.[1)]

---

1) For $g_\nu=1(\nu=1,\dots,K)$ this also follows from

$$H(\vec{\eta})=-\tfrac{1}{2}\sum_{i\in E} n_i\ln\, 2\pi\sigma_i^2-\tfrac{1}{2}\sum_{i\in E}\frac{\sum_{\gamma=1}^{n_i}(x'_{i,\gamma}-\eta_i)^2}{(\beta_i\sigma_i)^2}=$$

$$=-\tfrac{1}{2}\sum_{i\in E} n_i\ln\, 2\pi\sigma_i^2-\tfrac{1}{2}\sum_{i\in E}\frac{\sum_{\gamma=1}^{n_i}\Big(x'_{i,\gamma}-\frac{1}{n_i}\sum_{\lambda=1}^{n_i}x'_{i,\lambda}\Big)^2}{(\beta_i\sigma_i)^2}+$$

$$-\frac{1}{2\sigma_1^2}\sum_{i\in E}\frac{n_i}{\beta_i^2 K_i}\Big(\eta_i-\frac{1}{n_i}\sum_{\gamma=1}^{n_i}x'_{i,\gamma}\Big)^2.$$

Further the inequality (1.6;24) reduces to

$$(2.4;42)\quad \sum_{\nu=1}^{K} g_{\nu} \sum_{i\in E_{\nu}} \frac{n_i}{\beta_i^2 \kappa_i}\Big(u_i - \frac{1}{n_i}\sum_{\gamma=1}^{n_i} x'_{i,\gamma}\Big)(u_i - \eta_i^{\circ}) \leqq 0 \text{ for each point } \vec{\eta}^{\circ}\in B$$

and (2.4;42) is identical with

$$(2.4;43)\quad \sum_{\nu=1}^{K} g_{\nu} \sum_{i\in E_{\nu}} \frac{n_i}{\beta_i}\Big(t_i - \frac{1}{n_i}\sum_{\gamma=1}^{n_i} x_{i,\gamma}\Big)\Big(t_i - \frac{\eta_i^{\circ}-\alpha_i}{\beta_i}\Big) \leqq 0 \text{ for each point } \vec{\eta}^{\circ}\in B.$$

2.5 *A Poisson distribution with unknown parameter*

Let, for each $i\in E$, $\underline{x}_i$ possess a Poisson distribution with parameter $\theta_i > 0$.

A practical situation of ordered parameters of Poisson distributions might occur if several toxicants are to be investigated concerning their toxicity for a certain kind of bacterium.
Let T and T′ denote two toxicants added to cultures of bacteria, each in the two concentrations C_1 and C_2 ($C_1 < C_2$). Then four samples are obtained, the observations consisting of the numbers of survivors in each of the experiments. Let $\theta_1,\dots,\theta_4$ denote the expected values of the numbers of survivors, θ_1 and θ_2 for T in the concentrations C_1 and C_2 respectively, θ_3 and θ_4 for T′ in the concentrations C_1 and C_2 respectively.
Now let it be known that the toxicity of T is smaller than or equal to the toxicity of T′, if added in the same concentration; then we have $\theta_1 \leqq \theta_3$ and $\theta_2 \leqq \theta_4$. If moreover the toxicity of both toxicants is known to be a non-decreasing function of the concentration, then we have moreover: $\theta_1 \leqq \theta_2$ and $\theta_3 \leqq \theta_4$. Thus we have

$$\begin{cases}\theta_1 \leqq \theta_2 \leqq \theta_4,\\ \theta_1 \leqq \theta_3 \leqq \theta_4,\end{cases}$$

i.e. $\theta_1,\dots,\theta_4$ are partially ordered with

$$\begin{cases}\alpha_{1,2}=\alpha_{2,4}=\alpha_{1,3}=\alpha_{3,4}=1\\ \alpha_{2,3}=0\end{cases}$$,thus  .

i 1 2 3 4

In this case we have

$$(2.5;1)\qquad f_i(x|\theta_i)=\begin{cases}\dfrac{e^{-\theta_i}\theta_i^x}{x!} & x=0,1,\dots\\[2ex] 0 & x\neq 0,1\dots\end{cases}\qquad (i\in E).$$

Thus

$$(2.5;2)\quad L_i(\tau)=-n_i\tau+\sum_{\gamma=1}^{n_i} x_{i,\gamma}\ln\tau-\sum_{\gamma=1}^{n_i}\ln x_{i,\gamma}! \qquad (\tau \geqq 0;\ i\in E).$$

Further

$$(2.5;3)\qquad J_i=\begin{cases}[\alpha_i,\infty) \text{ if } \beta_i > 0,\\ (-\infty,\alpha_i] \text{ if } \beta_i < 0\end{cases}$$

and

$$(2.5;4)\quad H_M(\zeta)=\sum_{i\in M}\{-n_i\frac{\zeta-\alpha_i}{\beta_i}+\sum_{\gamma=1}^{n_i} x_{i,\gamma}\ln\frac{\zeta-\alpha_i}{\beta_i}-\sum_{\gamma=1}^{n_i}\ln x_{i,\gamma}!\} \qquad (\zeta\in J_M).$$

Consequently, for ζ in the open interval J_M,

$$(2.5;5)\qquad \begin{cases}\dfrac{dH_M(\zeta)}{d\zeta}=\sum\limits_{i\in M}\{-\dfrac{n_i}{\beta_i}+\dfrac{1}{\zeta-\alpha_i}\sum\limits_{\gamma=1}^{n_i} x_{i,\gamma}\},\\ \dfrac{d^2H_M(\zeta)}{d\zeta^2}=-\sum\limits_{i\in M}\dfrac{\sum_{\gamma=1}^{n_i} x_{i,\gamma}}{(\zeta-\alpha_i)^2}.\end{cases}$$

The following two cases may be distinguished

1. $\sum_{\gamma=1}^{n_i} x_{i,\gamma} > 0$ for each $i\in M$; then $\frac{d^2H_M(\zeta)}{d\zeta^2} < 0$ for each ζ in the open interval J_M; i.e. $H_M(\zeta)$ is strictly unimodal in J_M,

2. $\sum_{\gamma=1}^{n_i} x_{i,\gamma}=0$ for each $i\in M$; then $\frac{dH_M(\zeta)}{d\zeta}=-\sum_{i\in M}\frac{n_i}{\beta_i}$.

Thus in this case $H_M(\zeta)$ is strictly unimodal in J_M if and only if

$$(2.5;6)\qquad \sum_{i\in M}\frac{n_i}{\beta_i}\neq 0$$

and (2.5;6) is e.g. satisfied if $\beta_i > 0$ (or $\beta_i < 0$) for each $i\in M$. Further $H_i(\eta) < \infty$ for each $\eta\in J_i$ and each $i\in E$.
In general it is not possible to give an explicit formula for w_M. If a value of ζ exists satisfying

$$(2.5;7)\qquad \begin{cases}\dfrac{dH_M(\zeta)}{d\zeta}=0,\\ \zeta\in J_M,\end{cases}$$

then w_M is equal to this value if ζ. If e.g. $\beta_i > 0$ and $\sum_{\gamma=1}^{n_i} x_{i,\gamma}=0$

for each $i \in M$ then J_M is the interval $[\max_{i\in M} \alpha_i, \infty)$ and $\frac{dH_M(\zeta)}{d\zeta} = -\sum_{i\in M} \frac{n_i}{\beta_i} < 0$. Thus in this case we have

$$w_M = \max_{i\in M} \alpha_i . \tag{2.5;8}$$

Further we have, for M=i,

$$w_i = \alpha_i + \frac{\beta_i}{n_i} \sum_{\gamma=1}^{n_i} x_{i,\gamma} . \tag{2.5;9}$$

Now let (i,j) be a pair of values with $\alpha_i=\alpha_j(=\alpha)$ and $\beta_i=\beta_j(=\beta)$ then (cf. (2.5;4)), for $M=\{i,j\}$,

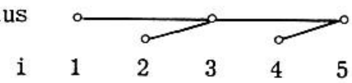
$$H_M(\zeta) = -\frac{\zeta-\alpha}{\beta} n_M + \sum_{\gamma=1}^{n_M} x_{M,\gamma} \ln\frac{\zeta-\alpha}{\beta} - \sum_{\gamma=1}^{n_M} \ln x_{M,\gamma}! \qquad (\zeta\in J_M), \tag{2.5;10}$$

where $n_M \overset{def}{=} n_i+n_j$ and where $x_{M,\gamma}(\gamma=1,\dots,n_M)$ denotes the pooled samples of $\underline{x}_i$ and $\underline{x}_j$. Thus if in this case H attains its maximum in D for $\eta_i=\eta_j$ then the problem reduces to the case of k-1 samples, i.e. the samples of $\underline{x}_h(h\neq i, h\neq j)$ and the pooled samples of $\underline{x}_i$ and $\underline{x}_j$.

Example 2.5;1

Let k=5,

$$\begin{cases}\alpha_{1,3}=\alpha_{2,3}=\alpha_{3,5}=\alpha_{4,5}=1\\ \alpha_{1,2}=\alpha_{1,4}=\alpha_{2,4}=\alpha_{3,4}=0\end{cases}; \text{ thus} \tag{2.5;11}$$



and

(2.5;12)

i	1	2	3	4	5
$x_{i,\gamma}$	0 4 2 0	3 2 0 7	2 1 5	0 0 1 2 1	0 1 3 2
$\sum_{\gamma=1}^{n_i} x_{i,\gamma}$	6	12	8	4	6
α_i	0	2	6	3	0
β_i	1	3	-2	-1	2
J_i	$[0;\infty)$	$[2;\infty)$	$(-\infty;6]$	$(-\infty;3]$	$[0;\infty)$
I_i	$[0;\infty)$	$[2;\infty)$	$(-\infty;4]$	$(-\infty;3]$	$[0;\infty)$

Then $\sum_{\gamma=1}^{n_i} x_{i,\gamma} > 0$ for each $i \in E$, thus $H_M(\zeta)$ is strictly unimodal in J_M for each $M \subset E$.
From (2.5;9) follows

$$(2.5;13) \qquad \begin{cases} i & 1 \quad 2 \quad 3 \quad 4 \quad 5 \\ w_i & 1,5 \quad 11 \quad 0,67 \quad 2,2 \quad 3 \end{cases}$$

and (cf. (1.6;14))

$$(2.5;14) \qquad \begin{cases} i & 1 \quad 2 \quad 3 \quad 4 \quad 5 \\ v_i & 1,5 \quad 11 \quad 0,67 \quad 2,2 \quad 3\ . \end{cases}$$

Thus $\alpha_{1,3}(v_1-v_3) > 0$ and $\alpha_{2,3}(v_2-v_3) > 0$, i.e. $\vec{v} \notin D$.
We now apply theorem 1.4;4 with $i=2, j=1$; we have

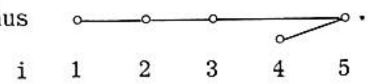
$$(2.5;15) \qquad \begin{cases} 1.\ v_1 < v_2 \\ 2.\ \alpha_{2,1}=0, \\ 3.\ \alpha_{2,3}=\alpha_{1,3}, \alpha_{2,4}=\alpha_{1,4}, \alpha_{2,5}=\alpha_{1,5}, \end{cases}$$

thus H attains its maximum in D for $\eta_1 \leqq \eta_2$. This reduces the problem to maximizing H in the set

$$(2.5;16) \qquad D' \quad \begin{cases} \alpha'_{i,j}(\eta_i-\eta_j) \leqq 0 \\ \eta_i \in I_i \end{cases} \qquad (i,j \in E)$$

with

$$(2.5;17) \quad \begin{cases} \alpha'_{1,2}=\alpha'_{2,3}=\alpha'_{3,5}=\alpha'_{4,5}=1 \\ \alpha'_{1,4}=\alpha'_{2,4}=\alpha'_{3,4}=0 \end{cases}, \text{ thus}$$



To this problem we apply theorem 1.4;2 with $i=2, j=3$; then we have

$$(2.5;18) \qquad \begin{cases} 1.\ \alpha'_{2,3}=1, v_2 > v_3, \\ 2.\ \alpha'_{1,2}=\alpha'_{1,3}, \\ 3.\ \alpha'_{2,4}=\alpha'_{3,4}, \alpha'_{2,5}=\alpha'_{3,5}, \end{cases}$$

consequently H attains its maximum in D' for $\eta_2=\eta_3$. This reduces the problem to maximizing

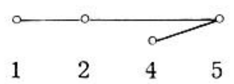
$$(2.5;19) \quad H^*=H^*(\zeta_1,\zeta_2,\zeta_4,\zeta_5) \overset{\text{def}}{=\!=} H_1(\zeta_1)+H_{\{2,3\}}(\zeta_2)+\sum_{i=4}^{5} H_i(\zeta_i)$$

in the set

$$(2.5;20) \qquad D_{4,4} \quad \begin{cases} \alpha''_{i,j}(\zeta_i-\zeta_j) \leqq 0 \quad (i,j=1,2,4,5) \\ \zeta_i \in I_i\,(i=1,4,5), \zeta_2 \in I_{\{2,3\}}, \end{cases}$$

where

(2.5;21) $\begin{cases}\alpha''_{1,2}=\alpha''_{2,5}=\alpha''_{4,5}=1\\ \alpha''_{1,4}=\alpha''_{2,4}=0\end{cases}$, thus



We first determine the values of $\zeta_1,\zeta_2,\zeta_4,\zeta_5$ which maximize H^* in the set

(2.5;22) $$G_4 \overset{def}{=\!=} I_1 \times I_{\{2,3\}} \times I_4 \times I_5.$$

This maximum is attained at the point $(v_1, v_{\{2,3\}}, v_4, v_5)$.

Now it follows from (2.5;5) that $\frac{dH_{\{2,3\}}(\zeta)}{d\zeta}=0$ for $\zeta=4,5$; thus $J_{\{2,3\}}$ being the interval $[2;6]$, we have $w_{\{2,3\}}=4,5$. Further $I_{\{2,3\}}$ being the interval $[2;4]$, we have $v_{\{2,3\}}=4$, i.e. H^* attains its maximum in G_4 at the point (1,5; 4; 2,2; 3) and this point does not lie in $D_{4,4}$.

We now apply theorem 1.4;4 with $i=2, j=4$; then (cf. (2.5;21))

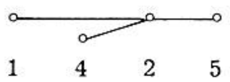
(2.5;23) $$\begin{cases}1.\ v_4 < v_{\{2,3\}},\\ 2.\ \alpha''_{2,4}=0,\\ 3.\ \alpha''_{2,5}=\alpha''_{4,5},\\ 4.\ \alpha''_{1,2} > \alpha''_{1,4},\end{cases}$$

thus H^* attains its maximum in $D_{4,4}$ for $\zeta_4 \leqq \zeta_2$. This reduces the problem to maximizing H^* in the set

(2.5;24) $$D'' \begin{cases}\alpha'''_{i,j}(\zeta_i-\zeta_j) \leqq 0 \quad (i,j=1,2,4,5),\\ \zeta_i \in I_i\,(i=1,4,5),\ \zeta_2 \in I_{\{2,3\}},\end{cases}$$

where

(2.5;25) $\begin{cases}\alpha'''_{1,2}=\alpha'''_{2,5}=\alpha'''_{4,2}=1\\ \alpha'''_{1,4}=0\end{cases}$, thus



We now apply theorem 1.4;2 with $i=2, j=5$; we have

(2.5;26) $$\begin{cases}1.\ \alpha'''_{2,5}=1,\ v_{\{2,3\}} > v_5,\\ 2.\ \alpha'''_{1,2}=\alpha'''_{1,5},\ \alpha'''_{4,2}=\alpha'''_{4,5},\end{cases}$$

thus H^* attains its maximum in D'' for $\zeta_2=\zeta_5$. This reduces the problem to maximizing the function

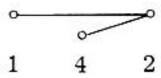
(2.5;27) $$H^{**}=H^{**}(\zeta_1,\zeta_2,\zeta_4) \overset{def}{=\!=} H_1(\zeta_1)+H_{\{2,3,5\}}(\zeta_2)+H_4(\zeta_4)$$

in the set

$$(2.5;28) \qquad D_{3,4} \begin{cases} \alpha''''_{i,j}(\zeta_i-\zeta_j) \leqq 0 & (i,j=1,2,4), \\ \zeta_i \in I_i\,(i=1,4),\ \zeta_2 \in I_{\{2,3,5\}}, \end{cases}$$

where

$$(2.5;29) \qquad \begin{cases} \alpha''''_{1,2}=\alpha''''_{4,2}=1 \\ \alpha''''_{1,4}=0 \end{cases}, \text{ thus}$$



The function H** attains its maximum in the set

$$(2.5;30) \qquad G_3 \overset{def}{=\!=} I_1 \times I_{\{2,3,5\}} \times I_4$$

at the point $(v_1, v_{\{2,3,5\}}, v_4)$. From (2.5;5) and (2.5;12) it follows that $\frac{dH_{\{2,3,5\}}(\zeta)}{d\zeta}$ is positive for each $\zeta \in I_{\{2,3,5\}}$, where $I_{\{2,3,5\}}$ is the interval [2;4]; thus $v_{\{2,3,5\}}=4$. Consequently H** attains its maximum in G_3 at the point (1,5; 4; 2,2) and this is a point in $D_{3,4}$. Thus H** attains its maximum in $D_{3,4}$ at the point (1,5; 4; 2,2), i.e.

$$(2.5;31) \qquad u_1=1{,}5;\ u_2=u_3=u_5=4;\ u_4=2{,}2$$

and from $t_i=\frac{u_i-\alpha_i}{\beta_i}\,(i\in E)$ then follows

$$(2.5;32) \qquad t_1=1{,}5;\ t_2=0{,}67;\ t_3=1{,}5;\ t_4=0{,}8;\ t_5=2.$$

It follows from (2.5;9) that the function Q (cf. section 1.6) reduces to

$$(2.5;33) \qquad Q(\vec{\eta})=\sum_{i\in E} q_i\Big(\eta_i-\alpha_i-\frac{\beta_i}{n_i}\sum_{\gamma=1}^{n_i} x_{i,\gamma}\Big)^2$$

with (cf. (1.6;15))

$$(2.5;34) \qquad w'_M=\frac{\sum_{i\in M} q_i\Big(\alpha_i+\frac{\beta_i}{n_i}\sum_{\gamma=1}^{n_i} x_{i,\gamma}\Big)}{\sum_{i\in M} q_i}$$

and (cf. also section 2.3) the point where H attains its maximum in B coincides with the point where Q attains its minimum in B if $q_1,\dots,q_k$ are chosen in such a way that (1.6;16) is satisfied. Concerning the existence of numbers $q_1,\dots,q_k$ satisfying (1.6;16)

the same holds as in section 2.3; only in the special case that $\alpha_i=\alpha$ for each $i\in E$ we are able to prove the existence of such numbers. For this special case an explicit formula for $q_1,\dots,q_k$ will be given.
Now let, in the special case that $\alpha_i=\alpha$ for each $i\in E$,

$$\begin{cases}\beta_i > 0 & \text{for each } i\in M_o,\\ \beta_i < 0 & \text{for each } i\in \overline{M}_o,\end{cases} \tag{2.5;35}$$

then (cf. (2.5;3)) for each pair of values (i,j) with $i\in M_o$ and $j\in\overline{M}_o$ the intersection $J_i\cap J_j$ contains exactly one point. Consequently the intersection $I_i\cap I_j$ contains at most one point, thus (cf. (1.1;3.2)) $\alpha_{i,j}=0$ for each pair of values (i,j) with $i\in M_o, j\in\overline{M}_o$. From theorem 1.4;1 then follows that the maximum of H in D may be found separately maximizing $\sum_{i\in M_o} H_i(\eta_i)$ and $\sum_{j\in\overline{M}_o} H_j(\eta_j)$.
Further, for $j\in\overline{M}_o$ and with $\eta' \overset{\text{def}}{=} 2\alpha-\eta$

$$H_j(2\alpha-\eta)=-n_j\frac{\eta'-\alpha}{-\beta_j}+\sum_{\gamma=1}^{n_j}x_{j,\gamma}\ln\frac{\eta'-\alpha}{-\beta_j}-\sum_{\gamma=1}^{n_j}\ln x_{j,\gamma}! \qquad (\eta' \geqq \alpha). \tag{2.5;36}$$

Thus we may suppose without any loss of generality that $\beta_i > 0$ for each $i\in E$. Then J_i is the interval $[\alpha,\infty)$ $(i\in E)$. Further (cf. (2.5;5))

$$\frac{dH_M(\zeta)}{d\zeta}=-\sum_{i\in M}\frac{n_i}{\beta_i}+\frac{\sum_{i\in M}\sum_{\gamma=1}^{n_i}x_{i,\gamma}}{\zeta-\alpha} \qquad (\zeta > \alpha), \tag{2.5;37}$$

thus

$$w_M=\alpha+\frac{\sum_{i\in M}\sum_{\gamma=1}^{n_i}x_{i,\gamma}}{\sum_{i\in M}\frac{n_i}{\beta_i}}. \tag{2.5;38}$$

The function Q (cf. section 1.6) reduces to

$$Q(\vec{\eta})=\sum_{i\in E}q_i\left(\eta_i-\alpha-\frac{\beta_i}{n_i}\sum_{\gamma=1}^{n_i}x_{i,\gamma}\right)^2 \tag{2.5;39}$$

with (cf. (1.6;15))

$$w'_M=\alpha+\frac{\sum_{i\in M}q_i\frac{\beta_i}{n_i}\sum_{\gamma=1}^{n_i}x_{i,\gamma}}{\sum_{i\in M}q_i}. \tag{2.5;40}$$

Thus if we take

$$(2.5;41)\qquad q_i=g_\nu \frac{n_i}{\beta_i}\quad (i\in E_\nu;\ \nu=1,\dots,K),$$

where $g_\nu(\nu=1,\dots,K)$ are positive numbers then $w_M=w'_M$ for each $M\subset E_\nu(\nu=1,\dots,K)$ and (1.6;19) then entails that (1.6;16) is satisfied. Consequently the point where H attains its maximum in B coincides with the point where

$$(2.5;42)\qquad \sum_{\nu=1}^{K} g_\nu \sum_{i\in E_\nu} \frac{n_i}{\beta_i}\Big(\eta_i-\alpha-\frac{\beta_i}{n_i}\sum_{\gamma=1}^{n_i} x_{i,\gamma}\Big)^2$$

attains its minimum in B. Further the inequality (1.6;24) reduces to

$$(2.5;43)\qquad \sum_{\nu=1}^{K} g_\nu \sum_{i\in E_\nu} \frac{n_i}{\beta_i}\Big(u_i-\alpha-\frac{\beta_i}{n_i}\sum_{\gamma=1}^{n_i} x_{i,\gamma}\Big)(u_i-\eta_i^\circ)\leqq 0 \text{ for each point } \vec{\eta}^\circ\in B,$$

which is identical with

$$(2.5;44)\qquad \sum_{\nu=1}^{K} g_\nu \sum_{i\in E_\nu} n_i\beta_i\Big(t_i-\frac{1}{n_i}\sum_{\gamma=1}^{n_i} x_{i,\gamma}\Big)\Big(t_i-\frac{\eta_i^\circ-\alpha}{\beta_i}\Big)\leqq 0 \text{ for each point } \vec{\eta}^\circ\in B.$$

Finally if $I_i=J_i$ for each $i\in E$ then $v_M=w_M$ for each $M\subset E$, thus in this case (1.5;4) reduces to

(2.5;45)

$$\alpha+\beta_i t_i=u_i=\alpha+\max_M \min_{M'}\Big\{\sum_{j\in T_M\cap S_{M'}} \sum_{\gamma=1}^{n_j} x_{j,\gamma}\Big/\sum_{j\in T_M\cap S_{M'}} \frac{n_j}{\beta_j}\ \Big|\ i\in T_M\cap S_{M'}\Big\}\ (i\in E).$$

2.6 *An exponential distribution with unknown parameter*

Let, for each $i\in E$, $\underline{x}_i$ possess an exponential distribution with parameter $\theta_i > 0$.

