Planar Critical Percolation: Large clusters and Scaling limits.

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1 Introduction

Probably the simplest stochastic model to define on a two-dimensional lattice is Bernoulli percolation. Consider the square lattice $\mathbb{Z}^2$. The edge set consists of all pairs of neighbouring vertices u, v (i.e. $|u - v| = 1$). Every edge is open with probability p and closed with probability $1 - p$, independently of each other. This model is called bond percolation. One can also put the randomness on the vertices. In that case every vertex is open (resp. closed) with probability p (resp. $1 - p$). We call the latter version site percolation.

Originally this model was introduced by Broadbent and Hammersley [18]. In their article they suggested that it could be a basic model for a wide variety of applications. They were right. Nowadays it is 'used' in models for spread of disease, forest fire processes, sol gel transition, flow of fluids through a porous medium. This list is far from complete. This wide variety of models where percolation-like models play a prominent role is an important motivation to study two-dimensional percolation. Our second motivation is of a different kind. Heuristic arguments and lots of simulations, which were mainly done by theoretical physicists, give quite a good understanding of the behaviour of the model. But it turned out that, in order to prove these statements in a mathematical rigorous way, completely new methods have to be developed.

Let us start with the first basic example of the behaviour of the percolation model: the critical value of p. If p is sufficiently small, the probability that the origin, O, is contained in an infinite open cluster is zero, and if p is sufficiently large, this probability is strictly positive. The critical probability is the value of p where this change of behaviour takes place. More precisely, with $\mathbb{P}_p$ the probability measure for percolation with parameter p and $\theta(p) := \mathbb{P}_p(O \text{ is contained in an infinite cluster})$, we define the critical point to be

$$p_c := \sup(p \geq 0 : \theta(p) = 0).$$

The existence of p_c follows from the monotonicity of $\theta(p)$.

Let us try to guess heuristically what p_c should be. An important and extremely useful property on the square lattice (and also of critical site percolation on the triangular lattice) is its self-duality. That is, the dual lattice is essentially the same as the original lattice. The events of an open crossing of an $(n + 1) \times n$ box and a closed dual crossing of the dual rectangle are complements of each other. By the self-duality, for $p = 1/2$, the probabilities of these two events are equal. Combining these two observations shows that the probability of an open crossing of an $(n+1) \times n$ box is $1/2$ if $p = 1/2$. Intuitively one might reason that, for $p \leq 1/2$, the probability that there is a closed circuit in the annulus $[-3n, 3n]^2 \setminus [-n, n]^2$ surrounding the origin is larger or equal to a positive constant which does not depend on n, since the probability of having a crossing of a square is independent of its size. Hence there will almost surely be a closed circuit surrounding the origin. Thus the origin is in an infinite open cluster with probability zero. On the other hand, when $p > 1/2$, the probability that there are arbitrarily large closed circuits around the origin should be zero. Namely, the probability that a box is crossed by an open crossing probably tends to 1 as the size of the box tends to infinity. Therefore large closed crossings are blocked by open crossings. The nonexistence of an arbitrarily large closed circuit implies that there is an infinite open cluster. One could expect that a positive fraction of the vertices is contained in this infinite cluster. That implies that $\theta(p) > 0$. So p_c should be $1/2$.

Although this may sound quite reasonable it is far from a complete proof. Actually it took years before this was made mathematically rigorous. Kesten proved in his famous paper [53] that for bond percolation on the square lattice $p_c \leq 1/2$. This combined with the earlier proof of $p_c \geq 1/2$ by Harris [46] completed the proof of $p_c = 1/2$. Similar arguments also prove that for site percolation on the triangular lattice $p_c = 1/2$. Recently Duminil-Copin and Tassion gave in [36] an elegant proof for the upper bound of p_c using generalizations of ideas from an old paper by Hammersley [45] combined with Russo's formula [70]. Both proofs provide exponential decay of the probability that the origin is connected with the boundary of an $(n \times n)$-box, for all $p < p_c$. The proof of exponential decay by Duminil-Copin and Tassion can easily be generalised to higher dimensions, which was not the case for Kesten's proof. More precise, let $p_c(\mathbb{Z}^d)$ be the critical probability for bond percolation on $\mathbb{Z}^d$. They prove, for all $d \geq 2$, exponential decay of the probability that the origin is connected with the boundary of a hyper-cube with side-length n, for all $p < p_c(\mathbb{Z}^d)$.

With this critical probability at hand, one can, in some sense divide the model in three classes: subcritical: $p < p_c$, supercritical: $p > p_c$ and critical: $p = p_c$. In subcritical percolation one has exponential decay of the cluster size distribution. In supercritical percolation there exists almost surely an infinite cluster. Moreover, this cluster turns out to be unique almost surely. We focus on the last class, critical percolation, where the cluster size distribution has a power law behaviour.

Let us describe in the rest of this introduction some recent developments in percolation which are relevant for this thesis. For a more general overview see the review paper by Grimmett and Kesten [44], the books [15, 41, 43] or the lecture notes [81, 82].

1.1 Largest clusters

Let us restrict ourselves to a box $\Lambda_n := [-n, n]^2 \cap \mathbb{Z}^2$. Previously we were looking for an infinite cluster, this is obviously impossible here. Instead we study the clusters which come close to this infinite cluster. Namely we will study the largest open clusters in terms of the number of vertices they contain.

Borgs, Chayes, Kesten and Spencer studied these clusters intensively in a pair of papers [16, 17] in the three different regimes, below, above- and at the critical point. About the same time Penrose proved in [67] a central limit theorem for the size of the largest cluster in supercritical percolation. Later Van der Hofstad and Kager [47] and

Van der Hofstad and Redig [48] studied the largest cluster for $p \neq 1/2$ in more detail. One of their main results for subcritical percolation is that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of integers, with $u_n \to \infty$ as $n \to \infty$, constants $a, \rho > 0$ and a bounded sequence $a_n \in [a, 1]$, such that, for all $x \in \mathbb{N}$, the size $\mathcal{M}_n$ of the largest cluster in Λ_n satisfies

$$\mathbb{P}(\mathcal{M}_n \leq u_n + x) = e^{-a_n \beta^x} + O(n^{-\rho}),$$

where

$$\beta = \lim_{n\to\infty} (\mathbb{P}(|\mathcal{C}(0)| = n))^{1/n}$$

with $\mathcal{C}(0)$ the cluster in $\mathbb{Z}^2$, of the origin. Thus the size of the maximal cluster has a Gumbel-like distribution. This is intuitively clear, since the clusters in subcritical percolation are small, and therefore the sizes of the different large clusters are approximately independent.

In Chapters 2, 3 and 5 we study the largest clusters at $p = 1/2$. Here the clusters are much larger, compared to those in the subcritical regime. Even more, they are fractal-like: the fraction of points of Λ_n which are in the largest cluster still tends to zero as n grows to infinity, but the large clusters do often have a diameter of order n. Hence the largest clusters are heavily dependent on each other, which is in contrast to the situation of the approximate independence in subcritical percolation. This makes the largest cluster in critical percolation much harder to study.

Let $\pi(n)$ denote the probability that there is an open path from the origin to the boundary of Λ_n. We define

$$s(n) := n^2 \pi(n)$$

which is widely believed (and is rigorously proved for site percolation on the triangular lattice, see Section 1.2) to behave like

$$s(n) \approx n^{2-\frac{5}{48}}. \tag{1.1}$$

Let $\mathcal{M}_n$ be the size of the largest open cluster in Λ_n. Borgs, Chayes, Kesten and Spencer proved in [17] that $\mathcal{M}_n$ is of the order $s(n)$. The proof is based on another very useful result in an earlier paper by the same authors [16], which states that one has exponential decay of the distribution of the 'appropriately scaled' size of the largest cluster.

Theorem 1.1.1. *There exist constants $C_1, C_2 > 0$ such that, for all $n \in \mathbb{N}$,*

$$\mathbb{P}(\mathcal{M}_n \geq x s(n)) \leq C_1 e^{-C_2 x}. \tag{1.2}$$

One might ask whether the l.h.s. of (1.2), for large x, actually tends to zero as $n \to \infty$. This question was answered negatively by Borgs et al. This brings us to our first contribution in the study of the largest cluster. We prove in Chapter 2 that for every interval (a, b) with $0 < a < b$

$$\liminf_{n\to\infty} \mathbb{P}\left(\frac{\mathcal{M}_n}{s(n)} \in (a, b)\right) > 0. \tag{1.3}$$

Interestingly the proof of (1.3) might give rise to a question about Theorem 1.1.1. As noted before, the bound in Theorem 1.1.1 is sufficient for the purposes of Borgs et al., but is it also optimal? The answer turned out to be negative. A first indication in this direction can be obtained from the proof of (1.3) in Chapter 2 in the form of a lower bound for $\mathbb{P}(\mathcal{M}_n \geq xs(n))$. Namely it follows implicitly from our arguments, assuming (1.1), that there exist constants, $C_3, C_4 > 0$ such that, for all $x > 0$ and $n \in \mathbb{N}$

$$\mathbb{P}(\mathcal{M}_n \geq xs(n)) \geq C_3 \exp(-C_4 x^{96/5}).$$

Now the question becomes: Is this lower bound optimal, or is it somewhere in between? Recently Kiss [57] proved an upper bound where the order matches with the order of the last mentioned lower bound. More precisely, he proved that there exist constants, $C_5, C_6 > 0$, such that, for all $x > 0$ and $n \geq N(x)$

$$\mathbb{P}(\mathcal{M}_n \geq xs(n)) \leq C_5 \exp(-C_6 x^{96/5}).$$

This answers the above questions.

When studying the large clusters in a box, one should not restrict attention to the largest one. Borgs, Chayes, Kesten and Spencer proved that not only the largest cluster has size of order $s(n)$ but also the sizes of the second largest, the third largest, etcetera are of this order.

Let us make a side step and introduce a notion called *Incipient Infinite Cluster*. This notion appeared frequently in the literature on percolation as being, informally: "the infinite cluster at p_c" or "the large clusters, which give birth to the infinite cluster when, uniformly at random, a small but strictly positive fraction of additional edges are opened." The first mathematically precise definitions were given by Kesten [54]. The first one is the weak limit, as $n \to \infty$, of the conditional probability measure of critical bond percolation conditioned on the event that the origin is connected to the boundary of Λ_n. For the second one, consider percolation with parameter $p > p_c$ and condition on the event that the origin is contained in the infinite cluster; then the incipient infinite cluster measure is the weak limit of these probability measures as $p \searrow p_c$. Kesten not only proved existence, but also showed that these two definitions are equivalent. A precise definition which is more suitable for simulations was proposed by Járai in [50] and is similar in spirit as the definition from Aizenman [3]. Under critical bond percolation, choose uniformly at random a site in the k-th largest cluster in Λ_n and translate it to the origin. Than take weak limits as $n \to \infty$. Járai proved that this definition is equivalent with those of Kesten.

An important ingredient for Járai's proof of the just mentioned equivalence is that the gap between the sizes of the k-th and $(k+1)$-th largest cluster is big. Namely the local neighbourhood of the chosen vertex should be more or less independent of the fact that this vertex is contained in the k-th largest cluster. Járai showed that these gaps are indeed large, at least of order $\sqrt{s(n)}$. He conjectured that the gaps are of the same order as the cluster sizes, i.e. of the order $s(n)$. In Chapter 3 we prove that this is indeed the case. We will use this result in Chapter 5.

Our contribution

We already mentioned our first result in (1.3) which is proved in Chapter 2. We prove a similar result for p sufficiently close to 1/2. The main subject of Chapter 3 is the proof of the fact that the differences in sizes of the largest clusters are of order $s(n)$. Another result, which follows from the same arguments, is the fact that, for any fixed

$x > 0$, the probability that the size of the largest cluster is between $(x - \varepsilon)s(n)$ and $(x + \varepsilon)s(n)$ goes to zero as $\varepsilon \to 0$, uniformly in n.

1.2 Scaling limits

Until here we did not touch the topic of the existence of limits of, for example crossing probabilities in two-dimensional critical percolation as n tends to infinity. Proofs of existence of these limits were longstanding open problems, and for bond percolation on the square lattice it is still open, despite serious attempts in this direction. (See for example [6].) For site percolation on the triangular lattice, there was a great breakthrough in 2001: Smirnov proved in [77] the conformal invariance and the existence of the limits of crossing probabilities. Actually he gave a rigorous proof of Cardy's formula [26], which gives the limit of the probability that a topological rectangle is crossed. We wrote "topological rectangle", since the result holds for any simply connected domain. One year earlier Schramm described in [71] a stochastic process which he believed, and is now widely believed, to be the scaling limit for certain observables of a wide class of discrete two-dimensional models. The models in this class satisfy a Domain Markov Property and have in some sense a conformal invariant scaling limit. The process, denoted by SLE_κ for a certain constant $\kappa > 0$, is a random curve in the upper half-plane starting at the origin. The initials SLE stand originally for Stochastic Löwner Evolution, nowadays it is usually called Schramm-Löwner Evolution.

Let us say a few words about the concepts behind Schramm's conjecture. The Löwner equation is a differential equation which pops up when one tries to describe a curve in a conformal invariant way, with respect to its past. More precisely, let $\gamma(t)$ be a curve in the upper half-plane $\mathbb{H}$, starting at 0. We set, for all $t > 0$,

- H_t to be the unbounded connected component of $\mathbb{H} \setminus \gamma([0, t])$ and
- g_t the conformal map from H_t onto $\mathbb{H}$, such that

$$g_t(z) = z + \frac{2t}{z} + o(1/z) \qquad \text{as } z \to \infty$$

  (This is possible by taking a suitable parametrization of the curve γ.)

Then g_t satisfies Löwner's equation:

$$\partial_t g_t(z) = \frac{2}{g_t(z) + w(t)},$$

where $w(t) = g_t(\gamma(t))$ is a continuous function $w : \mathbb{R} \to \mathbb{R}$ which we call the driving function. It turns out that the Domain Markov Property and conformal invariance imply that the driving function must be a continuous stochastic process with stationary independent increments, hence a Brownian motion with a certain speed. This speed is expressed in the parameter κ. Moreover, one can, almost surely, reconstruct the curve γ from the driving function if the latter one is a Brownian motion (see [62, 68]).

In the case of percolation the SLE path should be the limit of the exploration path starting at the origin. Here the exploration path is the path from O to infinity

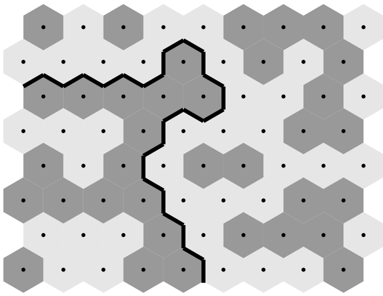

Figure 1.1: Exploration path

in between the closed cluster of the vertex at the left of O and the open cluster of the vertex at the right of O, under the assumption that all vertices on $(-\infty, 0)$ are closed and the vertices on $(0, \infty)$ are open, see Figure 1.1.

Lawler, Schramm, Werner and Smirnov argued that a combination of their results would imply that the exploration process converges to SLE_κ, with $\kappa = 6$. Namely the exploration path satisfies a certain property, named locality which is only satisfied by SLE_6. To explain the locality property we introduce an SLE process on the half-disk $\mathbb{U} \cap \mathbb{H}$ where $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk. This process starts at the origin and is defined similarly to the SLE process in $\mathbb{H}$ defined above, with the difference that $\mathbb{H}$ is replaced by $\mathbb{U} \cap \mathbb{H}$ and $g_t(\mathbf{i}) = \mathbf{i}$. Let us denote the corresponding SLE curve by $\tilde{\gamma}(t)$. The locality property means, more or less, that the distribution of the curves $\gamma(t)$ and $\tilde{\gamma}(t)$ are equal up to the stopping time T, when the curve hits the boundary of the unit circle for the first time.

A full proof of convergence to SLE_6 was given by Camia and Newman in [23]. (See also [60], where Lawler, Schramm and Werner proved the locality property of SLE_6.)

In the following subsections we consider different kinds of limiting objects, which are studied in Chapters 4, 5 and 6. The existence of all these limits, except the limit of FK-percolation in subsection 1.2.3, follow from the convergence results by Smirnov and Camia and Newman [22], together with SLE techniques. Explicit formulas and constants used in this thesis have been derived rigorously in the literature by SLE techniques. We use these results in this thesis, but for our purposes there is no need to go into the underlying SLE techniques.

1.2.1 The expected number of clusters

In Section 1.1 we mentioned the large clusters in the box Λ_n. What can we say about the total number of clusters in Λ_n? Zhang proved in [85] a central limit theorem for the number of clusters in Λ_n, denoted by K_n. That is

$$\frac{K_n - \mathbb{E}[K_n]}{\sqrt{\mathrm{Var}(K_n)}} \xrightarrow{\text{(d)}} \mathcal{N}(0,1) \quad \text{as } n \to \infty.$$

(Actually he considered bond percolation on the square lattice and general $p \in (0,1)$.) Similar results for $p \neq 1/2$ were obtained earlier by Cox and Grimmett in [32])

A related quantity is the expected number of clusters, in the full plane, crossing the boundary of Λ_n. Let us denote this number by $E_{\partial\Lambda_n}$. The leading term will be of order n, from the length of the boundary. The main 'error'-term turns out to be of the order $\log(n)$ and is caused by the corners. The leading order is nonuniversal: its prefactor depends on the lattice under consideration. The prefactor of the logarithm is believed not to depend on the precise choice of the lattice. Therefore we will start our study with the prefactor of the logarithmic term. Kovács, Iglói and Cardy obtained in [59] heuristically (with numerical verification) that the prefactor of the logarithmic term of $E_{\partial\Lambda_n}$ should be $\frac{5\sqrt{3}}{36\pi}$. Furthermore they gave a formula for the prefactor of the logarithm in $E_{\partial P_n}$ for any polygon P_n.

Our contribution
In Chapter 4 we study the expected number of clusters intersecting a line segment. For the case of site percolation on the triangular lattice in the half-plane, with the line segment on the boundary, we derive rigorously the logarithmic prefactor predicted by Kovács et al. For the case of the full plane we derive a rigorous upper bound.

1.2.2 Convergence of the largest clusters in an $n \times n$ box

In Section 1.2.1 we considered the number of clusters intersecting the boundary. Now we get back to the situation in Section 1.1. It is natural to ask whether convergence of the clusters themselves holds. First of all we need to decide what it means for the clusters to converge. Instead of sending the size n of the box to infinity it is more convenient to send the lattice spacing η (which is also called the mesh) to zero. Roughly speaking, the small microscopic clusters will disappear and what remains are the large macroscopic clusters. Aizenman and Burchard introduced in [5] a description of the full discrete percolation configuration in terms of paths which would survive in a scaling limit, where the mesh goes to zero. Later Camia and Newman proposed a description of the full scaling limit in terms of loops. The loops are the boundaries of the closed and open clusters. Aizenman and Burchard proved the existence of sub-sequential limits of the random collection of paths. In the aforementioned paper [22] by Camia and Newman the uniqueness of the limit in terms of loops was proved, using results and ideas from Lawler, Schramm, Werner and Smirnov described above. Although this collection of loops contains almost all information about the large clusters, it is not immediately clear how to obtain the convergence of the clusters as closed subsets of the plane from it, which is more natural. Furthermore it would be interesting to be able to prove that the size of the largest cluster in the 2 by 2 box Λ_1 converges in distribution after scaling.

Our contribution
In Chapter 5 we prove that the clusters, as closed subsets of the plane, indeed converge in distribution. Even more we show that the large clusters in a bounded domain converge. Finally, to continue the results in Section 1.1, we prove the convergence of the 'scaled' counting measures of the clusters. By 'scaled' we mean the counting measure divided by $s(1/\eta)$. (If we did not scale, the measure would blow up as $\eta \to 0$.) Based on the convergence of these counting measures we prove the convergence in

distribution of the scaled size of the largest cluster in Λ_1, the second largest cluster in Λ_1 etc.

1.2.3 Magnetization in FK-Ising model

The Ising-model, introduced by Lenz and first studied by his student Ising, is a widely studied model in statistical mechanics. It is, in contrast to ordinary percolation, one of the models which are in some sense *exactly solvable*, for example, using combinatorial arguments, important quantities such as the partition function can be computed exactly. A property which is probably related to this is the so called 'discrete holomorphicity' of some observables. (In contrast: ordinary percolation does not have this property.) However, the existence of discrete holomorphic observables does not make it easy to prove conformal invariance of the interfaces, i.e. boundaries of "+"-spin clusters, and convergence to the corresponding SLE curve.

Let us briefly define the model. We will not use the original definition of the Ising model, using Hamiltonians on spin configurations, but define it via the random cluster model, which we will call FK-percolation in the following. The abbreviation 'FK' stands for Fortuin and Kasteleijn who introduced the random cluster model as a class of models, which includes Percolation, Ising, Potts and electrical networks. The random cluster model has two parameters p, q. The Edwards-Sokal coupling can be used to prove that the model we define is actually the ordinary Ising-model. The reason to do it in this way will become clear at the end of this section.

Let G be a finite graph with vertex set $V(G)$ and edge set $E(G)$. Let $p \in (0,1)$ and $q \geq 1$. For a configuration $\omega \in \{0,1\}^{E(G)}$ we denote by $o(\omega)$ the number of open (1) edges, by $c(\omega)$ the number of closed (0) edges and by $k(\omega)$ the number of clusters (where isolated vertices also count as clusters.) We define the measure

$$\phi^{\xi}_{G,p,q}(\omega) := \frac{1}{Z^{\xi}_{G,p,q}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega)},$$

where $Z^{\xi}_{G,p,q}$ is the normalizing constant and ξ denotes the boundary condition. The latter means that ξ determines which vertices on the boundary are assumed to be connected with each other "outside G". So $k(\omega)$ depends on ξ. If the graph G has no boundary we omit ξ. Note that $q = 1$ is ordinary bond percolation. See [G06] for a general introduction to the random cluster model. The Ising model is defined by first drawing at random an ω from the random cluster model with $q = 2$ and then assigning to every cluster, independent of the other clusters, with probability $1/2$ a plus and with probability $1/2$ a minus. In the rest of this section we will stick to the Ising model, and therefore fix $q = 2$. The clusters from the random cluster model we will call FK-clusters in what follows.

Let us briefly see how to obtain the original definition of the Ising-model from this. Let us denote by $\sigma \in \{-1,+1\}^{V(G)}$ the spin configuration where σ_x has the sign of the FK-cluster of x. It is easy to see that the procedure described above gives us a configuration (ω, σ) according to the measure

$$\mu_p(\omega, \sigma) := \frac{1}{Z_{G,p,2}} p^{o(\omega)}(1-p)^{c(\omega)} \mathbf{1}_{\sigma,\omega}(F),$$

where F consists of the pairs (ω, σ), such that for every edge $e = (x, y)$ with $\omega_e = 1$ we have $\sigma_x = \sigma_y$. In words, an open edge implies equal spins on both sides of the edge. The Ising model is given by the marginal distribution $\mu_p(\cdot, \sigma) := \sum_\omega \mu_p(\omega, \sigma)$. To compute $\mu_p(\cdot, \sigma)$ we split the product $p^{o(\omega)}(1-p)^{c(\omega)}$ in two parts

$$p^{|\{e=(x,y):\, \omega_e=1, \sigma_x=\sigma_y\}|} \cdot (1-p)^{|\{e=(x,y):\, \omega_e=0, \sigma_x=\sigma_y\}|} \tag{1.4}$$
$$p^{|\{e=(x,y):\, \omega_e=1, \sigma_x\neq\sigma_y\}|} \cdot (1-p)^{|\{e=(x,y):\, \omega_e=0, \sigma_x\neq\sigma_y\}|}. \tag{1.5}$$

The last one is, for $(\omega, \sigma) \in F$, equal to $(1-p)^{|\{e=(x,y):\, \omega_e=0, \sigma_x\neq\sigma_y\}|}$. Hence

$$\begin{aligned}
&\sum_{\omega:(\omega,\sigma)\in F} \mu_p(\omega, \sigma) \\
&= \frac{1}{Z_{G,p,2}} \cdot \sum_{\omega:(\omega,\sigma)\in F} \Big\{ \prod_{e=(x,y):\sigma_x=\sigma_y} (p \cdot \mathbf{1}\{\omega_e = 1\} + (1-p) \cdot \mathbf{1}\{\omega_e = 0\}) \\
&\qquad \cdot \prod_{e=(x,y):\sigma_x\neq\sigma_y} (1-p) \cdot \mathbf{1}\{\omega_e = 0\} \Big\} \\
&= \frac{1}{Z_{G,p,2}} \cdot \prod_{e=(x,y):\sigma_x\neq\sigma_y} (1-p) \\
&= \frac{1}{Z_{G,p,2}} \cdot \exp(\beta \sum_e -\mathbf{1}\{\sigma_x \neq \sigma_y\}),
\end{aligned}$$

where $p = 1 - e^{-\beta}$ (in physical terms the constant β denotes the inverse temperature). The last equation is similar to the usual definition of the Ising model. We began this section with a definition of the Ising model, which makes it an extension of ordinary bond percolation. This was not the original motivation for Lenz and Ising. Their aim was to describe the phase transition between the existence and nonexistence of so called "spontaneous magnetization" in a ferromagnet. The plus and minus spins represent the direction of the magnetic moments of the atoms of the ferromagnet.

Like ordinary percolation, the Ising model has a critical point, which we may define as a critical point of the random cluster model. However we defined the random cluster model only on finite graphs. So we first need to know whether we can define the random cluster model on $\mathbb{Z}^2$. A natural way to construct this is by considering the weak limit of

$$\phi_{\mathbb{Z}^2,p} := \lim_{n\to\infty} \phi^{free}_{\Lambda_n,p,2},$$

where Λ_n denotes the square lattice restricted to $[-n, n]^2$ and *free* stands for free boundary conditions, that is no vertices on the boundary are assumed to be connected outside the domain. For a proof of the existence of this limit see for example [40]. Furthermore Grimmett proved that the limit is independent of the used boundary conditions. To define the critical point, let

$$\hat{\theta}(p) := \phi_{\mathbb{Z}^2,p}(O \text{ is connected to infinity }).$$

Then

$$\hat{p}_c := \sup(p \geq 0 : \hat{\theta}(p) = 0).$$

It is well known that $\hat{p}_c = \frac{\sqrt{2}}{1+\sqrt{2}}$, see for example [66] or [42]. (See [7] for general $q \geq 1$.) It is worth mentioning that also in this case $\hat{p}_c$ is a self dual point in some sense, which we do not discuss further, because that is outside the scope of this introduction. As for percolation, we are mainly interested in the critical case.

One of the main objects of interest is the magnetization field in terms of a random signed measure (or distribution) on the plane

$$\Phi := \sum_{v \in \mathbb{Z}^2} \sigma_v \delta_v,$$

where δ_v denotes the Dirac measure with mass at v. Hence for any Borel set B

$$\Phi(B) = \sum_{v \in \mathbb{Z}^2 \cap B} \sigma_v$$

is the sum over all spins in B. Note that we can also write it as

$$\Phi(B) = \sum_{C \in \mathcal{C}} X_C \cdot \mu_C(B), \tag{1.6}$$

where $\mathcal{C}$ is the set of all FK-clusters, μ_C is the counting measure of the FK-cluster C, i.e. $\mu_C := \sum_{v \in C} \delta_v$ and the X_C's are the independent Bernoulli(1/2) random variables with outcome '+1' or '−1'. At criticality the distribution of the total magnetization of the Ising model in Λ_n (i.e. $\Phi(\Lambda_n)$) obeys a power law behaviour as n tends to infinity. Similar to the convergence of the largest cluster as a measure we described in the previous section, a scaling limit of the magnetization field as the mesh size η tends to zero can be defined. A first step in this direction was made by Camia and Newman [24] who proposed to use the representation in (1.6). They noticed that, in order to obtain a meaningful scaling limit, Φ and the counting measures μ_C should be scaled by the same power of the mesh size η.

Camia, Garban and Newman proved in [21] that a scaling limit of the magnetization field, with scaling factor $\eta^{-15/8}$, exists in the Sobolev space $\mathcal{H}^{-3}$ of generalized functions. However they used a more direct approach instead of using the representation (1.6)

Let us give a short overview of the results preceding the existence proof of Camia et al. The first substantial step was made by Smirnov in [78] where he introduced the so called Fermionic observables and proved that they exhibit a conformal invariant scaling limit. Building on this result Chelkak, Hongler and Izyurov proved in [30] the existence and conformal covariance of the scaling limit of n-point spin correlations in any simply connected domain with + boundary condition. More precisely they proved the convergence, for points $a_1, \cdots, a_n$ in the domain, of $\eta^{-n/8} \cdot \mathbb{E}_\eta[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]$ as the mesh-size (η)/lattice spacing tends to zero. Here σ_{a_1} is defined as the sign of the nearest vertex. Moreover they obtained an explicit formula for the limit.

Our contribution

A basic tool in critical percolation is the RSW lemma [72, 69], which states that, for every $k > 0$ there exists a $\delta > 0$ such that the probability that a kn by n rectangle has a horizontal open crossing is larger than δ for all n. Quite recently Duminil-Copin, Hongler and Nolin [35] and Chelkak, Duminil-Copin and Hongler [28] proved that also the random cluster model with $q = 2$ at the critical point satisfies a general RSW kind result. By this result many proofs for critical percolation can be transformed into proofs for the random cluster model. This makes it possible to translate, under a single widely believed assumption which we will discuss next, the results of the convergence of clusters in the previous subsection to the convergence of the appropriately scaled counting measures of the FK-clusters as the mesh (i.e. the distance between the vertices) decreases to zero. Consequently we can provide a continuum analogue of (1.6). Note that this scaling limit contains more information than the scaling limit of the magnetization field, therefore our result does not follow immediately from the existence result by Camia, Garban and Newman. See Chapter 5, in particular Section 5.2.2.

The assumption mentioned above is that the full scaling limit in terms of the boundaries of the FK-clusters is given by SLE loops. A lot of progress in proving this assumption has been made. Furthermore the people who are intensively involved in this project believe that they will be able to prove this full scaling limit. Informally stated, the following has been proved. Consider a rectangle with the following boundary condition: On the top and left side wired and on the bottom and right side open. The exploration path from lower-left corner towards the upper-right corner converges, as the mesh-size tends to zero, to an SLE_κ curve with $\kappa = 16/3$. See [29].

1.2.4 Factorization formulas

Interestingly enough, physicists still discover new results in two-dimensional critical percolation. Nowadays they are partly based on SLE but also on extremely powerful, but non-rigorous, techniques from Conformal Field Theory. A third technique is to study the random-cluster model with $p = 1/2$, $q > 1$ and then take limits as $q \to 1$.