An example of ordered parameters of exponential distributions may e.g. be found in waitingtime problems. Let the intervals between the arrivals of two succesive customers at a server be exponentially distributed. Let observations of these intervals be available for two or more periods $P_1,\dots,P_k$ and let θ_i denote the parameter of the exponential distribution for period $P_i (i\in E)$. Then knowledge concerning the intensity of the arrivals of customers in the periods $P_1,\dots,P_k$ may be available, leading to a partial ordering of the parameters $\theta_1,\dots,\theta_k$.

In this case we have

$$(2.6;1)\qquad f_i(x|\theta_i)=\begin{cases}\theta_i e^{-\theta_i x} & x \geqq 0\\ 0 & x < 0\end{cases}\qquad (i\epsilon E).$$

Thus

$$(2.6;2)\qquad L_i(\tau)=n_i \ln \tau-\tau \sum_{\gamma=1}^{n_i} x_{i,\gamma}\qquad (\tau \geqq 0;\, i\epsilon E)$$

and

$$(2.6;3)\qquad J_i=\begin{cases}[\alpha_i,\infty) & \text{if } \beta_i > 0,\\ (-\infty,\alpha_i] & \text{if } \beta_i < 0.\end{cases}$$

Further

$$(2.6;4)\qquad H_M(\zeta)=\sum_{i\epsilon M}\{n_i \ln\frac{\zeta-\alpha_i}{\beta_i}-\frac{\zeta-\alpha_i}{\beta_i}\sum_{\gamma=1}^{n_i} x_{i,\gamma}\}\qquad (\zeta\epsilon J_M)$$

and, for ζ in the open interval J_M,

$$(2.6;5)\qquad \begin{cases}1.\ \dfrac{dH_M(\zeta)}{d\zeta}=\sum\limits_{i\epsilon M}\{\dfrac{n_i}{\zeta-\alpha_i}-\dfrac{1}{\beta_i}\sum\limits_{\gamma=1}^{n_i} x_{i,\gamma}\},\\ 2.\ \dfrac{d^2H_M(\zeta)}{d\zeta^2}=-\sum\limits_{i\epsilon M}\dfrac{n_i}{(\zeta-\alpha_i)^2} < 0,\end{cases}$$

where (2.6;5.2) entails that $H_M(\zeta)$ is strictly unimodal in J_M. Further $H_i(\eta) < \infty$ for each $\eta\epsilon J_i$ and each $i\epsilon E$.

This example is analogous to the one treated in section 2.5; (2.6;5) reduces to (2.5;5) by interchanging n_i and $\sum_{\gamma=1}^{n_i} x_{i,\gamma}$ $(i\epsilon M)$. Therefore we do not give a numerical example.
Further (cf. also section 2.3 and 2.5) the existence of numbers $q_1,\ldots,q_k$ satisfying (1.6;16) can only be proved for the special case that $\alpha_i=\alpha$ for each $i\epsilon E$. In this special case we may (cf. section 2.5) again suppose that $\beta_i > 0$ for each $i\epsilon E$. Then J_i is the interval $[\alpha,\infty)$ and (cf. (2.6;5.1))

$$(2.6;6)\qquad w_M=\alpha+\{\sum_{i\epsilon M} n_i \Big/ \sum_{i\epsilon M}\frac{\sum_{\gamma=1}^{n_i} x_{i,\gamma}}{\beta_i}\}.$$

Further

$$w_i = \alpha + \frac{\beta_i n_i}{\sum_{\gamma=1}^{n_i} x_{i,\gamma}} \qquad (i \in E), \tag{2.6;7}$$

thus (cf. (1.6;15))

$$w'_M = \alpha + \left\{ \sum_{i \in M} q_i \frac{\beta_i n_i}{\sum_{\gamma=1}^{n_i} x_{i,\gamma}} \Big/ \sum_{i \in M} q_i \right\}. \tag{2.6;8}$$

Consequently if we take

$$q_i = g_\nu \frac{1}{\beta_i} \sum_{\gamma=1}^{n_i} x_{i,\gamma} \qquad (i \in E_\nu;\ \nu = 1, \ldots, K), \tag{2.6;9}$$

where $g_\nu (\nu=1,\ldots,K)$ are positive numbers, then $w_M = w'_M$ for each $M \subset E_\nu$ $(\nu=1,\ldots,K)$. Thus (cf. (1.6;19)) the point where H attains its maximum in B coincides with the point where

$$\sum_{\nu=1}^{K} g_\nu \sum_{i \in E_\nu} \frac{\sum_{\gamma=1}^{n_i} x_{i,\gamma}}{\beta_i} \left(\eta_i - \alpha - \frac{n_i \beta_i}{\sum_{\gamma=1}^{n_i} x_{i,\gamma}} \right)^2 \tag{2.6;10}$$

attains its minimum in B. Further the inequality (1.6;24) reduces to

$$\sum_{\nu=1}^{K} g_\nu \sum_{i \in E_\nu} \frac{\sum_{\gamma=1}^{n_i} x_{i,\gamma}}{\beta_i} \left(u_i - \alpha - \frac{\beta_i n_i}{\sum_{\gamma=1}^{n_i} x_{i,\gamma}} \right)(u_i - \eta_i^\circ) \leqq 0 \text{ for each point } \vec{\eta}^\circ \in B \tag{2.6;11}$$

and (2.6;11) is identical with

(2.6;12)

$$\sum_{\nu=1}^{K} g_\nu \sum_{i \in E_\nu} \beta_i \sum_{\gamma=1}^{n_i} x_{i,\gamma} \left(t_i - \frac{n_i}{\sum_{\gamma=1}^{n_i} x_{i,\gamma}} \right)\left(t_i - \frac{\eta_i^\circ - \alpha}{\beta_i} \right) \leqq 0 \quad \text{for each point } \vec{\eta}^\circ \in B.$$

2.7 *A rectangular distribution between* 0 *and* θ_i

Let, for each $i \in E$, $\underline{x}_i$ possess a rectangular distribution between 0 and $\theta_i > 0$; then

$$(2.7;1)\qquad f_i(x|\theta_i)=\begin{cases}\dfrac{1}{\theta_i} & 0\leqq x\leqq\theta_i\\ 0 & x<0 \text{ and } x>\theta_i\end{cases}\qquad (i\in E)$$

and

$$(2.7;2)\qquad L_i(\tau)=-n_i\ln\tau\qquad (\tau\geqq\max_{1\leqq\gamma\leqq n_i}x_{i,\gamma};\ \ i\in E).$$

Now let

$$(2.7;3)\qquad X_i\overset{def}{=\!=}\alpha_i+\beta_i\max_{1\leqq\gamma\leqq n_i}x_{i,\gamma}.\qquad (i\in E)$$

and

$$(2.7;4)\qquad \begin{cases}\beta_i>0 \text{ for each } i\in M_o,\\ \beta_i<0 \text{ for each } i\in\bar{M}_o,\end{cases}$$

then

$$(2.7;5)\qquad J_i=\begin{cases}[X_i,\infty) & \text{for each } i\in M_o,\\ (-\infty,X_i] & \text{for each } i\in\bar{M}_o.\end{cases}$$

Thus in this case J_i depends on the observations $x_{i,\gamma}(\gamma=1,\dots,n_i)$. Further

$$(2.7;6)\qquad H_M(\zeta)=-\sum_{i\in M}n_i\ln\frac{\zeta-\alpha_i}{\beta_i}\qquad (\zeta\in J_M)$$

and, for ζ in the open interval J_M,

$$(2.7;7)\qquad \begin{cases}1.\ \dfrac{dH_M(\zeta)}{d\zeta}=-\sum\limits_{i\in M}\dfrac{n_i}{\zeta-\alpha_i},\\[2ex] 2.\ \dfrac{d^2H_M(\zeta)}{d\zeta^2}=\sum\limits_{i\in M}\dfrac{n_i}{(\zeta-\alpha_i)^2}>0\end{cases}$$

and (2.7;7.2) entails that $\dfrac{dH_M(\zeta)}{d\zeta}=0$ for at most one value of $\zeta\in J_M$, i.e. $H_M(\zeta)$ is strictly unimodal in J_M. Further $H_i(\eta)<\infty$ for each $\eta\in J_i$ and each $i\in E$.

Further, for each $\eta\in J_i$ with $\eta\neq\alpha_i$ we have

$$(2.7;8)\qquad \frac{dH_i(\eta)}{d\eta}=-\frac{n_i}{\eta-\alpha_i}\begin{cases}<0 & \text{for } i\in M_o,\\ >0 & \text{for } i\in\bar{M}_o,\end{cases}$$

thus

$$(2.7;9)\qquad w_i=X_i\qquad (i\in E).$$

Now let (i,j) be a pair of values with $\alpha_i=\alpha_j(=\alpha)$ and $\beta_i=\beta_j(=\beta)$. Let further, for $M=\{i,j\}$,

$$(2.7;10)\qquad \begin{cases} n_M \overset{def}{=\!=} n_i+n_j, \\ X_M \overset{def}{=\!=} \begin{cases} \max\limits_{i\in M} X_i & \text{if } \beta>0, \\ \min\limits_{i\in M} X_i & \text{if } \beta<0, \end{cases} \end{cases}$$

then

$$(2.7;11)\qquad X_M=\alpha+\beta \max_{1\leqq\gamma\leqq n_M} x_{M,\gamma}$$

if $x_{M,\gamma}(\gamma=1,\ldots,n_M)$ denotes the pooled samples of $\underline{x}_i$ and $\underline{x}_j$. Further

$$(2.7;12)\qquad H_M(\zeta)=-n_M\ln\frac{\zeta-\alpha}{\beta}\qquad (\zeta\in J_M),$$

where

$$(2.7;13)\qquad J_M=\begin{cases} [X_M,\infty) & \text{if } \beta>0, \\ (-\infty,X_M] & \text{if } \beta<0. \end{cases}$$

Let further H attain its maximum in D for $\eta_i=\eta_j$ then (2.7;12) entails that the problem reduces to the case of k-1 samples, i.e. the samples of $\underline{x}_h(h\neq i, h\neq j)$ and the sample $x_{M,\gamma}(\gamma=1,\ldots,n_M)$.

We now consider the following example.

Example 2.7;1

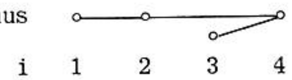
Let k=4,

(2.7;14)

i	1	2	3	4
$x_{i,\gamma}$	0,28 0,94 0,03 0,68	1,65 2,36 1,87 1,55 0,30	0,80 1,29 1,38 1,26	0,32 1,96 2,28
α_i	0	4	2	-4
β_i	1	-1	-2	1
J_i	$[0,94;\infty)$	$(-\infty;1,64]$	$(-\infty;-0,76]$	$[-1,72;\infty)$
$X_i=w_i$	0,94	1,64	-0,76	-1,72

and

$$(2.7;15)\qquad \begin{cases} \alpha_{1,2}=\alpha_{2,4}=\alpha_{3,4}=1 \\ \alpha_{1,3}=\alpha_{2,3}=0 \end{cases},\ \text{thus}$$



Let further $I_i=J_i$ for each $i\epsilon E$, then $v_M=w_M$ for each $M\subset E$.
Now we have: $\alpha_{2,4}(w_2-w_4) > 0$ and $\alpha_{3,4}(w_3-w_4) > 0$, thus $\vec{w}\notin D$.
In order to find $u_1,\dots,u_4$ we apply theorem 1.4;4 with $i=2, j=3$; we have

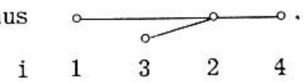
$$\text{(2.7;16)} \qquad \begin{cases} 1.\ w_2 > w_3, \\ 2.\ \alpha_{2,3}=0, \\ 3.\ \alpha_{2,4}=\alpha_{3,4}, \\ 4.\ \alpha_{1,2} > \alpha_{1,3}, \end{cases}$$

thus H attains its maximum in D for $\eta_3 \leqq \eta_2$. This reduces the problem to the problem of maximizing H in the set

$$\text{(2.7;17)} \qquad D' \quad \begin{cases} \alpha'_{i,j}(\eta_i-\eta_j) \leqq 0 \\ \eta_i \epsilon J_i \end{cases} \qquad (i,j\epsilon E),$$

where

$$\text{(2.7;18)} \qquad \begin{cases} \alpha'_{1,2}=\alpha'_{2,4}=\alpha'_{3,2}=1 \\ \alpha'_{1,3}=0 \end{cases}, \text{ thus}$$



Now we again apply theorem 1.4;4 with $i=1, j=3$; we have

$$\text{(2.7;19)} \qquad \begin{cases} 1.\ w_1 > w_3, \\ 2.\ \alpha'_{1,3}=0, \\ 3.\ \alpha'_{1,2}=\alpha'_{3,2}, \alpha'_{1,4}=\alpha'_{3,4}, \end{cases}$$

thus H attains its maximum in D' for $\eta_3 \leqq \eta_1$. This reduces the problem to maximizing H in the set

$$\text{(2.7;20)} \qquad D'' \quad \begin{cases} \eta_3 \leqq \eta_1 \leqq \eta_2 \leqq \eta_4, \\ \eta_i \epsilon J_i\ (i=1,\dots,4) \end{cases}$$

and this problem may be solved by means of the theorems 1.4;3 and 1.3;4. We have $w_2 > w_4$, thus (cf. theorem 1.4;3) H attains its maximum in D'' for $\eta_2=\eta_4$. This reduces the problem to maximizing the function

$$\text{(2.7;21)} \qquad H^*=H^*(\zeta_1,\zeta_2,\zeta_3) \stackrel{\text{def}}{=\!=} H_1(\zeta_1)+H_{\{2,4\}}(\zeta_2)+H_3(\zeta_3)$$

in the set

$$\text{(2.7;22)} \qquad D_{3,3} \quad \begin{cases} \zeta_3 \leqq \zeta_1 \leqq \zeta_2, \\ \zeta_i \epsilon J_i\ (i=1,3), \zeta_2 \epsilon J_{\{2,4\}}, \end{cases}$$

where (cf. (2.7;14)) $J_{\{2,4\}}$ is the interval $[-1,72;1,64]$. Further, for $\zeta \in J_{\{2,4\}}$,

(2.7;23) $$\frac{dH_{\{2,4\}}(\zeta)}{d\zeta} = -\left\{\frac{n_2}{\zeta-\alpha_2}+\frac{n_4}{\zeta-\alpha_4}\right\} = -\frac{8\zeta+8}{\zeta^2-16}$$

Thus $\frac{dH_{\{2,4\}}(\zeta)}{d\zeta}=0$ for $\zeta=-1$, i.e. $w_{\{2,4\}}=-1$. Consequently H^* attains its maximum in the set $J_1 x J_{\{2,4\}} x J_3$ at the point (0,94;-1;-0,76) and this is not a point in $D_{3,3}$: we have $w_1 > w_{\{2,4\}}$. Thus, according to theorem 1.4;3 H^* attains its maximum in $D_{3,3}$ for $\zeta_1=\zeta_2$, which reduces the problem to the problem of maximizing the function

(2.7;24) $$H^{**}=H^{**}(\zeta_1,\zeta_3)\overset{def}{=}H_{\{1,2,4\}}(\zeta_1)+H_3(\zeta_3)$$

in the set

(2.7;25) $$D_{2,3} \begin{cases}\zeta_3 \leqq \zeta_1, \\ \zeta_1 \in J_{\{1,2,4\}}, \zeta_3 \in J_3,\end{cases}$$

where $J_{\{1,2,4\}}$ is the interval $[0,94;1,64]$. Further (cf. (2.7;7))

(2.7;26) $$\frac{dH_{\{1,2,4\}}(\zeta)}{d\zeta} < 0 \quad \text{for } \zeta=1,64,$$

thus we have

(2.7;27) $$\frac{dH_{\{1,2,4\}}(\zeta)}{d\zeta} < 0 \quad \text{for } 0,94 \leqq \zeta \leqq 1,64$$

and (2.7;27) entails that $w_{\{1,2,4\}}=0,94$. Thus H^{**} attains its maximum in the set $J_{\{1,2,4\}} x J_3$ at the point (0,94;-0,76) and this is a point in $D_{2,3}$, i.e.

(2.7;28) $$u_1=u_2=u_4=0,94; \quad u_3=-0,76.$$

From $t_i=\frac{u_i-\alpha_i}{\beta_i}$ $(i \in E)$ then follows

(2.7;29) $$t_1=0,94; \quad t_2=3,06; \quad t_3=1,38; \quad t_4=4,94.$$

The existence of numbers $q_1,\ldots,q_k$ satisfying (1.6;16) will again only be proved for a special case. In this section we consider the special case that $\beta_i > 0$ for each $i \in E$. Then, for each $i \in E$, J_i is the interval $[X_i,\infty)$, thus

(2.7;30) $$J_M=[\max_{i \in M} X_i,\infty).$$

Further

$$(2.7;31)\qquad \frac{dH_M(\zeta)}{d\zeta} = -\sum_{i\in M}\frac{n_i}{\zeta-\alpha_i} < 0 \quad \text{for each } \zeta\in J_M,$$

i.e.

$$(2.7;32)\qquad \begin{cases} w_M = \max\limits_{i\in M} X_i, \\ w_i = X_i \quad (i\in E) \end{cases}$$

and if $q_1,\dots,q_k$ are positive numbers, then (cf. (1.6;15))

$$(2.7;33)\qquad w'_M = \frac{\sum\limits_{i\in M} q_i w_i}{\sum\limits_{i\in M} q_i} < w_M .$$

Thus (cf. fig. 2.7;1)

$$(2.7;34)\qquad \zeta \geqq w_M > w'_M \quad \text{for each } \zeta\in J_M .$$

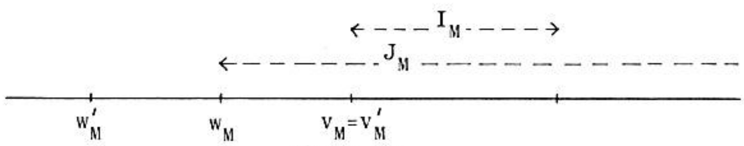


fig.2.7;1

$v_M=v'_M$ for each $M\subset E$

Consequently, I_M being a closed subinterval of J_M, we have

$$(2.7;35)\qquad c_M \geqq w_M > w'_M.$$

Thus (cf. (1.6;14))

$$(2.7;36)\qquad v_M=v'_M \quad \text{for each } M\subset E$$

and (2.7;36) entails that (1.6;16) is satisfied.
Consequently, for each set of positive numbers $q_1,\dots,q_k$, the point where H attains its maximum in B coincides with the point where

$$(2.7;37)\qquad Q(\vec{\eta}) = \sum_{i\in E} q_i(\eta_i - X_i)^2$$

attains its minimum in B. Further the inequality (1.6;24) reduces to

$$(2.7;38)\qquad \sum_{i\in E} q_i(u_i-X_i)(u_i-\eta_i^\circ) \leqq 0 \quad \text{for each point } \vec{\eta}^\circ\in B,$$

which is identical with

$$(2.7;39)\quad \sum_{i\in E} q_i\beta_i^2(t_i-\max_{1\leqq\gamma\leqq n_i} x_{i,\gamma})\,(t_i-\frac{\eta_i^\circ-\alpha_i}{\beta_i})\leqq 0 \text{ for each point } \vec{\eta}^\circ\in B.$$

Finally, if $\alpha_i=\alpha$ for each $i\in E$ then we may suppose (cf. the foregoing sections) that $\beta_i > 0$ for each $i\in E$.