In the end of Section 1.2.3 we mentioned that n-point spin correlations were used to prove convergence of the magnetization. However computing them, in particular for percolation, is extremely difficult. Kleban, Simmons and Ziff made an important step in the study of 3-point correlation functions in [58]. They did not obtain exact formulas for correlation functions, but instead they found 'factorizations' of a 3-point function in terms of 2-point functions. However their result does not apply to every set of 3 points. Let us make it more precise. Consider percolation on the upper half-plane. Fix 3 points: $0, u \in \mathbb{Z}$ and $w = x + y\mathbf{i}$, where $x \in \mathbb{Z}, y \in \mathbb{N}$. Let us denote the cluster of a point u by $\mathcal{C}(u)$. Kleban et al. heuristically derived that there exists a universal constant C such that

$$\frac{\mathbb{P}(nw \text{ and } nu \text{ are contained in } \mathcal{C}(0))}{\sqrt{\mathbb{P}(nu \in \mathcal{C}(0))\mathbb{P}(nw \in \mathcal{C}(0))\mathbb{P}(nw \in \mathcal{C}(nu))}} \to C \qquad \text{as } n \to \infty. \tag{1.7}$$

Moreover they claimed that the constant C can be obtained from a certain differential equation. Shortly after the publication of the previous mentioned paper the same authors published an article [74] where they computed the constant C explicitly. It turned out to be equal to

$$C = \frac{2^{7/2}\pi^{5/2}}{3^{3/4}\Gamma(1/3)^{9/2}},$$

where Γ denotes the Gamma-function.

Apart from the computation of C, Simmons et al. computed a constant C_2 which corresponds with a second ratio. Namely, with u and w as before, they also consider the product of the probability that nu, nw are both contained in $\mathcal{C}(0)$ and the probability that there is an open path from nw to $\mathbb{Z}$. They heuristically derived that

$$\frac{\mathbb{P}\big(nw \text{ and } nu \text{ are contained in } \mathcal{C}(0)\big)\,\mathbb{P}\big(\mathcal{C}(nw)\cap\mathbb{Z}\neq\emptyset\big)}{\mathbb{P}\big(nu\in\mathcal{C}(0)\big)\,\mathbb{P}\big(\mathcal{C}(nw)\cap[0,nu]\neq\emptyset\big)\,\mathbb{P}\big(\mathcal{C}(nw)\cap(\mathbb{Z}\setminus[0,nu])\neq\emptyset\big)}\to C_2$$

as n tends to infinity, where $\mathcal{C}(nw)\cap[0,nu]\neq\emptyset$ denotes the event that there is an open path from nw to a vertex between 0 and nu, similarly for the other events.

Our contribution
Although the derivation of the constant C from the differential equation was more or less rigorous, the derivation of the differential equation itself was not. A rigorous derivation of the differential equation and of the existence of a similar limit as (1.7) was given by Beliaev and Izyurov in [9]. ‘Similar’, since the vertices in the probabilities in (1.7) are in their result replaced by neighbourhoods. It is interesting to know whether one can prove the existence of the ratio (1.7) itself. This is the content of Chapter 6 building on the result of Beliaev and Izyurov and using coupling arguments which go back to Kesten’s construction of the incipient infinite cluster in [54].

1.3 Overview of the thesis and list of publications

In Chapters 2 and 3 we study the largest clusters in an $n \times n$-box in critical bond percolation. In Chapter 4 we state and prove our result for the logarithmic term in the number of clusters touching a line segment. The convergence of clusters in the scaling limit is considered in Chapter 5. Finally our results on factorization formulas can be found in Chapter 6. This thesis is based on the following papers

- Chapter 2: [10] J. van den Berg and R.P. Conijn, *On the size of the largest cluster in 2D critical percolation*, Electron. Commun. Probab. 17 (2012) no. 58. DOI:10.1214/ECP.v17-2263
- Chapter 3: [11] J. van den Berg and R.P. Conijn, *The gaps between the sizes of large clusters in 2D critical percolation*, Electron. Commun. Probab. 18 (2013) no. 92. DOI:10.1214/ECP.v18-3065
- Chapter 4: [12] J. van den Berg and R.P. Conijn, *The expected number of critical percolation clusters intersecting a line segment*, arXiv:1505.08046 (2015).
- Chapter 5: [20] F. Camia, R.P. Conijn and D. Kiss, *Conformal measure ensembles for percolation and the FK-Ising model*, arXiv:1507.01371 (2015).
- Chapter 6: [31] R.P. Conijn, *Factorization Formulas for 2D Critical Percolation, Revisited*, to appear in Stochastic Process. Appl. (2015). DOI:10.1016/j.spa.2015.05.017

2 | The size of the largest cluster

This chapter is based on [10] with Rob van den Berg.

We consider (near-)critical percolation on the square lattice. Let $\mathcal{M}_n$ be the size of the largest open cluster contained in the box $[-n,n]^2$, and let $\pi(n)$ be the probability that there is an open path from O to the boundary of the box. It is well-known (see [17]) that for all $0 < a < b$ the probability that $\mathcal{M}_n$ is smaller than $an^2\pi(n)$ and the probability that $\mathcal{M}_n$ is larger than $bn^2\pi(n)$ are bounded away from 0 as $n \to \infty$. It is a natural question, which arises for instance in the study of so-called frozen-percolation processes, if a similar result holds for the probability that $\mathcal{M}_n$ is *between* $an^2\pi(n)$ and $bn^2\pi(n)$. By a suitable partition of the box, and a careful construction involving the building blocks, we show that the answer to this question is affirmative. The 'sublinearity' of $1/\pi(n)$ appears to be essential for the argument.

2.1 Introduction and main result

Consider bond percolation on $\mathbb{Z}^2$ with parameter p. (See [41] for a general introduction to percolation theory.) Let $\Lambda_n = [-n,n]^2 \cap \mathbb{Z}^2$ and let, for $v \in \Lambda_n$, $\mathcal{C}_n(v)$ denote the size of the open cluster of v inside the box Λ_n:

$$\mathcal{C}_n(v) = |\{w \in \Lambda_n : v \leftrightarrow w, \quad \text{inside } \Lambda_n\}|,$$

where we use the standard notation $v \leftrightarrow w$ for the existence of an open path from v to w, and where the addition 'inside Λ_n' means that we require the existence of such a path which is located entirely in Λ_n. For a set $A \subset \mathbb{Z}^2$ we denote by ∂A the (internal) boundary of A:

$$\partial A = \{v \in A : \exists w \notin A : (v,w) \text{ is an edge } \}.$$

The remaining part of A will be called the interior of A. Let $\pi_p(n)$ be the probability $\mathbb{P}_p(O \leftrightarrow \partial\Lambda_n)$. For simplicity we write $\pi(n)$ for $\pi_{\frac{1}{2}}(n)$.

We are interested in the size of 'large' open clusters in Λ_n for the case where p is equal (or close) to the critical value 1/2. It is known in the literature that, informally speaking, the size of the largest open cluster is typically of order $n^2\pi(n)$: For any $c > 0$, there is a 'reasonable' probability that it is larger (smaller) than $cn^2\pi(n)$, and this probability goes to 0 uniformly in n as $c \to \infty$ ($c \to 0$). (See [16], [17]; see also [50] Section 3.) However, the question whether for all $0 < a < b$ there is a 'reasonable' probability that there is an open cluster with size *between* $an^2\pi(n)$ and $bn^2\pi(n)$ has not been investigated in the literature.

This question, which is also natural by itself, arises e.g. in the study of finite-parameter frozen-percolation models. In these models each edge is closed at time 0 and 'tries' to become open at some random time, independently of the other edges. However, an open cluster stops growing as soon as its size has reached a certain (large) value M, the parameter of the model. (See [13] where this was studied for the case where the 'size' of a cluster is defined as its diameter instead of its volume.) The investigation of such processes leads to the question how two open clusters which both have size of order M but smaller than M, merge to a cluster of size bigger than M, which in turn leads to the question at the end of the previous paragraph. To state our main result, an affirmative answer to that question, we first need a few more definitions.

For $k, l \in \mathbb{N}$, we denote by $HC(k,l)$ the event that there is an open horizontal crossing in the box $[0,k] \times [0,l]$. (This is an open path from the left side to the right side of the box, of which all vertices, except the starting and end point, are in the interior of the box). Let the "characteristic length" be as defined in e.g. [65] and [55]: For a fixed $\epsilon \in (0, \frac{1}{2})$:

$$L(p) = L_\epsilon(p) = \begin{cases} \min\{n \in \mathbb{N} : \mathbb{P}_p(HC(n,n)) \le \epsilon\} & \text{if } p < \frac{1}{2}, \\ \min\{n \in \mathbb{N} : \mathbb{P}_p(HC(n,n)) > 1-\epsilon\} & \text{if } p > \frac{1}{2}, \end{cases} \tag{2.1}$$

and $L(\frac{1}{2}) = \infty$. The precise value of ϵ is not essential. Throughout this chapter we will consider it as being fixed, and therefore we omit it from our notation.

As said before, our main question concerns the existence of *some* open cluster in Λ_n with size in some specific interval. The proof we obtained gives, with only a tiny bit of extra work, something stronger; it shows that with 'reasonable' probability the *maximal* open cluster has this property. Therefore we state our main result in this stronger form (and remark that we do not know an essentially simpler proof of the original weaker form):
Denote by $\mathcal{M}_n$ the size of the maximal open cluster in Λ_n. More precisely,

$$\mathcal{M}_n = \max_{v \in \Lambda_n} \mathcal{C}_n(v).$$

Theorem 2.1.1. *Let $0 < a < b$. There exist $\delta > 0$ and $N \in \mathbb{N}$ such that, for all $n \ge N$ and all p with $L(p) \ge n$,*

$$\mathbb{P}_p\left(\mathcal{M}_n \in (an^2\pi(n), bn^2\pi(n))\right) > \delta.$$

The proof is given in Section 2.3. Section 2.2 will list the main ingredients used in the proof. The proof involves a suitable partition of Λ_n in smaller boxes and annuli. A brief and informal summary is given in the beginning of Section 2.3, after the description of these objects.

Acknowledgement. We thank Antal Járai for a useful and pleasant discussion on these an related problems.

2.2 Ingredients

We will make amply use of standard RSW results of the following form: For all $l > 0$ there exists $\delta(l) > 0$ such that, for all k and all p with $L(p) \geq k$, $\mathbb{P}_p(HC(k, \lceil lk \rceil)) \geq \delta(l)$. For a set W of vertices define

$$\tilde{\mathcal{C}}(W) = |\{v \in W : v \leftrightarrow \partial W\}|. \tag{2.2}$$

For $k, n \in \mathbb{N}$, we use the notation $\Lambda_{k,n}$ for the rectangle $[-k, k] \times [-n, n]$. We will use the following properties of $\pi_p(n)$ from the literature.

Theorem 2.2.1. *There exist $\alpha, C_1, \cdots, C_6 > 0$ such that:*

(i) For all $m \leq n$:

$$C_1(\frac{n}{m})^\alpha \leq \frac{\pi(m)}{\pi(n)} \leq C_2(\frac{n}{m})^{\frac{1}{2}}.$$

(ii) For all $n \in \mathbb{N}$:

$$\sum_{k=0}^{n} \pi(k) \leq C_3 \cdot n\pi(n).$$

(iii) For all $p \in (0, 1)$ and all $n \leq L(p)$,

$$C_4\pi(n) \leq \pi_p(n) \leq C_5\pi(n).$$

(iv) For all $k, n \in \mathbb{N}$ and p with $L(p) \geq k \wedge n$,

$$\mathbb{E}_p[\tilde{\mathcal{C}}(\Lambda_{k,n})] \leq C_6kn\,\pi(k \wedge n).$$

Proof. The inequalities in (i) are well-known. (The first follows easily from RSW arguments, and the second goes back to [14]; see also for example [17]). Part (ii) follows from (7) in [54]. Part (iii) is Theorem 1 in [55]. Part (iv), of which versions are explicitly in the literature (see e.g. [54], [55] and [65]), is proved as follows (where we assume that $k \leq n$):

$$\mathbb{E}_p[\tilde{\mathcal{C}}(\Lambda_{k,n})] = \sum_{v \in \Lambda_{k,n}} \mathbb{P}_p(v \leftrightarrow \partial\Lambda_{k,n})$$

$$\leq \sum_{v \in \Lambda_{k,n}} \pi_p(d(v, \partial\Lambda_{k,n})) \leq 8n \sum_{l=0}^{k} \pi_p(l) \leq C_6nk\pi(k),$$

where the last inequality uses part (ii) and (iii). □

Define

$$Y(m) = |\{v \in \Lambda_m : v \leftrightarrow \partial\Lambda_{2m}\}|.$$

We need the following result for the distribution of $Y(m)$, which is essentially in [16] and (for the special case $p = 1/2$) [54].

Theorem 2.2.2. *There exist $\delta_1, C_7 > 0$ such that, for all $p \in (0,1)$ and all $m \leq L(p)$:*

$$\mathbb{P}_p(Y(m) \geq C_7 m^2 \pi(m)) \geq \delta_1. \tag{2.3}$$

Proof. By Lemma 6.1 in [16] there exists $C_8 > 0$ such that for all $p \in (0,1)$ and $m \leq L(p)$, $\mathbb{E}_p[(Y(m))^2] \leq C_8(m^2\pi(m))^2$. Further, by the definition of $\pi(n)$ and parts (i) and (iii) of Theorem 2.2.1, there exists $C_9 > 0$ such that $\mathbb{E}_p[Y(m)] \geq C_9 m^2 \pi(m)$. These two inequalities, and the one-sided Chebyshev's inequality, give Theorem 2.2.2. □

It was shown in [54] (and extended/generalized in [16] and [17]) that $\mathcal{M}_n$, the size of the largest open cluster in Λ_n, is typically of order $n^2\pi(n)$. In particular, its expectation has an upper and a lower bound which are linear in $n^2\pi(n)$. In the proof of Theorem 2.1.1 we use the following result from [17].

Theorem 2.2.3. ([17] Thm. 3.1 (i), Thm. 3.3 (ii))
Let p_n be a sequence, such that $n \leq L(p_n)$ for all n. Then for all $K > 0$,

$$\liminf_{n\to\infty} \mathbb{P}_{p_n}\left(\frac{\mathcal{M}_n}{n^2\pi(n)} < K\right) > 0.$$

Finally, to streamline the arguments in Section 2.3.5 at the end of the proof of Theorem 2.1.1, we state here the following fact about 'steering' the outcome of the sum of independent random variables. It is a simple observation rather than a lemma, and versions of it have without doubt been used in the probability literature in various contexts.

Lemma 2.2.4. *Let $0 < \alpha < \beta$, and let $k \in \mathbb{N}$ be such that $\alpha/k < (\beta-\alpha)/2$. Further, let $\eta_1, \eta_2 > 0$ and let $(X_i)_{1\leq i\leq k}$ be independent random variables, (not necessarily identically distributed) which satisfy the following:*

$$\begin{aligned} \mathbb{P}\left(X_i \in (\frac{\alpha}{k}, \frac{\beta-\alpha}{2})\right) &\geq \eta_1; \\ \mathbb{P}\left(X_i \leq \frac{\beta-\alpha}{2k}\right) &\geq \eta_2. \end{aligned}$$

Then

$$\mathbb{P}\left(\sum_{i=1}^k X_i \in (\alpha,\beta)\right) \geq (\eta_1 \wedge \eta_2)^k.$$

Proof. For $1 \leq i \leq k$ we say that 'step i is proper' if

$$X_i \begin{cases} \in (\frac{\alpha}{k}, \frac{\beta-\alpha}{2}) & \text{if } \sum_{j=1}^{i-1} X_j < \alpha \\ \leq \frac{\beta-\alpha}{2k} & \text{otherwise.} \end{cases}$$

It is clear that if all steps $i = 1, \cdots, k$ are proper, then $\sum_{i=1}^k X_i \in (\alpha,\beta)$. It is also easy to see that, for each i, the conditional probability that step i is proper, given that all steps $1, \cdots, i-1$ are proper, is at least $\min(\eta_1, \eta_2)$. □

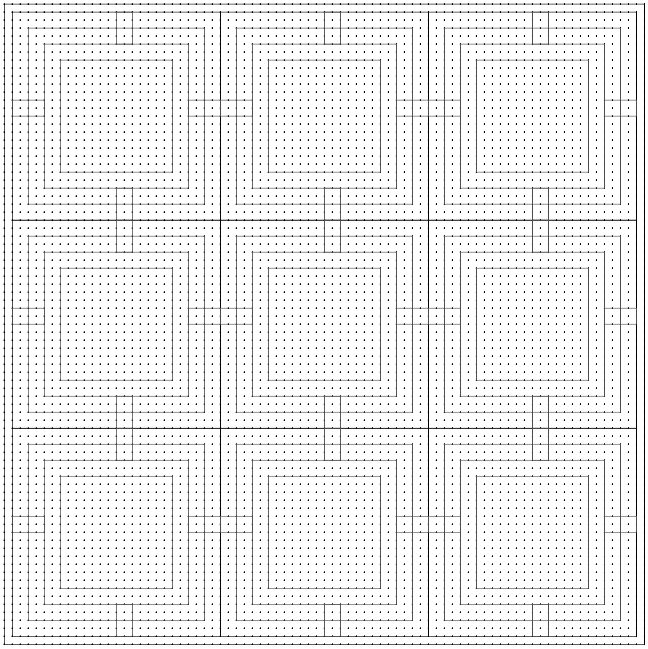

Figure 2.1: Partition of the box Λ_n. Here $n = 40, m = 3, s = 13, t = 2$.

2.3 Proof of Theorem 2.1.1

We first give a proof for the special case $p = 1/2$ and therefore drop the subscript p from the notation $\mathbb{P}_p$ and $\mathbb{E}_p$. At the end of Section 2.3.5 we point out that (due to the 'uniformity' of the ingredients stated in Section 2.2) the proof for the general case is essentially the same.

2.3.1 More definitions, and brief outline of the proof

Let $s, t \in \mathbb{N}$ with $t \leq \frac{1}{3}s$. The proof involves a construction using the following boxes and annuli.

$$\begin{aligned}
B_{0,0} &= \Lambda_s. \\
A^I_{0,0} &= \Lambda_s \setminus \Lambda_{s-t}; \quad A^{II}_{0,0} = \Lambda_{s-t} \setminus \Lambda_{s-2t}; \quad A^{III}_{0,0} = \Lambda_{s-2t} \setminus \Lambda_{s-3t}. \\
A'_{0,0} &= A^I_{0,0} \cup A^{II}_{0,0} \cup A^{III}_{0,0}. \\
H_{0,0} &= ([0, 4t] \times [0, t] + (s - 2t, 0)) \cap \mathbb{Z}^2. \\
V_{0,0} &= ([0, t] \times [0, 4t] + (0, s - 2t)) \cap \mathbb{Z}^2.
\end{aligned}$$

More generally, for all $i, j \in \mathbb{Z}$ we define $B_{i,j} = B_{0,0} + (2is, 2js)$, $A^I_{i,j} = A^I_{0,0} + (2is, 2js)$, etcetera.

Before we go on, we give a very brief and informal summary of the proof (see Figure 2.1): The box Λ_n in the statement of the theorem will be (roughly) partitioned

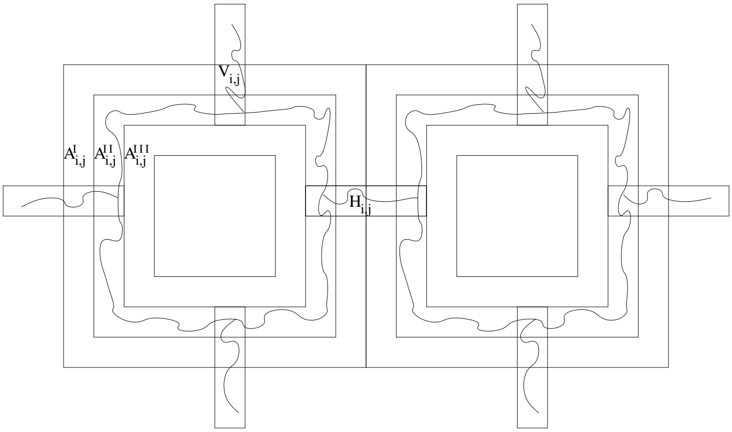

Figure 2.2: Illustration of the event $\tilde{O}^{s,t}$.

in m^2 boxes $B_{i,j}$ defined above, where the s (and hence m) and t will be chosen appropriately, depending on n, a and b. (For elegance/symmetry we take the number m odd). We will 'construct' an open cluster of which the 'skeleton' consists of circuits in the annuli $A^{II}_{i,j}$, 'glued' together by connections in the 'corridors' $V_{i,j}$ and $H_{i,j}$. (The other annuli defined above will be used for technical reasons in the proof). The setup is such that the contributions from the different $B_{i,j}$'s to the total cluster size are roughly independent, and that these contributions can be 'steered' to get the total sum inside the desired interval. In some sense this replaces the original problem for the box Λ_n by a similar problem, but now for the smaller boxes $B_{i,j}$. Apart from the technicalities involving the control of local dependencies, there is a subtle aspect in the proof related to the asymptotic behaviour of $\pi(n)$: Although the precise power-law behaviour of $\pi(n)$ is not important, it seems to be essential for the arguments that the exponent in a power-law upper bound is strictly smaller than 1 (see the note at the end of the proof of Lemma 2.3.6)).

Now we continue with the precise constructions mentioned above. First we give some more notation and definitions. Let E denote the set of edges of $\mathbb{Z}^2$ and $\Omega = \{0,1\}^E$. For $\omega \in \Omega$ and $F \subset E$ we will write $\omega_F \in \{0,1\}^F$ for the 'restriction' $(\omega_e, e \in F)$ of ω to F. Let $A \subset \Omega$ and $W \subset \mathbb{Z}^2$. We write $E(W)$ for the set of all edges of which both endpoints are in W. Informally, we use the notation $A(W)$ for the set of all configurations $\omega \in \Omega$ that belong to A or can be turned to an element of A by modifying ω outside $E(W)$. More precisely,

$$A(W) = \{\omega_{E(W)} : \omega \in A\} \times \{0,1\}^{E \setminus E(W)}. \tag{2.4}$$

We denote by $\tilde{O}^{s,t}$ the event that (i) - (iii) below occur (see Figure 2.2):

(i) $\forall i,j \in \mathbb{Z}$: the annulus $A^{II}_{i,j}$ contains an open circuit;

(ii) $\forall i,j \in \mathbb{Z}$: $H_{i,j}$ contains an open connection between the two widest open circuits in the annuli $A^{II}_{i,j}$ and $A^{II}_{i+1,j}$;

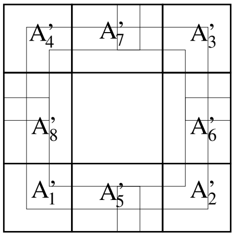

Figure 2.3: The subdivision of $A'_{0,0}$ in $A'_1, \cdots, A'_8$.

(iii) $\forall i, j \in \mathbb{Z} : V_{i,j}$ contains an open connection between the two widest open circuits in the annuli $A^{II}_{i,j}$ and $A^{II}_{i,j+1}$.

The introduction of this event looks meaningless since it has probability 0. It will only be used to give a 'compact' description of the following events (which do play a key role in the proof).

Definition 2.3.1. *Let $m, s, t \in \mathbb{N}$, with $t \leq \frac{1}{3}s$ and m odd. Let $i, j \in \mathbb{Z}$. We define, using notation* (2.4)*, the following events:*

$$\begin{aligned} O^{m,s,t} &= \tilde{O}^{s,t}(\Lambda_{ms}). \\ O^{s,t}_{i,j} &= \tilde{O}^{s,t}(B_{i,j}). \end{aligned}$$

Remark: From now on, for given m, s, t, the indices i, j under consideration will always be assumed to be in the set $\{-\frac{1}{2}(m-1), \cdots, 0, \cdots, \frac{1}{2}(m-1)\}$.

2.3.2 Expected cluster size in a narrow annulus

For a circuit γ in $\mathbb{Z}^2$ we denote by $\mathrm{Int}(\gamma)$ the bounded connected component of $\mathbb{Z}^2 \setminus \gamma$, and define

$$\mathcal{C}^\gamma = |\{v \in \mathrm{Int}(\gamma) : v \leftrightarrow \gamma\}|. \tag{2.5}$$

Further, for all i, j, let $\gamma_{i,j}$ denote the widest open circuit in the annulus $A^{II}_{i,j}$, and define, for $W \subset \Lambda_n$,

$$\mathcal{C}_{i,j}(W) = |\{v \in W : v \leftrightarrow \gamma_{i,j}\}|. \tag{2.6}$$

If there is no open circuit in $A^{II}_{i,j}$, then $\mathcal{C}_{i,j}(W) = 0$.

Recall the definition of $\tilde{\mathcal{C}}$ in (2.2).

Lemma 2.3.2. *There exists a constant $C_{10} > 0$ such that for all $s \in \mathbb{N}$, $t \leq \frac{1}{3}s$ and all i, j:*

$$\mathbb{E}[\mathcal{C}_{i,j}(A'_{i,j}) | O^{s,t}_{i,j}] \leq \mathbb{E}[\tilde{\mathcal{C}}(A'_{i,j}) | O^{s,t}_{i,j}] \leq C_{10} s t \pi(t).$$

Proof. The first inequality follows immediately from the fact that, on the event $O^{s,t}_{i,j}$, the circuit $\gamma_{i,j}$ is connected to the boundary of $B_{i,j}$ and hence $\mathcal{C}_{i,j}(A'_{i,j})$ is smaller than or equal to $\tilde{\mathcal{C}}(A'_{i,j})$. We prove the second inequality. Without loss of generality we take $i = j = 0$. We subdivide the annulus $A' = A'_{0,0} = \bigcup_{l=1}^{8} A'_l$, where A'_1, A'_2, A'_3, A'_4 are the $(3t \times 3t)$-squares in the four corners, and A'_5, A'_6, A'_7, A'_8 the remaining rectangles (see Figure 2.3). Note that $\tilde{\mathcal{C}}(A') \leq \sum_{l=1}^{8} \tilde{\mathcal{C}}(A'_l)$. Hence it is sufficient to show that there is a constant C_{11} such that for each $l = 1, \cdots, 8$,

$$\mathbb{E}[\tilde{\mathcal{C}}(A'_l) | O^{s,t}_{0,0}] \leq C_{11} st\pi(t). \tag{2.7}$$

By symmetry we only have to handle the cases $l = 1$ and $l = 5$. For each l the l.h.s. of (2.7) is

$$\mathbb{E}[\tilde{\mathcal{C}}(A'_l) | O^{s,t}_{0,0}] = \frac{1}{\mathbb{P}(O^{s,t}_{0,0})} \sum_{v \in A'_l} \mathbb{P}(v \leftrightarrow \partial A'_l; O^{s,t}_{0,0}). \tag{2.8}$$

Recall the notation (2.4). For each $v \in A'_1$, obviously,

$$\mathbb{P}(v \leftrightarrow \partial A'_1; O^{s,t}_{0,0}) \leq \mathbb{P}(v \leftrightarrow \partial A'_1) \mathbb{P}(O^{s,t}_{0,0}(A' \setminus A'_1)). \tag{2.9}$$

Further, informally speaking, the event $O^{s,t}_{0,0}(A' \setminus A'_1)$ can, with a 'local surgery involving a bounded cost in terms of probability', be turned into the event $O^{s,t}_{0,0}$. More precisely, if $O^{s,t}_{0,0}(A' \setminus A'_1)$ holds, and there is a horizontal open crossing of the rectangle $[-s, -s+6t] \times [-s+t, -s+2t]$ and of the square $[-s, -s+3t] \times [-s+3t, -s+6t]$, and a vertical open crossing of the rectangle $[-s+t, -s+2t] \times [-s, -s+6t]$ and of the square $[-s+3t, -s+6t] \times [-s, -s+3t]$, then the event $O^{s,t}_{0,0}$ holds. Hence, by RSW (and FKG) we have a positive constant C_{12} such that $\mathbb{P}(O^{s,t}_{0,0}(A' \setminus A'_1)) \leq C_{12} \mathbb{P}(O^{s,t}_{0,0})$. Combining this with (2.8) and (2.9) gives

$$\mathbb{E}[\tilde{\mathcal{C}}(A'_1) | O^{s,t}_{0,0}] \leq C_{12} \sum_{v \in A'_1} \mathbb{P}(v \leftrightarrow \partial A'_1). \tag{2.10}$$

For the case $l = 5$ let, for $v \in A'_5$, $R = ([v_1 - t, v_1 + t] \times [-s, -s+3t]) \cup V_{0,-1}$ and let $G(v)$ be the event that there are vertical open crossings in $[v_1 - t, v_1 - \frac{1}{2}t] \times [-s, -s+3t]$ and $[v_1 + \frac{1}{2}t, v_1 + t] \times [-s, -s+3t]$. By RSW (and FKG) arguments we have a positive constant C_{13} such that

$$\mathbb{P}(v \leftrightarrow \partial A'_5; O^{s,t}_{0,0}) \leq C_{13} \mathbb{P}(v \leftrightarrow \partial A'_5; O^{s,t}_{0,0}; G(v)) \leq C_{13} \mathbb{P}(v \leftrightarrow \partial A'_5) \mathbb{P}(O^{s,t}_{0,0}(R)).$$

Again some 'local surgery' gives a constant C'_{13} such that $\mathbb{P}(O^{s,t}_{0,0}(R)) \leq C'_{13} \mathbb{P}(O^{s,t}_{0,0})$. Hence

$$\mathbb{E}[\tilde{\mathcal{C}}(A'_5) | O^{s,t}_{0,0}] \leq C_{13} C'_{13} \sum_{v \in A'_5} \mathbb{P}(v \leftrightarrow \partial A'_5). \tag{2.11}$$

Application of part (iv) of Theorem 2.2.1 to the right-hand sides of (2.10) and (2.11) gives (2.7). □

2.3.3 Properties of nice circuits

Let m, s, t be as in Definition 2.3.1, and recall the Remark about the values of the indices i, j at the end of Section 2.3.1. Let, for each i, j, $\gamma_{i,j}$ be as in the beginning of Section 2.3.2, and let $\tilde{\gamma}_{i,j}$ be a deterministic circuit in the annulus $A^{II}_{i,j}$. Further we will denote the collection of all $\gamma_{i,j}$'s by (γ), and the collection of al $\tilde{\gamma}_{i,j}$'s by $(\tilde{\gamma})$.

Definition 2.3.3. *We say that $\tilde{\gamma}_{i,j}$ is (s,t)-nice if*

$$\mathbb{E}[\tilde{\mathcal{C}}(A'_{i,j}) \mid O^{s,t}_{i,j}; \gamma_{i,j} = \tilde{\gamma}_{i,j}] \leq 2C_{10} st\pi(t), \tag{2.12}$$

with C_{10} as in Lemma 2.3.2. Further, the collection $(\tilde{\gamma})$ is called (m,s,t)-nice if each circuit in the collection is (s,t)-nice.