Remark 2.7;1

In an analogous way it may be proved that the conditions A″ and B′ (cf. p. 34) are satisfied if, for each $i\in E$, $\underline{x}_i$ possesses an exponential distribution for $x_i \geqq \theta_i$. Then

$$(2.7;40)\qquad f_i(x|\theta_i)=\begin{cases}\mu_i e^{-\mu_i(x-\theta_i)} & x\geqq\theta_i,\\ 0 & x<\theta_i,\end{cases}$$

with known parameter $\mu_i > 0$ $(i\in E)$.
In this case we have

$$(2.7;41)\quad L_i(\tau)=n_i\ln\mu_i-\mu_i\sum_{\gamma=1}^{n_i}x_{i,\gamma}+\tau\, n_i\mu_i\quad(\tau\leqq\min_{1\leqq\gamma\leqq n_i}x_{i,\gamma};\ i\in E)$$

and it may easily be verified that (cf. (2.7;3) and (2.7;9))

$$(2.7;42)\qquad w_i=\alpha_i+\beta_i\min_{1\leqq\gamma\leqq n_i}x_{i,\gamma}\qquad(i\in E).$$

2.8 *A normal distribution with unknown variance*

Let, for each $i\in E$, $\underline{x}_i$ possess a normal distribution with variance $\theta_i > 0$ and known mean μ_i; then

$$(2.8;1)\qquad f_i(x|\theta_i)=\frac{1}{\sqrt{2\pi\theta_i}}e^{-\frac{1}{2}\frac{(x-\mu_i)^2}{\theta_i}}\qquad(-\infty<x<\infty;\ i\in E),$$

thus

$$(2.8;2)\qquad L_i(\tau)=-\tfrac{1}{2}n_i\ln 2\pi\tau-\tfrac{1}{2}\frac{\sum_{\gamma=1}^{n_i}(x_{i,\gamma}-\mu_i)^2}{\tau}\quad(\tau\geqq 0;\ i\in E).$$

Further

$$(2.8;3)\qquad J_i=\begin{cases}[\alpha_i,\infty) & \text{if } \beta_i>0,\\ (-\infty,\alpha_i] & \text{if } \beta_i<0\end{cases}$$

and

$$(2.8;4)\qquad H_M(\zeta)=-\tfrac{1}{2}\sum_{i\in M} n_i \ln 2\pi\frac{\zeta-\alpha_i}{\beta_i}-\tfrac{1}{2}\sum_{i\in M}\frac{\beta_i\sum_{\gamma=1}^{n_i}(x_{i,\gamma}-\mu_i)^2}{\zeta-\alpha_i}\quad(\zeta\epsilon J_M),$$

thus, for ζ in the open interval J_M,

$$(2.8;5)\qquad \frac{dH_M(\zeta)}{d\zeta}=-\tfrac{1}{2}\sum_{i\in M}\frac{n_i(\zeta-\alpha_i)-\beta_i\sum_{\gamma=1}^{n_i}(x_{i,\gamma}-\mu_i)^2}{(\zeta-\alpha_i)^2}\ .$$

Further $H_i(\eta)<\infty$ for each $\eta\epsilon J_i$ and each $i\epsilon E$.

We first consider the special case that $\alpha_i=\alpha$ for each $i\epsilon E$, then (in the same way as in the foregoing sections) we may suppose that $\beta_i>0$ for each $i\epsilon E$ and then J_i is the interval $[\alpha,\infty)$ for each $i\epsilon E$. Further

$$(2.8;6)\qquad \frac{dH_M(\zeta)}{d\zeta}=-\tfrac{1}{2}\frac{(\zeta-\alpha)\sum_{i\in M}n_i-\sum_{i\in M}\beta_i\sum_{\gamma=1}^{n_i}(x_{i,\gamma}-\mu_i)^2}{(\zeta-\alpha)^2}\quad(\zeta\epsilon J_M,\zeta\neq\alpha),$$

thus, β_i being positive for each $i\epsilon E$, $H_M(\zeta)$ is strictly unimodal in J_M and

$$(2.8;7)\qquad w_M=\alpha+\frac{\sum_{i\in M}\beta_i\sum_{\gamma=1}^{n_i}(x_{i,\gamma}-\mu_i)^2}{\sum_{i\in M}n_i}=\frac{\sum_{i\in M}n_iw_i}{\sum_{i\in M}n_i}\ .$$

Further if $I_i=J_i$ for each $i\epsilon E$ then (1.5;4) reduces to

$$(2.8;8)\qquad \alpha+\beta_it_i=\max_M\ \min_{M'}\Big\{\sum_{j\in T_M\cap S_{M'}}n_jw_j\Big/\sum_{j\in T_M\cap S_{M'}}n_j\ \Big|\ i\epsilon T_M\cap S_{M'}\Big\}$$

and if

$$(2.8;9)\qquad q_i=g_\nu n_i\qquad(i\epsilon E_\nu;\ \nu=1,\dots,K),$$

where $g_\nu(\nu=1,\dots,K)$ are positive numbers, then $q_1,\dots,q_k$ satisfy (1.6;16). Consequently the point where H attains its maximum in B coincides with the point where

$$(2.8;10)\qquad \sum_{\nu=1}^{K}g_\nu\sum_{i\in E_\nu}n_i\left(\eta_i-\alpha-\frac{\beta_i}{n_i}\sum_{\gamma=1}^{n_i}(x_{i,\gamma}-\mu_i)^2\right)^2$$

attains its minimum in B. Further the inequality (1.6;24) reduces to

(2:8;11)

$$\sum_{\nu=1}^{K} g_\nu \sum_{i\in E_\nu} n_i\left(u_i-\alpha-\frac{\beta_i}{n_i}\sum_{\gamma=1}^{n_i}(x_{i,\gamma}-\mu_i)^2\right)(u_i-\eta_i^\circ) \leqq 0 \text{ for each point } \vec{\eta}^\circ\epsilon B$$

and (2.8;11) is identical with

(2.8;12)

$$\sum_{\nu=1}^{K} g_\nu \sum_{i\in E_\nu} n_i\beta_i^2\left(t_i-\frac{1}{n_i}\sum_{\gamma=1}^{n_i}(x_{i,\gamma}-\mu_i)^2\right)\left(t_i-\frac{\eta_i^\circ-\alpha}{\beta_i}\right) \leqq 0 \text{ for each point } \vec{\eta}^\circ\epsilon B.$$

Now let (i,j) be a pair of values with $\beta_i=\beta_j(=\beta)$ and $\mu_i=\mu_j(=\mu)$ then, for $M=\{i,j\}$, we have (cf. (2.8;4))

$$H_M(\zeta)=-\tfrac{1}{2}n_M\ln 2\pi\frac{\zeta-\alpha}{\beta}-\tfrac{1}{2}\frac{\beta\sum_{\gamma=1}^{n_M}(x_{M,\gamma}-\mu)^2}{\zeta-\alpha}\quad(\zeta\epsilon J_M), \tag{2.8;13}$$

where $n_M\overset{def}{=\!=}n_i+n_j$ and where $x_{M,\gamma}(\gamma=1,\ldots,n_M)$ denotes the pooled samples of $\underline{x}_i$ and $\underline{x}_j$. Consequently if in this case H attains its maximum in D for $\eta_i=\eta_j$ then the problem reduces to the case of k-1 samples, i.e. the samples of $\underline{x}_h(h\neq i, h\neq j)$ and the sample $x_{M,\gamma}$ $(\gamma=1,\ldots,n_M)$.

If $\alpha_i\neq\alpha_j$ for at least one pair of values $(i,j)\epsilon M$ then $H_M(\zeta)$ need not be strictly unimodal in J_M. This may be seen from the following example.

Suppose $k=2$, $n_1=n_2(=m)$ and $\alpha_1=0$. Let further $\beta_1>0, \beta_2<0$ and $\alpha_2>0$ then

$$\begin{cases} J_1=[0,\infty), J_2=(-\infty,\alpha_2], \\ J_{\{1,2\}}=[0,\alpha_2]. \end{cases} \tag{2.8;14}$$

Now let

$$\begin{cases} A_1\overset{def}{=\!=}\dfrac{\beta_1}{m}\displaystyle\sum_{\gamma=1}^{m}(x_{1,\gamma}-\mu_1)^2, \\ A_2\overset{def}{=\!=}-\dfrac{\beta_2}{m}\displaystyle\sum_{\gamma=1}^{m}(x_{2,\gamma}-\mu_2)^2, \end{cases} \tag{2.8;15}$$

then $A_1>0, A_2>0$ and, for $M=\{1,2\}$,

$$\frac{dH_M(\zeta)}{d\zeta}=-\tfrac{1}{2}m\left\{\frac{\zeta-A_1}{\zeta^2}+\frac{\zeta-\alpha_2+A_2}{(\zeta-\alpha_2)^2}\right\}\quad(0<\zeta<\alpha_2). \tag{2.8;16}$$

Further

$$(2.8;17)\qquad \begin{cases} \lim\limits_{\zeta \searrow 0} \dfrac{dH_M(\zeta)}{d\zeta} = +\infty, \\ \lim\limits_{\zeta \nearrow \alpha_2} \dfrac{dH_M(\zeta)}{d\zeta} = -\infty. \end{cases}$$

Now let

$$(2,8;18)\qquad \begin{cases} \left(\dfrac{dH_M(\zeta)}{d\zeta}\right)_{\zeta=\frac{1}{3}\alpha_2} < 0 \quad \text{i.e. } 4A_1 - A_2 < \frac{2}{3}\alpha_2, \\ \left(\dfrac{dH_M(\zeta)}{d\zeta}\right)_{\zeta=\frac{1}{2}\alpha_2} > 0 \quad \text{i.e. } A_1 > A_2, \end{cases}$$

then it follows from (2.8;17) that $\frac{dH_M(\zeta)}{d\zeta}=0$ for three values of $\zeta \epsilon J_M$. Further (2.8;18) is e.g. satisfied if

$$(2.8;19)\qquad A_2 < A_1 < \tfrac{1}{6}\alpha_2.$$

From the previous sections it follows that the conditions A″ and B′ are also satisfied for some combinations of two or more of the examples considered in the sections 2.3-2.8, e.g. if $\underline{x}_i$ possesses a normal distribution with mean θ_i and known variance σ_i^2 for $i \epsilon M_o$ and an exponential distribution with parameter θ_i for $i \epsilon \overline{M}_o (M_o \neq o, \overline{M}_o \neq o)$.

CHAPTER 3

THE CONSISTENCY OF THE MAXIMUM LIKELIHOOD ESTIMATES OF ORDERED PARAMETERS

3.1 *Introduction*

In this chapter the consistency of the maximum likelihood estimates of $\theta_1,\dots,\theta_k$ under the restrictions (2.1;3) will be investigated. In the proof of the consistency no assumptions are made on the differentiability of the likelihood function; the method used is based on a proof by A. WALD [14], but is simplified by condition A′ (cf. p. 22) which does not occur in WALD's paper. In the sections 3.3-3.8 the consistency-theorem will be illustrated by means of the examples of chapter 2.

3.2 *The consistency of* $\underline{t}_1,\dots,\underline{t}_k$

In this chapter we suppose that the following condition is satisfied.
Condition C: For each $i\in E$ *the interval* I_i *and the function* φ_i *does not depend on the observations* $x_{1,1},\dots,x_{k,n_k}$.

Now let, for $\eta\in I_i, Z_i(\Phi_i(\eta))$ denote the set of all values of x for which $f_i(x|\Phi_i(\eta)) > 0$. Then we have, for all $\gamma=1,\dots,n_i$ and all $i\in E$,

$$P[\underline{x}_{i,\gamma}\in Z_i(\theta_i)|\theta_i]=1, \tag{3.2;1}$$

thus the n observations $x_{1,1},\dots,x_{k,n_k}$ may be considered as the coordinates of a point, say $\vec{x}$, in the Cartesian product

$$Z(\theta_1,\dots,\theta_k;n_1,\dots,n_k)\overset{\text{def}}{=\!=}\prod_{i\in E}\prod_{\gamma=1}^{n_i} Z_i(\theta_i), \tag{3.2;2}$$

where

$$n\overset{\text{def}}{=\!=}\sum_{i\in E} n_i\,. \tag{3.2;3}$$

We now suppose that the following condition is satisfied.
Condition D: The conditions A′ *and* B *are satisfied for each set of numbers* $n_1,\dots,n_k$ *with* $\min_{i\in E} n_i \geqq 1$ *and for each point* $\vec{x}\in Z(\theta_1,\dots,\theta_k;n_1,\dots,n_k)$.

Now consider, for $i\in E$, the function

$$(3.2;4) \qquad \ln\frac{f_i(x|\Phi_i(\eta))}{f_i(x|\theta_i)}.$$

This function is defined for each $\eta\in I_i$ and each $x\in Z_i(\theta_i)$. This may be seen as follows. For each $x\in Z_i(\theta_i)$ we have: $f_i(x|\theta_i) > 0$, thus $\ln f_i(x|\theta_i) > -\infty$. Further for a fixed value of $x\in Z_i(\theta_i)$ we have by condition B: $\ln f_i(x|\Phi_i(\eta)) < \infty$ for each $\eta\in I_i$. Thus, for each $i\in E$,

$$(3.2;5) \quad -\infty \leqq \ln\frac{f_i(x|\Phi_i(\eta))}{f_i(x|\theta_i)} < \infty \quad \text{for each } \eta\in I_i \text{ and each } x\in Z_i(\theta_i).$$

Now let, for each $i\in E$, $[c_i,d_i]$ (with $c_i < d_i$) denote the interval I_i and let $\delta_1,\dots,\delta_k$ be numbers satisfying (cf. fig. 3.2;1)

(3.2;6)

$$\begin{cases} 1.\ 0<\delta_i\leqq\min\{\varphi_i(\theta_i)-c_i,d_i-\varphi_i(\theta_i)\} & \text{if } \varphi_i(\theta_i) \text{ is an inner point of } I_i, \\ 2.\ 0<\delta_i\leqq d_i-c_i & \text{if } \varphi_i(\theta_i) \text{ is a borderpoint of } I_i. \end{cases}$$

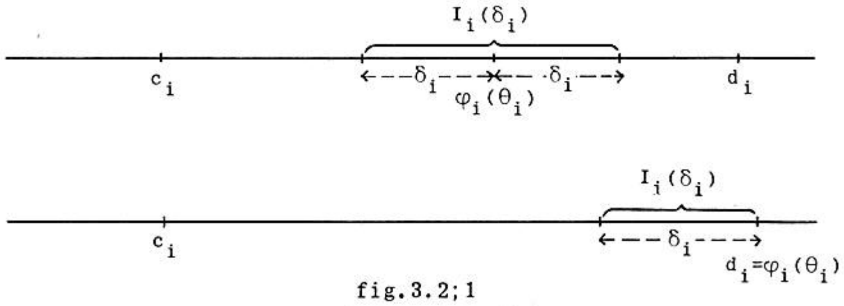


fig.3.2;1
The interval $I_i(\delta_i)$

Let further, for $\delta > 0$, $I_i(\delta)$ denote the set of all values of $\eta\in I_i$ satisfying

$$(3.2;7) \qquad |\eta-\varphi_i(\theta_i)| \leqq \delta \quad (i\in E)$$

and let, for $\eta\in I_i(\delta_i)$,

$$(3.2;8) \quad \begin{cases} g_i(\eta)\overset{\text{def}}{=\!=}\mathcal{E}\{\ln\frac{f_i(\underline{x}_i|\Phi_i(\eta))}{f_i(\underline{x}_i|\theta_i)}|\theta_i\} \\ \\ G_i(\eta)\overset{\text{def}}{=\!=}\sigma^2\{\ln\frac{f_i(\underline{x}_i|\Phi_i(\eta))}{f_i(\underline{x}_i|\theta_i)}|\theta_i\} \end{cases} \quad (i\in E)$$

Then we suppose that the following condition is satisfied.
Condition E: For each $i \in E$ *a number* δ_i *exists satisfying (3.2;6) such that for each* $\eta \in I_i(\delta_i)$ *with* $\eta \neq \varphi_i(\theta_i)$

$$\begin{cases} 1. & -\infty \leqq g_i(\eta) < 0, \\ 2. & G_i(\eta) < \infty. \end{cases}$$ [1]

WALD's [14] assumptions 2, 4 and 6 are sufficient for condition E 1. In our case, where condition C and D are satisfied, the following lemma holds.

Lemma 3.2;1: If condition C *and* D *are satisfied, if*

(3.2;9) $$-\infty < \mathcal{E}\{\ln f_i(\underline{x}_i|\theta_i)|\theta_i\} < \infty$$

and if a number δ_i *exists satisfying (3.2;6) such that*

(3.2;10)

$$-\infty \leqq \mathcal{E}\{\ln f_i(\underline{x}_i|\Phi_i(\eta))|\theta_i\} < \infty \qquad \text{for each } \eta \in I_i(\delta_i) \text{ with } \eta \neq \varphi_i(\theta_i)$$

then

(3.2;11) $$-\infty \leqq g_i(\eta) < 0 \qquad \text{for each } \eta \in I_i(\delta_i) \text{ with } \eta = \varphi_i(\theta_i).$$

Proof:

Consider any $\eta \in I_i(\delta_i)$ with $\eta \neq \varphi_i(\theta_i)$ then (cf. (3.2;10)) $\mathcal{E}\{\ln f_i(\underline{x}_i|\Phi_i(\eta))|\theta_i\} < \infty$. Clearly $g_i(\eta) < 0$ if $\mathcal{E}\{\ln f_i(\underline{x}_i|\Phi_i(\eta))|\theta_i\} = -\infty$. Thus it is sufficient to prove (3.2;11) for the case that $\mathcal{E}\{\ln f_i(\underline{x}_i|\Phi_i(\eta))|\theta_i\} > -\infty$. Then (cf. (3.2;9))

(3.2;12) $$-\infty < g_i(\eta) < \infty$$

and from (3.2;12) follows[2]

---

1) In most cases (cf. the sections 3.3-3.8) this condition is satisfied for each η in the open interval J_i with $\eta \neq \varphi_i(\theta_i)$.

2) If $\underline{X}$ is a random variable with

$$P[\underline{X} \geqq 0] = 1,$$
$$|\mathcal{E} \ln \underline{X}| < \infty, \ |\ln \mathcal{E}\underline{X}| < \infty,$$

then

$$\mathcal{E} \ln \underline{X} \leqq \ln \mathcal{E}\underline{X} .$$

The equality sign holds if and only if a number c exists with

$$P[\underline{X} = c] = 1.$$

$$(3.2;13)\qquad g_i(\eta)=\int_{x\in Z_i(\theta_i)} \ln \frac{f_i(x|\Phi_i(\eta))}{f_i(x|\theta_i)}\, dF_i(x|\theta_i) \leqq$$

$$\leqq \ln \int_{x\in Z_i(\theta_i)} \frac{f_i(x|\Phi_i(\eta))}{f_i(x|\theta_i)}\, dF_i(x|\theta_i)=$$

$$= \ln \int_{x\in Z_i(\theta_i)} f_i(x|\Phi_i(\eta))dx \leqq \ln 1=0.$$

Thus the lemma is proved if we show that in at least one of the two inequalities of (3.2;13) the equality sign does not hold. The first inequality in (3.2;13) is an equality if and only if a number c exists with

$$(3.2;14)\qquad P[\ln \frac{f_i(\underline{x}_i|\Phi_i(\eta))}{f_i(\underline{x}_i|\theta_i)}=c|\theta_i]=1$$

and then we have (cf. (3.2;12) and (3.2;13))

$$(3.2;15)\qquad -\infty < g_i(\eta)=c \leqq 0.$$

Thus it is sufficient to prove that each number c satisfying (3.2;14) satisfies $c\neq 0$. This may be proved as follows. Suppose $c=0$, then (3.2;14) reduces to

$$(3.2;16)\qquad P[\ln f_i(\underline{x}_i|\Phi_i(\eta))=\ln f_i(\underline{x}_i|\theta_i)|\theta_i]=1.$$

The fact that $\ln f_i(x|\Phi_i(\eta))$ is, for each $x\in Z_i(\theta_i)$, strictly unimodal in I_i then entails that

$$(3.2;17)\qquad P[\ln f_i(\underline{x}_i|\Phi_i(\eta')) > \ln f_i(\underline{x}_i|\theta_i)|\theta_i]=1$$

for each η' between η and $\varphi_i(\theta_i)$.
Further (3.2;17) is identical with

$$(3.2;18)\qquad P[f_i(\underline{x}_i|\Phi_i(\eta')) > f_i(\underline{x}_i|\theta_i)|\theta_i]=1$$

for each η' between η and $\varphi_i(\theta_i)$, which is in contradiction with

$$(3.2;19)\qquad \int_{x\in Z_i(\Phi_i(\eta'))} dF_i(x|\Phi_i(\eta'))= \int_{x\in Z_i(\theta_i)} dF_i(x|\theta_i)=1.$$

Thus if c is a number satisfying (3.2;14) then $c\neq 0$.

Now let

$$(3.2;20)\qquad \theta_i' \overset{\text{def}}{=\!=} \varphi_i(\theta_i) \quad (i\in E),$$

then

(3.2;21) $$\theta_i = \Phi_i(\theta_i') \qquad (i \in E)$$

and we have

Theorem 3.2;1: If the conditions C, D *and* E *are satisfied then we have, for each* $i \in E$,

(3.2;22) $$\lim_{n_i \to \infty} P[|\underline{v}_i - \theta_i'| \leqq \varepsilon | \theta_i'] = 1 \qquad \text{for each } \varepsilon > 0,$$

i.e. the maximum likelihood estimate of $\varphi_i(\theta_i)$ *under the restriction* $\varphi_i(\theta_i) \in I_i$ *is, for* $n_i \to \infty$, *a consistent estimate of* $\varphi_i(\theta_i)$.

Proof:

Consider a fixed value of $i \in E$ and let ε_1 be a positive number satisfying

(3.2;23) $$\varepsilon_1 \leqq \delta_i,$$

then (cf. fig.3.2;1)

(3.2;24)

$$\begin{cases} \theta_i' + \varepsilon_1 \in I_i(\delta_i) \text{ and } \theta_i' - \varepsilon_1 \in I_i(\delta_i) \text{ if } \theta_i' \text{ is an inner point of } I_i, \\ \theta_i' + \varepsilon_1 \in I_i(\delta_i) \text{ or } \theta_i' - \varepsilon_1 \in I_i(\delta_i) \text{ if } \theta_i' \text{ is a borderpoint of } I_i. \end{cases}$$

Thus the following two cases may be distinguished

(3.2;25) $$\begin{cases} 1. \ \theta_i' + \varepsilon_1 \in I_i(\delta_i), \\ 2. \ \theta_i' + \varepsilon_1 \notin I_i(\delta_i). \end{cases}$$

In case (3.2;25.1) we have

(3.2;26) $$\begin{cases} \mathcal{E}\{\sum_{\gamma=1}^{n_i} \ln \dfrac{f_i(\underline{x}_{i,\gamma} | \Phi_i(\theta_i' + \varepsilon_1))}{f_i(\underline{x}_{i,\gamma} | \Phi_i(\theta_i'))} | \theta_i'\} = n_i g_i(\theta_i' + \varepsilon_1) < 0, \\ \sigma^2\{\sum_{\gamma=1}^{n_i} \ln \dfrac{f_i(\underline{x}_{i,\gamma} | \Phi_i(\theta_i' + \varepsilon_1))}{f_i(\underline{x}_{i,\gamma} | \Phi_i(\theta_i'))} | \theta_i'\} = n_i G_i(\theta_i' + \varepsilon_1) < \infty \end{cases}$$

and BIENAYMÉ's inequality then entails that

(3.2;27) $$P[\sum_{\gamma=1}^{n_i} \ln f_i(\underline{x}_{i,\gamma} | \Phi_i(\theta_i' + \varepsilon_1)) > \sum_{\gamma=1}^{n_i} \ln f_i(\underline{x}_{i,\gamma} | \Phi_i(\theta_i')) | \theta_i'] =$$

$$= P[\sum_{\gamma=1}^{n_i} \ln \frac{f_i(\underline{x}_{i,\gamma} | \Phi_i(\theta_i' + \varepsilon_1))}{f_i(\underline{x}_{i,\gamma} | \Phi_i(\theta_i'))} > 0 | \theta_i'] \leqq \frac{G_i(\theta_i' + \varepsilon_1)}{n_i \{g_i(\theta_i' + \varepsilon_1)\}^2}$$

and thus

$$(3.2;28)\quad P\Big[\sum_{\gamma=1}^{n_i} \ln f_i(\underline{x}_{i,\gamma}|\Phi_i(\theta_i'+\varepsilon_1)) \leqq \sum_{\gamma=1}^{n_i} \ln f_i(\underline{x}_{i,\gamma}|\Phi_i(\theta_i'))\Big|\theta_i'\Big] \geqq$$

$$\geqq 1-\frac{G_i(\theta_i'+\varepsilon_1)}{n_i\{g_i(\theta_i'+\varepsilon_1)\}^2}.$$

Further (cf. fig. 3.2;2) the fact that $H_i(\eta)=\sum_{\gamma=1}^{n_i} \ln f_i(x_{i,\gamma}|\Phi_i(\eta))$ is a strictly unimodal function of η in the interval I_i entails that

(3.2;29)

$$v_i \leqq \theta_i'+\varepsilon_1 \text{ if } \sum_{\gamma=1}^{n_i} \ln f_i(x_{i,\gamma}|\Phi_i(\theta_i'+\varepsilon_1)) \leqq \sum_{\gamma=1}^{n_i} \ln f_i(x_{i,\gamma}|\Phi_i(\theta_i')).$$

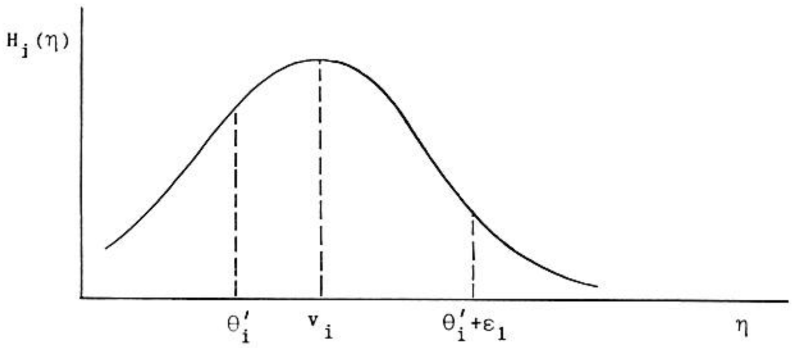


fig.3.2;2

$H_i(\theta_i') \geqq H_i(\theta_i'+\varepsilon_1)$

Consequently

$$(3.2;30)\qquad P[\underline{v}_i \leqq \theta_i'+\varepsilon_1|\theta_i'] \geqq 1-\frac{G_i(\theta_i'+\varepsilon_1)}{n_i\{g_i(\theta_i'+\varepsilon_1)\}^2}$$

and condition E then entails

$$(3.2;31)\qquad \lim_{n_i\to\infty} P[\underline{v}_i \leqq \theta_i'+\varepsilon_1|\theta_i']=1.$$

Consequently

$$(3.2;32)\qquad \lim_{n_i\to\infty} P[\underline{v}_i \leqq \theta_i'+\varepsilon|\theta_i']=1 \quad \text{for each } \varepsilon > 0.$$

In case (3.2;25.2) we have

$$(3.2;33)\qquad v_i \leqq \theta_i',$$

i.e. (3.2;32) holds in both cases (3.2;25.1) and (3.2;25.2).

In an analogous way it may be proved that

(3.2;34) $$\lim_{n_i\to\infty} P[\underline{v}_i \geqq \theta_i'-\varepsilon|\theta_i']=1 \quad \text{for each } \varepsilon > 0.$$

Theorem 3.2;2: If the conditions C, D *and* E *are satisfied, if, for each* $i\in E$ *and for each* $\eta\in I_i(\delta_i), \Phi_i(\eta)$ *is continuous*[1] *and if*

(3.2;35) $$\lim_{n\to\infty} n_i=\infty \quad \text{for each } i\in E$$

then

(3,2;36)

$$\lim_{n\to\infty} P[|\underline{t}_i-\theta_i| \leqq \varepsilon \text{ for each } i\in E|\theta_1,\dots,\theta_k]=1 \quad \text{for each } \varepsilon > 0,$$

i.e. the maximum likelihood estimates of $\theta_1,\dots,\theta_k$ *under the restrictions (2.1;3), are, for* $n\to\infty$, *consistent estimates of* $\theta_1,\dots,\theta_k$.

Proof:

Let $M_\nu(\nu=1,\dots,N)$ be subsets of E with

(3.2;37) $$\begin{cases} 1. & \bigcup_{\nu=1}^{N} M_\nu=E, \\ 2. & \theta_i' < \theta_j' \text{ for each pair of values } (i,j) \text{ with } i\in M_\nu, j\in M_\mu (\nu < \mu; \nu,\mu=1,\dots,N), \\ 3. & \theta_i'=\theta_j' \quad \text{for each pair of values } (i,j)\in M_\nu(\nu=1,\dots,N) \end{cases}$$

and let θ_ν'' denote the value of θ_i' for $i\in M_\nu(\nu=1,\dots,N)$.
Let further ε_1 denote a positive number satisfying

(3.2;38) $$\varepsilon_1 \leqq \min_{i\in E} \delta_i$$

and let

(3.2;39) $$A_\nu \overset{def}{=} \text{Ens}\{i\in M_\nu|\theta_\nu''+\varepsilon_1\in I_i(\delta_i)\} \quad (\nu=1,\dots,N).$$

Then, if A_ν' denotes the complement of A_ν in $M_\nu(\nu=1,\dots,N)$, we have (cf. (3.2;25))

(3.2;40) $$\begin{cases} 1. & \theta_\nu''+\varepsilon_1\in I_i(\delta_i) \quad \text{for } i\in A_\nu \\ 2. & \theta_\nu''+\varepsilon_1\notin I_i(\delta_i) \quad \text{for } i\in A_\nu' \end{cases} \quad (\nu=1,\dots,N).$$

---

1) The continuity of $\Phi_i(\eta)$ in $I_i(\delta_i)$ is a necessary condition for the consistency of $\underline{t}_1,\dots,\underline{t}_k$, but not for the existence of $t_1,\dots,t_k$ (cf. chapter 2). Further if $\Phi_i(\eta)$ is continuous in $I_i(\delta_i)$ then $\Phi_i(\eta)$ is monotone in $I_i(\delta_i)$ (cf. condition (2.1;2)).

Thus (cf. (3.2;33))

$$v_i \leqq \theta''_\nu \qquad (i \in A'_\nu;\ \nu=1,\ldots,N) \tag{3.2;41}$$

and (cf. (3.2;28))

$$P\Big[\sum_{\gamma=1}^{n_i} \ln f_i(\underline{x}_{i,\gamma}|\Phi_i(\theta''_\nu+\varepsilon_1)) \leqq \tag{3.2;42}$$

$$\leqq \sum_{\gamma=1}^{n_i} \ln f_i(\underline{x}_{i,\gamma}|\Phi_i(\theta''_\nu)) \text{ for each } i\in A_\nu \Big| \theta''_\nu\Big] \geqq$$

$$\geqq 1-\sum_{i\in A_\nu} \frac{G_i(\theta''_\nu+\varepsilon_1)}{n_i\{g_i(\theta''_\nu+\varepsilon_1)\}^2} \quad (\nu=1,\ldots,N).$$

Further (cf. (3.2;29)) we have, for $\nu=1,\ldots,N$,

$$v_i \leqq \theta''_\nu+\varepsilon_1 \quad \text{for each } i\in A_\nu \text{ if} \tag{3.2;43}$$

$$\sum_{\gamma=1}^{n_i} \ln f_i(x_{i,\gamma}|\Phi_i(\theta''_\nu+\varepsilon_1)) \leqq \sum_{\gamma=1}^{n_i} \ln f_i(x_{i,\gamma}|\Phi_i(\theta''_\nu)) \quad \text{for each } i\in A_\nu.$$

Consequently (cf. (3.2;41))

$$P[\max_{i\in M_\nu} \underline{v}_i \leqq \theta''_\nu+\varepsilon_1 | \theta''_\nu] \geqq 1-\sum_{i\in A_\nu} \frac{G_i(\theta''_\nu+\varepsilon_1)}{n_i\{g_i(\theta''_\nu+\varepsilon_1)\}^2} \ (\nu=1,\ldots,N). \tag{3.2;44}$$

Now let (cf. (3.2;39))

$$B_\nu \overset{\text{def}}{=\!=} \text{Ens}\{i\in M_\nu | \theta''_\nu-\varepsilon_1 \in I_i(\delta_i)\} \qquad (\nu=1,\ldots,N) \tag{3.2;45}$$

then it may be proved in an analogous way that

$$P[\min_{i\in M_\nu} \underline{v}_i \geqq \theta''_\nu-\varepsilon_1 | \theta''_\nu] \geqq 1-\sum_{i\in B_\nu} \frac{G_i(\theta''_\nu-\varepsilon_1)}{n_i\{g_i(\theta''_\nu-\varepsilon_1)\}^2} \ (\nu=1,\ldots,N). \tag{3.2;46}$$

From (3.2;44) and (3.2;46) then follows

(3.2; 47)

$$P[\theta''_\nu-\varepsilon_1 \leqq \min_{i\in M_\nu} \underline{v}_i \leqq \max_{i\in M_\nu} \underline{v}_i \leqq \theta''_\nu+\varepsilon_1 \text{ for each } \nu=1,\ldots,N | \theta''_1,\ldots,\theta''_N] \geqq$$

$$\geqq 1-\sum_{\nu=1}^{N}\Big\{\sum_{i\in A_\nu} \frac{G_i(\theta''_\nu+\varepsilon_1)}{n_i\{g_i(\theta''_\nu+\varepsilon_1)\}^2}+\sum_{i\in B_\nu} \frac{G_i(\theta''_\nu-\varepsilon_1)}{n_i\{g_i(\theta''_\nu-\varepsilon_1)\}^2}\Big\}.$$

Now take for ε_1 a positive number satisfying (3.2;38) and

$$\varepsilon_1 < \tfrac{1}{2} \min_{1\leqq\nu\leqq N-1} (\theta''_{\nu+1}-\theta''_\nu), \tag{3.2;48}$$

then

(3.2;49) $\min_{i\in M_\nu} v_i \leqq u_j \leqq \max_{i\in M_\nu} v_i$ for each $j\in M_\nu$ and each $\nu=1,\dots,N$

if $\theta''_\nu-\varepsilon_1 \leqq \min_{i\in M_\nu} v_i \leqq \max_{i\in M_\nu} v_i \leqq \theta''_\nu+\varepsilon_1$ for each $\nu=1,\dots,N$.

Consequently, if ε_1 satisfies (3.2;38) and (3.2;48) then

(3.2;50)

$$P[\theta''_\nu-\varepsilon_1 \leqq \underline{u}_j \leqq \theta''_\nu+\varepsilon_1 \text{ for each } j\in M_\nu \text{ and each } \nu=1,\dots,N|\theta''_1,\dots,\theta''_N] \geqq$$

$$\geqq 1-\sum_{\nu=1}^{N}\Big\{\sum_{i\in A_\nu}\frac{G_i(\theta''_\nu+\varepsilon_1)}{n_i\{g_i(\theta''_\nu+\varepsilon_1)\}^2}+\sum_{i\in B_\nu}\frac{G_i(\theta''_\nu-\varepsilon_1)}{n_i\{g_i(\theta''_\nu-\varepsilon_1)\}^2}\Big\}.$$

From condition E and (3.2;35) then follows

(3.2;51) $\lim_{n\to\infty} P[|\underline{u}_j-\theta'_j| \leqq \varepsilon_1 \text{ for each } j\in E|\theta'_1,\dots,\theta'_k]=1,$

consequently

(3.2;52)

$$\lim_{n\to\infty} P[|\underline{u}_j-\theta'_j| \leqq \varepsilon \text{ for each } j\in E|\theta'_1,\dots,\theta'_k]=1 \quad \text{for each } \varepsilon > 0.$$

Further (3.2;36) follows from (3.2;52) and the continuity of $\Phi_i(\eta)$ for $\eta\in I_i(\delta_i)(i\in E)$.

In the sections 3.3-3.8 the theorems of this section will be illustrated by means of the examples of chapter 2. Thus we take $\varphi_i(\tau)=\alpha_i+\beta_i\tau$, where $\beta_i\neq 0$ and α_i are given numbers ($i\in E$). For this special case condition E may be written in a more simple form as follows. Let (cf. fig. 3.2;3)

(3.2;53)
$$\begin{cases} c'_i \overset{\text{def}}{=\!=} \dfrac{c_i-\alpha_i}{\beta_i} \\ d'_i \overset{\text{def}}{=\!=} \dfrac{d_i-\alpha_i}{\beta_i} \end{cases} \quad (i\in E),$$

then (cf. section 2.1) I'_i is the interval $[\min(c'_i,d'_i),\max(c'_i,d'_i)]$ with $c'_i\neq d'_i$ and $\theta_i\in I'_i(i\in E)$.
Let further (cf. (3.2;6)) $\delta'_1,\dots,\delta'_k$ be numbers satisfying

(3.2;54)

$$\begin{cases} 1.\ 0 < \delta'_i \leqq \min\{|c'_i-\theta_i|,|d'_i-\theta_i|\} & \text{if } \theta_i \text{ is an inner point of } I'_i, \\ 2.\ 0 < \delta'_i \leqq |d'_i-c'_i| & \text{if } \theta_i \text{ is a borderpoint of } I'_i. \end{cases}$$

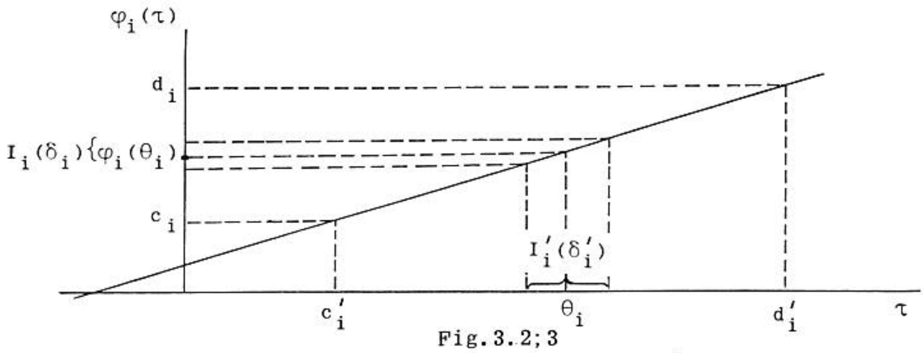


Fig. 3.2; 3
Relation between δ_i and δ_i'

and let, for $\delta > 0$, $I_i'(\delta)$ denote the set of all values of $\tau \in I_i'$ satisfying

(3.2;55) $$|\tau - \theta_i| \leqq \delta.$$

Finally let

(3.2;56) $$\begin{cases} g_i'(\tau) \overset{\text{def}}{=\!=} g_i\{\varphi_i(\tau)\} \\ G_i'(\tau) \overset{\text{def}}{=\!=} G_i\{\varphi_i(\tau)\} \end{cases} \qquad (\tau \in I_i'(\delta_i');\, i \in E),$$

then

(3.2;57) $$\begin{cases} g_i'(\tau) = \mathcal{E}\{\ln \dfrac{f_i(\underline{x}_i|\tau)}{f_i(\underline{x}_i|\theta_i)}|\theta_i\} \\ G_i'(\tau) = \sigma^2\{\ln \dfrac{f_i(\underline{x}_i|\tau)}{f_i(\underline{x}_i|\theta_i)}|\theta_i\} \end{cases} \qquad (\tau \in I_i'(\delta_i');\, i \in E)$$

and condition E is equivalent with the following condition.
Condition E': *For each* $i \in E$ *a number* δ_i' *exists satisfying (3.2;54) such that for each* $\tau \in I_i'(\delta_i')$ *with* $\tau \neq \theta_i$

$$\begin{cases} 1. \ -\infty \leqq g_i'(\tau) < 0, \\ 2. \ G_i'(\tau) < \infty. \end{cases}$$

3.3 *The binomial case*

Let, for each $i \in E$,

(3.3;1) $$P[\underline{x}_i = 1] = \theta_i, \quad P[\underline{x}_i = 0] = 1 - \theta_i \qquad (0 < \theta_i < 1),$$

then (cf. (2.3;2))

$$(3.3;2)\qquad f_i(x|\theta_i)=\begin{cases}\theta_i^x(1-\theta_i)^{1-x} & x=0 \text{ and } x=1\\ 0 & x\neq 0 \text{ and } \neq 1\end{cases}\qquad (i\epsilon E).$$

Thus, θ_i being $\neq 0$ and $\neq 1$, $Z_i(\theta_i)$ consists of the numbers 0 and 1 and from section 2.3 it then follows that condition D is satisfied. Further

$$(3.3;3)\qquad \ln f_i(\underline{x}_i|\tau)=\underline{x}_i\ln\tau+(1-\underline{x}_i)\ln(1-\tau)\qquad (0\leqq\tau\leqq 1; i\epsilon E).$$

Consequently

$$(3.3;4)\qquad g_i'(\tau)=\theta_i\ln\frac{\tau}{\theta_i}+(1-\theta_i)\ln\frac{1-\tau}{1-\theta_i}\qquad (0<\tau<1; i\epsilon E).$$

The function $g_i'(\tau)$ possesses a unique maximum in the interval $0<\tau<1$ and this maximum is attained for $\tau=\theta_i$.
Further

$$(3.3;5)\qquad g_i'(\theta_i)=0$$

consequently

$$(3.3;6)\qquad g_i'(\tau)<0\quad \text{for each } \tau\neq\theta_i \text{ with } 0<\tau<1 (i\epsilon E).$$

Further

$$(3.3;7)\qquad G_i'(\tau)=\theta_i(1-\theta_i)\{\ln\frac{\tau(1-\theta_i)}{\theta_i(1-\tau)}\}^2\qquad (0<\tau<1; i\epsilon E),$$

thus

$$(3.3;8)\qquad G_i'(\tau)<\infty\quad \text{for each } \tau \text{ with } 0<\tau<1 (i\epsilon E)$$

and (3.3;6) and (3.3;8) entail that condition E′ is satisfied.

3.4 *A normal distribution with unknown mean*

Let, for each $i\epsilon E$, $\underline{x}_i$ possess a normal distribution with mean θ_i and variance $\sigma_i^2(0<\sigma_i<\infty)$, where σ_i^2/σ_j^2 is known for each pair of values $(i,j\epsilon E)$. Then $Z_i(\theta_i)$ is the interval $(-\infty,\infty)$ and (cf. section 2.4) condition D is satisfied. Further

$$(3.4;1)\qquad \ln f_i(\underline{x}_i|\tau)=-\tfrac{1}{2}\ln 2\pi\sigma_i^2-\tfrac{1}{2}\frac{(\underline{x}_i-\tau)^2}{\sigma_i^2}\qquad (-\infty<\tau<\infty; i\epsilon E).$$

Consequently

$$(3.4;2)\qquad g_i'(\tau)=-\tfrac{1}{2}\frac{(\theta_i-\tau)^2}{\sigma_i^2}\qquad (-\infty<\tau<\infty; i\epsilon E)$$

and

$$(3.4;3) \qquad G_i'(\tau)=\frac{(\theta_i-\tau)^2}{\sigma_i^2} \qquad (-\infty < \tau < \infty;\ i\in E).$$

Thus condition E′ is satisfied.