We define $\Gamma^{s,t}_{i,j}$ as the event that $\gamma_{i,j}$ is (s,t)-nice, and define

$$\Gamma^{m,s,t} = \bigcap_{i,j} \Gamma^{s,t}_{i,j}. \tag{2.13}$$

Recall the notation (2.6). We will study the open cluster $\mathcal{C}_{0,0}(Q)$, where

$$Q = (\Lambda_n \setminus \Lambda_{ms}) \cup (\bigcup_{i,j} A'_{i,j}).$$

Lemma 2.3.4. *There exist positive constants C_{14} and C_{15} such that, for all m, s and t,*

(i)

$$\mathbb{P}(\Gamma^{m,s,t} | O^{m,s,t}) \geq (C_{14})^{m^2}.$$

(ii) For all (m,s,t)-nice $(\tilde{\gamma})$ and all n with $n - ms \leq t$,

$$\mathbb{P}\left(\mathcal{C}_{0,0}(Q) \leq C_{15} m^2 st\pi(t) \mid O^{m,s,t}; (\gamma) = (\tilde{\gamma})\right) \geq \frac{1}{2}.$$

Proof. We claim that there is a constant $C_{16} > 0$ such that for all i, j:

$$\mathbb{P}(O^{m,s,t}; (\gamma) = (\tilde{\gamma})) \geq C_{16} \mathbb{P}(D) \mathbb{P}(O^{s,t}_{i,j}; \gamma_{i,j} = \tilde{\gamma}_{i,j}), \tag{2.14}$$

where (with the notation (2.4))

$$D = O^{m,s,t}(\Lambda_n \setminus B_{i,j}) \cap \bigcap_{\tilde{i},\tilde{j}:(\tilde{i},\tilde{j}) \neq (i,j)} \{\gamma_{\tilde{i},\tilde{j}} = \tilde{\gamma}_{\tilde{i},\tilde{j}}\}.$$

To prove this claim we write

$$G = B_{i,j} \setminus A^I_{i,j}, \qquad J = A^I_{i,j} \cup \bigcup_{(\tilde{i},\tilde{j}) \in M_{i,j}} A^I_{\tilde{i},\tilde{j}}, \qquad K = \Lambda_n \setminus (G \cup J),$$

where $M_{i,j} = \{(i-1,j), (i,j-1), (i+1,j), (i,j+1)\} \cap \{-(m-1)/2, \cdots, (m-1)/2\}^2$. Let $D_1 = O^{s,t}_{i,j} \cap \{\gamma_{i,j} = \tilde{\gamma}_{i,j}\}$. We also need an event D_2 which, informally speaking, connects the structures in the definition of D_1 with those in D. More precisely,

$$D_2 = \bigcap_{(\tilde{i},\tilde{j}) \in M_{i,j}} D_2^{\tilde{i},\tilde{j}},$$

where $D_2^{i+1,j}$ is the event that (i) $H_{i,j} \cap J$ contains a horizontal crossing and (ii) $H_{i,j} \cap A^I_{i,j}$ and $H_{i,j} \cap A^I_{i+1,j}$ both contain a vertical crossing. The other $D_2^{\tilde{i},\tilde{j}}$'s are defined similarly. By RSW (and FKG) there is a positive constant C_{16} such that $\mathbb{P}(D_2) > C_{16}$. Note that D_2 is increasing and (with the notation in Section 2.3.1), the event D is increasing with respect to the edges outside $E(K)$, and D_1 is increasing with respect to the edges outside $E(G)$. We get

$$\begin{aligned}
&\mathbb{P}(O^{m,s,t};(\gamma)=(\tilde{\gamma})) \geq \mathbb{P}(D\cap D_1\cap D_2)\\
&= \sum_{\omega_1\in\{0,1\}^{E(K)}} \mathbb{P}(\omega_1) \sum_{\omega_2\in\{0,1\}^{E(G)}} \mathbb{P}(D\cap D_1\cap D_2|\omega_1,\omega_2)\mathbb{P}(\omega_2)\\
&\geq \sum_{\omega_1\in\{0,1\}^{E(K)}} \mathbb{P}(\omega_1) \sum_{\omega_2\in\{0,1\}^{E(G)}} \mathbb{P}(D|\omega_1,\omega_2)\mathbb{P}(D_1|\omega_1,\omega_2)\mathbb{P}(D_2|\omega_1,\omega_2)\mathbb{P}(\omega_2)\\
&= \sum_{\omega_1\in\{0,1\}^{E(K)}} \mathbb{P}(\omega_1)\mathbb{P}(D|\omega_1) \sum_{\omega_2\in\{0,1\}^{E(G)}} \mathbb{P}(D_1|\omega_2)\mathbb{P}(D_2)\mathbb{P}(\omega_2)\\
&\geq C_{16}\mathbb{P}(D)\mathbb{P}(D_1),
\end{aligned}$$

where we used FKG in the third line, and in the fourth line we used that D doesn't depend on the configuration on G, D_1 doesn't depend on the configuration on K, and D_2 doesn't depend on the configuration on $G\cup K$. This proves the claim.

By repeating the same arguments for each $B_{i,j}$, we eventually get the following 'extension' of (2.14):

$$\mathbb{P}\left((\gamma)=(\tilde{\gamma});O^{m,s,t}\right) \geq C_{16}^{m^2}\prod_{i,j}\mathbb{P}(\gamma_{i,j}=\tilde{\gamma}_{i,j};O^{s,t}_{i,j}). \tag{2.15}$$

Now we are ready to prove part (i):

$$\begin{aligned}
\mathbb{P}(\Gamma^{m,s,t}|O^{m,s,t}) &= \frac{1}{\mathbb{P}(O^{m,s,t})}\sum_{\tilde{\gamma}:(m,s,t)\text{-nice}}\mathbb{P}\left((\gamma)=(\tilde{\gamma});O^{m,s,t}\right)\\
&\geq C_{16}^{m^2}\prod_{i,j}\mathbb{P}(\Gamma^{s,t}_{i,j}|O^{s,t}_{i,j}),
\end{aligned} \tag{2.16}$$

where the inequality follows from (2.15) and the obvious inequality $\mathbb{P}(O^{m,s,t}) \leq \prod_{i,j}\mathbb{P}(O^{s,t}_{i,j})$. This gives part (i) of the lemma because for each factor in the product of the last expression in (2.16) we have, by Definition 2.3.3, Markov's inequality and Lemma 2.3.2,

$$\begin{aligned}
\mathbb{P}(\Gamma^{s,t}_{i,j}|O^{s,t}_{i,j}) &= \mathbb{P}\left(\mathbb{E}[\tilde{\mathcal{C}}(A'_{i,j})\mid O^{s,t}_{i,j};\gamma_{i,j}]\leq 2C_{10}st\pi(t)\mid O^{s,t}_{i,j}\right)\\
&\geq 1-\frac{\mathbb{E}[\tilde{\mathcal{C}}(A'_{i,j})\mid O^{s,t}_{i,j}]}{2C_{10}st\pi(t)}\geq\frac{1}{2}.
\end{aligned}$$

To prove part (ii) first note that

$$\begin{aligned}
\mathbb{E}\left[\tilde{\mathcal{C}}(A'_{i,j})|O^{m,s,t};(\gamma)=(\tilde{\gamma})\right] &= \frac{\sum_{v\in A'_{i,j}}\mathbb{P}\left(v\leftrightarrow\partial A'_{i,j};O^{m,s,t};(\gamma)=(\tilde{\gamma})\right)}{\mathbb{P}(O^{m,s,t};(\gamma)=(\tilde{\gamma}))}\\
&\leq \frac{\sum_{v\in A'_{i,j}}\mathbb{P}(v\leftrightarrow\partial A'_{i,j};O^{s,t}_{i,j};\gamma_{i,j}=\tilde{\gamma}_{i,j})\mathbb{P}(D)}{C_{16}\mathbb{P}(D)\mathbb{P}(O^{s,t}_{i,j};\gamma_{i,j}=\tilde{\gamma}_{i,j})},
\end{aligned}$$

where we used (2.14) in the denominator. Hence

$$\mathbb{E}\left[\tilde{\mathcal{C}}(A'_{i,j})|O^{m,s,t};(\gamma)=(\tilde{\gamma})\right] \leq \frac{1}{C_{16}}\mathbb{E}[\tilde{\mathcal{C}}(A'_{i,j})|O^{s,t}_{i,j};\gamma_{i,j}=\tilde{\gamma}_{i,j}] \leq \frac{2C_{10}}{C_{16}}st\pi(t), \quad (2.17)$$

where the last inequality is just the 'niceness' property (Definition 2.3.3) of $(\tilde{\gamma})$. To finish the proof of part (ii), note that, for each $K > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{C}_{0,0}(Q) \leq Km^2 st\pi(t)|O^{m,s,t};(\gamma)=(\tilde{\gamma})\right) \\
&\geq 1 - \frac{\mathbb{E}\left[\mathcal{C}_{0,0}(Q)|O^{m,s,t};(\gamma)=(\tilde{\gamma})\right]}{Km^2 st\pi(t)} \\
&\geq 1 - \frac{\mathbb{E}\left[\tilde{\mathcal{C}}(\Lambda_n \setminus \Lambda_{ms})\right] + \sum_{i,j}\mathbb{E}\left[\tilde{\mathcal{C}}(A'_{i,j})|O^{m,s,t};(\gamma)=(\tilde{\gamma})\right]}{Km^2 st\pi(t)}.
\end{aligned}$$

Applying part (iv) of Theorem 2.2.1 to the first expectation in the r.h.s. of the last expression, and (2.17) to each of the other expectations gives, by choosing K sufficiently large, the desired result. This completes the proof of part (ii) of Lemma 2.3.4. □

2.3.4 Cluster-size contributions inside the circuits

In this section we write the value t (the width of the relevant annuli and 'corridors' in the construction) as $\lfloor \varepsilon s \rfloor$. A suitable value for ε (depending on the values of a and b in the statement of Theorem 2.1.1) will be determined in the next section. The main result in the current section concerns the contribution from the interior of a nice circuit to the cluster of that circuit. Recall the notation (2.5).

Lemma 2.3.5. *There exist constants $C_{17}, C_{18}, \delta_2 > 0$, and for every $\varepsilon < \frac{1}{12}$ there exists $\delta_3(\varepsilon) > 0$, such that for all $s \in \mathbb{N}$ and all $(s, \lfloor \varepsilon s \rfloor)$-nice circuits $\tilde{\gamma}_{0,0}$ in $A^{II}_{0,0}$,*

(i)

$$\mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{0,0}} \in (C_{17}s^2\pi(s), C_{18}s^2\pi(s)) \mid \gamma_{0,0} = \tilde{\gamma}_{0,0}\right) \geq \delta_2. \quad (2.18)$$

(ii)

$$\mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{0,0}} < 4C_{10}s\lfloor \varepsilon s \rfloor \pi(\lfloor \varepsilon s \rfloor) \mid \gamma_{0,0} = \tilde{\gamma}_{0,0}\right) \geq \delta_3(\varepsilon). \quad (2.19)$$

Proof. Let $B'_{0,0} = B_{0,0} \setminus A'_{0,0}$. Let $\hat{Y} = |\{v \in B'_{0,0} : v \leftrightarrow \partial B_{0,0}\}|$. Clearly,

$$\mathbb{P}(\mathcal{C}^{\tilde{\gamma}_{0,0}} \geq C_{17}s^2\pi(s) \mid \gamma_{0,0} = \tilde{\gamma}_{0,0}) \geq \mathbb{P}(\hat{Y} \geq C_{17}s^2\pi(s)), \quad (2.20)$$

which (for a suitable choice of C_{17}) by Theorem 2.2.2 is at least a positive constant, which we write as $2\delta_2$. To complete the proof we need to find a $C_{18} > 0$ such that

$$\mathbb{P}(\mathcal{C}^{\tilde{\gamma}_{0,0}} \geq C_{18}s^2\pi(s) \mid \gamma_{0,0} = \tilde{\gamma}_{0,0}) \leq \delta_2. \quad (2.21)$$

To do this we look for an upper bound for $\mathbb{E}[\mathcal{C}^{\tilde{\gamma}_{0,0}} \mid \gamma_{0,0} = \tilde{\gamma}_{0,0}]$. We have

$$\begin{aligned}
\mathbb{E}[\mathcal{C}^{\tilde{\gamma}_{0,0}} \mid \gamma_{0,0} = \tilde{\gamma}_{0,0}] &= \mathbb{E}[\mathcal{C}^{\tilde{\gamma}_{0,0}} \mid O^{s,\lfloor \varepsilon s \rfloor}_{0,0}; \gamma_{0,0} = \tilde{\gamma}_{0,0}] \quad (2.22)\\
&\leq \mathbb{E}[\tilde{\mathcal{C}}(B'_{0,0})] + \mathbb{E}[\tilde{\mathcal{C}}(A'_{0,0}) \mid O^{s,\lfloor \varepsilon s \rfloor}_{0,0}; \gamma_{0,0} = \tilde{\gamma}_{0,0}]
\end{aligned}$$

Applying part (iv) of Theorem 2.2.1 to the first expectation in the last line, and the niceness property of $\tilde{\gamma}_{0,0}$ to the other expectation, shows that the l.h.s. of (2.22) is at most $C_{19}s^2\pi(s)$. Finally, Markov's inequality gives (2.21) with $C_{18} = \frac{C_{19}}{\delta_2}$. This completes the proof of part (i).

Now we prove part (ii). Let G be the event that there is a closed dual circuit in $A^{III}_{0,0}$. On this event, let $\beta_{0,0}$ denote the innermost of such circuits. Observe that, conditioned on $\beta_{0,0}$, the configuration outside $\beta_{0,0}$ is independent of the configuration inside. Also observe that, on the event G, all vertices in the interior of $\gamma_{0,0}$ that are connected to $\gamma_{0,0}$ are in $A'_{0,0}$. By these and related simple observations we have that

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{0,0}} < 4C_{10}s\lfloor\varepsilon s\rfloor\pi(\lfloor\varepsilon s\rfloor) \mid \gamma_{0,0} = \tilde{\gamma}_{0,0}; \beta_{0,0} = \tilde{\beta}\right)\\
&= \mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{0,0}} < 4C_{10}s\lfloor\varepsilon s\rfloor\pi(\lfloor\varepsilon s\rfloor) \mid O^{s,\lfloor\varepsilon s\rfloor}_{0,0}; \gamma_{0,0} = \tilde{\gamma}_{0,0}; \beta_{0,0} = \tilde{\beta}\right)\\
&\geq \mathbb{P}\left(\tilde{\mathcal{C}}(A'_{0,0}) < 4C_{10}s\lfloor\varepsilon s\rfloor\pi(\lfloor\varepsilon s\rfloor) \mid O^{s,\lfloor\varepsilon s\rfloor}_{0,0}; \gamma_{0,0} = \tilde{\gamma}_{0,0}\right),
\end{aligned}$$

which, by Markov's inequality and because $\tilde{\gamma}_{0,0}$ is nice, is at least $1/2$. Hence, the l.h.s. of (2.19) is at least $(1/2)\mathbb{P}(G)$, which by RSW is larger than some positive constant which depends only on ε. □

2.3.5 Completion of the proof of Theorem 2.1.1

We are now ready to prove Theorem 2.1.1. First we still restrict to the case $p = 1/2$. Let $0 < a < b$ be given. See the brief outline in Section 2.3.1. The lengths of the building blocks $B_{i,j}$ and the widths of the annuli and 'corridors' in the partition of Λ_n, will be taken proportional to n, say (roughly) xn and εxn respectively, with suitably chosen x and ε. For this purpose we will use the following lemma:

Lemma 2.3.6. *There exist $x > 0, \varepsilon \in (0, \frac{1}{12})$ and $N \in \mathbb{N}$, with $\frac{1}{x}$ an odd integer, such that for all $n \geq N$ the following inequalities hold:*

$$C_{17}\lfloor xn\rfloor^2\pi(\lfloor xn\rfloor)\cdot(\frac{1}{x})^2 \geq an^2\pi(n); \tag{2.23}$$

$$C_{18}\lfloor xn\rfloor^2\pi(\lfloor xn\rfloor) \leq \frac{1}{3}(b-a)n^2\pi(n); \tag{2.24}$$

$$(4C_{10}\vee C_{15})(\frac{1}{x})^2\lfloor\varepsilon\lfloor xn\rfloor\rfloor\lfloor xn\rfloor\pi(\lfloor\varepsilon\lfloor xn\rfloor\rfloor) \leq \frac{1}{3}(b-a)n^2\pi(n). \tag{2.25}$$

Proof. It is easy to see (a weak form of the lower bound in part (i) of Theorem 2.2.1 suffices) that if x is sufficiently small (depending on a), then (2.23) holds for all sufficiently large n. It also easily follows (now from the upper bound in the same Theorem) that if x is sufficiently small (depending on $b - a$), (2.24) holds for all sufficiently large n. Finally, for x fixed, it follows (again from the upper bound in part (i) of Theorem 2.2.1) that if ε is sufficiently small (depending on $b - a$ and x), then (2.25) holds for all sufficiently large n.
Note that for this last step it is essential that the exponent (1/2) in part (i) of Theorem 2.2.1 is strictly smaller than 1.
This completes the proof of Lemma 2.3.6. □

Now let x, ε and N be as in Lemma 2.3.6. Moreover we assume (which we may, because we can enlarge N if necessary) that $n - \frac{1}{x}\lfloor xn \rfloor \leq \lfloor \varepsilon \lfloor xn \rfloor \rfloor$ for all $n \geq N$. Denote by D_n the event

$$D_n = \{\exists v \in \Lambda_n : \mathcal{C}_n(v) \in (an^2\pi(n), bn^2\pi(n))\}.$$

Let $n \geq N$ and let $m = \frac{1}{x}$, $s = \lfloor xn \rfloor$ and $t = \lfloor \varepsilon \lfloor xn \rfloor \rfloor$. By straightforward RSW and FKG arguments, there is a $\delta_4(x, \varepsilon) > 0$ such that $\mathbb{P}(O^{m,s,t}) > \delta_4(x, \varepsilon)$. Hence

$$\mathbb{P}(D_n) \geq \delta_4(x, \varepsilon)\mathbb{P}(D_n | O^{m,s,t}). \tag{2.26}$$

From Lemma 2.3.4 (i) it follows that

$$\mathbb{P}(D_n | O^{m,s,t}) \geq (C_{14})^{\frac{1}{x^2}} \mathbb{P}(D_n | O^{m,s,t}; \Gamma^{m,s,t}). \tag{2.27}$$

The next step is conditioning on the widest open circuits.

$$\begin{aligned} &\mathbb{P}(D_n | O^{m,s,t}; \Gamma^{m,s,t}) \\ &= \sum_{(\tilde{\gamma}):(m,s,t)\text{-nice}} \mathbb{P}(D_n | (\gamma) = (\tilde{\gamma}); O^{m,s,t})\, \mathbb{P}((\gamma) = (\tilde{\gamma}) | O^{m,s,t}; \Gamma^{m,s,t}). \end{aligned} \tag{2.28}$$

For each $(\tilde{\gamma})$ we denote by $\mathcal{C}^{in}_{(\tilde{\gamma})}$ the number of all vertices that are in the interior of a circuit in the collection $(\tilde{\gamma})$ and connected to that circuit, and by $\mathcal{C}^{out}_{(\tilde{\gamma})}$ the number of vertices outside these circuits that are connected to one or more of these circuits, plus the number of vertices on these circuits. We have

$$\begin{aligned} &\mathbb{P}(D_n | (\gamma) = (\tilde{\gamma}); O^{m,s,t}) \\ \geq\; &\mathbb{P}\left(\frac{\mathcal{C}^{in}_{(\tilde{\gamma})}}{n^2\pi(n)} \in (a, b - \frac{1}{3}(b-a)) \mid \frac{\mathcal{C}^{out}_{(\tilde{\gamma})}}{n^2\pi(n)} \leq \frac{1}{3}(b-a); (\gamma) = (\tilde{\gamma}); O^{m,s,t} \right) \\ &\cdot \mathbb{P}\left(\mathcal{C}^{out}_{(\tilde{\gamma})} \leq \frac{1}{3}(b-a)n^2\pi(n) \mid (\gamma) = (\tilde{\gamma}); O^{m,s,t} \right) \\ \geq\; &\frac{1}{2}\mathbb{P}\left(\frac{\mathcal{C}^{in}_{(\tilde{\gamma})}}{n^2\pi(n)} \in (a, \frac{1}{3}a + \frac{2}{3}b) \right), \end{aligned} \tag{2.29}$$

where the last inequality holds by (2.25) and Lemma 2.3.4 (ii), and because the configurations in the interiors of the $\tilde{\gamma}_{i,j}$'s are obviously independent of the event conditioned on in the expression in the r.h.s. of the first inequality. Note that $\mathcal{C}^{in}_{(\tilde{\gamma})} = \sum_{i,j} \mathcal{C}^{\tilde{\gamma}_{i,j}}$. The $\mathcal{C}^{\tilde{\gamma}_{i,j}}$'s are independent and for each i, j we have

$$\begin{aligned} &\mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{i,j}} \in (ax^2n^2\pi(n), \frac{1}{3}(b-a)n^2\pi(n)) \right) \\ &\overset{(2.23),(2.24)}{\geq} \mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{i,j}} \in \left(C_{17}\lfloor xn \rfloor^2 \pi(\lfloor xn \rfloor), C_{18}\lfloor xn \rfloor^2 \pi(\lfloor xn \rfloor) \right) \right) \overset{\text{Lem.2.3.5}(i)}{\geq} \delta_2, \end{aligned}$$

and

$$\begin{aligned} &\mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{i,j}} \leq x^2 \frac{1}{3}(b-a)n^2\pi(n) \right) \\ &\overset{(2.25)}{\geq} \mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{i,j}} \leq 4C_{10}\lfloor \varepsilon \lfloor xn \rfloor \rfloor \lfloor xn \rfloor \pi(\lfloor \varepsilon \lfloor xn \rfloor \rfloor) \right) \overset{\text{Lem.2.3.5}(ii)}{\geq} \delta_3(\varepsilon). \end{aligned}$$

Hence the conditions of Lemma 2.2.4 (with $k = (1/x)^2$, $\alpha = an^2\pi(n)$, $\beta = (\frac{1}{3}a + \frac{2}{3}b)n^2\pi(n)$) are satisfied. Hence, by that lemma the l.h.s. of (2.29) is at least $\frac{1}{2}(\delta_2 \wedge \delta_3(\varepsilon))^{(1/x)^2}$. Together with (2.26) - (2.28) this shows that

$$\mathbb{P}(D_n) > \delta_5, \tag{2.30}$$

with δ_5 a positive constant which depends only on a and b.

Now we will show that, by the way we 'constructed' the open cluster, a similar result holds for the *maximal* open cluster in Λ_n. First note that the 'constructed' cluster has the property that it contains an open horizontal and an open vertical crossing of the box Λ_{ms}. Also note that there is at most one open cluster with this property. Given the exact location of the (unique) open cluster with this property, the conditional probability that it is the maximal cluster in Λ_n is, if its size is larger than $an^2\pi(n)$, clearly larger than or equal to the probability that the remaining part of Λ_n contains no open cluster of size larger than $an^2\pi(n)$. By obvious monotonicity this probability is at least $\mathbb{P}(\mathcal{M}_n \leq an^2\pi(n))$, which by Theorem 2.2.3 is at least some positive constant δ_2 (which depends only on a). This argument gives

$$\mathbb{P}\left(\mathcal{M}_n \in (an^2\pi(n), bn^2\pi(n))\right) \geq \delta_2\delta_5,$$

which completes the proof of Theorem 2.1.1 for $p = 1/2$.

Now let, more generally, p be such that $L(p) \geq n$. It is straightforward to check that (due to the 'uniformity' in p of the results in Section 2.2) each step in the proof remains essentially valid. For instance, it is easy to see from the arguments used that Lemma 2.3.2 (now with $\mathbb{P}$ replaced by $\mathbb{P}_p$) remains valid as long as $L(p) \geq s$. Since we take $s \leq n$ (application of) this lemma (and, similarly, the other lemma's) can be carried out as before. This completes the proof of Theorem 2.1.1.

3 | Gaps between cluster sizes

This chapter is based on [11] with Rob van den Berg.

Consider critical bond percolation on a large $2n \times 2n$ box on the square lattice. It is well-known (see [16]) that the size (i.e. number of vertices) of the largest open cluster is, with high probability, of order $n^2\pi(n)$, where $\pi(n)$ denotes the probability that there is an open path from the center to the boundary of the box. The same result holds for the second-largest cluster, the third largest cluster etcetera.

Járai [50] showed that the differences between the sizes of these clusters is, with high probability, at least of order $\sqrt{n^2\pi(n)}$. Although this bound was enough for his applications (to incipient infinite clusters), he believed, but had no proof, that the differences are in fact of the same order as the cluster sizes themselves, i.e. $n^2\pi(n)$. Our main result is a proof that this is indeed the case.

3.1 Introduction and statement of main results

For general background on percolation we refer to [41] and [15]. We consider bond percolation on the square lattice with parameter p equal to its critical value $p_c = 1/2$. Let $\Lambda_n = [-n, n]^2 \cap \mathbb{Z}^2$ be the $2n \times 2n$ box centered at $O = (0, 0)$ and let $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$ be the (inner) boundary of the box. For each vertex $v \in \mathbb{Z}^2$, we write $\Lambda_n(v) = \Lambda_n + v$. Further, the open cluster in Λ_n of the vertex v is denoted by $\mathcal{C}_n(v)$. More precisely,

$$\mathcal{C}_n(v) := \{u \in \Lambda_n : u \leftrightarrow v \text{ inside } \Lambda_n\},$$

where '$u \leftrightarrow v$ inside Λ_n' means that there is an open path from u to v of which all vertices are in Λ_n. We write $\pi(n)$ for the probability $\mathbb{P}(O \leftrightarrow \partial\Lambda_n)$, the probability that there is an open path from O to $\partial\Lambda_n$. Further, we write

$$s(n) := n^2\pi(n). \tag{3.1}$$

By the *size* of a cluster we mean the number of vertices in the cluster. Let, for $i = 1, 2, \cdots$, $\mathcal{C}_n^{(i)}$ denote the i-th largest open cluster in Λ_n, and let $|\mathcal{C}_n^{(i)}|$ denote its size. (If two clusters have the same size, we order them in some deterministic way).

In [16] it was proved that $|\mathcal{C}_n^{(1)}|$ is of order $s(n)$. In the later paper [17] by the same authors it is shown that also $|\mathcal{C}_n^{(2)}|$, $|\mathcal{C}_n^{(3)}|$ etcetera are of order $s(n)$. They also proved an extension of this result for the case where the parameter p is not equal but close to p_c.

It was shown by Járai (see Section 1, Proposition 1, and its proof in section 3.1 in [50]) that for each i the difference $|\mathcal{C}_n^{(i)}|-|\mathcal{C}_n^{(i+1)}| \to \infty$ in probability as $n \to \infty$. In fact he showed that this difference is at least of order $\sqrt{s(n)}$. He suggested that it should be of order $s(n)$, but did not have a proof. In this chapter we show that his conjecture is correct. We became interested in such problems through our investigation of frozen-percolation processes. Our main theorem is as follows.

Theorem 3.1.1. *For all $k \in \mathbb{N}, \delta > 0$, there exist $\varepsilon > 0, N \in \mathbb{N}$ such that for all $n \geq N$:*

$$\mathbb{P}\left(\exists i \leq k-1 \,:\, |\mathcal{C}_n^{(i)}| - |\mathcal{C}_n^{(i+1)}| \leq \varepsilon s(n)\right) < \delta. \tag{3.2}$$

Remarks: *(i) The analog of Theorem 3.1.1 can be proved for site and bond percolation on other common two-dimensional lattices, e.g. site percolation on the square or the triangular lattice. In this latter model (site percolation on the triangular lattice) one of the last steps of the proof can be made a little bit shorter (see the Remark below the proof of Proposition 3.3.2).*
(ii) The proof, which is given in Section 3.3, follows the main line of Járai's proof of the weaker bound: We divide the box Λ_n in boxes of smaller length (denoted by $2t$), and condition on the configuration outside certain open circuits in these smaller boxes. Conditioned on this information, the 'contributions' (to the sizes of certain open clusters) from the interiors of these circuits are independent random variables. This leads to a problem concerning the concentration function of a sum of independent random variables, to which a general ('classical') theorem is applied. The main difference with Járai's arguments is that we take t proportional to n, with a proportionality factor chosen as a suitable function of the 'parameters' k and δ in the theorem. This makes the arguments more powerful (and also somewhat more complicated). Moreover, the theorem on concentration functions we used (see Theorem 3.2.6 below) is somewhat stronger than the one used in Járai's arguments.

Furthermore, with essentially the same argument we can show that the probability that there exists a cluster with size in a given interval of length $\varepsilon s(n)$ goes to zero as $\varepsilon \to 0$ uniformly in n:

Theorem 3.1.2. *For all $x, \delta > 0$, there exists an $\varepsilon > 0$ such that, for all $n \in \mathbb{N}$:*

$$\mathbb{P}\left(\exists u \in \Lambda_n : xs(n) < |\mathcal{C}_n(u)| < (x+\varepsilon)s(n)\right) < \delta. \tag{3.3}$$

This last theorem is in some sense complementary to the result in Chapter 2, where we proved that, for any interval (a,b), the probability that $|\mathcal{C}_n^{(1)}|/s(n) \in (a,b)$ is bounded away from zero as $n \to \infty$.

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3.2 Notation and Preliminaries

3.2.1 Preliminaries

First we need some more notation. For a cluster $\mathcal{C}_n(u)$ we define its (left-right) diameter by

$$\text{diam}(\mathcal{C}_n(u)) = \max_{v,w\in\mathcal{C}_n(u)} |v_1 - w_1|.$$

For a box Λ_n we define the spanning cluster by

$$SC_n = \{u \in \Lambda_n : u \leftrightarrow L(\Lambda_n) \text{ and } u \leftrightarrow R(\Lambda_n)\}, \tag{3.4}$$

where $L(\Lambda_n) = \{-n\} \times [-n,n] \cap \mathbb{Z}^2$ and $R(\Lambda_n) = \{n\} \times [-n,n] \cap \mathbb{Z}^2$. We use the notation $A_{m,n}$ for the annulus $\Lambda_n \setminus \Lambda_m$ and, for a vertex $v \in \mathbb{Z}^2$, the notation $A_{m,n}(v)$ for $A_{m,n} + v$.

In our proof of Theorem 3.1.1 and 3.1.2 we will use the following results from the literature, Theorems 3.2.1 - 3.2.6 below. The first one is well known, see for example [17], [14].