3.5 *A Poisson distribution with unknown parameter*

Let, for each $i\in E$, $\underline{x}_i$ possess a Poisson distribution with parameter θ_i $(0 < \theta_i < \infty)$. Then (cf. (2.5;1))

$$(3.5;1) \qquad f_i(x|\theta_i)=\begin{cases} \dfrac{e^{-\theta_i}\theta_i^x}{x!} & x=0,1,\ldots \\ 0 & x\neq 0,1,\ldots \end{cases} \qquad (i\in E),$$

thus, θ_i being positive and finite, $Z_i(\theta_i)$ consists of the numbers $0,1,\ldots$. Consequently (cf. section 2.5) condition D is satisfied if and only if $\sum_{i\in M}\frac{n_i}{\beta_i}\neq 0$ for each $M\subset E$.

Further

$$(3.5;2) \qquad \ln f_i(\underline{x}_i|\tau)=-\tau+\underline{x}_i\ln\tau-\ln\underline{x}_i! \qquad (\tau \geqq 0;\ i\in E).$$

Consequently

$$(3.5;3) \qquad g_i'(\tau)=\theta_i-\tau+\theta_i\ln\frac{\tau}{\theta_i} \qquad (\tau \geqq 0;\ i\in E).$$

The function $g_i'(\tau)$ possesses a unique maximum for $\tau \geqq 0$ attained for $\tau=\theta_i$; further

$$(3.5;4) \qquad g_i'(\theta_i)=0 \qquad (i\in E),$$

thus

$$(3.5;5) \qquad g_i'(\tau) < 0 \quad \text{for each } \tau \geqq 0 \text{ with } \tau\neq\theta_i (i\in E).$$

Further

$$(3.5;6) \qquad G_i'(\tau)=\theta_i\{\ln\frac{\tau}{\theta_i}\}^2 \qquad (\tau > 0;\ i\in E)$$

and (3.5;5) and (3.5;6) entail that condition E′ is satisfied.

3.6 *An exponential distribution with unknown parameter*

Let, for each $i\in E$, $\underline{x}_i$ possess an exponential distribution with parameter $\theta_i (0 < \theta_i < \infty)$; then $Z_i(\theta_i)$ is the interval $[0,\infty)$

and (cf. section 2.6) condition D is satisfied. Further

(3.6;1) $$\ln f_i(\underline{x}_i|\tau) = \ln \tau - \tau\, \underline{x}_i \qquad (\tau \geqq 0;\ i \in E).$$

Consequently

(3.6;2) $$g_i'(\tau) = \ln\frac{\tau}{\theta_i} - \frac{\tau - \theta_i}{\theta_i} \qquad (\tau \geqq 0;\ i \in E).$$

The (unique) maximum of $g_i'(\tau)$ in the interval $\tau \geqq 0$ is attained at the point $\tau = \theta_i$ and $g_i'(\theta_i) = 0$; consequently

(3.6;3) $$g_i'(\tau) < 0 \quad \text{for each } \tau \geqq 0 \text{ with } \tau \neq \theta_i\ (i \in E).$$

Further

(3.6;4) $$G_i'(\tau) = \frac{(\theta_i - \tau)^2}{\theta_i^2} \qquad (\tau > 0;\ i \in E),$$

thus condition E$'$ is satisfied.

3.7 *A rectangular distribution between* 0 *and* θ_i

Let, for each $i \in E$, $\underline{x}_i$ possess a rectangular distribution between 0 and θ_i ($\theta_i > 0$), then (cf. (2.7;5) and (2.7;9))

(3.7;1) $$J_i = \begin{cases} [w_i, \infty) & \text{if } \beta_i > 0, \\ (-\infty, w_i] & \text{if } \beta_i < 0, \end{cases}$$

where (cf. (2.7;3))

(3.7;2) $$w_i = \alpha_i + \beta_i \max_{1 \leqq \gamma \leqq n_i} x_{i,\gamma} \qquad (i \in E).$$

Consequently J_i depends on $x_{i,1}, \ldots, x_{i,n_i}$, i.e. I_i may depend on $x_{i,1}, \ldots, x_{i,n_i}$ ($i \in E$). This entails that the theorems of section 3.2 cannot be applied.

In this case the consistency of $\underline{t}_1, \ldots, \underline{t}_k$ may be proved as follows.

We have, for each $i \in E$, (cf. (3.2;20))

(3.7;3) $$\theta_i' = \alpha_i + \beta_i \theta_i.$$

Now let ε be a positive number with

(3.7;4) $$\varepsilon \leqq \min_{i \in E} |\beta_i| \theta_i,$$

then we have, for each $i \in E$,

(3.7;5) $$P[|\underline{w}_i - \theta_i'| \leqq \varepsilon |\theta_i'] = P[|\max_{1 \leqq \gamma \leqq n_i} \underline{x}_{i,\gamma} - \theta_i| \leqq \frac{\varepsilon}{|\beta_i|} |\theta_i].$$

Further we have

$$(3.7;6) \qquad \max_{1\leq\gamma\leq n_i} x_{i,\gamma} \leq \theta_i \quad \text{for each } i\epsilon E,$$

thus, for each $i\epsilon E$,

$$(3.7;7) \qquad P[|\underline{w}_i-\theta_i'| \leq \varepsilon|\theta_i'] = P[\max_{1\leq\gamma\leq n_i} \underline{x}_{i,\gamma} \geq \theta_i - \frac{\varepsilon}{|\beta_i|}|\theta_i].$$

Further we have, for $0 \leq x \leq \theta_i$ and for each $i\epsilon E$,

$$(3.7;8) \qquad P[\max_{1\leq\gamma\leq n_i} \underline{x}_{i,\gamma} \geq x|\theta_i] = 1-P[\max_{1\leq\gamma\leq n_i} \underline{x}_{i,\gamma} \leq x|\theta_i] =$$

$$=1-P[\underline{x}_{i,\gamma} \leq x \text{ for each } \gamma=1,\dots,n_i|\theta_i] = 1-\prod_{\gamma=1}^{n_i} P[\underline{x}_{i,\gamma} \leq x|\theta_i] = 1-\left(\frac{x}{\theta_i}\right)^{n_i}.$$

Consequently, for each $i\epsilon E$,

$$(3.7;9) \qquad P[|\underline{w}_i-\theta_i'| \leq \varepsilon|\theta_i'] = 1-\left(1-\frac{\varepsilon}{|\beta_i|\theta_i}\right)^{n_i}.$$

Further we have, for each $i\epsilon E$, (cf. (1.6;14))

$$(3.7;10) \qquad v_i = \begin{cases} w_i \text{ if } c_i < w_i < d_i, \\ c_i \text{ if } w_i \leq c_i, \\ d_i \text{ if } w_i \geq d_i \end{cases}$$

and

$$(3.7;11) \qquad c_i \leq \theta_i' \leq d_i.$$

Consequently

$$(3.7;12) \qquad P[|\underline{v}_i-\theta_i'| \leq \varepsilon|\theta_i'] \geq 1-\left(1-\frac{\varepsilon}{|\beta_i|\theta_i}\right)^{n_i} \quad \text{for each } i\epsilon E$$

and (3.7;12) implies that

$$(3.7;13) \quad P[|\underline{v}_i-\theta_i'| \leq \varepsilon \text{ for each } i\epsilon E|\theta_1',\dots,\theta_k'] \geq 1-\sum_{i\epsilon E}\left(1-\frac{\varepsilon}{|\beta_i|\theta_i}\right)^{n_i}.$$

Now let $M_\nu(\nu=1,\dots,N)$ be subsets of E satisfying (3.2;37) and let θ_ν'' denote the value of θ_i' for $i\epsilon M_\nu(\nu=1,\dots,N)$. Let further ε_1 denote a positive number satisfying (cf. (3.2;48))

$$(3.7;14) \qquad \varepsilon_1 < \tfrac{1}{2} \min_{1\leq\nu\leq N-1} (\theta_{\nu+1}''-\theta_\nu''),$$

then (cf. (3.2;49))

(3.7;15) $\quad |u_j-\theta''_\nu| \leqq \varepsilon_1 \quad$ for each $j\epsilon M_\nu$ and each $\nu=1,\ldots,N$ if

$$|v_i-\theta'_i| \leqq \varepsilon_1 \quad \text{for each } i\epsilon E.$$

Thus for each positive $\varepsilon_1 \leqq \min\limits_{i\epsilon E}|\beta_i|\theta_i$ satisfying (3.7;14) we have (cf. (3.7;12))

(3.7;16) $\quad P[|\underline{u}_i-\theta'_i| \leqq \varepsilon_1 \text{ for each } i\epsilon E|\theta'_1,\ldots,\theta'_k] \geqq$

$$\geqq 1-\sum_{i\epsilon E}\left(1-\frac{\varepsilon_1}{|\beta_i|\theta_i}\right)^{n_i}.$$

Consequently, if

(3.7;17) $$\lim_{n\to\infty} n_i=\infty \quad \text{for each } i\epsilon E,$$

then

(3.7;18) $\lim\limits_{n\to\infty} P[|\underline{u}_i-\theta'_i| \leqq \varepsilon$ for each $i\epsilon E|\theta'_1,\ldots,\theta'_k]=1$ for each $\varepsilon>0$

and this is equivalent with

(3.7;19)

$$\lim_{n\to\infty} P[|\underline{t}_i-\theta_i| \leqq \varepsilon \text{ for each } i\epsilon E|\theta'_1,\ldots,\theta'_k]=1 \quad \text{for each } \varepsilon > 0.$$

3.8 *A normal distribution with unknown variance*

Let, for each $i\epsilon E$, $\underline{x}_i$ possess a normal distribution with known mean μ_i and variance $\theta_i (0 < \theta_i < \infty)$; then $Z_i(\theta_i)$ is the interval $(-\infty,\infty)$ and (cf. section 2.8) condition D is satisfied if $\alpha_i=\alpha$ for each $i\epsilon E$. Further

(3.8;1) $$\ln f_i(\underline{x}_i|\tau)=-\tfrac{1}{2}\ln 2\pi\tau-\tfrac{1}{2}\frac{(\underline{x}_i-\mu_i)^2}{\tau} \quad (\tau \geqq 0;\ i\epsilon E).$$

Consequently

(3.8;2) $$g'_i(\tau)=-\tfrac{1}{2}\ln\frac{\tau}{\theta_i}-\tfrac{1}{2}\frac{\theta_i-\tau}{\tau} \quad (\tau \geqq 0; i\epsilon E)$$

and from (3.8;2) follows

(3.8;3) $\quad g'_i(\tau) < 0 \quad$ for each $\tau\neq\theta_i$ with $0 \leqq \tau < \infty (i\epsilon E)$.

Further

(3.8;4) $$G'_i(\tau)=\tfrac{3}{4}(\theta_i-\tau)^2 \quad (\tau > 0; i\epsilon E),$$

thus condition E$'$ is satisfied.

CHAPTER 4

A CLASS OF TESTS FOR THE HYPOTHESIS THAT k PARAMETERS $\theta_1, \dots, \theta_k$ SATISFY THE INEQUALITIES $\theta_1 \leqq \dots \leqq \theta_k$

4.1 *Introduction*

Consider k independent random variables $\underline{x}_1, \dots, \underline{x}_k$ and, for each $i \in E \overset{def}{=} \{1, \dots, k\}$, n_i independent observations $x_{i,1}, \dots, x_{i,n_i}$ of $\underline{x}_i$. Let for each $i \in E$, θ_i denote an unknown parameter of the distribution of $\underline{x}_i$.

In this chapter a class of tests will be described for the hypothesis

(4.1;1) $$H_o: \theta_1 \leqq \dots \leqq \theta_k$$

against the alternative hypothesis

(4.1;2) H: at least one value of i exists with $\theta_i > \theta_{i+1}$.

These tests possess the following properties. Let α_o denote the size of the critical region (i.e. let α_o denote the probability, if H_o is true, of rejecting H_o). Let further α be a positive number < 1, let

(4.1;3) $$n \overset{def}{=} \sum_{i \in E} n_i$$

and let the limit $n \to \infty$ be taken under the conditions

(4.1;4) $$\lim_{n \to \infty} n_i = \infty \quad \text{for each } i \in E.$$

Then the tests are constructed in such a way that

(4.1;5)

1. $\alpha_o \leqq \alpha$,
2. the probability of rejecting H_o under the hypothesis $\theta_1 = \dots = \theta_k$ is $\geqq \alpha - \frac{1}{2}\alpha^2$,
3. the probability of rejecting H_o under the hypothesis $\theta_1 < \dots < \theta_k$ tends to zero for $n \to \infty$,
4. the probability of rejecting H_o under the hypothesis H tends to 1 for $n \to \infty$.

A description of the tests will be given in section 4.2. In section 4.3 the properties (4.1;5) will be proved and the sections 4.4-4.8 contain some special cases.

4.2 Description of the tests

The n observations $x_{1,1},\dots,x_{k,n_k}$ may be considered as the coordinates of a point $\vec{x}$ in an n-dimensional space R_n. Let further

(4.2;1) $$N_i \overset{def}{=} n_i+n_{i+1} \quad (i=1,\dots,k-1).$$

Now let, for $i=1,\dots,k-1$, Z_i [1] denote a subspace of R_n with the following properties

I. Let $\vec{x}$ denote any point in Z_i, then if $\vec{x}'$ is any point in R_n with

(4.2;2) $$\begin{cases} x'_{i,\gamma}=x_{i,\gamma} & \text{for each } \gamma=1,\dots,n_i, \\ x'_{i+1,\gamma}=x_{i+1,\gamma} & \text{for each } \gamma=1,\dots,n_{i+1}, \end{cases}$$

we have $\vec{x}'\epsilon Z_i$.

II. Let the limit $N_i\to\infty$ be taken under the conditions

(4.2;3) $$\begin{cases} \lim\limits_{N_i\to\infty} n_i=\infty, \\ \lim\limits_{N_i\to\infty} n_{i+1}=\infty \end{cases}$$

and let

(4.2;4) $$\alpha_i \overset{def}{=} P[\underline{\vec{x}}\epsilon Z_i \mid \theta_i=\theta_{i+1}]$$

then Z_i satisfies

(4.2;5) $$\begin{cases} 1.\ P[\underline{\vec{x}}\epsilon Z_i \mid \theta_i < \theta_{i+1}] \leqq \alpha_i, \\ 2.\ \lim\limits_{N_i\to\infty} P[\underline{\vec{x}}\epsilon Z_i \mid \theta_i < \theta_{i+1}]=0, \\ 3.\ \lim\limits_{N_i\to\infty} P[\underline{\vec{x}}\epsilon Z_i \mid \theta_i > \theta_{i+1}]=1. \end{cases}$$

III. Z_i and Z_j satisfy, for each pair of values (i,j) with $i < j$,

(4.2;6) $$P[\underline{\vec{x}}\epsilon Z_i \cap Z_j \mid \theta_i=\theta_{i+1} \text{ and } \theta_j=\theta_{j+1}] \leqq$$
$$\leqq P[\underline{\vec{x}}\epsilon Z_i \mid \theta_i=\theta_{i+1}].P[\underline{\vec{x}}\epsilon Z_j \mid \theta_j=\theta_{j+1}].$$

---

1) Z_i is an abbreviation of $Z_{i,n_i,n_{i+1}}$.

Now let

$$(4.2;7)\qquad Z \overset{def}{=} \bigcup_{i=1}^{k-1} Z_i,$$

then the hypothesis H_o is rejected if and only if $\vec{x} \in Z$.

These tests for the hypothesis H_o may also be described as follows. Let, for $i=1,\ldots,k-1$, T_i denote a test for the hypothesis

$$(4.2;8)\qquad H_{o,i}: \theta_i \leqq \theta_{i+1}$$

against the alternative hypothesis

$$(4.2;9)\qquad H_i: \theta_i > \theta_{i+1}$$

based on the observations of $\underline{x}_i$ and $\underline{x}_{i+1}$. Let $H_{o,i}$ be rejected if and only if $\vec{x} \in Z_i$, where $Z_1,\ldots,Z_{k-1}$ satisfy the abovementioned properties I, II and III. Then the hypothesis H_o is rejected if and only if a value of i exists for which $H_{o,i}$ is rejected.

4.3 *Some properties of the tests*

In this section the properties (4.1;5) will be proved. Let α_o denote the size of the critical region, i.e. let

$$(4.3;1)\qquad \alpha_o \overset{def}{=} P[\vec{\underline{x}} \in Z | H_o],$$

then

Theorem 4.3;1:

$$(4.3;2)\qquad \alpha_o \leqq \sum_{i=1}^{k-1} \alpha_i .$$

Proof:

We have

$$(4.3;3)\qquad \alpha_o = P[\vec{\underline{x}} \in Z | H_o] = P[\vec{\underline{x}} \in \bigcup_{i=1}^{k-1} Z_i | H_o] \leqq$$

$$\leqq \sum_{i=1}^{k-1} P[\vec{\underline{x}} \in Z_i | \theta_i \leqq \theta_{i+1}] \leqq \sum_{i=1}^{k-1} \alpha_i .$$

Theorem 4.3;2: If (4.2;6) is satisfied for each pair of values (i,j) *with* i < j *then*

$$(4.3;4)\qquad P[\vec{\underline{x}} \in Z | \theta_1 = \ldots = \theta_k] \geqq \sum_{i=1}^{k-1} \alpha_i - \tfrac{1}{2}\{\sum_{i=1}^{k-1} \alpha_i\}^2 .$$

This theorem has been proved by R.DOORNBOS and H.J.PRINS [6].

Thus, if $Z_1, \ldots, Z_{k-1}$ are chosen in such a way that

$$0 < \sum_{i=1}^{k-1} \alpha_i < 1,$$

then the tests possess the properties (4.1;5.1) and (4.1;5.2).

Theorem 4.3;3: If (4.2;5.2) is satisfied for each $i=1,\ldots,k-1$ *and if*

(4.3;5) $$\lim_{n\to\infty} n_i = \infty \quad \text{for each } i \in E$$

then

(4.3;6) $$\lim_{n\to\infty} P[\underline{\vec{x}} \in Z | \theta_1 < \ldots < \theta_k] = 0.$$

Proof:

We have

(4.3;7) $$\lim_{n\to\infty} P[\underline{\vec{x}} \in Z | \theta_1 < \ldots < \theta_k] \leqq \lim_{n\to\infty} \sum_{i=1}^{k-1} P[\underline{\vec{x}} \in Z_i | \theta_i < \theta_{i+1}] =$$

$$= \sum_{i=1}^{k-1} \lim_{N_i\to\infty} P[\underline{\vec{x}} \in Z_i | \theta_i < \theta_{i+1}] = 0.$$

Theorem 4.3;4: If (4.2;5.3) is satisfied for each $i=1,\ldots,k-1$ *and if*

(4.3;8) $$\lim_{n\to\infty} n_i = \infty \quad \text{for each } i \in E$$

then

(4.3;9) $$\lim_{n\to\infty} P[\underline{\vec{x}} \in Z | H] = 1.$$

Proof:

Let i_o be a value of i with $\theta_i > \theta_{i+1}$, then

(4.3;10) $$\lim_{n\to\infty} P[\underline{\vec{x}} \in Z | H] = 1 - \lim_{n\to\infty} P[\underline{\vec{x}} \notin Z | H] \geqq$$

$$\geqq 1 - \lim_{N_{i_o}\to\infty} P[\underline{\vec{x}} \notin Z_{i_o} | \theta_{i_o} > \theta_{i_o+1}] = 1.$$

Remarks:

4.3;1. If, for each $i=1,\ldots,k-1$, (4.2;5.2) and (4.2;5.3) also hold if $\lim_{N_i\to\infty} n_i = C_i < \infty$, then it follows easily from the proofs of theorem 4.3;3 and 4.3;4 that these theorems also hold if

$$(4.3;11)\qquad \begin{cases} \lim\limits_{n\to\infty} n_i=\infty & \text{for each } i=2,\dots,k, \\ \lim\limits_{n\to\infty} n_1=C_1. \end{cases}$$

Analogously, if for each $i=1,\dots,k-1$, (4.2;5.2) and (4.2;5.3) also hold if $\lim\limits_{N_i\to\infty} n_{i+1}=C_{i+1} < \infty$, then the theorems 4.3;3 and 4.3;4 also hold if

$$(4.3;12)\qquad \begin{cases} \lim\limits_{n\to\infty} n_i=\infty & \text{for each } i=1,\dots,k-1, \\ \lim\limits_{n\to\infty} n_k=C_k. \end{cases}$$

In section 4.4 an example of this situation will be given.
4.3;2. In the special cases, which will be considered in the sections 4.4-4.8, $Z_1,\dots,Z_{k-1}$ are chosen as follows.
Let, for each $i=1,\dots,k-1$, t_i denote a test statistic for the hypothesis $H_{o,i}$, then we choose for Z_i the set of all values of $x_{1,1},\dots,x_{k,n_k}$ satisfying

$$(4.3;13)\qquad t_i \geqq t_{i,\alpha_i},$$

where t_{i,α_i} satisfies

$$(4.3;14)\qquad P[\underline{t}_i \geqq t_{i,\alpha_i} | \theta_i=\theta_{i+1}]=\alpha_i.$$

4.4 *A rectangular distribution between* 0 *and* θ_i

Let, for each $i\epsilon E$, $\underline{x}_i$ possess a rectangular distribution between 0 and θ_i with $0 < \theta_i < \infty$. Then (cf. (2.7;9)) $z_i \overset{\text{def}}{=\!=} \max\limits_{1\leqq\gamma\leqq n_i} x_{i,\gamma}$ is the maximum likelihood estimate of $\theta_i (i\epsilon E)$. In this case we take, for $i=1,\dots,k-1$, as a test statistic for the hypothesis $H_{o,i}$

$$(4.4;1)\qquad \underline{t}_i=\frac{\underline{z}_i}{\underline{z}_{i+1}}$$

and $H_{o,i}$ is rejected if and only if

$$(4.4;2)\qquad t_i \geqq t_{i,\alpha_i},$$

where t_{i,α_i} satisfies

$$(4.4;3)\qquad P[\underline{t}_i \geqq t_{i,\alpha_i} | \theta_i=\theta_{i+1}]=\alpha_i,$$

We first prove that this test satisfies (4.2;5). In order to simplify the notation we take i=1. Further we suppose that $0 < \alpha_1 < 1$.
According to (3.7;8) we have, for $0 \leqq z \leqq \theta_j$ and j=1,2,

$$P[\underline{z}_j \leqq z] = \left(\frac{z}{\theta_j}\right)^{n_j}, \tag{4.4;4}$$

thus

(4.4;5)

$$P[\underline{t}_1 \geqq t] = \begin{cases} n_1 n_2 \int_0^1 x^{n_1-1} dx \int_0^{\frac{x\theta_1}{t\theta_2}} y^{n_2-1} dy = \frac{n_1}{N_1 t^{n_2}} \left(\frac{\theta_1}{\theta_2}\right)^{n_2} & \text{if } t \geqq \frac{\theta_1}{\theta_2}, \\ 1 - n_1 n_2 \int_0^{t\frac{\theta_2}{\theta_1}} x^{n_1-1} dx \int_{\frac{x\theta_1}{t\theta_2}}^{1} y^{n_2-1} dy = 1 - \frac{n_2}{N_1} t^{n_1} \left(\frac{\theta_2}{\theta_1}\right)^{n_1} & \text{if } t \leqq \frac{\theta_1}{\theta_2}. \end{cases}$$

Consequently (cf. (4.4;3))

$$t_{1,\alpha_1} = \begin{cases} \left(\frac{n_1}{N_1\alpha_1}\right)^{\frac{1}{n_2}} & \text{if } \alpha_1 \leqq \frac{n_1}{N_1}, \\ \left\{\frac{N_1}{n_2}(1-\alpha_1)\right\}^{\frac{1}{n_1}} & \text{if } \alpha_1 \geqq \frac{n_1}{N_1}. \end{cases} \tag{4.4;6}$$

Further (4.4;5) entails that $P[\underline{t}_1 \geqq t]$ is, for fixed t, a monotone increasing function of $\frac{\theta_1}{\theta_2}$, i.e. (4.2;5.1) is satisfied.

We now prove (4.2;5.2) and we first consider the case that

$$\lim_{N_1\to\infty} n_1 = \infty. \tag{4.4;7}$$

Let c_1 be a positive number < 1, then (cf. (4.4;5))

$$P[\underline{t}_1 \geqq c_1 | \theta_1 = \theta_2] = 1 - \frac{n_2}{N_1} c_1^{n_1}, \tag{4.4;8}$$

thus

(4.4;9) $P[\underline{t}_1 \geqq c_1 | \theta_1 = \theta_2] > \alpha_1$ for sufficiently large n_1

and (4.4;9) entails

(4.4;10) $\quad t_{1,\alpha_1} > c_1$ for sufficiently large n_1.

Now suppose $\theta_1 < \theta_2$, then substitution of $c_1=\frac{\theta_1}{\theta_2}$ in (4.4;10) gives

(4.4;11) $\quad t_{1,\alpha_1} > \frac{\theta_1}{\theta_2}$ for sufficiently large n_1,

thus we have (cf. (4.4;5))

(4.4;12) $$\lim_{n_1\to\infty} P[\underline{t}_1 \geqq t_{1,\alpha_1}|\theta_1 < \theta_2] = \lim_{n_1\to\infty} \frac{n_1}{N_1}\left(\frac{\theta_1}{t_{1,\alpha_1}\theta_2}\right)^{n_2}.$$

Thus if moreover

(4.4;13) $$\lim_{N_1\to\infty} n_2=\infty$$

then (4.4;11) and (4.4;12) entail that

(4.4;14) $$\lim_{N_1\to\infty} P[\underline{t}_1 \geqq t_{1,\alpha_1}|\theta_1 < \theta_2]=0.$$

Thus (4.2;5.2) is satisfied if $\lim_{N_i\to\infty} n_i=\infty$ and $\lim_{N_i\to\infty} n_{i+1}=\infty$.
If on the other hand (cf. remark 4.3;1), $\lim_{N_1\to\infty} n_2=C_2 < \infty$, then

(4.4;15) $\quad \alpha_1 \leqq \frac{n_1}{N_1}$ for sufficiently large N_1,

thus (cf. (4.4;6))

(4.4;16) $\quad t_{1,\alpha_1}=\left(\frac{n_1}{N_1\alpha_1}\right)^{\frac{1}{n_2}}$ for sufficiently large N_1.

Consequently in this case we have (cf. (4.4;12))

(4.4;17) $$\lim_{N_1\to\infty} P[\underline{t}_1 \geqq t_{1,\alpha_1}|\theta_1 < \theta_2] = \lim_{N_1\to\infty} \frac{n_1}{N_1}\left(\frac{\theta_1}{t_{1,\alpha_1}\theta_2}\right)^{n_2} = \alpha_1\left(\frac{\theta_1}{\theta_2}\right)^{C_2} > 0.$$

Thus (4.2;5.2) is not satisfied if $\lim_{N_i\to\infty} n_{i+1}=C_{i+1} < \infty$.
Finally, let $\lim_{N_1\to\infty} n_1=C_1 < \infty$. Then if $0 < c_1 < 1$ we have (cf. (4.4;8))

(4.4;18) $$0 < \lim_{N_1\to\infty} P[\underline{t}_1 \geqq c_1|\theta_1=\theta_2]=1-c_1^{C_1} < 1,$$

thus, for sufficiently small α_1,

$$(4.4;19) \qquad 0 < \alpha_1 < \lim_{N_1\to\infty} P[\underline{t}_1 \geqq c_1 | \theta_1=\theta_2].$$

Substituting $\frac{\theta_1}{\theta_2}$ for c_1 1), (4.4;19) entails that

$$(4.4;20) \qquad t_{1,\alpha_1} > \frac{\theta_1}{\theta_2} \quad \text{for sufficiently large } N_1,$$

thus in this case we have

$$(4.4;21) \qquad \lim_{N_1\to\infty} P[\underline{t}_1 \geqq t_{1,\alpha_1} | \theta_1 < \theta_2] = \lim_{N_1\to\infty} \frac{n_1}{N_1}\left(\frac{\theta_1}{t_{1,\alpha_1}\theta_2}\right)^{n_2} = 0.$$

Consequently, if $\lim_{N_i\to\infty} n_i=C_i < \infty$, then (4.2;5.2) holds if α_i is sufficiently small, i.e. (cf. remark 4.3;1) theorem 4.3;3 also holds if

$$(4.4;22) \qquad \begin{cases} \lim_{n\to\infty} n_i=\infty & \text{for } i=2,\dots,k, \\ \lim_{n\to\infty} n_1=C_1 < \infty, \\ \alpha_1 \text{ is sufficiently small.} \end{cases}$$

We now prove (4.2;5.3) and we first consider the case that

$$(4.4;23) \qquad \lim_{N_1\to\infty} n_2=\infty.$$

Let c_2 be a number > 1, then

$$(4.4;24) \qquad P[\underline{t}_1 \geqq c_2 | \theta_1=\theta_2] = \frac{n_1}{N_1 c_2^{n_2}},$$

thus

$$(4.4;25) \qquad P[\underline{t}_1 \geqq c_2 | \theta_1=\theta_2] < \alpha_1 \quad \text{for sufficiently large } n_2.$$

Now suppose that $\theta_1 > \theta_2$, then (4.4;25) entails that

$$(4.4;26) \qquad t_{1,\alpha_1} < \frac{\theta_1}{\theta_2} \quad \text{for sufficiently large } n_2.$$

---

1) If $c_1=\frac{\theta_1}{\theta_2}$, (4.4;19) is identical with

$$0 < \alpha_1 < 1-\left(\frac{\theta_1}{\theta_2}\right)^{C_1}.$$

Thus (cf. (4.4;5))

$$\text{(4.4;27)} \quad \lim_{n_2\to\infty} P[\underline{t}_1 \geqq t_{1,\alpha_1} | \theta_1 > \theta_2] = 1 - \lim_{n_2\to\infty} \frac{n_2}{N_1}\left(t_{1,\alpha_1}\frac{\theta_2}{\theta_1}\right)^{n_1}.$$

Consequently, if moreover

$$\text{(4.4;28)} \quad \lim_{N_1\to\infty} n_1 = \infty,$$

then (cf. (4.4;26) and (4.4;27))

$$\text{(4.4;29)} \quad \lim_{N_1\to\infty} P[\underline{t}_1 \geqq t_{1,\alpha_1} | \theta_1 > \theta_2] = 1.$$

Thus (4.2;5.3) is satisfied if $\lim_{N_i\to\infty} n_i = \infty$ and $\lim_{N_i\to\infty} n_{i+1} = \infty$.

If, on the other hand, $\lim_{N_1\to\infty} n_1 = C_1 < \infty$ then (cf. (4.4;6))

$$\text{(4.4;30)} \quad t_{1,\alpha_1} = \left\{\frac{N_1}{n_2}(1-\alpha_1)\right\}^{\frac{1}{n_1}} \quad \text{for sufficiently large } N_1,$$

thus in this case we have

$$\text{(4.4;31)} \quad \lim_{N_1\to\infty} P[\underline{t}_1 \geqq t_{1,\alpha_1} | \theta_1 > \theta_2] = 1 - (1-\alpha_1)\left(\frac{\theta_2}{\theta_1}\right)^{C_1} < 1.$$

Consequently (4.2;5.3) is not satisfied if $\lim_{N_i\to\infty} n_i = C_i < \infty$.

Finally let $\lim_{N_1\to\infty} n_2 = C_2 < \infty$. Then, if $c_2 > 1$, we have (cf. (4.4;24))

$$\text{(4.4;32)} \quad 0 < \lim_{N_1\to\infty} P[\underline{t}_1 \geqq c_2 | \theta_1 = \theta_2] = \frac{1}{c_2^{\,C_2}} < 1,$$

thus, for sufficiently large α_1,

$$\text{(4.4;33)} \quad \lim_{N_1\to\infty} P[\underline{t}_1 \geqq c_2 | \theta_1 = \theta_2] < \alpha_1 < 1.$$

Substituting $\frac{\theta_1}{\theta_2}$ for c_2, (4.4;33) entails

$$\text{(4.4;34)} \quad t_{1,\alpha_1} < \frac{\theta_1}{\theta_2} \quad \text{for sufficiently large } N_1,$$

thus in this case we have

$$(4.4;35)\quad \lim_{N_1\to\infty} P[\underline{t}_1 \geqq t_{1,\alpha_1}|\theta_1 > \theta_2]=1-\lim_{N_1\to\infty}\frac{n_2}{N_1}\left(t_{1,\alpha_1}\frac{\theta_2}{\theta_1}\right)^{n_1}=1.$$

Consequently, if $\lim\limits_{N_i\to\infty} n_{i+1}=C_{i+1} < \infty$, then (4.2;5.3) holds if α_i is sufficiently large, i.e. (cf. remark 4.3;1) theorem 4.3;3 also holds if

$$(4.4;36)\quad \begin{cases} \lim\limits_{n\to\infty} n_i=\infty \quad \text{for } i=1,\dots,k-1, \\ \lim\limits_{n\to\infty} n_k=C_k < \infty, \\ \alpha_{k-1} \text{ is sufficiently large.} \end{cases}$$

We now prove (4.2;6), i.e. we prove that, for each pair of values (i,j) with $i < j$

$$(4.4;37)\quad P[\underline{t}_i \geqq t_{i,\alpha_i} \text{ and } \underline{t}_j \geqq t_{j,\alpha_j}|\theta_i=\theta_{i+1} \text{ and } \theta_j=\theta_{j+1}] \leqq$$

$$\leqq P[\underline{t}_i \geqq t_{i,\alpha_i}|\theta_i=\theta_{i+1}].P[\underline{t}_j \geqq t_{j,\alpha_j}|\theta_j=\theta_{j+1}].$$

It will be clear that (4.4;37) holds if $j > i+1$. Thus it is sufficient to prove (4.4;37) for $j=i+1$. In order to simplify the notation we take $i=1$ and we omit the index α_i in t_{1,α_1} and t_{2,α_2}.

First consider the case that $t_1 \geqq 1$ and $t_2 \geqq 1$; then

$$(4.4;38)\quad P[\underline{t}_1 \geqq t_1 \text{ and } t_2 \geqq t_2|\theta_1=\theta_2=\theta_3]=$$

$$=n_1n_2n_3\int_0^1 z^{n_1-1}dz\int_0^{\frac{z}{t_1}} u^{n_2-1}du\int_0^{\frac{u}{t_2}} v^{n_3-1}dv=$$

$$=\frac{n_1n_2}{(n_1+n_2+n_3)N_2t_1^{N_2}t_2^{n_3}}.$$

Thus (4.4;37) is, for i=1 and j=2, identical with (cf. (4.4;5))

$$(4.4;39)\quad \frac{n_1n_2}{(n_1+n_2+n_3)N_2t_1^{N_2}t_2^{n_3}} \leqq \frac{n_1n_2}{N_1t_1^{n_2}N_2t_2^{n_3}}.$$

which is identical with

$$t_1^{n_3} \geqq \frac{N_1}{n_1+n_2+n_3}, \tag{4.4;40}$$

which is evidently true.

In an analogous way (4.4;37) may be proved for the case that $t_i < 1$ for at least one value of $i=1,2$.

Remark 4.4;1

If

$$n_1=\ldots=n_k \tag{4.4;41}$$

and

$$\alpha_1=\ldots=\alpha_{k-1}, \tag{4.4;42}$$

then (cf. (4.4;6))

$$t_{1,\alpha_1}=\ldots=t_{k-1,\alpha_{k-1}} (=t_\alpha, \text{ say}). \tag{4.4;43}$$

Thus in this case H_o is rejected if and only if

$$\max_{1\leqq i\leqq k-1} t_i \geqq t_\alpha. \tag{4.4;44}$$

4.5 *An exponential distribution with unknown parameter*

Let, for each $i\epsilon E$, $\underline{x}_i$ possess an exponential distribution with parameter θ_i $(0 < \theta_i < \infty)$, i.e. let

$$P[\underline{x}_i \geqq x]=e^{-\theta_i x} \quad (x \geqq 0; i\epsilon E). \tag{4.5;1}$$

Now let

$$\bar{x}_i \overset{\text{def}}{=\!=} \frac{1}{n_i}\sum_{\gamma=1}^{n_i} x_{i,\gamma} \quad (i\epsilon E), \tag{4.5;2}$$

then we take, as test statistic for the hypothesis $H_{o,i}$ (cf. e.g. [11])

$$\underline{t}_i=\frac{\bar{\underline{x}}_{i+1}}{\bar{\underline{x}}_i} \quad (i=1,\ldots,k-1). \tag{4.5;3}$$

Now (4.5;1) entails that, for each $i\epsilon E$, $2\theta_i n_i \bar{\underline{x}}_i$ possesses a χ^2-distribution with $2n_i$ degrees of freedom, i.e. $\frac{\theta_{i+1}}{\theta_i}\underline{t}_i$ (and thus $\underline{t}_i$ under the hypothesis $\theta_i=\theta_{i+1}$) possesses an F-distribution with $2n_{i+1}$ and $2n_i$ degrees of freedom $(i=1,\dots,k-1)$. Further we take critical regions of the form $t_i \geqq t_{i,\alpha_i}$, where t_{i,α_i} satisfies

$$(4.5;4)\quad P[\underline{t}_i \geqq t_{i,\alpha_i} | \theta_i=\theta_{i+1}]=\alpha_i \quad (i=1,\dots,k-1).$$

Thus, for $i=1,\dots,k-1$, t_{i,α_i} may be found from a table of the F-distribution.

Remark 4.5;1

If $n_1=\dots=n_k$ then $\underline{t}_1,\dots,\underline{t}_{k-1}$ possess under the hypothesis $\theta_1=\dots=\theta_k$ the same probability distribution. Thus, if moreover $\alpha_1=\dots=\alpha_{k-1}$, then $t_{1,\alpha_1}=\dots=t_{k-1,\alpha_{k-1}}$ ($=t_\alpha$, say). Consequently in this case H_o is rejected if and only if

$$(4.5;5)\qquad \max_{1\leqq i\leqq k-1} t_i \geqq t_\alpha.$$

We now prove (4.2;5). In order to simplify the notation we take $i=1$. Let

$$(4.5;6)\qquad \underline{t}_1' \overset{\text{def}}{=} \frac{\theta_2}{\theta_1}\underline{t}_1,$$

then

$$(4.5;7)\qquad P[\underline{t}_1 \geqq t_{1,\alpha_1}]=P[\underline{t}_1' \geqq \frac{\theta_2}{\theta_1}t_{1,\alpha_1}].$$

Consequently $P[\underline{t}_1 \geqq t_{1,\alpha_1}]$ is a monotone increasing function of $\frac{\theta_1}{\theta_2}$, i.e. (4.2;5.1) is satisfied.

We now prove (4.2;5.2)[1]. We have, $\frac{\theta_2}{\theta_1}\underline{t}_1$ possessing an F-distribution with $2n_2$ and $2n_1$ degrees of freedom,

$$(4.5;8)\qquad \begin{cases} \mathcal{E}\underline{t}_1=\dfrac{\theta_1}{\theta_2}\,\dfrac{n_1}{n_1-1}, \\[2ex] \sigma^2(\underline{t}_1)=\left(\dfrac{\theta_1}{\theta_2}\right)^2 \dfrac{n_1^2(N_1-1)}{n_2(n_1-2)(n_1-1)^2}, \end{cases}$$

1) The proof of (4.2;5.2) and (4.2;5.3) is based on a method which may be found in D. VAN DANTZIG [3].

thus

$$(4.5;9) \qquad \mathcal{E}(\underline{t}_1|\theta_1=\theta_2)=\frac{n_1}{n_1-1},$$

i.e., for sufficiently small α_1, we have

$$(4.5;10) \qquad t_{1,\alpha_1} \geqq \frac{n_1}{n_1-1}.$$

Now consider the case that $\theta_1 < \theta_2$, then

$$(4.5;11) \qquad P[\underline{t}_1 \geqq t_{1,\alpha_1}|\theta_1 < \theta_2] = $$

$$=P[\underline{t}_1-\mathcal{E}\underline{t}_1 \geqq t_{1,\alpha_1}-\frac{\theta_1}{\theta_2}\frac{n_1}{n_1-1}|\theta_1 < \theta_2],$$

where (cf. (4.5;10))

$$(4.5;12) \qquad \frac{\theta_1}{\theta_2}\frac{n_1}{n_1-1} < \frac{n_1}{n_1-1} \leqq t_{1,\alpha_1}.$$

Thus, if

$$(4.5;13) \qquad \begin{cases} \lim\limits_{N_1\to\infty} n_1=\infty, \\ \lim\limits_{N_1\to\infty} n_2=\infty, \end{cases}$$

then, according to BIENAYMÉ's inequality,

$$(4.5;14) \quad \lim_{N_1\to\infty} P[\underline{t}_1 \geqq t_{1,\alpha_1}|\theta_1 < \theta_2] \leqq$$

$$\leqq \left(\frac{\theta_1}{\theta_2}\right)^2 \lim_{N_1\to\infty} \frac{n_1^2(N_1-1)}{n_2(n_1-2)(n_1-1)^2(t_{1,\alpha_1}-\frac{\theta_1}{\theta_2}\frac{n_1}{n_1-1})^2} \leqq$$

$$\leqq \left(\frac{\theta_1}{\theta_2-\theta_1}\right)^2 \lim_{N_1\to\infty} \frac{N_1}{n_1 n_2} = 0.$$

Consequently (4.2;5.2) is satisfied if $\lim\limits_{N_i\to\infty} n_i=\infty$ and $\lim\limits_{N_i\to\infty} n_{i+1}=\infty$.

We now prove (4.2;5.3). Let c be a constant > 1, then, according to BIENAYMÉ's inequality,

$$(4.5;15) \qquad P[\underline{t}_1 \geqq c\frac{n_1}{n_1-1}|\theta_1=\theta_2] \leqq \frac{N_1-1}{n_2(n_1-2)(c-1)^2},$$

thus for each $c > 1$ we have

(4.5;16) $\quad t_{1,\alpha_1} < c\frac{n_1}{n_1-1}$ for sufficiently large n_1 and n_2.

Now let $\theta_1 > \theta_2$, then

(4.5;17) $$P[\underline{t}_1 < t_{1,\alpha_1} | \theta_1 > \theta_2] =$$

$$= P[\underline{t}_1 - \mathcal{E}\underline{t}_1 < t_{1,\alpha_1} - \frac{\theta_1}{\theta_2}\frac{n_1}{n_1-1} | \theta_1 > \theta_2],$$

where (cf. (4.5;16))

(4.5;18) $\quad t_{1,\alpha_1} < \frac{\theta_1}{\theta_2}\frac{n_1}{n_1-1}$ for sufficiently large n_1 and n_2.