Theorem 3.2.1. ***([17],[14])*** *There exist constants $c_1, c_2, c_3 > 0$, such that for all $m \le n$:*

$$c_1(\frac{n}{m})^{c_2} \le \frac{\pi(m)}{\pi(n)} \le c_3(\frac{n}{m})^{\frac{1}{2}}.$$

As we already mentioned in the introduction, the largest clusters in Λ_n are of order $s(n)$. This is stated in the following result.

Theorem 3.2.2. ***([17] Thm. 3.1(i), 3.3, 3.6)*** *For all $i \in \mathbb{N}$,*

$$\mathbb{E}[|\mathcal{C}_n^{(i)}|] \asymp s(n), \tag{3.5}$$

and,

$$\liminf_{n\to\infty} \mathbb{P}\left(\varepsilon < \frac{|\mathcal{C}_n^{(i)}|}{\mathbb{E}[|\mathcal{C}_n^{(i)}|]} < \frac{1}{\varepsilon}\right) \to 1 \qquad \text{as } \varepsilon \to 0. \tag{3.6}$$

In an earlier paper Borgs, Chayes, Kesten and Spencer showed exponential decay for the probability that there exists a cluster with large volume, but a small diameter:

Theorem 3.2.3. ***([16] Remark (xiii))*** *There exist $C_1, C_2 > 0$ such that for all $x > 0$, $\alpha \in (0,1]$ and $n \ge 4/\alpha$ we have*

$$\mathbb{P}(\exists u \in \Lambda_n : |\mathcal{C}_n(u)| \ge xs(n);\ \text{diam}(\mathcal{C}_n(u)) \le \alpha n) \le C_1\alpha^{-2}\exp(-C_2 x/\alpha). \tag{3.7}$$

An easy consequence of Theorems 3.2.2 and 3.2.3 is the following.

Corollary 3.2.4. *Let $k \in \mathbb{N}$. For all $\delta > 0$ there exist $\alpha > 0$ and $N \in \mathbb{N}$ such that for all $n \ge N$:*

$$\mathbb{P}\left(\exists i \le k : \text{diam}(\mathcal{C}_n^{(i)}) < \alpha n\right) < \delta. \tag{3.8}$$

In [50] a version of Theorem 3.2.2 for the spanning cluster is given:

Theorem 3.2.5. ***([50] Thm. 8)***

$$\mathbb{E}[|SC_n|] \asymp s(n); \tag{3.9}$$

moreover,

$$\lim_{\varepsilon\to 0}\inf_{n\in\mathbb{N}} \mathbb{P}\left(\varepsilon < \frac{|SC_n|}{\mathbb{E}[|SC_n|]} < \frac{1}{\varepsilon} \mid SC_n \neq \emptyset\right) = 1. \tag{3.10}$$

In the proof of our main theorem we use the following inequality concerning the concentration function $Q(X,\lambda)$ of a random variable X, which is defined by

$$Q(X,\lambda) = \sup_{x\in\mathbb{R}} \mathbb{P}(x \le X \le x+\lambda), \tag{3.11}$$

for $\lambda > 0$.

Theorem 3.2.6. ***([63]; [37] (B))*** *Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of independent random variables, and $0 < \tilde{\lambda} \le \lambda$. Let $a > 0$ and let $(b_k)_{k\in\mathbb{N}}$ be a sequence of real numbers such that, for all $k \in \mathbb{N}$,*

$$\mathbb{P}(X_k \le b_k - \frac{\tilde{\lambda}}{2}) \ge a, \qquad \mathbb{P}(X_k \ge b_k + \frac{\tilde{\lambda}}{2}) \ge a.$$

There exists a universal constant $C > 0$ such that, for all $m \in \mathbb{N}$

$$Q(S_m,\lambda) \le \frac{C\lambda}{\tilde{\lambda}\sqrt{m\,a}},$$

where $S_m = X_1 + X_2 + \cdots + X_m$.

3.2.2 Large clusters contain many good boxes

In the proof of our main theorem we need the following lemma, which is essentially already in [50]. First some definitions. Recall the notation $A_{m,n}$ in the beginning of Section 3.2.. Let $t \in 3\mathbb{N}$. (Later we will choose a suitable value for t). For any $i,j \in \mathbb{Z}$ we say that the box $\Lambda_t(2ti,2tj)$ is 'good' if there is an open circuit in the annulus $A_{\frac{2}{3}t,t}(2ti,2tj)$; in that case we denote the widest open circuit in that annulus by $\gamma_{i,j}$. (Although $\gamma_{i,j}$ depends on t, we omit that parameter from the notation). For each vertex u we denote by $G_t(\mathcal{C}_n(u))$ the set of good boxes in Λ_n of which the corresponding $\gamma_{i,j}$ is contained in the open cluster of u. More precisely,

$$G_t(\mathcal{C}_n(u)) = \{(i,j) : \Lambda_t(2ti,2tj) \subset \Lambda_n \text{ is good }; \gamma_{i,j} \subset \mathcal{C}_n(u)\}. \tag{3.12}$$

Lemma 3.2.7. *Let $\alpha > 0$. For any $\delta,\beta > 0$ there exist $\eta > 0$ and $N \in \mathbb{N}$ such that, for all $n \ge N$ and $t \in (0,\eta n) \cap 3\mathbb{N}$*

$$\mathbb{P}\left(\exists u \in \Lambda_n : \operatorname{diam}(\mathcal{C}_n(u)) \ge \alpha n;\ |G_t(\mathcal{C}_n(u))| < \beta\right) < \delta. \tag{3.13}$$

Járai proved a somewhat stronger statement (see (3.15) in [50] and Proposition 3.6 in [39]), but we only need this weaker statement and give a (short) proof.

Proof. The C_i's in this proof denote universal constants larger than 0. Their existence is important but their precise value does not matter for the proof. First note that we can cover the box Λ_n by at most

$$\frac{C_1}{\alpha^2} \tag{3.14}$$

rectangles of width $\frac{1}{4}\alpha n$ and length $\frac{1}{2}\alpha n$, such that every cluster with diameter at least αn crosses at least one of these rectangles in the easy direction. We consider one such rectangle, namely $Q_0 := [0, \frac{1}{4}\alpha n] \times [0, \frac{1}{2}\alpha n]$. (The argument for each of the other rectangles in Λ_n is, a rotated, reflected and/or translated version of that for Q_0.) By RSW and the BK inequality we have that the probability that there are more than C_2 disjoint horizontal open crossings of Q_0 is less than

$$\frac{\delta}{2}\frac{\alpha^2}{C_1}. \tag{3.15}$$

Let R_l denote the l-th lowest open crossing of Q_0. We claim that there exist $\eta \in (0, 1/2)$ and $N \in \mathbb{N}$ such that, for any $n \geq N$, deterministic crossing r_0 of Q_0, and $t \in (0, \eta n)$,

$$\mathbb{P}\left(|G_t(\mathcal{C}_n(r_0))| < \beta \mid R_l = r_0\right) < \frac{\delta\alpha^2}{2C_1C_2}, \tag{3.16}$$

where $\mathcal{C}_n(r_0)$ denotes the open cluster which contains the crossing r_0. From this claim we get (see (3.14) and (3.15)) that the l.h.s. of (3.13) is less than

$$\frac{C_1}{\alpha^2}\left(\frac{\delta}{2}\frac{\alpha^2}{C_1} + C_2\frac{\delta\alpha^2}{2C_1C_2}\right) = \delta,$$

and the lemma follows.

It remains to prove the claim concerning the inequality (3.16): The objects defined below involve a parameter i. We will always assume that i is such that the corresponding object is contained in the rectangle $[0, \frac{1}{4}\alpha n] \times [0, \alpha n]$ (Note that this rectangle is contained in Λ_n). Consider all rectangles of the form $A(i) := [2ti - t, 2ti + t] \times [0, \alpha n]$. For every i we let $j(i)$ be the smallest integer j for which the box $\Lambda_t(2ti, 2tj)$ is located above r_0. Let $E(i)$ be the event that $\Lambda_t(2ti, 2tj(i))$ is good and $\gamma_{i,j(i)}$ is connected with r_0 inside $A(i)$. The events $E(i)$ are conditionally independent of each other (where we condition on the event $R_l = r_0$), and, by RSW, each has probability larger than C_3. Hence, when η is small enough (that is, n/t and thus the number of events $E(i)$ is large enough), the probability that at most β of the $E(i)$'s occur is smaller than the r.h.s. of (3.16). This proves the claim and completes the proof of Lemma 3.2.7. □

Remark: *In one of the steps of Járai's proof (see the lines below our statement of Lemma 3.2.7), he shows that with large probability the l-th lowest crossing in Q_0 is contained in $[0, \frac{1}{4}\alpha n] \times [0, \frac{1}{2}\alpha n(1-a)]$, for some constant $a < 1$. He used this to guarantee that the good boxes obtained are inside Q_0. However, as the above arguments show, this (and hence the introduction of the extra constant a) is not needed in our argument.*

3.3 Proof of Theorems 3.1.1 and 3.1.2

3.3.1 Gaps between sizes of clusters with large diameter

The following lemma will be used later to show that the conditions for Theorem 3.2.6 are satisfied in our situation. First we define, for each circuit γ, $\text{int}(\gamma)$ as the interior of γ (that is, the bounded connected component of $\mathbb{R}^2 \setminus \gamma$, where γ is seen as subset of the plane), and

$$X_\gamma := |\{u \in \text{int}(\gamma) \cap \mathbb{Z}^2 : u \leftrightarrow \gamma\}|. \quad (3.17)$$

Lemma 3.3.1. *There exist universal constants $\chi, \xi > 0$ and, for all $t \in 3\mathbb{N}$ and for any circuit γ in $A_{\frac{2}{3}t,t}$, a value $c(t,\gamma) \geq 0$ such that*

$$\mathbb{P}(X_\gamma \leq c(t,\gamma)) \geq \chi; \quad (3.18)$$
$$\mathbb{P}(X_\gamma \geq c(t,\gamma) + \xi s(t)) \geq \chi. \quad (3.19)$$

Proof. Fix some $a \in (0, \frac{1}{2})$. Define the random variable $Z = |\{u \in A_{\frac{1}{3}t,t} \cap \text{int}(\gamma) : u \leftrightarrow \gamma\}|$. Let $c(t,\gamma)$ be defined by

$$c(t,\gamma) = \min\{z \in \mathbb{N} \cup \{0\} : \mathbb{P}(Z \leq z) > a\}.$$

By RSW, the probability that there is a closed dual circuit in $A_{\frac{1}{3}t,\frac{2}{3}t}$ is larger than some universal constant $C_1 > 0$. Moreover, if there is such a circuit, then $X_\gamma = Z$. Hence, $\mathbb{P}(X_\gamma \leq c(t,\gamma))$ is larger than or equal to the probability that there is such a circuit and that $Z \leq c(t,\gamma)$. By the above and FKG this is larger than $C_1 a$.

To prove (3.19) recall the notation (3.4) and define the random variable $Y = |SC_{\frac{1}{3}t}|$. Theorem 3.2.5 implies that there exist constants $C_2, \xi > 0$ such that, for all t, we have $\mathbb{P}(Y \geq \xi s(t)) > C_2$. Let E be the event that there is an open crossing in $\Lambda_{\frac{1}{3}t}$ from top to bottom and that this crossing is connected to γ. On E we have that $X_\gamma \geq Z + Y$, since the spanning cluster is connected to γ. By RSW, $\mathbb{P}(E)$ is larger than some universal constant C_3. Hence

$$\mathbb{P}\left(X_\gamma \geq c(t,\gamma) + \xi s(t)\right) \geq \mathbb{P}(E; Z \geq c(t,\gamma); Y \geq \xi s(t)) \geq C_3 (1-a) C_2, \quad (3.20)$$

where the last inequality uses FKG. This proves Lemma 3.3.1. □

Now we prove the following proposition, from which, as we show in the next subsection, Theorem 3.1.1 follows almost immediately. The set of clusters with diameter larger than αn is denoted by $\mathbf{C}_{\alpha,n}$. More precisely,

$$\mathbf{C}_{\alpha,n} = \{\mathcal{C}_n(u) : u \in \Lambda_n; \text{diam}(\mathcal{C}_n(u)) \geq \alpha n\}. \quad (3.21)$$

Proposition 3.3.2. *For all $\alpha, \delta > 0$ there exist $\varepsilon = \varepsilon(\alpha,\delta) > 0, N = N(\alpha,\delta) \in \mathbb{N}$ such that, for all $n \geq N$*

$$\mathbb{P}(\exists \textit{ distinct } \mathcal{D}_1, \mathcal{D}_2 \in \mathbf{C}_{\alpha,n} : ||\mathcal{D}_1| - |\mathcal{D}_2|| < \varepsilon s(n)) < \delta. \quad (3.22)$$

Proof. Let $\alpha, \delta > 0$ be given. By a standard RSW argument, the probability that $|\mathbf{C}_{\alpha,n}| \geq 1$ is smaller than some constant < 1 which depends only on α. Hence, by the BK inequality we can choose a $\kappa = \kappa(\alpha, \delta) \in \mathbb{N}$ such that, for all n:

$$\mathbb{P}(|\mathbf{C}_{\alpha,n}| > \kappa) < \frac{\delta}{3}. \tag{3.23}$$

Let ξ and χ as in Lemma 3.3.1 and C as in Theorem 3.2.6. Take β so large that

$$\frac{\xi}{2} \leq \frac{\delta\xi\sqrt{\chi}}{6C\binom{\kappa}{2}} \cdot \sqrt{\beta}. \tag{3.24}$$

(For the time being, this property of β will play no role; it will become essential at (3.32) for a suitable choice of ε). Let η be as in Lemma 3.2.7 (but with $\delta/3$ instead of δ in (3.13)). It is clear from that lemma that without loss of generality we may assume that

$$\eta < \frac{\alpha}{2}. \tag{3.25}$$

For each n we take $t = t(n) = 3\lfloor \frac{1}{3}\eta n \rfloor$. Hence, by the above choice of η we have, for all sufficiently large n,

$$\mathbb{P}\left(\exists \mathcal{D} \in \mathbf{C}_{\alpha,n} : |G_t(\mathcal{D})| < \beta\right) < \frac{\delta}{3}. \tag{3.26}$$

Denote by W the event that there are at most κ clusters in Λ_n with diameter at least αn and all these clusters have at least β good boxes. Note that the complement of W is the union of the event in the l.h.s. of (3.23) and the event in the l.h.s. of (3.26), and hence has probability smaller than $2\delta/3$. Therefore, to prove Proposition 3.3.2 it is sufficient to show that there exists $\varepsilon > 0$ such that for all sufficiently large n,

$$\mathbb{P}(W \cap \{\exists \text{ distinct } \mathcal{D}_1, \mathcal{D}_2 \in \mathbf{C}_{\alpha,n} : ||\mathcal{D}_1| - |\mathcal{D}_2|| < \varepsilon s(n)\}) < \frac{\delta}{3}. \tag{3.27}$$

We define (compare with (3.12))

$$G_{t,n} = \{(i,j) \in \mathbb{Z}^2 : \Lambda_t(2ti, 2tj) \subset \Lambda_n \text{ is good }\}.$$

Recall that we denote the outermost open circuit in $A_{\frac{2}{3}t,t}(2ti, 2tj)$ (if it exists) by $\gamma_{i,j}$. Denote the configuration on the edges in the set

$$H := [-n,n]^2 \setminus \left(\bigcup_{(i,j) \in G_{t,n}} \operatorname{int}(\gamma_{i,j}) \right) \tag{3.28}$$

by ω_H.

To estimate the l.h.s. of (3.27) we condition first on the $\gamma_{i,j}$'s and the configuration ω_H. Therefore, let $\tilde{G}$ be an arbitrary set of vertices (i,j) with $\Lambda_t(2ti, 2tj) \subset \Lambda_n$, and let, for each $(i,j) \in \tilde{G}$, $\tilde{\gamma}_{i,j}$ be a (deterministic) circuit in $A_{\frac{2}{3}t,t}(2ti, 2tj)$. Let $\tilde{H}$ be the analog of (3.28), with γ replaced by $\tilde{\gamma}$ and let $\tilde{\omega}$ be a configuration on $\tilde{H}$. We will consider the conditional distribution $\mathbb{P}(\cdot | G_{t,n} = \tilde{G};\ \gamma_{i,j} = \tilde{\gamma}_{i,j} \, \forall (i,j) \in G_{t,n};\ \omega_H =$

$\tilde{\omega}$). Note that the information we condition on allows us to distinguish all the clusters in $\mathbf{C}_{\alpha,n}$ and their good boxes. (Here we used that (3.25) implies that no cluster of $\mathbf{C}_{\alpha,n}$ fits entirely in the interior of one of the above mentioned $\gamma_{i,j}$'s). We may assume that $\tilde{\omega}$ is such that W holds. Let $\mathcal{D}_1, \mathcal{D}_2$ be two open clusters in $\mathbf{C}_{\alpha,n}$ for the configuration $\tilde{\omega}$. Their sizes can be decomposed as follows:

$$|\mathcal{D}_1| = a_1 + \sum_{(i,j)\in G_t(\mathcal{D}_1)} X_{\tilde{\gamma}_{i,j}}, \qquad (3.29)$$
$$|\mathcal{D}_2| = a_2 + \sum_{(i,j)\in G_t(\mathcal{D}_2)} X_{\tilde{\gamma}_{i,j}},$$

where $a_1 = |\mathcal{D}_1 \cap H|$ and $a_2 = |\mathcal{D}_2 \cap H|$, and the X variables are as defined in (3.17). The terms a_1 and a_2 can be considered as 'fixed' (namely, determined by $\tilde{\omega}$), and the $X_{\tilde{\gamma}_{i,j}}$'s as independent random variables. Therefore, and because there are at most $\binom{\kappa}{2}$ choices for $\mathcal{D}_1$ and $\mathcal{D}_2$, to prove (3.27) it is enough to show that there exists $\varepsilon > 0$, which does not depend on $a_1, a_2, G_t(\mathcal{D}_1), G_t(\mathcal{D}_2)$ and the $\tilde{\gamma}_{i,j}$'s, such that

$$\mathbb{P}\left(\left|\left(a_1 + \sum_{(i,j)\in G_t(\mathcal{D}_1)} X_{\tilde{\gamma}_{i,j}}\right) - \left(a_2 + \sum_{(i,j)\in G_t(\mathcal{D}_2)} X_{\tilde{\gamma}_{i,j}}\right)\right| < \varepsilon s\,(n)\right) < \frac{\delta}{3\binom{\kappa}{2}}, \qquad (3.30)$$

On the event W we have that $|G_t(\mathcal{D}_1)| \geq \beta$. So we can mark β of the good boxes in $G_t(\mathcal{D}_1)$, and condition (in addition to the earlier mentioned information) also on the values of $X_{\gamma_{i,j}}$ for the remaining good boxes in $G_t(\mathcal{D}_1)$ and all the good boxes in $G_t(\mathcal{D}_2)$. Hence it is enough to show that there exists an $\varepsilon > 0$ such that

$$\mathbb{P}\left(|\sum_{m=1}^{\beta} X_{\gamma_m} - b| < \varepsilon s(n)\right) < \frac{\delta}{3\binom{\kappa}{2}}, \qquad (3.31)$$

uniformly in $b \in \mathbb{N}$, $\gamma_1, \cdots, \gamma_\beta$ and $G_t(\mathcal{D}_1)$, where the γ_m's are circuits in distinct annuli $A_{\frac{2}{3}t,t}(2ti, 2tj)$. We will do this by application of Theorem 3.2.6, where Lemma 3.3.1 (and our choice (3.24) for β) enables a suitable application of that theorem:

From (3.24) it follows immediately that for all n there is an $\varepsilon(n)$ such that

$$\frac{\xi}{2} \cdot \frac{s(t)}{s(n)} \leq \varepsilon(n) \leq \frac{\delta \xi \sqrt{\chi}}{6C\binom{\kappa}{2}} \cdot \sqrt{\beta} \cdot \frac{s(t)}{s(n)}. \qquad (3.32)$$

By the lower bound in Theorem 3.2.1, the l.h.s. of (3.32) is bounded away from 0, uniformly in n. Hence, $\inf_n \varepsilon(n) > 0$. Take ε equal to this infimum. We get (with Q as in (3.11),

$$\begin{aligned}\mathbb{P}\left(|\sum_{m=1}^{\beta} X_{\gamma_m} - b| < \varepsilon s(n)\right) &\leq Q(\sum_{m=1}^{\beta} X_{\gamma_m}, 2\varepsilon s(n)) \\ &\leq Q(\sum_{m=1}^{\beta} X_{\gamma_m}, 2\varepsilon(n)s(n)) \\ &\leq \frac{2C}{\xi\sqrt{\beta\chi}} \cdot \frac{s(n)}{s(t)} \cdot \varepsilon(n), \qquad (3.33)\end{aligned}$$

where in the last inequality we used Lemma 3.3.1 and applied Theorem 3.2.6 (with $\tilde{\lambda} = \xi s(t)$, $a = \chi$, $m = \beta$ and $\lambda = 2\varepsilon(n)s(n)$). Note that the condition $\tilde{\lambda} \leq \lambda$ in that theorem is satisfied because $\xi s(t) \leq 2\varepsilon(n)s(n)$ by the first inequality in (3.32).
Now, by the second inequality of (3.32) we have that the r.h.s. of (3.33) is at most $\frac{\delta}{3\binom{\kappa}{2}}$. This shows (3.31) and completes the proof of Proposition 3.3.2. □

Remark *In the case of site percolation on the triangular lattice we can, in equation* (3.32) *and the line above it, skip the introduction of* $\varepsilon(n)$*, and choose* ε *itself such that it is (for all sufficiently large* n*) between the l.h.s. and r.h.s. of* (3.32)*. For that percolation model such* ε *exists because (see [38], Proposition 4.9 and the last part of the proof of Theorem 5.1 in that paper)* $\pi(t)/\pi(n)$*, and hence* $s(t)/s(n)$*, has a limit as* $n \to \infty$ *(with* t/n *fixed).*

3.3.2 Proof of Theorem 3.1.1

Let δ and k be fixed. By Corollary 3.2.4 we can choose $\alpha = \alpha(\delta, k)$ and $N_1 = N_1(\delta, k)$ such that, for all $n \geq N_1$

$$\mathbb{P}(\exists i \leq k : \text{diam}(\mathcal{C}_n^{(i)}) < \alpha n) < \frac{\delta}{2}.$$

Further, by Proposition 3.3.2 there is an $\varepsilon > 0$ such that the probability that there are two clusters with diameter larger than αn of which the sizes differ less than $\varepsilon s(n)$ is smaller than $\delta/2$. Hence the l.h.s. of (3.2) is less than $\delta/2 + \delta/2$. □

3.3.3 Proof of Theorem 3.1.2

Let x and δ be given. By Theorem 3.2.3 we can find an α such that

$$\mathbb{P}\left(\exists u \in \Lambda_n : |\mathcal{C}_n(u)| \geq xs(n);\ \text{diam}(\mathcal{C}_n(u)) \leq \alpha n\right) < \frac{\delta}{2}.$$

Let $\mathbf{C}_{\alpha,n}$ be defined as in (3.21). It is enough to show that there exist $\varepsilon = \varepsilon(\alpha, \delta) > 0, N = N(\alpha, \delta) \in \mathbb{N}$ such that, for all sufficiently large n,

$$\mathbb{P}(\exists \mathcal{D} \in \mathbf{C}_{\alpha,n} : |\mathcal{D} - xs(n)| < \varepsilon s(n)) < \frac{\delta}{2}. \qquad (3.34)$$

This can be proved in practically the same way as Proposition 3.3.2. (And, in fact, a bit easier, because now we deal with single clusters instead of pairs of clusters. In particular the factor $\binom{\kappa}{2}$ is replaced by κ in the proof.) □

4 Expected number of clusters intersecting a line segment

This chapter is based on [12] with Rob van den Berg.

We study critical percolation on a regular planar lattice. Let $E_G(n)$ be the expected number of open clusters intersecting or hitting the line segment $[0,n]$. (For the subscript G we either take $\mathbb{H}$, when we restrict to the upper halfplane, or $\mathbb{C}$, when we consider the full lattice).

Cardy [25] (see also Yu, Saleur and Haas [84]) derived heuristically that $E_{\mathbb{H}}(n) = An + \frac{\sqrt{3}}{4\pi}\log(n) + o(\log(n))$, where A is some constant. Recently Kovács, Iglói and Cardy derived in [59] heuristically (as a special case of a more general formula) that a similar result holds for $E_{\mathbb{C}}(n)$ with the constant $\frac{\sqrt{3}}{4\pi}$ replaced by $\frac{5\sqrt{3}}{32\pi}$.

In this chapter we give, for site percolation on the triangular lattice, a rigorous proof for the formula of $E_{\mathbb{H}}(n)$ above, and a rigorous upper bound for the prefactor of the logarithm in the formula of $E_{\mathbb{C}}(n)$.

4.1 Introduction

Consider critical bond percolation on $\mathbb{Z}^2$. Kovács, Iglói and Cardy [59] studied the expected number of clusters which intersect the boundary of a polygon. The leading order is the size n of the boundary. The prefactor of this term is lattice dependent. Their main interest is in the first correction term (of order $\log n$). Their motivation came from relations with entanglement entropy in a diluted quantum Ising model. Using indirect and non-rigorous methods from conformal field theory and the q-state Potts model (letting $q \to 1$), they derived a (universal) formula for the prefactor of the logarithmic term.

A special case of their result is that of a line segment (treated in Section F of their paper). In their setup the line segment was placed in the full plane and they claim that the prefactor is equal to $\frac{5\sqrt{3}}{32\pi}$. Furthermore they refer to an earlier obtained result by Cardy in [25] (see also Yu, Saleur and Haas [84]) where the line segment was placed on the boundary of the half-plane. In the latter case the claim is that the prefactor equals $\frac{\sqrt{3}}{4\pi}$. Also this latter result was obtained by non-rigorous arguments using q-state Potts models.

This motivated us to try to find rigorous and more direct proofs of these results (starting with the case of line segments). Since the prefactors are believed to be universal it is natural to consider the most well studied percolation model, site percolation on the triangular lattice.

Because conformal invariance plays a role, it is convenient to identify the plane with the set $\mathbb{C}$ of complex numbers. We embed the triangular lattice $\mathbb{T}$ in the half-plane $\mathbb{H} = \{z : \Im z \geq 0\}$ or the full plane $\mathbb{C}$ with vertex set $\{m + n\mathbf{j} : m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$ (resp. $\{m + n\mathbf{j} : m, n \in \mathbb{Z}\}$), where $\mathbf{j} = e^{\frac{\pi}{3}\mathbf{i}}$. We denote the probability measure by $\mathbb{P}_{\mathbb{H}}$ (resp. $\mathbb{P}_{\mathbb{C}}$) and the expectation by $\mathbb{E}_{\mathbb{H}}$ (resp. $\mathbb{E}_{\mathbb{C}}$). For subsets $A, B \subset \mathbb{C}$ we denote by $A \leftrightarrow B$ the event that there are vertices x, y on the triangular lattice, with $x \in A, y \in B$, which are connected by a path of open vertices. with some abuse of notation we denote, for any $x \in \mathbb{C}$, the set $\{x\}$ by x. A cluster is a collection connected vertices. Consider the line segment $[1, n]$ on $\mathbb{R}$, containing n vertices. We are interested in

$$E_G(n) := \mathbb{E}_G[\,|\{C \in \mathcal{C}_G : C \cap [1, n] \neq \emptyset\}|\,],$$

where $\mathcal{C}_G$ is the collection of all clusters in the triangular lattice on the lattice $G = \mathbb{H}, \mathbb{C}$.

It is easy to derive the leading (of order n) term: see the Remark a few paragraphs below Theorem 4.1.1. In the case of the half-plane we could obtain a rigorous proof for the earlier mentioned logarithmic correction term. In the case of the full plane we only obtained a logarithmic upper bound for the correction term. (We do not see a method how to prove the precise prefactor $\frac{5\sqrt{3}}{32\pi}$ given in [59]; even finding a non-trivial lower bound is, in our opinion, a challenging problem).

More precisely, our main contribution is a rigorous proof of the following:

Theorem 4.1.1.

$$\text{(a)} \quad E_{\mathbb{H}}(n) = n \cdot (\mathbb{P}_{\mathbb{H}}(1 \not\leftrightarrow (-\infty, 0]) - \frac{1}{2}) + \frac{\sqrt{3}}{4\pi} \log(n) + o(\log(n))$$

and

$$\text{(b)} \qquad \limsup_{n\to\infty} \frac{E_{\mathbb{C}}(n) - n \cdot (\mathbb{P}_{\mathbb{C}}(1 \not\leftrightarrow (-\infty, 0]) - \frac{1}{2})}{\log(n)} \leq \frac{8}{5} \cdot \frac{\sqrt{3}}{4\pi}.$$

We now describe the first steps of the strategy to derive the result above. This will also give some insight, where the log comes from. First rewrite the number of clusters as follows

$$\begin{aligned} |\{C \in \mathcal{C}_G : C \cap [1, n] \neq \emptyset\}| &= \mathbf{1}\{1 \text{ open}\} + \sum_{k=2}^{n} \mathbf{1}\{k \not\leftrightarrow [1, k-1], k \text{ open}\} \\ &= 1 + \sum_{k=2}^{n} \mathbf{1}\{k \not\leftrightarrow [1, k-1]\} - \sum_{k=1}^{n} \mathbf{1}\{k \text{ closed}\} \end{aligned}$$

So

$$
\begin{aligned}
&E_G(n)\\
&= 1-\frac{1}{2}n+\sum_{k=2}^{n}\big(\mathbb{P}_G(k \not\leftrightarrow (-\infty,k-1]) + \mathbb{P}_G(\{k \not\leftrightarrow [1,k-1]\}\cap\{k \leftrightarrow (-\infty,0]\})\big)\\
&= 1-\frac{1}{2}n+(n-1)\cdot(\mathbb{P}_G(1 \not\leftrightarrow (-\infty,0]))\\
&\quad+\sum_{k=2}^{n}\mathbb{P}_G(\{k \not\leftrightarrow [1,k-1]\}\cap\{k \leftrightarrow (-\infty,0]\}).
\end{aligned}
$$

Remark: Since $\mathbb{P}_G((-\infty,0] \leftrightarrow [k,\infty)) \to 0$ as $k\to\infty$, this implies that the leading term of $E_G(n)$ is $n(\mathbb{P}_G(1 \not\leftrightarrow (-\infty,0]) - \frac{1}{2})$.