Consequently if (4.5;13) is satisfied then we have, according to BIENAYMÉ's inequality

(4.5;19) $$\lim_{N_1\to\infty} P[\underline{t}_1 < t_{1,\alpha_1} | \theta_1 > \theta_2] \leqq$$

$$\leqq \left(\frac{\theta_1}{\theta_2}\right)^2 \lim_{N_1\to\infty} \frac{n_1^2(N_1-1)}{n_2(n_1-2)(n_1-1)^2(t_{1,\alpha_1} - \frac{\theta_1}{\theta_2}\frac{n_1}{n_1-1})^2} = 0,$$

thus

(4.5;20) $$\lim_{N_1\to\infty} P[\underline{t}_1 \geqq t_{1,\alpha_1} | \theta_1 > \theta_2] = 1.$$

Consequently (4.2;5.3) is satisfied if $\lim_{N_i\to\infty} n_i = \infty$ and $\lim_{N_i\to\infty} n_{i+1} = \infty$.

We now prove (4.2;6). It will be clear that this inequality holds if $j > i+1$. Thus it is sufficient to prove (4.2;6) for the case that $j=i+1$. In order to simplify the notation we omit the index α_i in t_{i,α_i} and we take $i=1$. Then, $2\theta_i n_i \underline{\overline{x}}_i$ possessing, for each $i \in E$, a χ^2-distribution with $2n_i$ degrees of freedom, we have

(4.5;21) $$P[\underline{t}_1 \geqq t_1 \text{ and } \underline{t}_2 \geqq t_2 | \theta_1 = \theta_2 = \theta_3] =$$

$$= \frac{1}{2^{n_1+n_2+n_3}\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)} \cdot$$

$$\cdot \int_0^\infty x^{n_1-1} e^{-\frac{1}{2}x} dx \int_{t_1\frac{n_2}{n_1}x}^\infty y^{n_2-1} e^{-\frac{1}{2}y} dy \int_{t_2\frac{n_3}{n_2}y}^\infty z^{n_3-1} e^{-\frac{1}{2}z} dz.$$

Thus if $f(t_1,t_2)$ is the simultaneous density function of $\underline{t}_1$ and $\underline{t}_2$, then

(4.5;22)

$$f(t_1,t_2)=\frac{1}{2^{n_1+n_2+n_3}\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)}\left(\frac{n_3}{n_2}\right)^{n_3}\left(\frac{n_2}{n_1}\right)^{n_2+n_3} t_1^{N_2-1} t_2^{n_3-1}.$$

$$.\int_0^\infty x^{n_1+n_2+n_3-1} e^{-\frac{1}{2}x(1+t_1\frac{n_2}{n_1}+t_1t_2\frac{n_3}{n_1})} dx=$$

$$=\frac{\Gamma(n_1+n_2+n_3)}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)}\left(\frac{n_3}{n_2}\right)^{n_3}\left(\frac{n_2}{n_1}\right)^{n_2+n_3}\frac{t_1^{N_2-1}t_2^{n_3-1}}{(1+t_1\frac{n_2}{n_1}+t_1t_2\frac{n_3}{n_1})^{n_1+n_2+n_3}}.$$

According to theorem 6.1 in [5] condition (4.2;6) is satisfied if

(4.5;23) $$f(t_1,t_2).f(s_1,s_2) \leqq f(t_1,s_2).f(s_1,t_2)$$

for each t_1, t_2, s_1 and s_2 with $t_1 \geqq s_1$ and $t_2 \geqq s_2$.
From (4.5;22) it follows that (4.5;23) is identical with

(4.5;24) $$(1+t_1\frac{n_2}{n_1}+t_1t_2\frac{n_3}{n_1})(1+s_1\frac{n_2}{n_1}+s_1s_2\frac{n_3}{n_1}) \geqq$$

$$\geqq (1+t_1\frac{n_2}{n_1}+t_1s_2\frac{n_3}{n_1})(1+s_1\frac{n_2}{n_1}+s_1t_2\frac{n_3}{n_1}) \quad \text{for } t_1 \geqq s_1 \text{ and } t_2 \geqq s_2,$$

which is identical with

(4.5;25) $$\frac{n_3}{n_1}(t_1-s_1)(t_2-s_2) \geqq 0 \quad \text{for } t_1 \geqq s_1 \text{ and } t_2 \geqq s_2.$$

Consequently condition (4.2;6) is satisfied.

4.6 *A normal distribution with unknown variance*

Let, for each $i \in E$, $\underline{x}_i$ possess a normal distribution with unknown mean μ_i and unknown variance θ_i. Then if

(4.6;1) $$\begin{cases} \overline{x}_i \overset{def}{=\!=} \frac{1}{n_i}\sum_{\gamma=1}^{n_i} x_{i,\gamma} \\ s_i^2 \overset{def}{=\!=} \frac{1}{n_i-1}\sum_{\gamma=1}^{n_i}(x_{i,\gamma}-\overline{x}_i)^2 \end{cases} \qquad (i \in E)$$

we take as a test statistic for the hypothesis $H_{o,i}$

$$\underline{t}_i=\frac{\underline{s}_i^2}{\underline{s}_{i+1}^2} \quad (i=1,\dots,k-1) \tag{4.6;2}$$

Now $\frac{(n_i-1)\underline{s}_i^2}{\theta_i}$ possesses a χ^2-distribution with n_i-1 degrees of freedom, i.e. $\frac{\theta_{i+1}}{\theta_i}\underline{t}_i$ possesses an F-distribution with n_i-1 and $n_{i+1}-1$ degrees of freedom. We again take critical regions of the form $t_i \geqq t_{i,\alpha_i}$ $(i=1,\dots,k-1)$ and the proofs of (4.2;5) and (4.2;6) are identical with the proofs given in section (4.5).

Remark 4.6;1

If μ_i is known then $\underline{s}_i^2$ is replaced by $\frac{1}{n_i}\sum_{\gamma=1}^{n_i}(\underline{x}_{i,\gamma}-\mu_i)^2$, possessing a χ^2-distribution with n_i degrees of freedom $(i\in E)$.

4.7 *A normal distribution with unknown mean*

Let, for each $i\in E$, $\underline{x}_i$ possess a normal distribution with mean θ_i and known variance σ_i^2. Then if

$$\overline{x}_i \stackrel{\text{def}}{=} \frac{1}{n_i}\sum_{\gamma=1}^{n_i} x_{i,\gamma} \quad (i\in E) \tag{4.7;1}$$

we take, for $i=1,\dots,k-1$, as a test statistic for the hypothesis $H_{o,i}$

$$\underline{t}_i=\overline{\underline{x}}_i-\overline{\underline{x}}_{i+1}. \tag{4.7;2}$$

Now $\underline{t}_i$ possesses a normal distribution with

$$\begin{cases} \mathcal{E}\underline{t}_i=\theta_i-\theta_{i+1} \\ \sigma^2(\underline{t}_i)=\frac{\sigma_i^2}{n_i}+\frac{\sigma_{i+1}^2}{n_{i+1}} \end{cases} \quad (i=1,\dots,k-1). \tag{4.7;3}$$

The hypothesis $H_{o,i}$ is rejected if and only if $t_i \geqq t_{i,\alpha_i}$ and (4.7;3) then entails that

$$t_{i,\alpha_i}=\xi_{\alpha_i}\sqrt{\frac{\sigma_i^2}{n_i}+\frac{\sigma_{i+1}^2}{n_{i+1}}} \quad (i=1,\dots,k-1), \tag{4.7;4}$$

where ξ_α is defined by

$$\frac{1}{\sqrt{2\pi}}\int_{\xi_\alpha}^{\infty} e^{-\frac{1}{2}x^2} dx=\alpha. \tag{4.7;5}$$

It will be clear that, for each $i=1,\dots,k-1$, this test for $H_{o,i}$ satisfies (4.2;5) if

$$\begin{cases} \lim_{N_i\to\infty} n_i=\infty, \\ \lim_{N_i\to\infty} n_{i+1}=\infty. \end{cases} \tag{4.7;6}$$

Further it will be clear that the inequality (4.2;6) holds for each pair of values (i,j) with $j > i+1$. Thus it is sufficient to prove (4.2;6) for the case that $j=i+1$. In order to simplify the notation we take $i=1$. Then

$$\mathcal{E}\underline{t}_1\underline{t}_2=\mathcal{E}(\underline{\bar{x}}_1\underline{\bar{x}}_2-\underline{\bar{x}}_1\underline{\bar{x}}_3-\underline{\bar{x}}_2^2+\underline{\bar{x}}_2\underline{\bar{x}}_3). \tag{4.7;7}$$

Thus

$$\mathcal{E}(\underline{t}_1\underline{t}_2|\theta_1=\theta_2=\theta_3)=-\sigma^2(\underline{\bar{x}}_2|\theta_1=\theta_2=\theta_3)=-\frac{\sigma_2^2}{n_2}, \tag{4.7;8}$$

i.e.

$$\mathrm{cov}(\underline{t}_1,\underline{t}_2|\theta_1=\theta_2=\theta_3)=-\frac{\sigma_2^2}{n_2}. \tag{4.7;9}$$

Consequently, if $\theta_1=\theta_2=\theta_3$, $\underline{t}_1$ and $\underline{t}_2$ possess a two-dimensional normal probability distribution with negative correlation-coefficient and in [5] it has been proved that (4.2;6) holds in this case.

Remark 4.7;1

If

$$\frac{\sigma_i^2}{n_i}=\frac{\sigma_{i+2}^2}{n_{i+2}} \quad \text{for } i=1,\dots,k-2 \tag{4.7;10}$$

and

$$\alpha_1=\dots=\alpha_{k-1}, \tag{4.7;11}$$

then (cf. (4.7;4))

$$t_{1,\alpha_1}=\dots=t_{k-1,\alpha_{k-1}} (=t_\alpha, \text{ say}). \tag{4.7;12}$$

Thus in this case H_o is rejected if and only if

(4.7; 13) $$\max_{1\leqq i\leqq k-1} (\bar{x}_i-\bar{x}_{i+1}) \geqq t_\alpha.$$

4.8 *An analogous distribution-free test*

In this section an analogous distribution-free test based on WILCOXON's two sample test (cf. e.g. [12] and [15]) will be considered.

Let $\underline{x}_1,\dots,\underline{x}_k$ be k independent random variables possessing continuous probability distributions. Let further, for each $i\epsilon E$, $x_{i,1},\dots,x_{i,n_i}$ be n_i independent observations of $\underline{x}_i$ and let (cf. e.g. [4])

(4.8; 1) $$W_i \stackrel{def}{=} \sum_{\gamma=1}^{n_i} \sum_{\lambda=1}^{n_{i+1}} sgn(x_{i,\gamma}-x_{i+1,\lambda})^{1)} \qquad (i=1,\dots,k-1),$$

where

(4.8; 2) $$sgnz= \begin{cases} 1 \text{ if } z > 0, \\ 0 \text{ if } z=0, \\ 1 \text{ if } z < 0. \end{cases}$$

In this section a test will be described for the hypothesis H_o' that $\underline{x}_1,\dots,\underline{x}_k$ possess the same probability distribution, based on $W_1,\dots,W_{k-1}$. This test is performed as follows. The statistics $W_1,\dots,W_{k-1}$ may be considered as the coordinates of a point $\vec{W}$ in a (k-1)-dimensional space R_{k-1}'. Now let, for $i=1,\dots,k-1$, $H_{o,i}'$ denote the hypothesis that $\underline{x}_i$ and $\underline{x}_{i+1}$ possess the same probability distribution and let Z_i' denote the set of all points $\vec{W}\epsilon R_{k-1}'$ satisfying

(4.8; 3) $$W_i \geqq W_{i,\alpha_i},$$

where W_{i,α_i} is the smallest integer with

(4.8; 4) $$P[\underline{W}_i \geqq W_{i,\alpha_i} | H_{o,i}'] \leqq \alpha_i.$$

---

1) If U_i is the test statistic of WILCOXON's two sample test, applied to the samples of $\underline{x}_i$ and $\underline{x}_{i+1}$, according to H.B. MANN and D.R. WHITNEY [12], then

$$W_i=2U_i-n_in_{i+1} \qquad (i=1,\dots,k-1).$$

Let further $Z' \stackrel{def}{=} \bigcup_{i=1}^{k-1} Z'_i$, then the hypothesis H'_o is rejected if and only if $\vec{\underline{W}} \in Z'$, i.e. H'_o is rejected if and only if a value of i exists with $\vec{\underline{W}} \in Z'_i$.

For small values of n_i and n_{i+1} the critical values W_{i,α_i} may be found from a table of the exact probability distribution of $\underline{W}_i$ under the hypothesis $H'_{o,i}$ (cf. e.g. [12] and [13]). For large values of n_i and n_{i+1}, $\underline{W}_i$ is under the hypothesis $H'_{o,i}$ approximately normally distributed with

$$\text{(4.8;5)} \qquad \begin{cases} \mathcal{E}\,(\underline{W}_i \mid H'_{o,i}) = 0, \\ \sigma^2(\underline{W}_i \mid H'_{o,i}) = \frac{1}{3} n_i n_{i+1} (N_i + 1), \end{cases}$$

thus in this case an approximation to W_{i,α_i} may be found from a table of the normal distribution.

In the sequel of this section some properties of this test will be proved.

We first prove the following lemma.

Lemma 4.8;1: For each pair of values (i,j) *with* $i < j$ *we have*

$$\text{(4.8;6)} \qquad P[\vec{\underline{W}} \in Z'_i \cap Z'_j \mid H'_{o,i} \text{ and } H'_{o,j}] \leqq$$
$$\leqq P[\vec{\underline{W}} \in Z'_i \mid H'_{o,i}] . P[\vec{\underline{W}} \in Z'_j \mid H_{o,j}].$$

Proof:

It will be clear that (4.8;6) holds for each pair of values (i,j) with $j > i+1$. Thus it is sufficient to prove (4.8;6) for the case that $j=i+1$. In order to simplify the notation we omit the index α_i in W_{i,α_i} and we take i=1. The proof is analogous to the proof of the inequality (9.5) in [5].

Let

(4.8;7)
1. $P_{n_h, n_{h+1}}[W_h] \stackrel{def}{=} P[\underline{W}_h \geqq W_h \mid n_h, n_{h+1}; H'_{o,h}]$ (h=1,2),
2. $P_{n_h, n_{h+1}}[W_h \mid j] \stackrel{def}{=}$ the conditional probability of $\underline{W}_h \geqq W_h$ under $H'_{o,h}$ given that the largest observation in the samples of $\underline{x}_h$ and $\underline{x}_{h+1}$ belongs to the j-th sample for the sample sizes n_h and n_{h+1} (j=h, h+1; h=1,2)

and

(4.8;8)

1. $P_{n_1,n_2,n_3}[W_1,W_2] \overset{def}{=} P[\underline{W}_1 \geqq W_1$ and $\underline{W}_2 \geqq W_2 | n_1,n_2,n_3; H'_{o,1}, H'_{o,2}]$,
2. $P_{n_1,n_2,n_3}[W_1,W_2|j] \overset{def}{=}$ the conditional probability of $\underline{W}_1 \geqq W_1$ and $\underline{W}_2 \geqq W_2$ under $H'_{o,1}$ and $H'_{o,2}$ given that the largest observation in the samples of $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$ belongs to the j-th sample for the sample sizes n_1, n_2 and n_3 (j=1,2,3),

then we have to prove

(4.8;9) $$P_{n_1,n_2,n_3}[W_1,W_2] \leqq P_{n_1,n_2}[W_1].P_{n_2,n_3}[W_2].$$

We shall prove (4.8;9) by induction with respect to $n_1+n_2+n_3$. Clearly (4.8;9) holds for $n_1+n_2+n_3 \leqq 2$ (we take $\underline{W}_h=0$ with probability 1 if $n_h=0(h=1,2)$). Now suppose (4.8;9) holds for $n_1+n_2+n_3 \leqq N-1$, then we prove that the inequality holds for $n_1+n_2+n_3=N$.

We have

(4.8;10) $$P_{n_1,n_2,n_3}[W_1,W_2] = \sum_{j=1}^{3} \frac{n_j}{N} P_{n_1,n_2,n_3}[W_1,W_2|j] =$$

$$= \frac{n_1}{N} P_{n_1-1,n_2,n_3}[W_1-n_2,W_2] + \frac{n_2}{N} P_{n_1,n_2-1,n_3}[W_1+n_1,W_2-n_3] +$$

$$+ \frac{n_3}{N} P_{n_1,n_2,n_3-1}[W_1,W_2+n_2] \leqq$$

$$\leqq \frac{n_1}{N} P_{n_1-1,n_2}[W_1-n_2].P_{n_2,n_3}[W_2] + \frac{n_2}{N} P_{n_1,n_2-1}[W_1+n_1].$$

$$.P_{n_2-1,n_3}[W_2-n_3] + \frac{n_3}{N} P_{n_1,n_2}[W_1].P_{n_2,n_3-1}[W_2+n_2] =$$

$$= \frac{n_1}{N} P_{n_1,n_2}[W_1|1].P_{n_2,n_3}[W_2] + \frac{n_2}{N} P_{n_1,n_2}[W_1|2].P_{n_2,n_3}[W_2|2] +$$

$$+ \frac{n_3}{N} P_{n_1,n_2}[W_1].P_{n_2,n_3}[W_2|3] =$$

$$= \frac{n_1}{N} P_{n_1,n_2}[W_1|1]\{\frac{n_2}{N_2} P_{n_2,n_3}[W_2|2] + \frac{n_3}{N_2} P_{n_2,n_3}[W_2|3]\} +$$

$$+ \frac{n_2}{N} P_{n_1,n_2}[W_1|2].P_{n_2,n_3}[W_2|2] +$$

$$+\frac{n_3}{N}P_{n_2,n_3}[W_2|3]\{\frac{n_1}{N_1}P_{n_1,n_2}[W_1|1]+\frac{n_2}{N_1}P_{n_1,n_2}[W_1|2]\}.$$

Thus (4.8;9) holds if

(4.8;11)

$$\frac{n_1}{N}P_{n_1,n_2}[W_1|1]\{\frac{n_2}{N_2}P_{n_2,n_3}[W_2|2]+\frac{n_3}{N_2}P_{n_2,n_3}[W_2|3]\}+$$

$$+\frac{n_2}{N}P_{n_1,n_2}[W_1|2].P_{n_2,n_3}[W_2|2]+$$

$$+\frac{n_3}{N}P_{n_2,n_3}[W_2|3]\{\frac{n_1}{N_1}P_{n_1,n_2}[W_1|1]+\frac{n_2}{N_1}P_{n_1,n_2}[W_1|2]\} \leqq$$

$$\leqq \{\frac{n_1}{N_1}P_{n_1,n_2}[W_1|1]+\frac{n_2}{N_1}P[W_1|2]\}\{\frac{n_2}{N_2}P_{n_2,n_3}[W_2|2]+\frac{n_3}{N_2}P_{n_2,n_3}[W_2|3]\},$$

which is identical with

(4.8;12)

$$\frac{n_1 n_2 n_3}{N\ N_1 N_2}\{P_{n_1,n_2}[W_1|1]-P_{n_1,n_2}[W_1|2]\}\{P_{n_2,n_3}[W_2|2]-P_{n_2,n_3}[W_2|3]\} \geqq 0$$

and in [5] (cf. p. 34) it has been proved that

(4.8;13) $\quad P_{n_i,n_{i+1}}[W_i|i] \geqq P_{n_i,n_{i+1}}[W_i|i+1] \quad (i=1,2).$

Now let α_o denote the size of the critical region, i.e. let

(4.8;14) $$\alpha_o = P[\underline{\vec{W}} \epsilon Z'|H_o'],$$

then (cf. the proofs of theorem 4.3;1 and 4.3;2) lemma 4.8;1 entails that

Theorem 4.8;1

(4.8;15) $$\sum_{i=1}^{k-1}\alpha_i - \tfrac{1}{2}\{\sum_{i=1}^{k-1}\alpha_i\}^2 \leqq \alpha_o \leqq \sum_{i=1}^{k-1}\alpha_i.$$

Now let, for $i=1,\ldots,k-1$,

(4.8;16) $$\theta_i' \stackrel{def}{=} P[\underline{x}_i > \underline{x}_{i+1}]$$

and let H_1' denote the hypothesis

(4.8;17) $\quad$ for each value of $i:\theta_i' < \tfrac{1}{2}$,

then

Theorem 4.8;2: If

(4.8;18) $$\lim_{n\to\infty} n_i=\infty \quad \text{for each } i\in E$$

then

(4.8;19) $$\lim_{n\to\infty} P[\underline{\vec{W}}\in Z' | H_1']=0.$$

Proof:

We have

(4.8;20) $$P[\underline{\vec{W}}\in Z' | H_1'] \leqq \sum_{i=1}^{k-1} P[\underline{W}_i \geqq W_{i,\alpha_i} | \theta_i' < ½].$$

Further if

(4.8;21) $$\begin{cases} \lim_{N_i\to\infty} n_i=\infty, \\ \lim_{N_i\to\infty} n_{i+1}=\infty, \end{cases}$$

then (cf. D.VAN DANTZIG [3])

(4.8;22) $$\lim_{N_i\to\infty} P[\underline{W}_i \geqq W_{i,\alpha_i} | \theta_i' < ½]=0 \quad (i=1,\dots,k-1).$$

Thus (4.8;19) is satisfied.