Let us introduce the following notation:

$$L_G(n) := \frac{1}{\log(n)}\sum_{k=2}^{n}\mathbb{P}_G(\{k \not\leftrightarrow [1,k-1]\}\cap\{k \leftrightarrow (-\infty,0]\}).$$

That is,

$$L_G(n) = \frac{E_G(n)-1+\frac{1}{2}n-(n-1)\cdot(\mathbb{P}_G(1 \not\leftrightarrow (-\infty,0]))}{\log(n)}.$$

Hence Theorem 4.1.1 is equivalent to
(a) $\lim_{n\to\infty} L_{\mathbb{H}}(n) = \frac{\sqrt{3}}{4\pi}$ and
(b) $\limsup_{n\to\infty} L_{\mathbb{C}}(n) \le \frac{8}{5}\cdot\frac{\sqrt{3}}{4\pi}$.

Take $\varepsilon>0$. We will introduce $M = M(n,\varepsilon)\in\mathbb{N}$ and a sequence $a(i) = a(i,n,\varepsilon)$ for $1\le i\le M+1$, such that

$$a(M+1) = n.$$

With these values we split up the sum in $L_G(n)$ in the following terms. For all $1\le i\le M$,

$$f_i := \sum_{k=a(i)+1}^{a(i+1)}\mathbb{P}_G(\{k \not\leftrightarrow [1,k-1]\}\cap\{k \leftrightarrow (-\infty,0]\}). \tag{4.1}$$

and

$$f_0 := \sum_{k=2}^{a(1)}\mathbb{P}_G(\{k \not\leftrightarrow [1,k-1]\}\cap\{k \leftrightarrow (-\infty,0]\}). \tag{4.2}$$

Then

$$L_G(n) = \frac{f_0}{\log(n)} + \frac{1}{\log(n)}\sum_{i=1}^{M} f_i.$$

The idea is now, roughly speaking, to choose $a(i,n,\varepsilon)$ so that the ratio of two consecutive ones equals $1+\varepsilon$ and choose M such that $a(1,n,\varepsilon)$ goes to infinity as $n\to\infty$, but is of a smaller order than $\log(n)$.

Than obviously the term $f_0/\log(n)$ is negligible. We will see that M is more or less of the order $\log(n)/\varepsilon$. The existence of the limit $\lim_{n\to\infty} L_G(n)$ would follow if we can show that, for ε close to zero, f_i is approximately a constant times ε as $n \to \infty$.

In the case that $G = \mathbb{H}$, we will see in Section 4.3.1 that this strategy indeed leads to the existence, and even the value, of the limit of $L_{\mathbb{H}}(n)$ as $n \to \infty$. Unfortunately in the full-plane it only leads to the upper bound stated in Theorem 4.1.1 (b), as we will see in Section 4.3.2.

Now we make the above choices precise. We define

$$M := \left\lfloor \frac{\log(n) - \frac{1}{2}\log(\log(n))}{\log(1+\varepsilon)} \right\rfloor \tag{4.3}$$

and for $i \in \{-1, \cdots, M-1\}$

$$a(M-i, n, \varepsilon) := \left\lfloor \frac{n}{(1+\varepsilon)^{i+1}} \right\rfloor \tag{4.4}$$

or alternatively, for $j \in \{1, \cdots, M+1\}$

$$a(j, n, \varepsilon) := \left\lfloor \frac{n}{(1+\varepsilon)^{M-j+1}} \right\rfloor.$$

Note that than $a(1, n, \varepsilon)$ is of order $\sqrt{\log(n)}$. To examine f_i it is useful to rewrite it in terms of an expectation as follows. Let

$$T(i) := \sum_{k=a(i)+1}^{a(i+1)} \mathbf{1}\{k \not\leftrightarrow [1, k-1] \text{ and } k \leftrightarrow (-\infty, 0]\}. \tag{4.5}$$

Then $f(i) = \mathbb{E}_G[T(i)]$. Hence

$$L_G(n) = \frac{f_0}{\log(n)} + \frac{1}{\log(n)} \sum_{i=1}^{M} \mathbb{E}_G[T(i)]. \tag{4.6}$$

4.2 Preliminaries

In this section we state some results, which we will use in the proof of our main result, Theorem 4.1.1. First some additional notation. We use the following notation for the probabilities of so-called arm-events. Let, for $m < n \in \mathbb{N}$

$$\pi_1(m, n) := \mathbb{P}_{\mathbb{H}}([-m, m]^2 \leftrightarrow \mathbb{H} \setminus [-n, n]^2) \tag{4.7}$$

and let $\pi_3(m, n)$ be the probability of having two disjoint closed paths, and an open path, from $[-m, m]^2$ to $\mathbb{H} \setminus [-n, n]^2$. The following lemma is well known (see for example Theorems 21 and 22 in [65]).

Lemma 4.2.1. *There exist constants $C_1, C_2 > 0$ and $\alpha \leq 1/2$ such that, for all $m < n$*

$$\pi_1(m, n) \leq C_1 \left(\frac{m}{n}\right)^{\alpha}, \qquad \pi_3(m, n) \leq C_2 \left(\frac{m}{n}\right)^{2}.$$

In fact, much more precise results for these probabilities are known, but will not be used in this chapter.

In the rest of this section, for a domain $D \subsetneq \mathbb{C}$ and $n \in \mathbb{N}$ the notation nD denotes the set $\{n \cdot u : u \in D\}$. For points a_1, a_2 on the boundary of D we denote by $[a_1, a_2]$ the part of the boundary of D between a_1 and a_2 in the counter clockwise direction. Furthermore we generalize the notation slightly, namely by $\mathbb{P}_D$ (and $\mathbb{E}_D$) we will denote the probability measure for percolation restricted to the triangular lattice on D. In this setting two intervals $[a_1, a_2]$ and $[a_3, a_4]$ on the boundary are said to be connected if the there are vertices x, y on the lattice inside D, which are connected, and are such that x has an edge which crosses $[a_1, a_2]$ and y has an edge which crosses $[a_3, a_4]$.

The first theorem is the famous Cardy's formula, which was proved by Smirnov in [77].

Theorem 4.2.2 (Cardy's formula, [77]). *Let $D \subsetneq \mathbb{C}$ be a simply connected domain and $\phi : D \to \mathbb{H}$ a conformal map. Let a_1, a_2, a_3, a_4 be ordered points on the boundary of D. We have*

$$\lim_{n\to\infty} \mathbb{P}_{nD}([na_1, na_2] \leftrightarrow [na_3, na_4]) = \frac{2\pi\sqrt{3}}{\Gamma\left(\frac{1}{3}\right)^3} \lambda^{1/3} \cdot {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; \lambda\right),$$

where λ is the cross-ratio

$$\lambda = \frac{(\phi(a_1) - \phi(a_2))(\phi(a_4) - \phi(a_3))}{(\phi(a_1) - \phi(a_3))(\phi(a_4) - \phi(a_2))}. \tag{4.8}$$

This theorem concerns crossing probabilities of generalized rectangles in one 'direction'. The following theorem gives a formula for probabilities of crossings in two directions. It is called after Watts, who proposed the formula in 1996. The first rigorous proof was by Dubédat [34]. An alternative proof was obtained by Schramm (see [73]).

Theorem 4.2.3 (Watts' formula, [34, 73]). *Let $D \subsetneq \mathbb{C}$ be a simply connected domain and $\phi : D \to \mathbb{H}$ a conformal map. Let a_1, a_2, a_3, a_4 be ordered points on the boundary of D. We have*

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{P}_{nD}([na_1, na_2] \leftrightarrow [na_3, na_4] \textit{ and } [na_4, na_1] \leftrightarrow [na_2, na_3]) \\ &= \frac{2\pi\sqrt{3}}{\Gamma\left(\frac{1}{3}\right)^3} \lambda^{1/3} \cdot {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; \lambda\right) - \frac{\sqrt{3}}{2\pi} \lambda \cdot {}_3F_2\left(1, 1, \frac{4}{3}; \frac{5}{3}, 2; \lambda\right). \end{aligned}$$

where λ is the cross-ratio (4.8).

The last theorem we state here concerns the expected number of crossing clusters of a rectangle. It was predicted by Cardy [25] and by Simmons, Kleban and Ziff [75]. A proof was given by Smirnov and Hongler in [49]. Here $N(nD, a_1, a_2, a_3, a_4)$ denote the number of clusters in nD which connect $[na_1, na_2]$ with $[na_3, na_4]$.

Theorem 4.2.4 ([49]). *Let $D \subsetneq \mathbb{C}$ be a simply connected domain and $\phi : D \to \mathbb{H}$ a conformal map. Let a_1, a_2, a_3, a_4 be ordered points on the boundary of D. We have*

$$\lim_{n\to\infty} \mathbb{E}_{nD}[N(nD, a_1, a_2, a_3, a_4)]$$
$$= \frac{2\pi\sqrt{3}}{\Gamma\left(\frac{1}{3}\right)^3}\lambda^{1/3} \cdot {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; \lambda\right) - \frac{\sqrt{3}}{4\pi}\lambda \cdot {}_3F_2\left(1, 1, \frac{4}{3}; \frac{5}{3}, 2; \lambda\right) + \frac{\sqrt{3}}{4\pi}\log\left(\frac{1}{1-\lambda}\right).$$

where λ is the cross-ratio (4.8).

4.3 Proof of Theorem 4.1.1

Recall from the introduction that Theorem 4.1.1 is equivalent to
(a) $\lim_{n\to\infty} L_{\mathbb{H}}(n) = \frac{\sqrt{3}}{4\pi}$ and
(b) $\limsup_{n\to\infty} L_{\mathbb{C}}(n) \leq \frac{8}{5} \cdot \frac{\sqrt{3}}{4\pi}$.

Recall the definition (4.5) of $T(i)$. We begin this section with a lemma which says that, to prove the convergence of $L_G(n)$ as $n \to \infty$, it is sufficient to prove the convergence of $\varepsilon^{-1}\mathbb{E}_G[T(i)]$.

Lemma 4.3.1. *The following inequalities hold.*

$$\limsup_{n\to\infty} L_G(n) \leq \limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \max_{1\leq i\leq M} \frac{\mathbb{E}_G[T(i)]}{\varepsilon} \tag{4.9}$$

and

$$\liminf_{n\to\infty} L_G(n) \geq \liminf_{\varepsilon\to 0} \liminf_{n\to\infty} \min_{1\leq i\leq M} \frac{\mathbb{E}_G[T(i)]}{\varepsilon}. \tag{4.10}$$

Proof: Recall (4.6) and the definitions of $M, a(i), f_i$ in (4.1) - (4.4). To prove (4.9), first note that $0 \leq f_0 \leq a(1, n, \varepsilon)$ and M was chosen such that $a(1, n, \varepsilon) \approx \sqrt{\log(n)}$, hence

$$\lim_{n\to\infty} \frac{f_0}{\log(n)} = 0.$$

Thus it is enough to prove that

$$\limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \left(\sum_{i=1}^{M} \frac{\mathbb{E}_G[T(i)]}{\log(n)}\right) \leq \limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \max_{1\leq i\leq M} \frac{\mathbb{E}_G[T(i)]}{\varepsilon}. \tag{4.11}$$

Hereto, note that it is also easy to see from the definition of M that, for fixed $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{M}{\log(n)} = \frac{1}{\log(1+\varepsilon)}.$$

For all $\varepsilon > 0$ we have

$$\begin{aligned} \limsup_{n\to\infty} \sum_{i=1}^{M} \frac{\mathbb{E}_G[T(i)]}{\log(n)} &\leq \limsup_{n\to\infty} \left(\frac{M}{\log(n)} \max_{i\leq M} \mathbb{E}_G[T(i)]\right) \\ &\leq \frac{1}{\log(1+\varepsilon)} \cdot \varepsilon \cdot \limsup_{n\to\infty} \left(\max_{i\leq M} \frac{\mathbb{E}_G[T(i)]}{\varepsilon}\right). \end{aligned} \tag{4.12}$$

Next note that

$$\limsup_{\varepsilon\to 0}\left(\frac{\varepsilon}{\log(1+\varepsilon)}\cdot\limsup_{n\to\infty}\left(\max_{i\le M}\frac{\mathbb{E}_G[T(i)]}{\varepsilon}\right)\right)=\limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\left(\max_{i\le M}\frac{\mathbb{E}_G[T(i)]}{\varepsilon}\right).$$

This together with (4.12) implies (4.11) and completes the proof of (4.9).

The inequality in (4.10) follows in a similar way and we omit it. □

4.3.1 Proof of Theorem 4.1.1 (a)

First note that it is easy to see that $\{T(i)\ge 1\}$ if and only if there is an open and a closed path from $(-\infty,0]$ to $[a(i)+1,a(i+1)]$ and the closed path is below the open path. Furthermore the event $\{T(i)\ge m\}$ is equal to the event that there are $2m$ alternating paths between the aforementioned intervals, starting, from below, with a closed path. Thus the BK inequality implies that

$$\mathbb{P}_{\mathbb{H}}(T(i)\ge m)\le(\mathbb{P}_{\mathbb{H}}(T(i)\ge 1))^m. \tag{4.13}$$

Hence

$$\begin{aligned}\mathbb{E}_{\mathbb{H}}[T(i)] &= \sum_{m=1}^{\infty}\mathbb{P}_{\mathbb{H}}(T(i)\ge m) \qquad (4.14)\\ &\le \mathbb{P}_{\mathbb{H}}(T(i)\ge 1)+\sum_{m=2}^{\infty}(\mathbb{P}_{\mathbb{H}}(T(i)\ge 1))^m\\ &= \mathbb{P}_{\mathbb{H}}(T(i)\ge 1)+\frac{(\mathbb{P}_{\mathbb{H}}(T(i)\ge 1))^2}{1-\mathbb{P}_{\mathbb{H}}(T(i)\ge 1)}.\end{aligned}$$

It is well-known from standard RSW arguments that $\mathbb{P}_{\mathbb{H}}(T(i)\ge 1)$ goes, uniformly in i and n, to 0 as $\epsilon\to 0$. Hence the 'error term' (i.e. the second term in the r.h.s. of the equation array above) is negligible w.r.t. the main term (i.e. the first term in the r.h.s.). By this, Lemma 4.3.1, the fact that $a(1)\to\infty$ as $n\to\infty$, and the ratio between consecutive $a(i)$'s, it is sufficient to prove that

$$\lim_{k\to\infty}\mathbb{P}_{\mathbb{H}}(W_k)=\frac{\sqrt{3}}{4\pi}\varepsilon+o(\varepsilon), \tag{4.15}$$

where W_k denotes the event that there is an open and a closed path from $(-\infty,0]$ to $[k,k(1+\varepsilon)]$ and the closed path is below the open path.

Let W'_k be the event that there is an open and a closed path from $(-\infty,0]$ to $[k,k(1+\varepsilon)]$. (So, informally speaking, W'_k is the same as W_k without the condition on which path is above or below). Using that (by duality), there is either an open path from $[0,k]$ to $[k(1+\varepsilon),\infty)$ or a closed path from $(-\infty,0]$ to $[k,k(1+\varepsilon)]$, we have

$$\begin{aligned}&\mathbb{P}_{\mathbb{H}}((-\infty,0]\leftrightarrow[k,k(1+\varepsilon)]\text{ and }[0,k]\leftrightarrow[k(1+\varepsilon),\infty)) \qquad (4.16)\\ &\qquad= \mathbb{P}_{\mathbb{H}}((-\infty,0]\leftrightarrow[k,k(1+\varepsilon)])-\mathbb{P}_{\mathbb{H}}(W'_k).\end{aligned}$$

The limits as $k\to\infty$ of the first probability in the r.h.s. and the probability in the l.h.s. are obtained by Theorem 4.2.2 and Theorem 4.2.3 respectively, and we get

$$\begin{aligned}\lim_{k\to\infty} \mathbb{P}_{\mathbb{H}}(W_k') &= \frac{\sqrt{3}}{2\pi}\cdot\frac{\varepsilon}{1+\varepsilon}\cdot {}_3F_2\left(1,1,\frac{4}{3};\frac{5}{3},2;\frac{\varepsilon}{1+\varepsilon}\right) \\ &= 2\frac{\sqrt{3}}{4\pi}\cdot\varepsilon + o(\varepsilon).\end{aligned} \tag{4.17}$$

Finally, let $\tilde{W}_k$ denote the event obtained from W_k by replacing 'open' by 'closed' and vice versa. Since W_k and $\tilde{W}_k$ have the same probability and $W_k' = \tilde{W}_k \cup W_k$, we have

$$\mathbb{P}_{\mathbb{H}}(W_k') = 2\mathbb{P}_{\mathbb{H}}(W_k) - \mathbb{P}_{\mathbb{H}}(W_k \cap \tilde{W}_k).$$

Since $W_k \cap \tilde{W}_k$ is contained in the disjoint occurrence of W_k' and the event that there is an open or closed path from $(-\infty, 0]$ to $[k, k(1+\varepsilon)]$, its probability is negligible w.r.t. that of W_k', and we get

$$\lim_{k\to\infty} \mathbb{P}_{\mathbb{H}}(W_k) = \frac{1}{2}\lim_{k\to\infty} \mathbb{P}_{\mathbb{H}}(W_k'),$$

which by (4.17) is equal to $\frac{\sqrt{3}}{4\pi}\cdot\varepsilon + o(\varepsilon)$. As we saw (see the argument above (4.15)) this proves Theorem 4.1.1 (a). □

4.3.2 Proof of Theorem 4.1.1 (b)

We will bound the relevant probabilities (concerning the full plane) by the probabilities of certain connection events in the half-plane. We do this by cutting along the real line from $-\infty$ up to $a(i+1)$. Let us make the cutting precise. Let

$$L(i) := (-\infty, a(i+1)],$$

we define the new lattice to be the triangular lattice on $\mathbb{C} \setminus L(i)$. This is the full triangular lattice, without the vertices (and their edges) on $L(i)$. Let us denote the corresponding probability measure, concerning percolation on this sublattice, by $\tilde{\mathbb{P}}_i$ (and expectation by $\tilde{\mathbb{E}}_i$). Let the boundary $\partial_{\mathbb{T}}[a,b]$ of an interval $[a,b] \subset L(i)$ be the vertices v of $\mathbb{T}$ which are not in the interval $[a,b]$ but have a neighbouring vertex which is on the interval $[a,b]$. Let $\tilde{T}(i)$ be the number of clusters which connect $\partial_{\mathbb{T}}[a(i)+1, a(i+1)]$ with $\partial_{\mathbb{T}}(-\infty, 0]$ but are not connected with $\partial_{\mathbb{T}}[1, a(i)]$.

With this definition of $\tilde{T}(i)$ 'almost all' the open connections counted in $T(i)$ are counted in $\tilde{T}(i)$ as well; however, there are exceptions. In these exceptional cases there is an open connection from $(-\infty, 0]$ to $[a(i)+1, a(i+1)]$ which is not connected to $[1, a(i)]$ on $\mathbb{T}$, but ís connected to $\partial_{\mathbb{T}}[1, a(i)]$ on $\mathbb{C} \setminus L(i) \cap \mathbb{T}$. See Figure 3. More precisely, we define

$$B(i) := \bigcup_{k\in[1,a(i)]\cap\mathbb{T}} \left(B_u(i,k) \cup B_l(i,k)\right),$$

where $B_u(i,k)$ is the event that, on $\mathbb{H}\cap\mathbb{T}$, there are closed paths from k to $(-\infty, 1]$ and from k to $[a(i), \infty)$ and open paths from one of the vertices $k+\mathbf{j}$ and $k-1+\mathbf{j}$

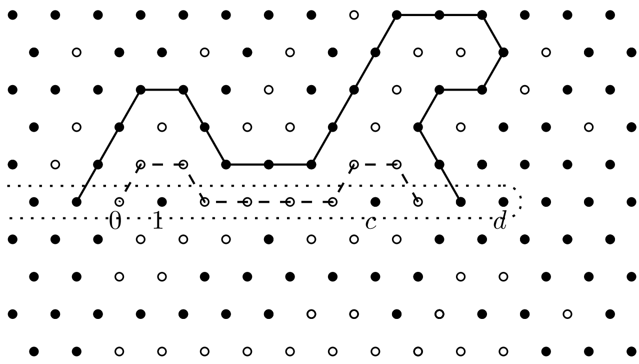

Figure 4.1: The event $B(i)$ occurs. Filled circles are open and empty circles are closed. $c = a(i), d = a(i+1)$.

to $(-\infty, 0]$ and to $[a(i)+1, \infty)$. (The open paths are not necessarily disjoint). The event $B_l(i,k)$ is defined similarly on the lower half-plane.

We have

$$\mathbb{E}_{\mathbb{C}}[T(i)] \leq \tilde{\mathbb{E}}_i[\tilde{T}(i)] + 2\mathbb{P}_{\mathbb{C}}(B(i)). \tag{4.18}$$

To bound $\mathbb{P}_{\mathbb{C}}(B(i))$ we use the first inequality of Lemma 4.2.1 for those k in the definition of $B(i)$ that are 'close to' 1 or $a(i)$, and the other inequality in that lemma for the other k's. More precisely, we fix a constant $\beta \in (0,1)$, and let $r(a(i)) := \lceil a(i)^\beta \rceil$. Then,

$$\begin{aligned}
\mathbb{P}_{\mathbb{C}}(B(i)) &\leq 4\pi_1(r(a(i)), a(i)) + 4 \sum_{k=r(a(i))+1}^{\lceil \frac{1}{2}a(i) \rceil} \pi_3(1,k) \\
&\leq 4C_1 \left(\frac{r(a(i))}{a(i)} \right)^\alpha + 4 \sum_{k=r(a(i))+1}^{\infty} C_2 \left(\frac{1}{k} \right)^2,
\end{aligned} \tag{4.19}$$

where the factor 4 comes from symmetry considerations. Hence, there exist constants $C_3, C_4 > 0$ such that

$$\mathbb{P}_{\mathbb{C}}(B(i)) \leq C_3 \left(a(i)^{\beta-1} \right)^\alpha + \frac{C_4}{a(i)^\beta} \tag{4.20}$$

Note that, since $a(1)$ (the smallest of the $a(i)$'s) tends to ∞ as $n \to \infty$, and $C_3(x^{\beta-1})^\alpha + \frac{C_4}{x^\beta}$ tends to 0 as $x \to \infty$, the contribution of $\mathbb{P}_{\mathbb{C}}(B(i))$ to the r.h.s. of (4.9) is 0.

Next we consider the term $\tilde{\mathbb{E}}_i[\tilde{T}(i)]$. Let $S(i)$ denote the number of closed clusters connecting $\partial_{\mathbb{T}}[a(i)+1, a(i+1)]$ with $\partial_{\mathbb{T}}(-\infty, 0]$. Observe that

$$\tilde{T}(i) = S(i) - \mathbf{1}\{S(i) \geq 1\}.$$

Thus it follows immediately, that

$$\tilde{\mathbb{E}}_i[\tilde{T}(i)] = \tilde{\mathbb{E}}_i[S(i)] - \tilde{\mathbb{P}}_i(S(i) \geq 1). \tag{4.21}$$

To complete the proof we will use Theorem 4.2.4. Therefore we consider the domain $\mathbb{C} \setminus L(i)$ and scale it by $a(i)$. (As noted before, $a(1)$ goes to ∞ as $n \to \infty$). This gives the conformal rectangle $\mathbb{C} \setminus (-\infty, 1+\varepsilon)$ with 'corners' 0^+, 0^-, 1^+ and 1^- (where, for $x < 1+\varepsilon$, x^+ and x^- denote the 'copy' of x in the upper and the lower half-plane respectively). To apply Theorem 4.2.4 we need the cross-ratio, which can be computed as follows: Consider the conformal map

$$\varphi(z) := \mathbf{i}\sqrt{z-1-\varepsilon}$$

which maps $\mathbb{C} \setminus (-\infty, 1+\varepsilon)$ onto the upper half-plane. The cross-ratio is

$$\lambda(\varepsilon) = \frac{(\varphi(1^+)-\varphi(1^-))(\varphi(0^-)-\varphi(0^+))}{(\varphi(0^+)-\varphi(1^-))(\varphi(0^-)-\varphi(1^+))}.$$

It is easy to see that

$$\varphi(0^-) = -\sqrt{1+\varepsilon}, \quad \varphi(1^-) = -\sqrt{\varepsilon}, \quad \varphi(1^+) = \sqrt{\varepsilon}, \quad \varphi(0^+) = \sqrt{1+\varepsilon}.$$

Hence

$$\begin{aligned} \lambda(\varepsilon)^2 &= \frac{16\varepsilon(1+\varepsilon)}{(\sqrt{1+\varepsilon}+\sqrt{\varepsilon})^4} \\ &= 16\varepsilon + o(\varepsilon). \end{aligned} \tag{4.22}$$

Applying Theorem 4.2.4 we conclude that, as $n \to \infty$, $\tilde{\mathbb{E}}_i[S(i)]$ converges (uniformly in the $i's$ with $1 \le i \le M(n)$), to

$$\begin{aligned} &\frac{2\pi\sqrt{3}}{\Gamma(\frac{1}{3})^3}\lambda(\varepsilon)^{1/3} \cdot {}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{4}{3};\lambda(\varepsilon)\right) \\ &- \frac{\sqrt{3}}{4\pi}\lambda(\varepsilon) \cdot {}_3F_2\left(1,1,\frac{4}{3};\frac{5}{3},2;\lambda(\varepsilon)\right) + \frac{\sqrt{3}}{4\pi}\log\left(\frac{1}{1-\lambda(\varepsilon)}\right). \end{aligned}$$

The first term is exactly the limit $\tilde{\mathbb{P}}_i(S(i) \ge 1)$ as $n \to \infty$ (Cardy's formula). Hence by noting that

$$-\frac{\sqrt{3}}{4\pi}\lambda \cdot {}_3F_2\left(1,1,\frac{4}{3};\frac{5}{3},2;\lambda\right) + \frac{\sqrt{3}}{4\pi}\log\left(\frac{1}{1-\lambda}\right) = \frac{\sqrt{3}}{4\pi}\cdot\frac{1}{10}\lambda^2 + o(\lambda^2),$$

and (4.21) and (4.22) we get that

$$\lim_{n\to\infty} \tilde{\mathbb{E}}_i[\tilde{T}(i)] = \frac{\sqrt{3}}{4\pi}\cdot\frac{16}{10}\varepsilon + o(\varepsilon) = \frac{8}{5}\cdot\frac{\sqrt{3}}{4\pi}\cdot\varepsilon + o(\varepsilon), \tag{4.23}$$

uniformly in the i's with $1 \le i \le M(n)$.

This, combined with (4.18) and the negligibility of $\mathbb{P}_{\mathbb{C}}(B(i))$ (see the line below (4.20)), gives

$$\limsup_{n\to\infty} \max_{1\le i\le M} \mathbb{E}_{\mathbb{C}}[T(i)] \le \frac{8}{5}\cdot\frac{\sqrt{3}}{4\pi}\cdot\varepsilon + o(\varepsilon).$$

By Lemma 4.3.1 this implies Theorem 4.1.1 (b). □

5 Conformal measure ensembles for percolation and FK-Ising

This chapter is based on [20] with Federico Camia and Demeter Kiss.

Under some general assumptions we construct the scaling limit of open clusters and their associated counting measures in a class of two-dimensional percolation models. Our results apply, in particular, to critical Bernoulli site percolation on the triangular lattice. We also provide conditional results for the critical FK-Ising model on the square lattice. Fundamental properties of the scaling limit, such as conformal covariance, are explored. Applications such as the scaling limit of the largest cluster in a bounded domain and a geometric representation of the magnetization field for the critical Ising model are presented.

5.1 Introduction

Several important models of statistical mechanics, such as percolation and the Ising and Potts models, can be described in terms of clusters. In the last fifteen years, there has been tremendous progress in the study of the geometric properties of such models in the scaling limit. Much of that work has focused on interfaces, that is, cluster boundaries, taking advantage of the introduction of the Schramm-Loewner Evolution (SLE) by Oded Schramm in [71]. In this chapter, we are concerned with the scaling limit of the clusters themselves and their "areas." More precisely, we analyze the scaling limit of the collection of clusters and the associated counting measures (rescaled by an appropriate power of the lattice spacing).

Our main results are valid under some general assumptions, which can be verified for Bernoulli site percolation on the triangular lattice. Most of the assumptions can be verified also for the FK-Ising model (FK percolation with $q = 2$), but in that case our results are conditional, since we need to assume that the critical FK-Ising percolation model has a unique, conformally invariant, full scaling limit in terms of loops. (The analogous result for Bernoulli percolation was proved in [22]). Such a scaling limit is conjectured to exist and to be described by the Conformal Loop Ensemble (CLE) with parameter 16/3. Recent progress in that direction has been reported in [29], [51].

Roughly speaking, our main results say that, under suitable assumptions, in a general two-dimensional percolation model, the collection of clusters and their associated counting measures, once appropriately rescaled, has a unique weak limit, in an appropriate topology. The collection of clusters converges to a collection of closed sets (the "continuum clusters"), while the collection of rescaled counting measures converges to a collection of continuum measures whose supports are the continuum clusters.

Our results are nontrivial at the critical point of the percolation model. For instance, in the case of critical site percolation on the triangular lattice, where a scaling limit in terms of cluster boundaries is known to exist and to be conformally invariant [22] (it can be described in terms of SLE_6 curves), we show that the continuum clusters are also conformally invariant, and that the associated measures are conformally covariant. The conformal covariance property of the collection of measures is a consequence of the conformal invariance of the critical scaling limit. Because of this property, we call the collection of measures arising in the scaling limit of a critical percolation model a Conformal Measure Ensemble, as proposed by Federico Camia and Charles M. Newman (see [24] and [19]). In the case of Bernoulli percolation, we also use our results to obtain the scaling limit of the largest clusters in a bounded domain.

The scaling limit of the rescaled counting measures is in the spirit of [38], and indeed we rely heavily on techniques and results from that paper. There is however a significant difference in that we distinguish between different clusters. In other words, we don't obtain a single measure that gives the combined size of all clusters inside a domain, but rather obtain a collection of measures, one for each cluster. This is the main technical difficulty in this chapter.

When applied to FK percolation, our results have an interesting application to the Ising model. Consider a critical Ising model on the scaled lattice $\eta\mathbb{Z}^2$. Using the FK representation, one can write the total magnetization in a domain D as $\sum_i \sigma_i \nu_i^\eta(D)$, where the σ_i's are (± 1)-valued, symmetric random variables independent of each other and everything else, and $\nu_i^\eta = \sum_{u \in \mathcal{C}_i} \delta_u$ is the counting measure associated to the i-th cluster (δ_u denotes the Dirac measure concentrated at u and the order of the clusters is irrelevant) and $\nu_i^\eta(D) = |\mathcal{C}_i \cap D|$, where $\mathcal{C}_i$ is the i-th cluster. Camia and Newman [24] noticed that the power of η by which one should rescale the magnetization to obtain a limit, as $\eta \to 0$, is the same as the power that should ensure the existence of a limit for the rescaled counting measures. They then predicted that one should be able to give a meaning to the expression "$\Phi^\infty = \sum_i \sigma_i \mu_i^0$", where Φ^∞ is the limiting magnetization field, obtained from the scaling limit of the renormalized lattice magnetization, and $\{\mu_i^0\}$ is the collection of measures obtained from the scaling limit of the collection of rescaled versions of the counting measures $\{\nu_i^\eta\}$. The existence and uniqueness of the limiting magnetization field was proved in [21], here we complete the program put forward in [24] for the two-dimensional critical Ising model by showing that the Ising magnetization field can indeed be expressed in terms of cluster measures, thus providing a geometric representation (a sort of continuum FK representation based on continuum clusters) for the limiting magnetization field.

5.1.1 Definitions and main results

Let $\mathbb{L}$ denote a regular lattice with vertex set $V(\mathbb{L})$ and edge set $E(\mathbb{L})$. For u and v in $V(\mathbb{L})$, we write $u \sim v$ if $(u,v) \in E(\mathbb{L})$. We are interested in Bernoulli percolation and FK-Ising percolation in $\mathbb{L}$ with parameter p. When we talk about FK-Ising

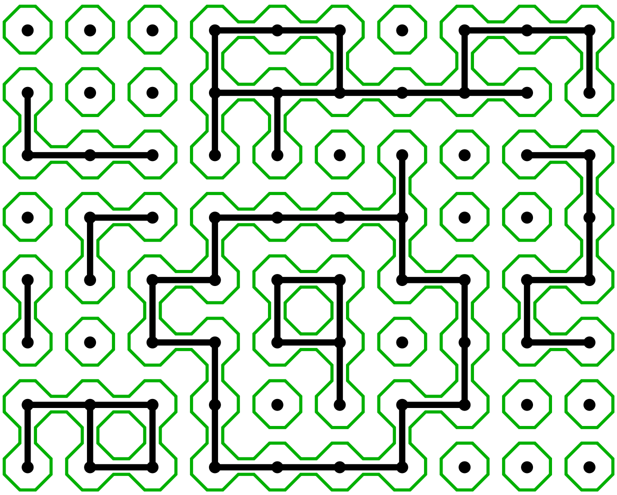

Figure 5.1: Illustration of FK clusters. Black dots represent vertices of $\mathbb{Z}^2$, black horizontal and vertical edges represent FK bonds. The FK clusters are highlighted by lighter (green) loops on the medial lattice.

percolation, $\mathbb{L}$ will be the square lattice $\mathbb{Z}^2$. The FK clusters are defined as illustrated in Figure 5.1, and we think of them as closed sets whose boundaries are the loops in the medial lattice shown in Figure 5.1 (see [42] for an introduction to FK percolation).

When dealing with Bernoulli percolation, $\mathbb{L}$ will be the triangular lattice $\mathbb{T}$, with vertex set

$$V(\mathbb{T}) := \{x + y\epsilon \in \mathbb{C} \mid x, y \in \mathbb{Z}\},$$

where $\epsilon = e^{\pi \mathbf{i}/3}$. The edge set $E(\mathbb{T})$ of $\mathbb{T}$ consists of the pairs $u, v \in V$ for which $\|u - v\|_2 = 1$. Further, let H_u denote the regular hexagon centered at $u \in V(\mathbb{T})$ with side length $1/\sqrt{3}$ with two of its sides parallel to the imaginary axis. Clusters are maximal connected components of open or closed hexagons (see [41] for an introduction to Bernoulli percolation).

Let $\eta > 0$ and consider Bernoulli percolation on $\eta\mathbb{T}$ or the FK-Ising model on $\eta\mathbb{Z}^2$. We think of open and closed clusters as compact sets. To distinguish between them, we will call open clusters 'red' and closed clusters 'blue' (we deviate from the usual terminology of open and closed clusters on purpose: we reserve the words 'open' and 'closed' to describe the topological properties of sets). Let σ_η denote the union of the red clusters in $\eta\mathbb{L}$.

Further, let

$$\Lambda_r := \{z \in \mathbb{C} \,|\, |\Re z| \le r, |\Im z| \le r\}$$

denote the ball of radius r around the origin in the L^∞ norm. We set $\Lambda_r(u) = u + \Lambda_r$.

Our aim is to understand the limit of the set σ_η as η tends to 0. It is easy to see that the limit of σ_η in the Hausdorff topology as $\eta \to 0$ is trivial: it is the empty set when $p = 0$ and a.s. $\mathbb{C}$ for $p > 0$. Hence we concentrate on the connected components, i.e. clusters, of σ_η with diameter at least δ for some fixed $\delta > 0$. It is well-known (see for instance [4]) that, again, we get trivial limits unless $p = p_c$. (For $p < p_c$ the limit of each of the clusters is the empty set, while for $p > p_c$ the limit of the unique largest clusters is dense in $\mathbb{C}$, with the other clusters having the empty set as a limit.) Hence we consider $p = p_c$ in the following, and state informal versions of our main results after some additional definitions. The precise versions of our results are postponed to later sections.

For a set $A \subset \mathbb{C}$ and $u, v \in \mathbb{C}$ we write $u \overset{A}{\leftrightarrow} v$ if there is a red path running in A which connects u to v. When A is omitted, it is assumed to be $\mathbb{C}$. Let $\mathrm{diam}(A)$ denote the L^∞ diameter of A. For $u \in \eta V$ denote by $\mathcal{C}^\eta(u)$ the connected component (i.e. cluster) of u in σ_η. For D a simply connected domain with piece-wise smooth boundary, let $\mathscr{C}^\eta_D(\delta)$ denote the collection of connected components of σ_η, which are contained in D and have diameter larger than δ. That is,

$$\mathscr{C}^\eta_D(\delta) := \{\mathcal{C}^\eta(u) \,|\, u \in \eta V,\, \mathcal{C}^\eta(u) \subset D,\, \mathrm{diam}(\mathcal{C}^\eta(u)) \geq \delta\}. \tag{5.1}$$

On many places D is taken to be Λ_k, in that case we simplify notation by writing $\mathscr{C}^\eta_k(\delta) := \mathscr{C}^\eta_{\Lambda_k}(\delta)$. Finally let

$$\mathscr{C}^\eta(\delta) = \bigcup_{k \in \mathbb{N}} \mathscr{C}^\eta_k(\delta) \tag{5.2}$$

denote the collection of all connected components of σ_η with diameter at least δ.

In the following theorem, distances between subsets of $\mathbb{C}$ will be measured by the Hausdorff distance built on the L^∞ distance in $\mathbb{C}$: For $A, B \subseteq \mathbb{C}$,

$$d_H(A, B) := \inf\{\varepsilon > 0 \,|\, A + \Lambda_\varepsilon \supseteq B \text{ and } B + \Lambda_\varepsilon \supseteq A\}, \tag{5.3}$$

where $A + \Lambda_\varepsilon := \{x + y \in \mathbb{C} : x \in A, y \in \Lambda_\varepsilon\}$.

Let $\hat{\mathbb{C}}$ be the one-point (Alexandroff) compactification of $\mathbb{C}$, i.e. the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. A distance between subsets of $\hat{\mathbb{C}}$ which is equivalent to d_H on bounded sets is defined via the metric on $\mathbb{C}$ with distance function

$$\Delta(u, v) := \inf_{\varphi} \int \frac{1}{1 + |\varphi(s)|^2} ds,$$

where we take the infimum over all curves $\varphi(s)$ in $\mathbb{C}$ from u to v and $|\cdot|$ denotes the Euclidean norm.

The distance D_H between sets is then defined by

$$D_H(A, B) := \inf\{\varepsilon > 0 \,|\, \forall u \in A : \exists v \in B : \Delta(u, v) \leq \varepsilon \text{ and vice versa}\}. \tag{5.4}$$

The distance between finite collections i.e., sets of subsets of $\mathbb{C}$, denoted by $\mathscr{S}, \mathscr{S}'$, is defined as

$$\min_{\phi} \max_{S \in \mathscr{S}} d_H(S, \phi(S)) \tag{5.5}$$

where the infimum is taken over all bijections $\phi : \mathscr{S} \to \mathscr{S}'$. In case $|\mathscr{S}| \neq |\mathscr{S}'|$ we define the distance to be infinite. To account for possibly infinite collections, $\mathscr{S}$ and $\mathscr{S}'$, of subsets of $\hat{\mathbb{C}}$, we define

$$dist(\mathscr{S}, \mathscr{S}') := \inf\{\varepsilon > 0 \,|\, \forall A \in \mathscr{S}\ \exists B \in \mathscr{S}' : D_H(A, B) \leq \varepsilon \text{ and vice versa}\}. \tag{5.6}$$

Our first result is the following, see Theorem 5.5.1 for a slightly stronger version.

Theorem 5.1.1. *Let $k > \delta > 0$. Then $\mathscr{C}_k^\eta(\delta)$ converges in distribution, in the topology* (5.5)*, to a collection of closed sets which we denote by $\mathscr{C}_k^0(\delta)$. Moreover, as $\delta \to 0$, $\mathscr{C}_k^0(\delta)$ has a limit in the metric* (5.6)*, which we denote by $\mathscr{C}_k^0$.*

The next natural question to ask is whether we can extract some more information from the scaling limit. In particular, can we count the number of vertices in each of the clusters in $\mathscr{C}^\eta(\delta)$ in the limit as η tends to 0? As we will see below, the number of vertices in the large clusters goes to infinity, hence we have to scale this number to get a non-trivial result. The correct factor is $\eta^{-2}\pi_1^\eta(\eta, 1)$, where $\pi_1^\eta(\eta, 1)$ denotes the probability that 0 is connected to $\partial\Lambda_1$ in σ_η. We arrive to the informal formulation of our next main result after some more notation.

For $S \subset \mathbb{C}$ let μ_S^η denote the normalized counting measure of its vertices, that is,

$$\mu_S^\eta := \frac{\eta^2}{\pi_1^\eta(\eta, 1)} \sum_{u \in S \cap \eta V} \delta_u, \tag{5.7}$$

where δ_u denotes the Dirac measure concentrated at u. Further, let $\mathscr{M}_k^\eta(\delta)$ denote the collection of normalized counting measures of the clusters in $\mathscr{C}_k^\eta(\delta)$. That is,

$$\mathscr{M}_k^\eta(\delta) := \{\mu_{\mathcal{C}}^\eta \,|\, \mathcal{C} \in \mathscr{C}_k^\eta(\delta)\}. \tag{5.8}$$

Similarly $\mathscr{M}^\eta(\delta) := \{\mu_{\mathcal{C}}^\eta \,|\, \mathcal{C} \in \mathscr{C}^\eta(\delta)\}$. We use the Prokhorov distance for the normalized counting measures. For finite Borel measures μ, ν on $\mathbb{C}$, it is defined as

$$d_P(\mu, \nu) := \inf\{\varepsilon > 0 \,|\, \mu(S) \leq \nu(S^\varepsilon),\, \nu(S) \leq \mu(S^\varepsilon) \text{ for all closed } S \subseteq \mathbb{C}\},$$

where $S^\varepsilon = S + \Lambda_\varepsilon$. Then we construct a metric on collections of Borel measures from d_P similarly to (5.5). We also introduce a distance *Dist* between (infinite) collections of measures which is the same as (5.6) but with collections of sets replaced by collections of measures and with the distance D_H replaced by the Prokhorov distance d_P.

We arrive to the following result. See Theorem 5.7.2 for a slightly stronger version.

Theorem 5.1.2. *Let $k > \delta > 0$, then $\mathscr{M}_k^\eta(\delta)$ converges in distribution to a collection of finite measures which we denote by $\mathscr{M}_k^0(\delta)$. Moreover, as $\delta \to 0$, $\mathscr{M}_k^0(\delta)$ has a limit in the metric Dist, which we denote by $\mathscr{M}_k^0$.*

The next theorem is a full-plane analogue of Theorems 5.1.1 and 5.1.2.

Theorem 5.1.3. *Let $\mathbb{P}_k$ denote the joint distribution of $(\mathscr{C}_k^0, \mathscr{M}_k^0)$. There exists a probability measure $\mathbb{P}$ on the space of collections of subsets of $\hat{\mathbb{C}}$ and collections of measures, which is the full plane limit of the probability measures $\mathbb{P}_k$ in the sense that, for every bounded domain D, the restriction $\mathbb{P}_k|_D$ of $\mathbb{P}_k$ to $(\mathscr{C}_D^0, \mathscr{M}_D^0)$ converges to the restriction $\mathbb{P}|_D$ of $\mathbb{P}$ to $(\mathscr{C}_D^0, \mathscr{M}_D^0)$ as $k \to \infty$.*

The next theorem shows that the collections of clusters and measures from the previous theorem are invariant under rotations and translations, and transform covariantly under scale transformations. (The theorem could be extended to include more general fractal linear (Möbius) transformations by restricting to the Riemann sphere minus a neighbourhood of the origin and of infinity. For simplicity, we restrict attention to linear transformations that map infinity to itself.) The random variables with distribution $\mathbb{P}$ introduced in the previous theorem are denoted by $(\mathscr{C}^0, \mathscr{M}^0)$.

Theorem 5.1.4. *Let f be a linear map from $\mathbb{C}$ to $\mathbb{C}$, that is $f(z) = rz + t$ with $r, t \in \mathbb{C}$. Assume that*

$$\lim_{\eta \to 0} \pi_1^\eta(a, b) = \left(\frac{a}{b}\right)^{\alpha_1 + o(1)}$$

for all $b > a > \eta$ and some $\alpha_1 \in [0, 1]$, where $o(1)$ is understood as $b/a \to \infty$. We set

$$f(\mathscr{C}^0) := \{f(\mathcal{C}) : \mathcal{C} \in \mathscr{C}^0\}, \text{ and}$$
$$f(\mathscr{M}^0) := \{\mu^{0*} : \mu^0 \in \mathscr{M}^0\}$$

where μ^{0*} is the modification of push-forward measure of μ^0 along f defined as

$$\mu^{0*}(B) := |r|^{2-\alpha_1} \mu^0(f^{-1}(B))$$

for Borel sets B. Then the pairs $(f(\mathscr{C}^0), f(\mathscr{M}^0))$ and $(\mathscr{C}^0, \mathscr{M}^0)$ have the same distribution.

Remark 5.1.5. In the case of Bernoulli percolation, we will prove invariance/covariance under all conformal maps between any two bounded domains with piecewise smooth boundaries (see Theorems 5.8.6 and 5.8.8).

Organization of this chapter

In the next section we discuss some applications of our results. First we consider applications for Bernoulli percolation on the triangular lattice. Secondly we provide a geometric representation for the magnetization field of the critical Ising model in terms of FK clusters.

In Section 5.3 we introduce the main tools and assumptions which we use throughout this chapter, namely the loop process, the quad-crossing topology, arm events and the general assumptions under which we prove our main results. We finish Section 5.3 with checking that the assumptions hold for critical Bernoulli percolation on $\mathbb{T}$ and comment on the validity of our assumptions in the critical FK-Ising model. In Sections 5.4 - 5.7 we give precise versions and proofs of Theorems 5.1.1, 5.1.2 and 5.1.3.

We investigate some fundamental properties of the continuum clusters and their normalized counting measures in Section 5.8. In particular, we also discuss the conformal invariance and covariance properties of the clusters in this section. We finish this chapter in Section 5.9 where we prove the convergence of the largest clusters for Bernoulli percolation in a bounded domain.

5.2 Applications

5.2.1 Largest Bernoulli percolation clusters and conformal invariance/covariance

Our first application concerns the scaling limit of the largest percolation clusters in a bounded domain with closed (blue) boundary condition. Denote by $\mathcal{M}^\eta_{(i)}$ the i-th largest cluster in $\Lambda_1 \cap \sigma_\eta$, where we measure clusters according to the number of vertices they contain.

In a sequence of papers, the behaviour of the normalized number of vertices,

$$\frac{|\mathcal{M}^\eta_{(i)}|}{\eta^{-2}\pi^\eta_1(\eta,1)} = \mu^\eta_{\mathcal{M}^\eta_{(i)}}(\Lambda_1), \tag{5.9}$$

was investigated for $\eta > 0$ and $i \geq 1$. Probably the first such results appeared in [16] and [17]. Using Theorems 5.1.1 and 5.1.2 and results in Section 5.6 about convergence of clusters and portions of clusters in bounded domains, we deduce the following theorem.

Theorem 5.2.1. *For all $i \in \mathbb{N}$, both the cluster $\mathcal{M}^\eta_{(i)}$ and its normalized counting measure $\mu^\eta_{\mathcal{M}^\eta_{(i)}}$ converges in distribution to a closed set $\mathcal{M}^0_{(i)}$ and a measure $\mu^0_{\mathcal{M}^0_{(i)}}$ as $\eta \to 0$.*

Recently some of the results from [16, 17] where sharpened [10, 11, 56]. These sharpened results, in combination with Theorem 5.2.1, imply that the distribution of $\mu^0_{\mathcal{M}^0_{(i)}}(\Lambda_1)$ has no atoms [11], that its support is $(0,\infty)$ [10] and that it has a stretched exponential upper tail [56].

It is a celebrated result of Smirnov [77] that the critical site percolation on the triangular lattice is conformally invariant in the limit as $\eta \to 0$. See also [22]. As we show, under certain technical conditions, that this implies that the collections of large clusters in the limit as $\eta \to 0$ are also conformally invariant, while their normalized counting measures are conformally covariant by the results in [38]. We arrive to the following, which is stated in a slightly stronger form as Theorems 5.8.6 and 5.8.8.

Theorem 5.2.2. *Let f be conformal map defined on an open neighbourhood of Λ_1, and $D = f(\Lambda_1)$. We set*

$$f(\mathscr{C}^0_{\Lambda_1}) := \{f(\mathcal{C}) \,:\, \mathcal{C} \in \mathscr{C}^0_{\Lambda_1}\}, \text{ and}$$
$$f(\mathscr{M}^0_{\Lambda_1}) := \{\mu^{0*} \,:\, \mu^0 \in \mathscr{M}^0_{\Lambda_1}\}$$

where μ^{0*} is the modification of push-forward measure of μ^0 along f defined as

$$\mu^{0*}(B) := \int_{f^{-1}(B)} |f'(z)|^{91/48} d\mu^0(z)$$

for Borel sets B.

Then the pairs $(f(\mathscr{C}^0_{\Lambda_1}), f(\mathscr{M}^0_{\Lambda_1}))$ and $(\mathscr{C}^0_D, \mathscr{M}^0_D)$ have the same distribution.

The proof of Theorem 5.2.1 will be presented in Section 5.9 and the proof of Theorem 5.2.2 in Section 5.8.2.

5.2.2 Geometric representation of the critical Ising magnetization field

In this section we give a geometric representation for the scaling limit of the critical Ising magnetization in two dimensions. The existence and uniqueness of the limiting magnetization field was proved in [21], but already in [24] it was heuristically argued that the Ising magnetization field should be expressible in terms of the limiting cluster measures of the FK-Ising clusters, giving a sort of continuum FK representation based on continuum clusters.

Consider a two-dimensional critical Ising model on $\eta\mathbb{Z}^2$ and its FK representation (see, e.g., [42]). We denote by Φ^∞ the limiting magnetization field constructed in [21] in the limit $\eta \to 0$; it is a random distribution acting on the Sobolev space $\mathcal{H}^3$. We also introduce the ε-cutoff magnetization Φ^∞_ε, define as

$$\Phi^\infty_\varepsilon := \sum_{j:\mathrm{diam}(\mathcal{C}_j)>\varepsilon} \sigma_j \mu^0_{\mathcal{C}_j} \, ,$$

where the sum is over all clusters of diameter larger than ε (the order of the sum is irrelevant), the σ_j's are i.i.d. symmetric (± 1)-valued random variables, the $\mu^0_{\mathcal{C}_j}$'s are the scaling limits of the FK-Ising normalized counting measures, and we think of Φ^∞_ε as a random measure acting on the space C^∞_0 of infinitely differentiable functions with bounded support. We will show that the cutoff magnetization Φ^∞_ε provides a good approximation of the magnetization field Φ^∞; since we will only apply Φ^∞_ε to functions with bounded support, the infinite sum in its definition will reduce to a finite sum, so we don't need to specify an order for the infinite sum.

Under the assumption that the critical FK-Ising percolation model has a unique, conformally invariant, full scaling limit in terms of loops we prove the following theorem (see Section 5.3.3 for a precise formulation of Assumption IV).

Theorem 5.2.3. *If Assumption IV holds for FK-Ising percolation, then for any $f \in C^\infty_0$, as $\varepsilon \to 0$, $\langle \Phi^\infty_\varepsilon, f\rangle$ is an L^2 random variable and moreover it converges to $\langle \Phi^\infty, f\rangle$ in the L^2 norm.*

Proof. As explained in Section 2.2.5 of [21], for any $f \in C^\infty_0$, $\langle \Phi^\infty, f\rangle$ can be approximated in the L^2 norm using functions that are linear combinations of indicator functions of dyadic squares. Therefore, without loss of generality, we can restrict our attention to the magnetization in the unit square: $\left\langle \Phi^\infty, \mathbf{1}_{[0,1]^2}\right\rangle$.

Using the triangle inequality, for any $\eta > 0$, we can write

$$\begin{aligned} ||\left\langle \Phi^\infty, \mathbf{1}_{[0,1]^2}\right\rangle - \left\langle \Phi^\infty_\varepsilon, \mathbf{1}_{[0,1]^2}\right\rangle||_2 &\leq ||\left\langle \Phi^\infty, \mathbf{1}_{[0,1]^2}\right\rangle - \left\langle \Phi^\eta, \mathbf{1}_{[0,1]^2}\right\rangle||_2 \\ &+ ||\left\langle \Phi^\eta, \mathbf{1}_{[0,1]^2}\right\rangle - \left\langle \Phi^\eta_\varepsilon, \mathbf{1}_{[0,1]^2}\right\rangle||_2 \\ &+ ||\left\langle \Phi^\eta_\varepsilon, \mathbf{1}_{[0,1]^2}\right\rangle - \left\langle \Phi^\infty_\varepsilon, \mathbf{1}_{[0,1]^2}\right\rangle||_2 \, , \end{aligned}$$

where $\Phi^\eta := \sum_j \sigma_j \mu^\eta_{\mathcal{C}_j}$ denotes the lattice field and $\Phi^\eta_\varepsilon := \sum_{j:\mathrm{diam}(\mathcal{C}_j)>\varepsilon} \sigma_j \mu^\eta_{\mathcal{C}_j}$ is the lattice field with a cutoff on the diameter of clusters. Note that the normalizing factor used in [21] to define the normalized lattice field is the same as the normalizing factor used in this chapter to define the normalized counting measures for FK-Ising clusters.

As $\eta \to 0$, the first term in the right hand side of the last inequality tends to zero by Theorem 2.6 of [21]. For fixed $\varepsilon > 0$, the last term can be expressed as a finite sum, containing the normalized counting measures of clusters of diameter larger than ε that intersect the unit square. As $\eta \to 0$, this term tends to zero because of the convergence in probability of normalized counting measures proved in Theorem 5.7.2 under Assumption IV, and the L^3 bounds provided by Lemma 5.3.15.

The remaining term can be made arbitrarily small by letting $\eta \to 0$ and taking ε small. This follows from results and calculations in [24]. For a proof of this statement, see the proof of Proposition 6.2 of [19]. This concludes the proof of the theorem. □

We remark that there has been recent progress [52, 29] on the full scaling limit of the critical Ising model in bounded domains with, say, plus boundary condition, corresponding to wired boundary condition for the FK-Ising model. Such a scaling limit is supposed to be unique and conformally invariant. Assuming that, the results and methods in this paper would be sufficient to prove conformal invariance/covariance away from the boundary. More precisely, assuming the uniqueness and conformal invariance of the full scaling limit in terms of loops for the critical FK-Ising percolation in a bounded domain D with wired boundary condition, our results and methods would imply that the collection of FK-Ising clusters completely contained in some smaller domain $D' \subset D$, with $\partial D'$ at positive distance from ∂D, has a conformally invariant scaling limit. Analogously, the corresponding collection of counting measures would be conformally covariant. In order to get a full analogue of Theorem 5.2.2, one would need additional arguments to deal with the wired boundary condition on ∂D.

5.3 Further notation and preliminaries

In the above we interpreted the union of red hexagons in a percolation configuration σ_η, as a (random) subset of $\mathbb{C}$. In the following, as an intermediate step, we will consider a percolation configuration as a (random) collection of loops. These loops form the boundaries of the clusters. We will describe this space first. In order to define the clusters as subsets of the plane, we will also consider the (random) collection of quads ('topological squares' with two marked opposing sides) which are crossed horizontally. This leads us to the Schramm-Smirnov [79] topological space, which we briefly recall in the second subsection.

5.3.1 Space of nonsimple Loops

The random collection of loops will be denoted by L_η for $\eta \geq 0$. The distance between two curves l, l' is defined as

$$d_c(l, l') := \inf \sup_{t \in [0,1]} \Delta(l(t), l'(t)), \tag{5.10}$$

where the infimum is over all parametrizations of the curves. The distance between closed sets of curves is defined similarly to the distance between collections of subsets of the Riemann sphere $\hat{\mathbb{C}}$. The space of closed sets of loops is a complete separable metric space.

For $\eta > 0$ the boundaries of the red clusters in σ_η is the closed set of loops, denoted by L_η. This set converges in distribution to L_0, called the *continuum nonsimple loop process.*

5.3.2 Space of quad-crossings

We borrow the notation and definitions from [38]. Let $D \subset \hat{\mathbb{C}}$ be open. A quad Q in D is a homeomorphism $Q : [0,1]^2 \to Q([0,1]^2) \subseteq D$. Let $\mathcal{Q}_D$ be the set of all quads, which we equip with the supremum metric

$$d\left(Q_1, Q_2\right) = \sup_{z \in [0,1]^2} \left|Q_1\left(z\right) - Q_2\left(z\right)\right|$$

for $Q_1, Q_2 \in \mathcal{Q}_D$.

A crossing of a quad Q is a closed connected subset of $Q\left([0,1]^2\right)$ which intersects $Q\left(\{0\} \times [0,1]\right)$ as well as $Q\left(\{1\} \times [0,1]\right)$. The crossings induce a natural partial order denoted by $\leq$ on $\mathcal{Q}_D$. We write $Q_1 \leq Q_2$ if all the crossings of Q_2 contain a crossing of Q_1. For technical reasons, we also introduce a slightly less natural partial order on $\mathcal{Q}_D$: we write $Q_1 < Q_2$ if there are open neighbourhoods $\mathcal{N}_i$ of Q_i such that for all $N_i \in \mathcal{N}_i$, $i \in \{1,2\}$, $N_1 \leq N_2$. We consider the collection of all closed hereditary subsets of $\mathcal{Q}_D$ with respect to $<$ and denote it by$\mathscr{H}_D$. It is the collection of the closed sets $\mathcal{S} \subset \mathcal{Q}_D$ such that if $Q \in \mathcal{S}$ and $Q' \in \mathcal{Q}_D$ with $Q' < Q$ then $Q' \in \mathcal{S}$.

For a quad $Q \in \mathcal{Q}_D$ let $\boxminus_Q$ denote the set

$$\boxminus_Q := \left\{\mathcal{S} \in \mathscr{H}_D \mid Q \in \mathcal{S}\right\},$$

which corresponds with the configurations where Q is crossed. For an open subset $\mathcal{U} \subset \mathcal{Q}_D$ let $\boxdot_\mathcal{U}$ denote the set

$$\boxdot_\mathcal{U} := \left\{\mathcal{S} \in \mathscr{H}_D \mid \mathcal{U} \cap \mathcal{S} = \emptyset\right\},$$

which corresponds with the configurations where none of the quads of $\mathcal{U}$ is crossed. We endow $\mathscr{H}_D$ with the topology $\mathscr{T}_D$ which is the minimal topology containing the sets $\boxminus_Q^c$ and $\boxdot_\mathcal{U}^c$ as open sets for all $Q \in \mathcal{Q}_D$ and $\mathcal{U} \subset \mathcal{Q}_D$ open. We have:

Theorem 5.3.1 (Theorem 1.13 of [79]). *Let D be an open subset of $\hat{\mathbb{C}}$. Then the topological space $(\mathscr{H}_D, \mathscr{T}_D)$ is a compact metrizable Hausdorff space.*

Using this topological structure, we construct the Borel σ-algebra on $\mathscr{H}_D$. We get:

Corollary 5.3.2 (Corollary 1.15 of [79]). Prob$(\mathscr{H}_D)$, *the space of Borel probability measures of $(\mathscr{H}_D, \mathscr{T}_D)$, equipped with the weak* topology is a compact metrizable Hausdorff space.*

Notational remarks 5.3.3. i) In the following we abuse the notation of a quad Q. When we refer to Q as a subset of $\hat{\mathbb{C}}$, we consider its range $Q([0,1]^2) \subset \hat{\mathbb{C}}$.

ii) Note that a percolation configuration σ_η, as defined in the introduction, naturally induces a quad-crossing configuration $\omega_\eta \in \mathscr{H}_{\hat{\mathbb{C}}}$, namely

$$\omega_\eta := \left\{Q \in \mathcal{Q}_{\hat{\mathbb{C}}} \,|\, \sigma_\eta \text{ contains a crossing of } Q\right\}. \tag{5.11}$$

Furthermore, $\mathbb{P}_\eta$ will denote the law governing $(\omega_\eta \times L_\eta)$.

Further we will need the following definitions for restrictions of the configuration to a subset of the Riemann Sphere.

Definition 5.3.4. *Let $D \subseteq \hat{\mathbb{C}}$ be an open set and $\omega \in \mathscr{H}_{\hat{\mathbb{C}}}$. Then $\omega|_D$, the restriction of ω to D, is defined as*

$$\omega|_D := \{Q \in \omega \,:\, Q \subset D\}.$$

The image of $\omega|_D$ under a conformal map $f : D \to \hat{\mathbb{C}}$ is defined as

$$f(\omega|_D) := \{f(Q) : Q \in \omega|_D\} \in \mathscr{H}_{f(D)}.$$

The restriction of the Loop process to D is defined as

$$L|_D := \{l \,:\, \exists \tilde{l} \in L \text{ s.t. } l \text{ is an excursion of } \tilde{l} \text{ in } D\}.$$

The image of $L|_D$ under a conformal map $f : D \to \hat{\mathbb{C}}$ is defined as

$$f(L|_D) := \{f(l) : l \in L|_D\}.$$

Furthermore, $\mathbb{P}_{\eta,D}$ denotes the law of $(\omega_{\eta,D}, L_{\eta,D}) := (\omega_\eta|_D, L_\eta|_D)$ for $\eta \geq 0$.

5.3.3 Assumptions

In the following we list the assumptions which are used throughout the article.