Now let H_2' denote the hypothesis

(4.8;23) at least one value of i exists with $\theta_i' > ½$,

then

Theorem 4.8;3: If

(4.8;24) $$\lim_{n\to\infty} n_i=\infty \quad \text{for each } i\in E$$

then

(4.8;25) $$\lim_{n\to\infty} P[\underline{\vec{W}}\in Z' | H_2']=1.$$

Proof:

Let i_o be a value of i with $\theta_i' > ½$, then

(4.8;26) $$P[\underline{\vec{W}}\in Z' | H_2']=1-P[\underline{\vec{W}}\notin Z' | H_2'] \geqq$$
$$\geqq 1-P[\underline{W}_{i_o} < W_{i_o,\alpha_{i_o}} | \theta_{i_o}' > ½].$$

Further if (4.8;21) is satisfied then we have (cf. D. VAN DANTZIG [3])

(4.8;27) $$\lim_{N_i\to\infty} P[\underline{W}_i < W_{i,\alpha_i} | \theta_i' > \tfrac{1}{2}] = 1 \qquad (i=1,\dots,k-1),$$

thus (4.8;25) is satisfied.

Finally let H_3' denote the hypothesis

(4.8;28) $$\begin{cases} \text{for each value of } i: \theta_i' \leqq \tfrac{1}{2}, \\ \text{at least one value of } i \text{ exists with } \theta_i' = \tfrac{1}{2} \end{cases}$$

and let

(4.8;29) $$\begin{cases} M \overset{\text{def}}{=\!=} \text{Ens}\{i | \theta_i' < \tfrac{1}{2}\}, \\ \overline{M} \overset{\text{def}}{=\!=} \text{Ens}\{i | \theta_i' = \tfrac{1}{2}\}. \end{cases}$$

Then $\overline{M} \neq o$ and we have

Theorem 4.8;4: If

(4.8;30) $$\lim_{n\to\infty} n_i = \infty \quad \text{for each } i \in E$$

and if α_i *is sufficiently small for each* $i \in \overline{M}$ *then*

(4.8;31) $$\lim_{n\to\infty} P[\underline{\vec{W}} \in Z' | H_3'] < 1.$$

Proof:

We have

(4.3;32) $$P[\underline{\vec{W}} \in Z' | H_3'] \leqq \sum_{i=1}^{k-1} P[\underline{W}_i \geqq W_{i,\alpha_i} | \theta_i' \leqq \tfrac{1}{2}] =$$

$$= \sum_{i\in M} P[\underline{W}_i \geqq W_{i,\alpha_i} | \theta_i' < \tfrac{1}{2}] + \sum_{i\in\overline{M}} P[\underline{W}_i \geqq W_{i,\alpha_i} | \theta_i' = \tfrac{1}{2}].$$

Thus (cf. (4.8;22))

(4.8;33) $$\lim_{n\to\infty} P[\underline{\vec{W}} \in Z' | H_3'] = \sum_{i\in\overline{M}} \lim_{N_i\to\infty} P[\underline{W}_i \geqq W_{i,\alpha_i} | \theta_i' = \tfrac{1}{2}].$$

Further if (4.8;21) is satisfied then (cf. D. VAN DANTZIG [3])

(4.8;34) $$\lim_{N_i\to\infty} P[\underline{W}_i \geqq W_{i,\alpha_i} | \theta_i' = \tfrac{1}{2}] \leqq$$

$$\leqq \lim_{N_i\to\infty} \frac{\sigma^2(\underline{W}_i | \theta_i = \tfrac{1}{2})}{\xi_{\alpha_i}^2 \sigma^2(\underline{W}_i | H_{o,i})} \leqq \frac{3}{\xi_{\alpha_i}^2} \qquad (i=1,\dots,k-1).$$

Thus

(4.8;35) $$\lim_{n\to\infty} P[\vec{\underline{W}}\in Z' | H_3'] \leqq 3 \sum_{i\in\overline{M}} \frac{1}{\xi_{\alpha_i}^2} .$$

Consequently, if

(4.8;36) $$\sum_{i\in\overline{M}} \frac{1}{\xi_{\alpha_i}^2} < \frac{1}{3} ,$$

then

(4.8;37) $$\lim_{n\to\infty} P[\vec{\underline{W}}\in Z' | H_3'] < 1.$$

Remark 4.8;1

Condition (4.8;36) is e.g. satisfied if

(4.8;38) $$\min_{i\in\overline{M}} \xi_{\alpha_i}^2 > 3(k-1).$$

APPENDIX

A.1 *Introduction*

In this appendix the results of MIRIAM AYER, H.D.BRUNK, G.M.EWING, W.T.REID and EDWARD SILVERMAN [1] and of H.D.BRUNK [2] will be compared with those given in the chapters 1-3.

In section A.2 a description will be given of the situation in which the methods of MIRIAM AYER, H.D.BRUNK et al. may be applied. The procedure itself will be given in section A.3.

Throughout this appendix we use the notation of the chapters 1-3.

A.2 *Description of the problem of MIRIAM AYER, H.D.BRUNK, et al.*

In [2] H.D.BRUNK describes the following problem. Let k and K be positive integers and let $\vec{r}^{\,i}=(r_1^i,\dots,r_K^i)\ (i\in E)$ denote k points in a K-dimensional space R_K. Let $\underline{x}_1,\dots,\underline{x}_k$ be independent random variables and let the distribution of $\underline{x}_i$ be completely specified by the knowledge of a single parameter $\theta_i\ (i\in E)$. Let the parameters $\theta_1,\dots,\theta_k$ be known to satisfy the following monotonicity condition: there is a real valued function $\theta(\vec{r})$ monotone non-decreasing in each of the separate variables $r_l\ (l=1,\dots,K)$, such that $\theta_1=\theta(\vec{r}^{\,i})\ (i\in E)$. Further let the distribution of $\underline{x}_i$ belong to (cf. [2]) the 'exponential family' $(i\in E)$ and let, for each pair of values $(i,j)\in E$, the distribution functions of $\underline{x}_i$ and $\underline{x}_j$ be identical if and only if $\theta_i=\theta_j$.

The distribution of a random variable $\underline{x}$ belongs to the 'exponential family' with parameter θ e.g. if

1. $\underline{x}$ possesses a binomial probability distribution:

$$(A.2;1)\qquad P[\underline{x}=x\,|\,\theta]=\binom{n}{x}\theta^x(1-\theta)^{n-x}\qquad (x=0,1,\dots,n),$$

2. $\underline{x}$ possesses a normal distribution with mean θ and variance 1:

$$(A.2;2)\qquad P[\underline{x}\leqq x\,|\,\theta]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}(u-\theta)^2}\,du\qquad (-\infty<x<\infty).$$

3. $\underline{x}$ possesses a normal distribution with mean 0 and variance θ:

$$(A.2;3)\qquad P[\underline{x}\leqq x\,|\,\theta]=\frac{1}{\sqrt{2\pi\theta}}\int_{-\infty}^{x} e^{-\frac{1}{2}\frac{u^2}{\theta}}\,du\qquad (-\infty<x<\infty),$$

4. $\underline{x}$ possesses a Poisson distribution with mean θ:

(A.2;4) $$P[\underline{x}=x|\theta]=\frac{e^{-\theta}\theta^x}{x!} \quad (x=0,1,\dots),$$

5. $\underline{x}$ possesses an exponential distribution:

(A.2;5) $$P[\underline{x} \leqq x|\theta]=1-e^{-\theta x} \quad (0 \leqq x < \infty).$$

Now let $x_{i,\gamma}(\gamma=1,\dots,n_i)$ be n_i independent observations of $\underline{x}_i$ ($i\epsilon E$). The problem solved by BRUNK [2] is the determination of the maximum likelihood estimates of $\theta_1,\dots,\theta_k$ subject to the above-mentioned monotonicity condition. In deriving these estimates BRUNK does not specify K and the function $\theta(\vec{r})$.
It will be clear that the monotonicity of the function $\theta(\vec{r})$ is equivalent with the partial (or complete) ordering of the parameters $\theta_1,\dots,\theta_k$ specified by the following set of inequalities. Let $\alpha_{i,j}(i,j\epsilon E)$ be numbers satisfying

(A.2;6)
1. $\alpha_{i,j}=-\alpha_{j,i}$,
2. $\alpha_{i,j}=1$ if no coordinate of $\vec{r}^i$ is greater than the corresponding coordinate of $\vec{r}^j$,
3. $\alpha_{i,j}=0$ in all other cases.

Then it follows from the fact that $\theta(\vec{r})$ is monotone non-decreasing in each of the separate variables r_l $(l=1,\dots,K)$ and that $\theta(\vec{r}^i)=\theta_i$ ($i\epsilon E$) that $\theta_1,\dots,\theta_k$ satisfy the inequalities

(A.2;7) $$\alpha_{i,j}(\theta_i-\theta_j) \leqq 0 \quad (i,j\epsilon E).$$

Further it may easily be seen (cf. [2], p. 611) that the conditions A′ and B (cf. p. 21, 22) are satisfied if
1. the distributions of $\underline{x}_1,\dots,\underline{x}_k$ belong to the 'exponential family',
2. for each pair of values $(i,j)\epsilon E$, the distribution functions of $\underline{x}_i$ and $\underline{x}_j$ are identical if and only if $\theta_i=\theta_j$.

Thus the problem solved by BRUNK is a special case of ours with, for each $i\epsilon E$,

(A.2;8)
1. the distribution of $\underline{x}_i$ belongs to the 'exponential family',
2. $\varphi_i(\tau)\equiv\tau$,
3. I_i is the set of *all* values of τ for which $F_i(x|\tau)$ is a distribution function.

On the other hand every partial (or complete) ordering of the parameters $\theta_1,\dots,\theta_k$ can be represented in the abovementioned way by means of a function $\theta(\vec{r})$ in a space of a sufficiently large number of dimensions K. However, the conditions for BRUNK's method as stated in [2] need not be satisfied if the conditions for our method are satisfied; we have e.g.

1. if $\underline{x}_i$ possesses, for each $i\epsilon E$, a rectangular distribution between 0 and θ_i then the conditions A' and B are satisfied (cf. section 2.7), but the distributions of $\underline{x}_1,\dots,\underline{x}_k$ do not belong to the 'exponential family',
2. if (cf. chapter 2) $\underline{x}_i$ possesses a normal distribution with mean θ_i and known variance for $i\epsilon M_o$ and an exponential distribution with parameter θ_i for $i\epsilon\overline{M}_o$ $(M_o\neq o, \overline{M}_o\neq o)$ then the conditions A' and B are satisfied. However, at least one pair of values $(i,j)\epsilon E$ exists such that, for $\theta_i=\theta_j$, $\underline{x}_i$ and $\underline{x}_j$ do not possess the same probability distribution,
3. BRUNK's method can only be applied if, for each $i\epsilon E$,
   a. $\varphi_i(\tau)\equiv\tau$,
   b. I_i is the set of *all* values of τ for which $F_i(x|\tau)$ is a distribution function.

Further MIRIAM AYER, H.D.BRUNK et al. consider in [1] a special case of BRUNK's problem, namely the case of completely ordered probabilities.

A.3 *The results of MIRIAM AYER, H.D.BRUNK et al.*

In [2] BRUNK gives an explicit formula for the maximum likelihood estimates $t_1,\dots,t_k$ of the parameters $\theta_1,\dots,\theta_k$. This formula is identical with the one derived in section 1.5 under the less stringent conditions A' and B, i.e. if S_M and T_M are defined by (1.5;2) and if v_M is the value of ζ which maximizes $H_M(\zeta)$ in I_M then

(A.3;1) $$t_i=\max_{M}\min_{M'}\{v_{T_M\wedge S_{M'}}\,\big|\, i\epsilon T_M\wedge S_{M'}\}\qquad (i\epsilon E).$$

In [1] formula (A.3;1) may be found for the special case of completely ordered probabilities (cf. also section 2.3). For this special case MIRIAM AYER, H.D.BRUNK et al. also give the procedure based on theorem 1.4;3 and the inequality (1.6;24).

SAMENVATTING

In de hoofdstukken 1-3 en de appendix wordt het volgende probleem behandeld: beschouw k onderling onafhankelijke stochastische grootheden $\underline{x}_1,\dots,\underline{x}_k$[1] en, voor iedere $i=1,\dots,k$, n_i onafhankelijke waarnemingen $x_{i,1},\dots,x_{i,n_i}$ van $\underline{x}_i$. Veronderstel dat, voor iedere $i=1,\dots,k$, de verdeling van $\underline{x}_i$ één onbekende parameter bevat en dat de parameters $\theta_1,\dots,\theta_k$ voldoen aan de volgende ongelijkheden: laat, vóór iedere $i=1,\dots,k$, $\varphi_i(\theta_i)$ een gegeven functie van θ_i zijn en laat I_i een gegeven interval zijn dan veronderstellen we dat $\varphi_1(\theta_1),\dots,\varphi_k(\theta_k)$ voldoen aan een aantal (niet strijdige) ongelijkheden van de vorm $\varphi_i(\theta_i)\leqq\varphi_j(\theta_j)$ (d.w.z. we onderstellen dat $\varphi_1(\theta_1),\dots,\varphi_k(\theta_k)$ partiëel of volledig geordend zijn) en bovendien dat, voor iedere $i=1,\dots,k$, $\varphi_i(\theta_i)$ voldoet aan de ongelijkheid $\varphi_i(\theta_i)\in I_i$. Het probleem is nu *het bepalen van de aannemelijkste schattingen van de parameters* $\theta_1,\dots,\theta_k$ *met de genoemde ongelijkheden als bijvoorwaarden.*

Dit probleem komt neer op het maximaliseren van de aannemelijkheidsfunctie in een deelgebied van de parameterruimte, waarbij dit deelgebied gedefiniëerd wordt door de voor $\theta_1,\dots,\theta_k$ gegeven ongelijkheden.

In hoofdstuk 1 worden voorwaarden gegeven waaronder dit maximum bestaat en uniek is[2]. Verder worden in dit hoofdstuk recurrente methoden beschreven met behulp waarvan men dit maximum kan vinden, terwijl ook een expliciete formule voor dit maximum gegeven wordt.

In hoofdstuk 2 worden de stellingen uit hoofdstuk 1 toegepast op het genoemde schattingsprobleem en wel voor de volgende speciale gevallen:

1. een binomiale verdeling met parameters n_i en θ_i,
2. een normale verdeling met gemiddelde θ_i en bekende spreiding,
3. een Poisson verdeling met parameter θ_i,

---

1) Stochastische grootheden worden onderscheiden van getallen (b.v. van de bij een experiment aangenomen waarden) door hun symbolen te onderstrepen.

2) In een deel van hoofdstuk 1 wordt een iets algemener probleem behandeld.

4. een exponentiële verdeling met parameter θ_i,
5. een homogene verdeling tussen 0 en θ_i,
6. een normale verdeling met bekend gemiddelde en met variantie θ_i.

In hoofdstuk 3 wordt bewezen dat de aannemelijkste schattingen van $\theta_1,\ldots,\theta_k$ asymptotisch rake schattingen van $\theta_1,\ldots,\theta_k$ zijn. Bij dit bewijs worden geen onderstellingen gemaakt over de differentiëerbaarheid van de aannemelijkheidsfunctie.
De appendix bevat een beschrijving van de oplossing die MIRIAM AYER, H.D.BRUNK, e.a. (zie [1] en [2]) gegeven hebben voor een speciaal geval van het genoemde schattingsprobleem.

In hoofdstuk 4 wordt een, met het voorafgaande nauw verwant, probleem behandeld. Laat weer $\underline{x}_1,\ldots,\underline{x}_k$ onderling onafhankelijke stochastische grootheden voorstellen en laat, voor iedere $i=1,\ldots,k$, $x_{i,1},\ldots,x_{i,n_i}$ onderling onafhankelijke waarnemingen van $\underline{x}_i$ zijn. Laat verder, voor iedere $i=1,\ldots,k$, θ_i een onbekende parameter van de verdeling van $\underline{x}_i$ voorstellen dan wordt in hoofdstuk 4 *een klasse van toetsen beschreven voor de hypothese dat* $\theta_1,\ldots,\theta_k$ *voldoen aan de ongelijkheden*

$$\theta_1 \leqq \ldots \leqq \theta_k$$

tegen de alternatieve hypothese dat $\theta_i > \theta_{i+1}$ *voor minstens één waarde van* $i=1,\ldots,k-1$.
Om deze toetsen te kunnen toepassen moet men, voor iedere $i=1,\ldots,k-1$, beschikken over een toets T_i voor de hypothese $\theta_i \leqq \theta_{i+1}$ tegen de alternatieve hypothese $\theta_i > \theta_{i+1}$ met de eigenschap

$$P[\underline{t}_i \epsilon Z_i \text{ en } \underline{t}_j \epsilon Z_j \mid \theta_i=\theta_{i+1} \text{ en } \theta_j=\theta_{j+1}] \leqq$$
$$\leqq P[\underline{t}_i \epsilon Z_i \mid \theta_i=\theta_{i+1}].P[\underline{t}_j \epsilon Z_j \mid \theta_j=\theta_{j+1}]$$

voor ieder paar (i,j) met $i < j$, waarbij $\underline{t}_i$ de toetsingsgrootheid en Z_i de kritieke zone van de toets T_i voorstellen ($i=1,\ldots,k-1$).
Toetsen T_i met de bovengenoemde eigenschappen worden in hoofdstuk 4 gegeven voor de volgende speciale gevallen:

1. een homogene verdeling tussen 0 en θ_i,
2. een exponentiële verdeling met parameter θ_i,
3. een normale verdeling met variantie θ_i,
4. een normale verdeling met gemiddelde θ_i en bekende variantie.

Verder wordt een analoge verdelingsvrije toets beschreven gebaseerd op de twee-steekproeven-toets van WILCOXON.

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STELLINGEN

STELLINGEN

1

De in hoofdstuk 1 van dit proefschrift beschreven recurrente methode voor het maximaliseren van een functie in het gebied (1.1;2) kan gegeneraliseerd worden tot een recurrente methode voor het maximaliseren van een functie in een deelgebied van G dat begrensd wordt door een aantal willekeurige hypervlakken.

2

De in hoofdstuk 4 van dit proefschrift beschreven toets voor de hypothese, dat de parameters $\theta_1,\ldots,\theta_k$ voldoen aan de ongelijkheden $\theta_1 \leqq \ldots \leqq \theta_k$ kan gegeneraliseerd worden tot een toets voor de hypothese H_o dat (in de notatie van hoofdstuk 1 van dit proefschrift) deze parameters voldoen aan de (essentiële) ongelijkheden: $\theta_{i_\lambda} \leqq \theta_{j_\lambda}$ voor iedere $\lambda=1,\ldots,s$. Bij deze generalisatie toetst men, voor iedere $\lambda=1,\ldots,s$, de hypothese $H_{o,\lambda}$: $\theta_{i_\lambda} \leqq \theta_{j_\lambda}$ tegen de alternatieve hypothese: $\theta_{i_\lambda} > \theta_{j_\lambda}$ en verwerpt H_o dan en slechts dan als er een λ is, waarvoor $H_{o,\lambda}$ verworpen wordt. De eigenschappen van deze toets zijn analoog aan die van de in hoofdstuk 4 beschreven toets. De ongelijkheid (4.3;4) geldt echter in het algemeen niet als er een paar $(\lambda_1,\lambda_2) \subset \{1,\ldots,s\}$ bestaat met $\lambda_1 \neq \lambda_2$ en $i_{\lambda_1}=i_{\lambda_2}$ of $j_{\lambda_1}=j_{\lambda_2}$.

3

Als men een hypothese wil toetsen op grond van een aantal steekproeven van verschillende grootten dient men er, indien mogelijk, voor te zorgen dat de alternatieve hypothesen, waarvoor de toets asymptotisch onderscheidend is, niet afhangen van de verhoudingen der steekproefgrootten.

C. VAN EEDEN, J. HEMELRIJK,
Proc. Kon. Ned. Akad. v. Wet. A **58**
Indag. Math. **17** } (1955) 191-198, 301-308

4

De twee-steekproeven-toetsen van E. J. G. PITMAN, F. WILCOXON, M. E. TERRY en B. L. VAN DER WAERDEN zijn, toegepast op twee steekproeven die tezamen twee knopen bevatten, identiek met de methode der 2x2-tabel.

C. VAN EEDEN, Statistica Neerlandica **10** (1956) 157-162

5

Het verdient aanbeveling in de toetsings- en schattingstheorie in de notatie onderscheid te maken tussen de 'ware ligging' van de onbekende parameter en de ligging van een veranderlijk punt in de parameterruimte.

6

De door J. E. WALSH in par. 8 van zijn artikel beschreven symmetrietoets is identiek met die van R. A. FISHER. De door hen beschreven methode ter bepaling van de verdeling van de toetsingsgrootheid onder de getoetste hypothese kan vereenvoudigd worden met behulp van de door C. VAN EEDEN en A. BENARD gegeven recursieformule.

J. E. WALSH, Ann. Math. Stat. **20** (1949) 64-81,

C. VAN EEDEN, A. BENARD,

Proc. Kon. Ned. Akad. v. Wet. A **60** }
Indag. Math. **19** } (1957) 381-408

7

De door J. H. PEEK ontwikkelde methode voor het aan ondernemingen toekennen van gevarencijfers, die afwijken van het gevarencijfer van het bedrijf, waartoe de onderneming behoort, moet als verouderd worden beschouwd. Het ware wenselijk deze methode door een andere, ook voor niet zeer grote ondernemingen toepasbare, methode te vervangen.

J. H. PEEK, Proc. of the fifth intern. congres of math. (1913) 395-406

8

Hoewel de stelling van A. DREWES inzake de uitspraken van de Centrale Raad van Beroep (S. V.) juist is, is een wijziging in de houding der Centrale Raad niet te verwachten voordat in deskundige kringen een goed gefundeerde statistische methode voor het aan ondernemingen toekennen van gevarencijfers algemeen aanvaard is.

A. DREWES, Dissertatie, Amsterdam (1945), stelling IX.