The edge set in the sublattice on $D \subset \mathbb{C}$ of $\eta\mathbb{L}$ is $(\eta E(\mathbb{L}))|_D := \{(u,v) \in \eta E(\mathbb{L}) : u, v \in \eta V(\mathbb{L}) \cap D\}$. The discrete boundary of $D \subset \mathbb{C}$ of the lattice $\eta\mathbb{L}$ is defined by:

$$\partial_\eta D := \{u \in \eta V(\mathbb{L}) \cap D \,:\, \exists v \in \eta\mathbb{L} \,:\, u \sim v \text{ and } v \in \eta\mathbb{L} \cap (\mathbb{C} \setminus D)\}.$$

A boundary condition ξ is a partition of the discrete boundary of D. A set in this partition denotes the vertices which are connected via red hexagons or edges (depending on the model) in $\mathbb{C} \setminus D$. When ξ is omitted, it means we are considering the full plane model and are not specifying any boundary conditions on the discrete boundary of D.

Assumption I (Domain Markov Property)**.** *Let $D \subset E \subset \mathbb{C}$ be open sets. Further let $S \subset \overline{E \setminus D}$ and $T \subset \overline{D}$ closed sets. Then*

$$\mathbb{P}_\eta(\sigma_D = T \cap D \,|\, \sigma_{\overline{E \setminus D}} = S) = \mathbb{P}_\eta(\sigma_D = T \,|\, \xi) =: \mathbb{P}^\xi_\eta(\sigma_D = T)$$

where $\sigma_D = \sigma_\eta \cap D$ and ξ is the discrete boundary condition on D induced by $\sigma_{\overline{E \setminus D}} = S$.

For some models the randomness is put on the vertices (e.g. Bernoulli site percolation) and for others on the edges (e.g. FK-Ising percolation). For the models of the first form we define $\Omega_{\eta,D} := \eta V(\mathbb{L}) \cap D$ and for models of the second form $\Omega_{\eta,D} := (\eta E(\mathbb{L}))|_D$.

Assumption II (Strong positive-association / FKG). *The finite measures are strongly positively-associated. More precisely, let $D \subset \mathbb{C}$ be a bounded closed set. For every boundary condition ξ on $\partial_\eta D$ and increasing functions $f, g : \{red, blue\}^{\Omega_{\eta,D}} \to \mathbb{R}$, we have*

$$\mathbb{E}^\xi_\eta[f \cdot g] \geq \mathbb{E}^\xi_\eta[f] \cdot \mathbb{E}^\xi_\eta[g].$$

Hence for increasing events A, B and boundary condition ξ on $\partial_\eta D$:

$$\mathbb{P}^\xi_\eta(A \cap B) \geq \mathbb{P}^\xi_\eta(A)\mathbb{P}^\xi_\eta(B).$$

It is well known that monotonicity in the boundary condition is equivalent to strongly positively-association, if the measure is strictly positive (has the finite energy property), i.e. every configuration has strictly positive probability. (See e.g. [42, Theorem 2.24].) Furthermore it is well known that positive association survives the limit as the lattice grows towards infinity. See for example [42, Proposition 4.10].

In the following assumption $l(Q)$ denotes the extremal length of Q, that is, let $\phi : Q \to [0, a] \times [0, 1]$ conformal such that $\phi(Q(\{0\} \times [0, 1])) = \{0\} \times [0, 1]$ and $\phi(Q(\{1\} \times [0, 1])) = \{a\} \times [0, 1]$, then $l(Q) = a$.

Assumption III (RSW). *Let $M > 0$. There exist $\delta > 0$ such that, for every quad Q with $l(Q) \leq M$ and every boundary condition ξ on the discrete boundary of $Q([0, 1]^2)$:*

$$\mathbb{P}^\xi_\eta(\omega_\eta \in \boxminus_Q) \geq \delta$$

and for every quad Q with $l(Q) \geq M$ and every boundary condition ξ on the discrete boundary of $Q([0, 1]^2)$:

$$\mathbb{P}^\xi_\eta(\omega_\eta \notin \boxminus_Q) \leq 1 - \delta.$$

Assumption IV (Full Scaling Limit). *As $\eta \to 0$, the law of L_η converges weakly to a random infinite collection of loops L_0 in the induced Hausdorff metric on collections of loops induced by the distance* (5.10). *Moreover, the limiting law is conformally invariant.*

5.3.4 Arm events

For $S \subset \hat{\mathbb{C}}$, let $\partial S, int\,(S)\,, \bar{S}$ denote the boundary, interior and the closure of S, respectively. We call the elements of $\{0, 1\}^k$, $k \geq 0$ as colour-sequences. For ease of notation, we omit the commas in the notation of the colour sequences, e.g. we write (101) for $(1, 0, 1)$.

Definition 5.3.5. *Let $l \in \mathbb{N}$, $\kappa \in \{0, 1\}^l$, $S \subseteq \hat{\mathbb{C}}$ and D, E be two disjoint open, simply connected subsets of $\hat{\mathbb{C}}$ with piecewise smooth boundary. Let $D \overset{\kappa,S}{\longleftrightarrow} E$ denote the event that there are $\delta > 0$ and quads $Q_i \in \mathcal{Q}_S$, $i = 1, 2, \ldots, l$ which satisfy the following conditions.*

1. *$\omega \in \boxminus_{Q_i}$ for $i \in \{1,2,\dots,l\}$ with $\kappa_i = 1$ and $\omega \in \boxminus^c_{Q_i}$ for $i \in \{1,2,\dots,l\}$ with $\kappa_i = 0$.*
2. *For all $i \neq j \in \{1,2,\dots,l\}$ with $\kappa_i = \kappa_j$, the quads Q_i and Q_j, viewed as subsets of $\hat{\mathbb{C}}$, are disjoint, and are at distance at least δ from each other and from the boundary of S;*
3. *$\Lambda_\delta + Q_i(\{0\} \times [0,1]) \subset D$ and $\Lambda_\delta + Q_i(\{1\} \times [0,1]) \subset E$ for $i \in \{1,2,\dots,l\}$ with $\kappa_i = 1$;*
4. *$\Lambda_\delta + Q_i([0,1] \times \{0\}) \subset D$ and $\Lambda_\delta + Q_i([0,1] \times \{1\}) \subset E$ for $i \in \{1,2,\dots,l\}$ with $\kappa_i = 0$;*
5. *the intersections $Q_i \cap D$, for $i = 1,2,\dots,l$, are at distance at least δ from each other, the same holds for $Q_i \cap E$;*
6. *a counterclockwise order of the quads Q_i $i = 1,2,\dots,l$ is given by ordering counterclockwise the connected components of $Q_i \cap D$ containing $Q_i(0,0)$.*

When the subscript S is omitted, it is assumed to be $\hat{\mathbb{C}}$.

Remark 5.3.6. It is a simple exercise to show that the events $D \overset{\kappa,S}{\longleftrightarrow} E$ are $Borel(\mathscr{T}_{\hat{\mathbb{C}}})$-measurable. See [38, Lemma 2.9] for more details.

In the following we consider some special arm events. For $z \in \mathbb{C}, a > 0$ let $H_1(z,a)$, $H_2(z,a)$, $H_3(z,a)$, $H_4(z,a)$ denote the left, lower, right, and upper half planes which have the right, top, left and bottom sides of $\Lambda_a(z)$ on its boundary, respectively. For $z \in \mathbb{C}$, $0 < a < b$ we set

$$A(z;a,b) := \Lambda_b(z) \setminus \Lambda_a(z).$$

Furthermore, for $i = 1,2,3,4$, $\kappa \in \{0,1\}^l$ and $\kappa' \in \{0,1\}^{l'}$ with $l, l' \geq 0$ we define the event where there are $l + l'$ disjoint arms with colour-sequence $\kappa \vee \kappa' := (\kappa_1, \dots, \kappa_l, \kappa'_1, \dots, \kappa'_{l'})$ in $A(z;a,b)$ so that the l' arms, with colour-sequence κ', are in the half-plane $H_i(z,a)$. That is,

$$\mathcal{A}^i_{\kappa,\kappa'}(z;a,b) := \left\{\Lambda_a(z) \overset{\kappa\vee\kappa'}{\longleftrightarrow} \left(\hat{\mathbb{C}} \setminus \Lambda_b(z)\right)\right\} \cap \left\{\Lambda_a(z) \overset{\kappa',H_i(z,a)}{\longleftrightarrow} \left(\hat{\mathbb{C}} \setminus \Lambda_b(z)\right)\right\} \tag{5.12}$$

In the notation above, when z is omitted, it is assumed to be 0.

Finally, for $0 < a < b$ and boundary condition ξ on $\partial_\eta \Lambda_b$ we set

$$\pi_1^{\eta,\xi}(a,b) := \mathbb{P}^\xi_\eta(\mathcal{A}^1_{(1),\emptyset}(a,b)), \qquad \pi_4^{\eta,\xi}(a,b) := \mathbb{P}^\xi_\eta(\mathcal{A}^1_{(1010),\emptyset}(a,b)),$$
$$\pi_6^{\eta,\xi}(a,b) := \mathbb{P}^\xi_\eta(\mathcal{A}^1_{(010101),\emptyset}(a,b)), \qquad \pi_{0,3}^{\eta,\xi}(a,b) := \mathbb{P}^\xi_\eta(\mathcal{A}^1_{\emptyset,(010)}(a,b)),$$
$$\pi_{1,3}^{\eta,\xi}(a,b) := \mathbb{P}^\xi_\eta(\mathcal{A}^1_{(1),(010)}(a,b)).$$

Remark 5.3.7. The (technical) reason to define $H_i(z,a)$ in this slightly unnatural way, will become clear in the proof of Lemma 5.4.7.

5.3.5 Consequences of RSW

Lemma 5.3.8 (Quasi multiplicativity). *Suppose that Assumptions I-III hold. There is a constant $C > 0$ so that*

$$\mathbb{P}_\eta^\xi(\mathcal{A}^1_{(1),\emptyset}(a,b)) \leq C \frac{\pi_1^{\eta,\xi}(a,c)}{\pi_1^{\eta,\xi}(b,c)}$$

for all $a, b, c, \eta > 0$ with $\eta < a < b < c$ and boundary condition ξ on $\partial_\eta \Lambda_c$.

Lemma 5.3.9. *Suppose that Assumptions I-III hold. There are constants $\lambda_1 \in (0, 1)$ and $C > 0$ so that*

$$\pi_1^{\eta,\xi}(\eta, b) \geq C \left(\frac{a}{b}\right)^{\lambda_1} \mathbb{P}_\eta^\xi(\mathcal{A}^1_{(1),\emptyset}(\eta, a))$$

for all $b > a > \eta$ and boundary condition ξ on $\partial_\eta \Lambda_b$.

Lemma 5.3.10. *Suppose that Assumptions I-III hold. There are positive constants C, λ_6 such that*

$$\pi_6^{\eta,\xi}(a,b) \leq C \left(\frac{a}{b}\right)^{2+\lambda_6}, \qquad \pi_{0,3}^{\eta,\xi}(a,b) \leq C \left(\frac{a}{b}\right)^2 \tag{5.13}$$

for all $0 < \eta < a < b$ and boundary condition ξ on $\partial_\eta \Lambda_b$.

Lemma 5.3.11. *Suppose that Assumptions I-III hold. There are positive constants $C, \lambda_{1,3}$ such that*

$$\pi_{1,3}^{\eta,\xi}(a,b) \leq C \left(\frac{a}{b}\right)^{2+\lambda_{1,3}} \tag{5.14}$$

for all $0 < \eta < a < b$ and boundary condition ξ on $\partial_\eta \Lambda_b$.

Lemma 5.3.12. *Suppose that Assumptions I-III hold. There are constants $C, \lambda > 0$ so that*

$$\frac{\pi_1^{\eta,\xi}(a,b)}{\pi_4^{\eta,\xi}(a,b)} \geq C \left(\frac{b}{a}\right)^{\lambda}$$

for all $b > a > \eta$ and boundary condition ξ on $\partial_\eta \Lambda_b$.

For the sake of generality, we have stated the bounds in the previous lemmas in the presence of boundary conditions. However, in the rest of this chapter only the full-plane versions of the bounds will appear, so the superscript ξ will be dropped. (The versions with boundary conditions are necessary to obtain results that we use in this paper, but whose proofs we do not reproduce.) For the next lemma we need some additional notation.

Definition 5.3.13. *For $\eta, a > 0$ let*

$$\mathcal{V}_a^\eta := \{v \in \Lambda_{a/2} \cap \eta V \mid v \overset{1}{\leftrightarrow} \partial\Lambda_a \text{ in } \omega_\eta\}$$

denote the number of vertices in $\Lambda_{a/2}$ connected to $\partial\Lambda_a$ in σ_η.

Lemma 5.3.14. *Suppose that Assumptions I-III hold. Then there are positive constants c, C such that*

$$\mathbb{P}_\eta(|\mathcal{V}_a^\eta| \geq x(a/\eta)^2 \pi_1^\eta(\eta, a)) \leq Ce^{-cx}$$

for all $a > \eta$ and $x \geq 0$.

Lemma 5.3.15. *Suppose that Assumptions I-III hold. Then there is a constant $C > 0$ such that*

$$\mathbb{E}_\eta[|\mathcal{W}_a^\eta|^3] \leq C\eta^{-6}\pi_1^\eta(\eta, a)^{-3}$$

for all $0 < \eta < a < 1/2$, where

$$\mathcal{W}_a^\eta := \{v \in \Lambda_1 \cap \eta V \,|\, v \overset{1}{\leftrightarrow} \partial\Lambda_a(v) \text{ in } \omega_\eta\}.$$

Proof of Lemmas 5.3.8 - 5.3.15. Lemmas 5.3.10 and 5.3.11 follow from Assumptions I - III, as explained in e.g. [65, 41] for the case of Bernoulli percolation and in [28, Corollary 1.5 and Remark 1.6] for the case of FK-Ising percolation. (The additional boundary conditions, which are not present in the above mentioned corollary and remark in [28], do not affect the results. This can easily be deduced from equation (5.1) in [28].)

Also Lemmas 5.3.9 and 5.3.12 follow from standard RSW, FKG arguments.

Lemma 5.3.8 is similar to [28, Theorem 1.3], which is shown to follow from our assumptions I-III. The boundary condition on $\partial_\eta\Lambda_c$ has no effect on the proof, because the RSW result is uniform in the boundary conditions. (Furthermore there is no need to "make" the arms well separated on $\partial_\eta\Lambda_c$.)

An easy proof of Lemma 5.3.14 for critical percolation can be found in [64]. It is easy to see that the same proof can be modified in such a way that the result follows from Lemmas 5.3.8 - 5.3.12, and hence from Assumptions I-III. For percolation, Lemma 5.3.14 can also be found in [16, Lemma 6.1], and for FK-Ising percolation in [21, Lemma 3.10].

Finally Lemma 5.3.15 can be proved easily using Lemma 5.3.8. See for example [38, Lemma 4.5] or the proof of Lemma 5.3.14. □

5.3.6 Additional preliminaries

Lemma 5.3.16. *Suppose that Assumptions I-IV hold. The set of crossed quads is, almost surely, measurable with respect to the collection of loops.*

Proof of Lemma 5.3.16. A proof of this can be found in [38, Section 2.3] and follows almost immediately from arguments given in [22, Section 5.2]. The proof of the measurability of quad crossings with respect to the collection of loops makes use of three properties of the loop process, which all follow from RSW techniques (see the first three items of Theorem 3 in [22, Section 5.2]). Because of this, the measurability is a simple consequence of our Assumptions I-IV. □

Remark 5.3.17. Assumption IV together with the separability of $\mathscr{H}_{\hat{\mathbb{C}}}$ shows that there is a coupling $\mathbb{P}$ so that $\omega_\eta \to \omega_0$ a.s. as $\eta \to 0$.

Before we proceed to the next lemma, we recall the following result on the scaling limits of arm events. A slightly weaker version of the following lemma appeared as [38, Lemma 2.9]. Its proof extends immediately to the more general case.

Lemma 5.3.18 (Lemma 2.9 of [38]). *Suppose that Assumptions I-IV hold. Then, under a coupling $\mathbb{P}$ of $(\mathbb{P}_\eta)_{\eta\geq 0}$ such that $\omega_\eta \to \omega_0$ almost surely, we have for events $\mathcal{D} \in \{\{A \xleftrightarrow{(1),S} B\}, \{A \xleftrightarrow{(010),S} B\}, \mathcal{A}^i_{\kappa,\kappa'}(z;a,b)\}$,*

$$\mathbf{1}_{\mathcal{D}}(\omega_\eta) \to \mathbf{1}_{\mathcal{D}}(\omega_0) \qquad \text{in } \mathbb{P}\text{-probability},$$

for $(\kappa,\kappa') \in \{((1),\emptyset),((1010),\emptyset),((010101),\emptyset),(\emptyset,(010)),((1),(010))\}$, rectangle $S \subseteq \mathbb{C}$, $i \in \{1,2,3,4\}$, $0 < a < b$ and A, B disjoint open subsets of $\mathbb{C}$ with piece-wise smooth boundary.

The lemma above implies that for all $a, b > 0$ with $a < b$ the probability $\pi_1^\eta(a,b)$ converges as $\eta \to 0$. We write $\pi_1^0(a,b)$ for the limit. General arguments [8, Section 4] using Lemma 5.3.8 above show that

$$\pi_1^0(a,b) = \left(\frac{a}{b}\right)^{\alpha_1 + o(1)} \tag{5.15}$$

for some $\alpha_1 \geq 0$ where $o(1)$ is understood as $b/a \to \infty$. Lemma 5.3.9 shows that $\alpha_1 < 1$.

We need some additional notation for the next theorems. For $z \in \mathbb{C}$ and $a > 0$ let $\Lambda'_a(z) := \{u \in \mathbb{C} \,|\, \Re(u-z), \Im(u-z) \in [-a,a)\}$. Note that $\Lambda_a(z)$ and $\Lambda'_a(z)$ differ only on their boundary. For an annulus $A = A(z;a,b)$ let

$$\mu_{1,A}^\eta := \frac{\eta^2}{\pi_1^\eta(\eta,1)} \sum_{v \in \Lambda'_a(z) \cap \eta V} \delta_v \mathbf{1}\{v \leftrightarrow \partial\Lambda_b(z) \text{ in } \omega_\eta\} \tag{5.16}$$

denote the counting measure of the vertices in $\Lambda'_a(z)$ with an arm to $\partial\Lambda_b(z)$ at scale η.

Theorem 5.3.19. *Suppose that Assumptions I-IV hold. Let $A = A(z;a,b)$ be an annulus, and $\mathbb{P}$ be a coupling such that $\omega_\eta \to \omega_0$ a.s. as $\eta \to 0$. Then the measures $\mu^\eta_{1,A}$ converge weakly to $\mu^0_{1,A}$ in probability under the coupling $\mathbb{P}$ as η tends to 0. Furthermore, $\mu^0_{1,A}$ is a measurable function of ω_0. In particular, the pair $(\omega_\eta, \mu^\eta_{1,A})$ converges to $(\omega_0, \mu^0_{1,A})$ in distribution as $\eta \to 0$.*

Theorem 5.3.19 is proved for site percolation on the triangular lattice in [38] where it is Theorem 5.1. Namely, it is easy to check that the proof of [38, Theorem 5.1] shows that the measures $\mu^\eta_{1,A} \xrightarrow{p} \mu^0_{1,A}$ under the coupling $\mathbb{P}$ converge weakly in probability as $\eta \to 0$. For FK-Ising, a sketch proof for a theorem similar to this was given in [21]. Unfortunately the proof contains a mistake, but luckily the mistake can be easily fixed. Below we give an informal sketch of the proof of Theorem 5.3.19, following the proof in [21] and briefly explaining how to fix it.

The strategy is to approximate, in the L^2-sense, the one-arm measure by the number of mesoscopic boxes connected to $\partial\Lambda_b(z)$, multiplied by a constant depending on the size of the boxes. Here mesoscopic means much larger than the mesh size η but much smaller than a.

In order to get L^2-bounds on the error terms, first we use a coupling argument to argue that the boxes which are far away from each other are almost independent. Namely, with high probability one can draw a red circuit around one of the boxes, which is also conditioned on having a long red arm (because of positive association, that event can only increase the probability of a red circuit). This red circuit makes, via the Domain Markov Property, the contribution of the surrounded box independent of that of the other boxes. The total contribution of the boxes which are close to each other is negligible. Secondly we use a ratio limit argument, based on the existence of the one-arm exponent α_1 from (5.15), to show that the contribution of a single box is approximately a constant, which only depends on the size of the mesoscopic box.

The small mistake in [21] mentioned above is in the assumption that the convergence in Lemma 5.3.18 is almost sure, as claimed in an earlier version of [38]. However, as noted in the final version of [38], one can only prove convergence in probability. Luckily, arguments in [38] show that convergence in probability, together with L^3 bounds from Lemma 5.3.15, is sufficient to prove convergence in L^2 of the number of mesoscopic boxes connected to $\partial\Lambda_b(z)$ times a constant depending on the size of these boxes.

5.3.7 Validity of the assumptions

The case of critical percolation

Now we check that the Assumptions above hold for critical site percolation on the triangular lattice.

Theorem 5.3.20. *For critical site percolation on the triangular lattice, the Assumptions I-IV hold.*

Proof of Theorem 5.3.20. The Domain Markov Property, Assumption I, is trivial, one even has independence. Assumption II is well known, see e.g. [42, Theorem 3.8]. RSW, Assumption III, is also well known, see for example [41, 65].

The existence of the full scaling limit in Assumption IV is proved by Camia and Newman in [22]. The value of α_1 is 5/48 as proved in [61]. □

The case of FK-Ising model

The Domain Markov Property and strongly positive association are standard and well known see e.g. [42]. The recent development of the RSW theory for the FK-Ising model proves Assumption III. Namely it follows from Theorem 1.1 in [28] combined with the fact that the discrete extremal length, used in [28] is comparable to its continuous counterpart, used here, see [27, Proposition 6.2].

Unfortunately, to our knowledge, Assumption IV has not yet been proved for the FK-Ising model. The fundamental reason is that the analogue of the results in [22] is missing, in particular, the uniqueness of the full scaling limit has not yet been proved for the FK-Ising model. The value of α_1 for the Ising model is 1/8. As shown in [24], this can be seen from the behaviour of the Ising two-point function at criticality [83].

5.4 Approximations of large clusters

In the following we give two approximations of open clusters with diameter at least $\delta > 0$, which are completely contained in Λ_k. The first one relies solely on the arm events described in the previous section, while the other is 'the natural' one, namely it is simply the union of ε-boxes which intersect the cluster. The advantage of the first approximation is that it can also be defined in the limit as the mesh size goes to 0. First we prove Proposition 5.4.3, which shows that on a certain event these two approximations coincide. Then in Section 5.4.1 we give a lower bound for the probability of the event above.

For simplicity, we set $k = 1$ from now on. The constructions and proofs for different values of k are analogous. Let $\mathbb{Z}[\mathbf{i}] = \{a + b\mathbf{i} \,|\, a, b \in \mathbb{Z}\}$. For $\varepsilon > 0$, let B_ε be the following collection of squares of side length ε:

$$B_\varepsilon := \left\{\Lambda_{\varepsilon/2}(\varepsilon z) \,|\, z \in \Lambda_{\lceil 1/\varepsilon \rceil} \cap \mathbb{Z}[\mathbf{i}]\right\}.$$

Fix $\omega \in \mathscr{H}_{\hat{\mathbb{C}}}$. We define the graph $G_\varepsilon = G_\varepsilon(\omega)$ as follows. Its vertex set is B_ε. The boxes $\Lambda_{\varepsilon/2}(\varepsilon z), \Lambda_{\varepsilon/2}(\varepsilon z') \in B_\varepsilon$ are connected by an edge if $||z - z'||_\infty = 1$ or if $\omega \in \{\Lambda_{\varepsilon/2}(\varepsilon z) \overset{(1)}{\longleftrightarrow} \Lambda_{\varepsilon/2}(\varepsilon z')\}$. For a graph H with $V(H) \subseteq B_\varepsilon$ we set

$$U(H) := \bigcup_{\Lambda \in V(H)} \Lambda \subseteq \Lambda_{1+2\varepsilon}. \tag{5.17}$$

Let $L(H)$ denote the set of leftmost vertices of H. That is,

$$L(H) := \{\Lambda_{\varepsilon/2}(\varepsilon z) \in V(H) \,|\, \forall z' \in \mathbb{Z}[\mathbf{i}] \text{ with } \Lambda_{\varepsilon/2}(\varepsilon z') \in V(H) \text{ we have } \Re z \leq \Re z'\}.$$

Similarly, we define $R(H), T(H), B(H)$ as the right-, top- and bottommost vertices of H, respectively. Let $SH(H)$ (resp. $SV(H)$) denote the most narrow double infinite horizontal (resp. vertical) strip containing $U(H)$. Finally, let $SR(H)$ denote the smallest rectangle containing $U(H)$ with sides parallel to one of the axes. Thus $SR(H) = SH(H) \cap SV(H)$.

Definition 5.4.1. *For $z, z' \in \mathbb{C}$, we set* $\mathrm{dist}_1(z, z') = |\Re(z - z')|$ *and* $\mathrm{dist}_2(z, z') = |\Im(z - z')|$. *We call* $\mathrm{dist}_1, \mathrm{dist}_2$ *as the distance in the horizontal and vertical directions, respectively. We also use the notation* $d_\infty(z, z') := ||z - z'||_\infty = \mathrm{dist}_1(z, z') \vee \mathrm{dist}_2(z, z')$ *for the L^∞ distance.*

For disjoint sets $A, B \subset \hat{\mathbb{C}}$ we set $\mathrm{dist}_i(A, B) := \inf\{\mathrm{dist}_i(z, z') : z \in A, z' \in B\}$ *for $i = 1, 2$.*

Let $\eta > 0$, $\Lambda = \Lambda_{\varepsilon/2}(z) \in B_\varepsilon$ and $\Lambda' = \Lambda_{\varepsilon/2}(z') \in B_\varepsilon$. Suppose there is a cluster which is completely contained in Λ_1, such that Λ contains a leftmost vertex of this cluster and Λ' a rightmost vertex. Then Λ and Λ' are connected by 2 blue arms and one red arm in between them.

This leads us to the following definition, which gives us a way to characterize the clusters using only arm events.

Definition 5.4.2. *Let $\omega \in \mathscr{H}_{\hat{\mathbb{C}}}$ and $G_\varepsilon = G_\varepsilon(\omega)$ the graph defined above. Let H be a subgraph of $G_\varepsilon(\omega)$. We say that H is good, if it satisfies the following conditions.*

1. *H is complete,*

2. *$U(H) \subseteq \Lambda_1$,*

3. *H is maximal, that is, if $\Lambda \in V(G_\varepsilon)$ and $(\Lambda, \Lambda') \in E(G_\varepsilon)$ for all $\Lambda' \in V(H)$, then $\Lambda \in V(H)$,*

4. *$\mathrm{diam}(U(H)) \geq \delta$,*

5. *for all $\Lambda \in L(H)$ and $\Lambda' \in R(H)$ we have $\omega \in \{\Lambda \xleftrightarrow{(010),SV(H)} \Lambda'\}$, a similar condition holds for $\Lambda \in T(H)$ and $\Lambda' \in B(H)$, with $SV(H)$ replaced by $SH(H)$.*

For a set $S \subseteq \mathbb{C}$ and $\varepsilon > 0$ let $K_\varepsilon(S)$ denote the complete graph on the vertex set

$$\left\{\Lambda_{\varepsilon/2}(\varepsilon z) \,|\, z \in \mathbb{Z}[\mathbf{i}] \text{ and } \Lambda_{\varepsilon/2}(\varepsilon z) \cap S \neq \emptyset\right\}.$$

Further, we use the shorthand

$$U_\varepsilon(S) := U(K_\varepsilon(S)) = \bigcup_{z \in \mathbb{Z}[\mathbf{i}] : \Lambda_{\varepsilon/2}(\varepsilon z) \cap S \neq \emptyset} \Lambda_{\varepsilon/2}(\varepsilon z).$$

For $\mathcal{C}_\eta \in \mathscr{C}_1^\eta(\delta)$, the graph $K_\varepsilon(\mathcal{C}_\eta)$ approximates $\mathcal{C}_\eta$ in the sense that $d_H(\mathcal{C}_\eta, U_\varepsilon(\mathcal{C}_\eta)) < \varepsilon$. This is the second approximation of large clusters we referred to in the beginning of this section. Our next aim is to find an event where the two approximations coincide.

In the following we use the quantities defined above in the case where $\omega = \omega_\eta$ for some $\eta \geq 0$. We denote the particular choice of η in the superscript, for example $G_\varepsilon^\eta := G_\varepsilon(\omega_\eta)$. We shall prove:

Proposition 5.4.3. *Let $\eta, \varepsilon, \delta > 0$ with $1/10 > \delta > 10\varepsilon$. Suppose that $\omega_\eta \in \mathcal{E}(\varepsilon, \delta)$, where $\mathcal{E}(\varepsilon, \delta)$ as in* (5.18) *below.*

i) Then for all good subgraphs $H \leq G_\varepsilon^\eta$ there is a unique cluster $\mathcal{C}^\eta \in \mathscr{C}_1^\eta(\delta)$ such that $H = K_\varepsilon(\mathcal{C}^\eta)$.

ii) Conversely, if $\mathcal{C}^\eta \in \mathscr{C}_1^\eta(\delta)$, then $K_\varepsilon(\mathcal{C}^\eta)$ is a good subgraph of G_ε^η.

Proof of Proposition 5.4.3. Proposition 5.4.3 follows from the combination of Lemma 5.4.5 and 5.4.7 with the definition (5.18) below. □

For $\varepsilon, \delta > 0$ we define the event as the intersection

$$\mathcal{E}(\varepsilon, \delta) := \mathcal{NA}(\varepsilon, \delta) \cap \mathcal{NC}(\varepsilon, \delta). \tag{5.18}$$

First we define the event $\mathcal{NC}(\varepsilon, \delta)$ below, then we introduce $\mathcal{NA}(\varepsilon, \delta)$ in Definition 5.4.6.

Definition 5.4.4. *Let $0 < 10\varepsilon < \delta < 1$. We write $\mathcal{NC}(\varepsilon, \delta)^c$ for the union of events*

$$\mathcal{A}^j_{\emptyset,(010)}(z; \varepsilon/2, \delta/2 - 3\varepsilon) \cap \mathcal{A}^{j+2}_{\emptyset,(010)}(z'; \varepsilon/2, \delta/2 - 3\varepsilon) \tag{5.19}$$

for $j = 1, 2$, and squares $\Lambda_{\varepsilon/2}(z), \Lambda_{\varepsilon/2}(z') \in B_\varepsilon$ with $\mathrm{dist}_j(z, z') \in (\delta - 3\varepsilon, \delta + 3\varepsilon)$.

Definition 5.4.4 implies the following lemma, which illuminates the choice of the event $\mathcal{NC}(\varepsilon, \delta)$.

Lemma 5.4.5. *Let $0 < 10\varepsilon < \delta < 1$. On $\omega_\eta \in \mathcal{NC}(\varepsilon, \delta)$ there is no cluster $\mathcal{C}^\eta$, which is completely contained in Λ_1 with diameter between $\delta - 2\varepsilon$ and δ.*

We define the event $\mathcal{NA}(\varepsilon, \delta)$ which will be crucial in the following.

Definition 5.4.6. *Let ε, δ with $0 < 10\varepsilon < \delta < 1$. We set $\mathcal{NA}_1(\varepsilon, \delta)$ for the complement of the event*

$$\bigcup_{z \in \Lambda_{\lceil 1/\varepsilon \rceil} \cap \mathbb{Z}[\mathbf{i}]} \bigcup_{j=1}^{4} \mathcal{A}^j_{1,(010)}(\varepsilon z; \varepsilon/2, \delta/2 - 3\varepsilon).$$

We write $\mathcal{NA}_2(\varepsilon, \delta)^c$ for the union of events

$$\mathcal{A}^j_{\emptyset,(010)}(z; \varepsilon/2, \delta/2 - 3\varepsilon) \tag{5.20}$$

for $j = 1, 2, 3, 4$, and squares $\Lambda_{\varepsilon/2}(z) \in B_\varepsilon$ with $\min_{i \in \{1,2\}} \operatorname{dist}_i(\Lambda_{\varepsilon/2}(z), \partial\Lambda_k) \leq \varepsilon$. We define $\mathcal{NA}(\varepsilon, \delta) := \mathcal{NA}_1(\varepsilon, \delta) \cap \mathcal{NA}_2(\varepsilon, \delta)$.

Lemma 5.4.7. *Let $\eta, \varepsilon, \delta > 0$ with $0 < 10\varepsilon < \delta < 1$. Suppose that $\omega_\eta \in \mathcal{NA}(\varepsilon, \delta)$. We have:*

i) If $\mathcal{C}^\eta \in \mathscr{C}^\eta_1(\delta)$, then $K_\varepsilon(\mathcal{C}^\eta)$ is a good subgraph of G^η_ε.

ii) Conversely, for any good subgraph $H \leq G^\eta_\varepsilon$, there is a unique cluster $\mathcal{C}^\eta \in \mathscr{C}^\eta_1(\delta - 2\varepsilon)$ such that $H = K_\varepsilon(\mathcal{C}^\eta)$.

Proof of Lemma 5.4.7. Let ε, δ as in the lemma above, and $\omega_\eta \in \mathcal{NA}(\varepsilon, \delta)$. First we prove part i) above. Apart from conditions (2) and (3), the conditions in Definition 5.4.2 are trivially satisfied. The fact that $\omega_\eta \in \mathcal{NA}_2(\varepsilon, \delta)$ implies that condition (2) is satisfied. We prove condition (3) by contradiction.

Suppose that condition (3) is violated. Then there is $\Lambda \in V(G^\eta_\varepsilon) \setminus V(K_\varepsilon(\mathcal{C}^\eta))$ such that $(\Lambda, \Lambda') \in E(G^\eta_\varepsilon)$ for all $\Lambda' \in V(K_\varepsilon(\mathcal{C}^\eta))$.

We can assume that the diameter of $\mathcal{C}^\eta$ is realized in the horizontal direction. Take $L \in L(K_\varepsilon(\mathcal{C}^\eta))$ and $R \in R(K_\varepsilon(\mathcal{C}^\eta))$. Let γ denote a path in $\mathcal{C}^\eta$ connecting L and R. We can further assume that $\operatorname{dist}_1(\Lambda, L) > \delta/2 - \varepsilon$. Note that γ is not connected to Λ. However, Λ is connected to L. Hence the blue boundary of $\mathcal{C}^\eta$ separates γ from the connection between Λ and L. We get, from L to distance $\delta/2 - \varepsilon$, three half plane arms with colour sequence (010), and a fourth red arm from the connection between Λ and L. In particular, $\omega_\eta \in \mathcal{NA}_1(\varepsilon, \delta)^c$, we deduce part i) of Lemma 5.4.7.

Now we proceed to the proof of part ii). We may assume that the diameter of $U(H)$ is realized between a leftmost and a rightmost point of it. Let $L \in L(H)$, $R \in R(H)$ and γ be a path in $SR(H)$ connecting L and R. Furthermore, let $\Lambda' \in V(G^\eta_\varepsilon)$ be such that γ is connected to Λ' by a path in $\sigma_\eta \cap \Lambda_1$.

We show that $(\Lambda, \Lambda') \in E(G^\eta_\varepsilon)$ for all $\Lambda \in V(H)$. Suppose the contrary, i.e. there is $\Lambda \in V(H)$ such that $(\Lambda, \Lambda') \notin E(G^\eta_\varepsilon)$. Then Λ is not connected to γ. Furthermore, we may assume that $\operatorname{dist}_1(\Lambda, L) > \delta/2 - \varepsilon$. Then as above, we find three half plane arms with colour sequence (010) and a fourth red arm starting at L to distance $\delta/2-\varepsilon$. In particular, $\omega_\eta \in \mathcal{NA}_1(\varepsilon,\delta)^c$, which contradicts the assumption on ω_η above.

Hence $\Lambda' \in V(H)$ since H is maximal. Thus $K_\varepsilon(\mathcal{C}^\eta(\gamma)) \leq H$, where $\mathcal{C}^\eta(\gamma)$ denotes the connected component of γ in σ_η. Note that $K_\varepsilon(\mathcal{C}^\eta(\gamma))$ is a good subgraph, since it satisfies condition 4 since $\mathrm{dist}_1(L,R) > \delta$, condition 3 by part i) of Lemma 5.4.7. This completes the proof of part ii) and that of Lemma 5.4.7. □

The proof above implies the following useful property of the event $\mathcal{NA}(\varepsilon,\delta)$.

Lemma 5.4.8. *Let $\eta,\varepsilon,\delta>0$ with $0<10\varepsilon<\delta<1$. If $\omega_\eta \in \mathcal{NA}(\varepsilon,\delta)$, then we have $|\mathscr{C}_1^\eta(\delta)| \leq 32\varepsilon^{-2}$.*

Proof of Lemma 5.4.8. Let $\mathcal{C},\mathcal{C}' \in \mathscr{C}_1^\eta(\delta)$ be clusters with diameter at least δ in the horizontal direction. The proof of Lemma 5.4.7 shows that on the event $\mathcal{NA}(\varepsilon,\delta)$ $L(K_\varepsilon(\mathcal{C}))$ and $L(K_\varepsilon(\mathcal{C}'))$ are disjoint. The same holds for pairs of clusters with vertical diameter at least δ. Thus $|\mathscr{C}_1^\eta(\delta)| \leq 2(2\lceil 1/\varepsilon\rceil)^2) \leq 32\varepsilon^{-2}$. □

5.4.1 Bounds on the probability of the events $\mathcal{NC}(\varepsilon,\delta)$ and $\mathcal{NA}(\varepsilon,\delta)$

Our aim in this section is to prove the following bound on the probability of $\mathcal{E}(\varepsilon,\delta)$.

Proposition 5.4.9. *Let ε,δ with $0<10\varepsilon<\delta<1$. Suppose that Assumptions I-III hold. Then there are positive constants $C=C(\delta),\lambda$ such that for all $\eta\in(0,\varepsilon)$ we have*

$$\mathbb{P}_\eta\left(\mathcal{E}(\varepsilon,\delta)^c\right) \leq C\varepsilon^\lambda.$$

The proof of the proposition above follows from Lemma 5.4.10 and 5.4.11 below. We start by an upper bound on the probability of $\mathcal{NA}(\varepsilon,\delta)$.

Lemma 5.4.10. *Suppose that Assumptions I-III hold. Let ε,δ with $0<10\varepsilon<\delta<1$. Then there are constants $C=C(\delta),\lambda>0$ such that*

$$\mathbb{P}_\eta(\mathcal{NA}\left(\varepsilon,\delta\right)^c) \leq C\varepsilon^\lambda \tag{5.21}$$

for all $\eta<\varepsilon$. In particular, $|\mathscr{C}^\eta(\delta)|$ is tight in η for all fixed $\delta>0$.

Proof of Lemma 5.4.10. For ε,δ with $0<10\varepsilon<\delta<1$ simple union bounds together with Lemmas 5.3.10 and 5.3.11 give

$$\begin{aligned}
\mathbb{P}_\eta(\mathcal{NA}_1\left(\varepsilon,\delta\right)^c) &\leq 10\varepsilon^{-2}\left(\frac{\varepsilon}{\delta}\right)^{2+\lambda_{1,3}} = 10\frac{\varepsilon^{\lambda_{1,3}}}{\delta^{2+\lambda_{1,3}}},\\
\mathbb{P}_\eta(\mathcal{NA}_2\left(\varepsilon,\delta\right)^c) &\leq 40\varepsilon^{-1}\left(\frac{\varepsilon}{\delta}\right)^2 = 40\frac{\varepsilon}{\delta^2}.
\end{aligned}$$

This combined with the definition of the event $\mathcal{NA}(\varepsilon,\delta)$ provides the desired upper bound.

The tightness of $|\mathscr{C}^\eta(\delta)|$ follows from the combination of Lemma 5.4.8 and (5.21). □

Lemma 5.4.11. *Suppose that Assumptions I-III hold. Let* ε, δ *with* $0 < 10\varepsilon < \delta < 1$. *Then there is a constant* $C > 0$ *such that for all* $\eta \in (0, \varepsilon)$ *we have*

$$\mathbb{P}_\eta(\mathcal{NC}(\varepsilon,\delta)^c) \le C\frac{\varepsilon}{\delta^2}.$$

Proof of Lemma 5.4.11. A simple union bound combined with Lemma 5.3.10 provides the desired bound. □

5.5 Construction of the set of large clusters in the scaling limit

Now we are ready to construct the limiting object from Theorem 5.1.1. Before we do so, Corollary 5.3.2 combined with Assumption IV and Lemma 5.3.16 implies that there is a coupling denoted by $\mathbb{P}$ of ω_η's for $\eta \ge 0$ such that

$$\mathbb{P}(\omega_\eta \to \omega_0 \text{ as } \eta \to 0) = 1,$$

where ω_0 has law $\mathbb{P}_0$, which we use in the following.

Fix some $\delta > 0$. Let $\omega \in \mathscr{H}$ be a quad-crossing configuration. We define

$$n_0(\omega) := \inf\left\{n \ge 0 \,|\, \omega \in \mathcal{E}(3^{-n'},\delta) \text{ for all } n' \ge n\right\},$$

where we use the convention that the infimum of the empty set is ∞. From the construction above, it is clear that the set $\mathcal{E}(3^{-n},\delta) \in Borel(\mathscr{T}_{\hat{\mathbb{C}}})$, hence the function n_0 is $Borel(\mathscr{T}_{\hat{\mathbb{C}}})$ measurable. Note that $\omega_\eta \in \mathcal{E}(\eta/10,\delta)$ for $0 < \eta < 10\delta$. Hence $n_0(\omega_\eta) < \infty$. Furthermore, we write $g_n(\omega,\delta)$ for the number of good subgraphs in $G_{3^{-n}}(\omega)$.

Let $\eta > 0$, $n \ge n_0(\omega_\eta)$, and H^η be a good subgraph in $G^\eta_{3^{-n}} = G_{3^{-n}}(\omega_\eta)$. Proposition 5.4.3 shows that for all $n' \ge n$, there is a unique good subgraph $H'^\eta \subseteq G^\eta_{3^{-n'}}$ such that $U(H^\eta) \supseteq U(H'^\eta)$.

Let $g^\eta_n = g_n(\omega_\eta,\delta)$. For each $n \ge 0$, we fix an ordering of the graphs with vertex sets in $B_{3^{-n}}$. For $j = 1, 2, \ldots, g^\eta_{n_0}$, let $H^\eta_{j,n_0} := H_{j,n_0(\omega_\eta)}(\omega_\eta)$ denote the jth good subgraph of $G^\eta_{3^{-n_0}}$. Then for $n \ge n_0(\omega_\eta)$, let $H^\eta_{j,n}$ denote the unique good subgraph of $G^\eta_{3^{-n}}$ such that $U(H^\eta_{j,n_0}) \supseteq U(H^\eta_{j,n})$.

For $\eta \ge 0$ and $j = 1, 2, \ldots, g^\eta_{n_0}$ we set

$$\mathcal{C}^\eta_j(\delta) := \bigcap_{n \ge n_0(\omega_\eta)} U(H^\eta_{j,n}) \tag{5.22}$$

on the event $n_0(\omega_\eta) < \infty$, while on the event $n_0(\omega_\eta) = \infty$ we set $\mathcal{C}^\eta_j(\delta) = \{-1/2, 1/2\}$ for all $j \ge 1$. (Note that we can replace $\{-1/2, 1/2\}$ by any disconnected subset of Λ_1.) Since the sequence of compact sets $U(H^\eta_{j,n})$ is decreasing, the intersection in (5.22) is non-empty on the event $n_0(\omega_\eta) < \infty$. Proposition 5.4.3 shows that for $\eta > 0$, we get the collection of clusters $\mathscr{C}^\eta_1(\delta)$, that is,

$$\mathscr{C}^\eta_1(\delta) = \{\mathcal{C}^\eta_j(\delta) \,:\, 1 \le j \le g^\eta_{n_0}\}.$$

Before we state and prove the following precise version of Theorem 5.1.1, let us comment on the topology used there. We employ a slightly different topology than the one in (5.5), defined as follows.

Let $\mathfrak{C}$ denote the set of non-empty closed subsets of Λ_1 endowed with the Hausdorff distance d_H as defined in (5.3). Let $l(\mathfrak{C})$ denote the space of sequences in $\mathfrak{C}$. We endow it with the metric d_l defined as

$$d_l(\underline{C}, \underline{C}') := \sum_{j=1}^{\infty} d_H(C_j, C_j')2^{-j} \tag{5.23}$$

for $\underline{C} = (C_j)_{j\geq 1}, \underline{C}' = (C_j')_{j\geq 1}$. Note that convergence in d_l is equivalent with coordinate-wise convergence. Furthermore, $l^\infty(\mathfrak{C})$ inherits the compactness from $\mathfrak{C}$.

For $\eta \geq 0$ we extend the definition (5.22), by setting $\mathcal{C}_j^\eta(\delta) := \{-1/2, 1/2\}$ for $j > g_{n_0}^\eta$. We write $\underline{\mathcal{C}}_1^\eta(\delta) := (\mathcal{C}_j^\eta(\delta))_{j\geq 1}$.

For a quad-crossing configuration ω, $\underline{\mathcal{C}}_1^\eta = \underline{\mathcal{C}}_1^\eta(\omega)$ denotes the vector of all macroscopic clusters in ω defined as follows. The first $g_{n_0}(\omega, 3^{-1})$ entries of $\underline{\mathcal{C}}_1^\eta(\omega)$ coincide with those of $\underline{\mathcal{C}}_1^\eta(\omega, 3^{-1})$. For $m \geq 4$, the next $g_{n_0}(\omega, m^{-1}) - g_{n_0}(\omega, (m-1)^{-1})$ entries coincide with those elements in $\underline{\mathcal{C}}_1^\eta(\omega, m^{-1})$ which are unlisted earlier in $\underline{\mathcal{C}}_1^\eta(\omega)$, with their relative ordering.

Now we are ready to state the following precise and slightly stronger version of Theorem 5.1.1.

Theorem 5.5.1. *Suppose that Assumptions I-IV hold. Let $\delta > 0$ and $\mathbb{P}$ be a coupling where $\omega_\eta \to \omega_0$ a.s. as $\eta \to 0$. Then $\underline{\mathcal{C}}_1^\eta(\delta) \to \underline{\mathcal{C}}_1^0(\delta)$ in probability in the metric d_l as $\eta \to 0$. In particular, the pair $(\omega_\eta, \underline{\mathcal{C}}_1^\eta(\delta))$ converges in distribution to $(\omega_0, \underline{\mathcal{C}}_1^0(\delta))$ as $\eta \to 0$. Moreover, same convergence result holds for $\underline{\mathcal{C}}_1^\eta$. Furthermore, $\underline{\mathcal{C}}_1^0(\delta)$ and $\underline{\mathcal{C}}_1^0$ are measurable functions of ω_0.*

Remark 5.5.2. Note that the connected sets of Λ_1 form a compact subspace of $\mathfrak{C}$. Hence $\{-1/2, 1/2\}$ is separated from the clusters $\mathcal{C}_j^\eta$ for $j = 1, \ldots, g_{n_0}^\eta$. Thus the convergence of the vectors $\underline{\mathcal{C}}_1^\eta(\delta)$ in the metric d_l implies the convergence of $\mathscr{C}_1^\eta(\delta)$ in the topology (5.5). Namely, the bijection is given by the ordering of the entries in the corresponding vectors, while the proof of Lemma 5.4.8 implies that, in the sequence, there is no pair of clusters converging to the same closed set. The convergence in the metric (5.6) follows from the equivalence of the metrics d_H and D_H.

Before we turn to the proof of Theorem 5.5.1, we prove the following lemma.

Lemma 5.5.3. *Suppose that Assumptions I-IV hold. Let $\mathbb{P}$ be a coupling such that $\omega_\eta \to \omega_0$ $\mathbb{P}$-a.s. as $\eta \to 0$. Then*

$$\mathbb{P}(n_0(\omega_0) = \infty) = 0.$$

Moreover, $n_0(\omega_\eta) \to n_0(\omega_0)$ in probability under $\mathbb{P}$ as $\eta \to 0$.

Proof of Lemma 5.5.3. For each fixed $\varepsilon, \delta > 0$ the event $\mathcal{E}(\varepsilon, \delta)$ can be written as a finite union of intersections of some events appearing in Lemma 5.3.18. Thus

$$\begin{aligned}\mathbb{P}_0(\mathcal{E}(\varepsilon, \delta)^c) &= \lim_{\eta\to 0} \mathbb{P}_\eta(\mathcal{E}(\varepsilon, \delta)^c) \\ &\leq C\varepsilon^\lambda\end{aligned}$$

where C, λ as in Proposition 5.4.9. Hence

$$\sum_{n=1}^{\infty} \mathbb{P}_0(\mathcal{E}(3^{-n}, \delta)^c) < \infty.$$

Thus the Borel-Cantelli lemma shows that $\mathbb{P}(n_0(\omega_0) = \infty) = 0$.

Let $k \geq 1$. Lemma 5.3.18 and Proposition 5.4.9 implies that

$$\begin{aligned}
\mathbb{P}(&|n_0(\omega_\eta) - n_0(\omega_0)| \geq 1) \\
&\leq \mathbb{P}(n_0(\omega_\eta) > k) + \mathbb{P}(n_0(\omega_0) > k) \\
&\quad + \mathbb{P}(|n_0(\omega_\eta) - n_0(\omega_0)| \geq 1, n_0(\omega_0) \vee n_0(\omega_\eta) \leq k) \\
&\leq \sum_{l \geq k+1} \big(\mathbb{P}_\eta(\mathcal{E}(3^{-l}, \delta)^c) + \mathbb{P}_0(\mathcal{E}(3^{-l}, \delta)^c)\big) \\
&\quad + \mathbb{P}(\exists l \leq k \text{ s.t } \mathbf{1}(\omega_\eta \in \mathcal{E}(3^{-l}, \delta)) \neq \mathbf{1}(\omega_0 \in \mathcal{E}(3^{-l}, \delta))) \\
&\leq C \sum_{l \geq k+1} 3^{-\lambda l} + \sum_{l=1}^{k} \mathbb{P}(\mathbf{1}(\omega_\eta \in \mathcal{E}(3^{-l}, \delta)) \neq \mathbf{1}(\omega_0 \in \mathcal{E}(3^{-l}, \delta)))
\end{aligned} \tag{5.24}$$

with some constant $C > 0$. Taking $\eta \to 0$ in (5.24) with a suitable constant C' we get

$$\lim_{\eta \to 0} \mathbb{P}(|n_0(\omega_\eta) - n_0(\omega_0)| \geq 1) \leq C' 3^{-\lambda k}$$

for all $k > 0$. This shows that $n_0(\omega_\eta) \to n_0(\omega_0)$ in probability as $\eta \to 0$, and concludes the proof of Lemma 5.5.3. □

Proof of Theorem 5.5.1. Let $\delta > 0$, and $\mathbb{P}$ be a coupling such that $\omega_\eta \to \omega_0$ a.s. We will work under $\mathbb{P}$ in the following. Note that for each $n \in \mathbb{N}$, the event $\mathcal{E}(3^{-n}, \delta)$, the graph $G_{3^{-n}}(\omega)$ and the good subgraphs of $G_{3^{-n}}(\omega)$ are functions of the outcomes of finitely many arm events appearing in Lemma 5.3.18. Thus each of

- $\mathbf{1}\{\omega_\eta \in \mathcal{E}(3^{-n}, \delta)\}$,
- $G_{3^{-n}}(\omega_\eta)$, and
- the ordered set of good subgraphs of $G_{3^{-n}}(\omega_\eta)$

converge to the same quantities with ω_η replaced by ω_0 in probability as $\eta \to 0$. This has the following consequences:

1) with Lemma 5.5.3, we have $n_0(\omega_\eta) \to n_0(\omega_0) < \infty$,

2) $g_n^\eta \to g_n^0$ for all $n \geq 1$, in particular, $g^\eta_{n_0(\omega_\eta)} \to g^0_{n_0(\omega_0)}$,

3) $H_{j,n}^\eta \to H_{j,n}^0$ for $j = 1, 2, \ldots, g_{n_0(\omega_0)}$ and $n \geq n_0(\omega_0)$

in probability as $\eta \to 0$. Let $n \geq n_0(\omega_\eta) \vee n_0(\omega_0)$, then

$$\begin{aligned}
d_H(\mathcal{C}_j^\eta, \mathcal{C}_j^0) &\leq d_H(\mathcal{C}_j^\eta, U(H_{j,n}^\eta)) + d_H(U(H_{j,n}^\eta), U(H_{j,n}^0)) + d_H(U(H_{j,n}^0), \mathcal{C}_j^0) \\
&\leq 3^{-n} + d_H(U(H_{j,n}^\eta), U(H_{j,n}^0)) + 3^{-n}
\end{aligned} \tag{5.25}$$

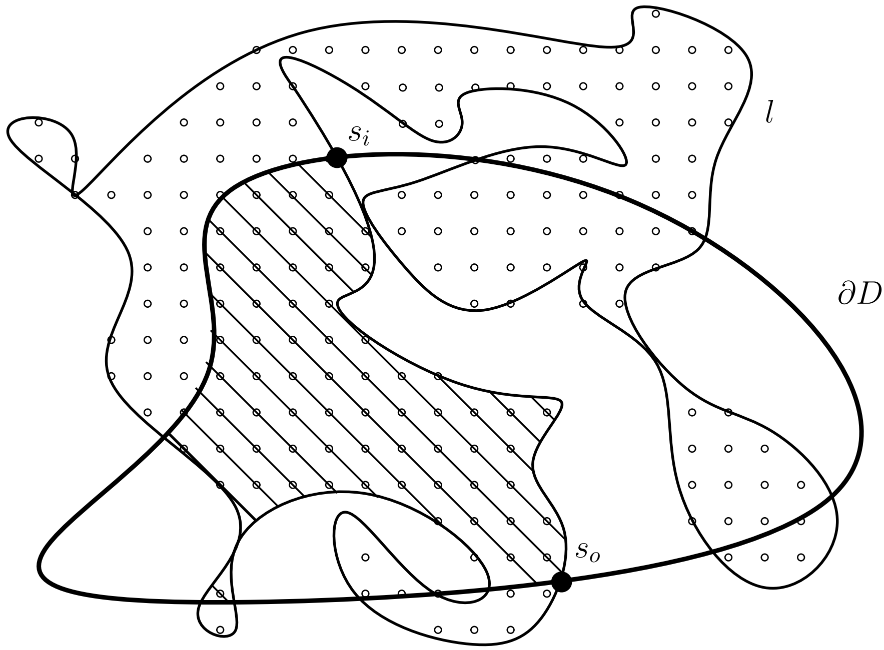

Figure 5.2: Illustration of a cluster in D. The small open circles denote the interior of the loop l. The shaded area intersected with the cluster of the loop is equal to $\mathcal{B}(\mathcal{E})$.

for $j = 1, 2, \ldots, g^{\eta}_{n_0} \wedge g^{0}_{n_0}$. Thus taking the limit $\eta \to 0$ in (5.25), by 1)-3) above, we get

$$\lim_{\eta \to 0} \mathbb{P}(d_H(\mathcal{C}^{\eta}_j, \mathcal{C}^0_j) > 3 \cdot 3^{-n}, n \geq n_0(\omega_0) \vee n_0(\omega_\eta)) = 0 \tag{5.26}$$

for $j \geq 1$. Then taking the limit $n \to \infty$, Lemma 5.5.3 shows that $\mathcal{C}^{\eta}_j \to \mathcal{C}^0_j$ in Hausdorff metric in probability as $\eta \to 0$ for all $j \geq 1$. Since convergence in $l^{\infty}(\mathfrak{C})$ coincides with coordinate-wise convergence, we get that $\lim_{\eta \to 0} \underline{\mathcal{C}}^{\eta}_1(\delta) = \underline{\mathcal{C}}^0_1(\delta)$ in probability, as required.

The proof of the claims of Theorem 5.5.1 for $\underline{\mathcal{C}}^{\eta}_1$ is analogous. It follows from the convergence of $\underline{\mathcal{C}}^{\eta}_1(\delta)$ with $\delta = 3^{-m}$ for $m \geq 1$. The measurability of $\underline{\mathcal{C}}^0_1(\delta)$ and $\underline{\mathcal{C}}^0_1$ with respect to ω_0 follows easily from their definition involving arm events (see Remark 5.3.6). Thus the proof of Theorem 5.5.1 is complete. □

5.6 Scaling limits in a bounded domain

In this section we will deduce the convergence of all clusters and "pieces" of clusters contained in a bounded domain D from the convergence of clusters and loops completely contained in $\Lambda_k \supset D$, for some k sufficiently large. We denote $\mathscr{B}^{\eta}_D(\delta)$ the collection of all clusters or portions of clusters of diameter at least δ contained in D^{η}, where D^{η} denotes an appropriate discretization of D. In the case of $\mathbb{Z}^2$, the boundary of D^η is a circuit in the medial lattice that surrounds all the vertices of $\mathbb{Z}^2$ contained in D and minimizes the distance to ∂D. Analogously, in the case of the triangular lattice, $\mathbb{T}$, the boundary of D^η is a circuit in the dual (hexagonal) lattice that surrounds all the vertices of $\mathbb{T}$ contained in D and minimizes the distance to ∂D. More precisely, for every cluster $\mathcal{C} \in \mathscr{C}^\eta(\delta)$ that intersect D^η, consider the set of all connected components $\mathcal{B}$ of $\mathcal{C} \cap D^\eta$ with diameter at least $\delta > 0$. For every $\eta, \delta > 0$, we let $\mathscr{B}^\eta_D(\delta)$ denote the union of $\mathscr{C}^\eta_D(\delta)$ with the set of all such connected components $\mathcal{B}$. (Note that clusters contained in Λ_k but not completely contained in D^η are split into different elements of $\mathscr{B}^\eta_D(\delta)$. See figure 5.2.) For the case of Bernoulli percolation, the collection $\mathscr{B}^\eta_D(\delta)$ is precisely the set of all clusters in D^η with closed boundary condition.

As in Section 5.5, instead of the collection $\mathscr{B}^\eta_D(\delta)$, we consider the sequence $\underline{\mathcal{B}}^\eta_D(\delta)$ of clusters with diameter at least δ, with the metric d_l. Now we are ready to state the theorem on the convergence of all portions of clusters in $\sigma_\eta \cap D$ for a bounded domain D.

Theorem 5.6.1. *Suppose that Assumptions I-IV hold. Let D be a simply connected bounded domain with piecewise smooth boundary. Let $\mathbb{P}$ be a coupling where $(\omega_\eta, L_\eta) \to (\omega_0, L_0)$ a.s. as $\eta \to 0$. Then, for any $\delta > 0$, $\underline{\mathcal{B}}^\eta_D(\delta) \to \underline{\mathcal{B}}^0_D(\delta)$ in probability in the metric d_l as $\eta \to 0$. In particular, the triple $(\omega_\eta, L_\eta, \underline{\mathcal{B}}^\eta_D(\delta))$ converges in distribution to $(\omega_0, L_0, \underline{\mathcal{B}}^0_D(\delta))$ as $\eta \to 0$. Moreover, the same convergence result holds for $\underline{\mathcal{B}}^\eta_D$. Furthermore, $\underline{\mathcal{B}}^0_D(\delta)$ and $\mathscr{B}^0_D$ are measurable functions of the pair (ω_0, L_0).*

Proof. Let (ω_η, L_η) and (ω_0, L_0) be as in the statement of Theorem 5.6.1. The probability that all the clusters that intersect D are completely contained in Λ_k is at least one minus the probability of having a red arm from the boundary of D to $\partial\Lambda_k$. The latter probability goes to zero as $k \to \infty$, hence there is a finite $k \in \mathbb{N}$ such that there is no red arm from D to $\partial\Lambda_{k-1}$ in ω_0. We take the smallest such k. With this choice, all clusters in $\mathscr{C}^\eta$ that intersect D are contained in Λ_k.

We first give an orientation to the loops contained in Λ_k in such a way that clockwise loops are the outer boundaries of red clusters and counterclockwise loops are the outer boundaries of blue clusters. For each clockwise loop ℓ intersecting ∂D, we consider all excursions $\mathcal{E}$ inside D of diameter at least δ. Each excursion $\mathcal{E}$ runs from a point s_{in} on ∂D to a point s_{out} on ∂D. We call the counterclockwise segment of ∂D from s_{in} to s_{out} the base of $\mathcal{E}$. We call $\overline{\mathcal{E}}$ the concatenation of $\mathcal{E}$ with its base. We define the interior $\mathrm{I}(\overline{\mathcal{E}})$ of $\overline{\mathcal{E}}$ to be the closure of the set of points with nonzero winding number for the curve $\overline{\mathcal{E}}$.

We call $\mathscr{E}_{\mathcal{E}}$ the collection of all clockwise excursions in D of the same loop ℓ with base contained inside the base of $\mathcal{E}$. If $\mathcal{C}$ is the cluster whose outer boundary is the loop ℓ, we define $\mathcal{B}(\mathcal{E})$ as follows:

$$\mathcal{B}(\mathcal{E}) := \overline{\mathrm{I}(\overline{\mathcal{E}}) \setminus \{\cup_{\mathcal{E}' \in \mathscr{E}_{\mathcal{E}}} \mathrm{I}(\overline{\mathcal{E}'})\}} \cap \mathcal{C},$$

where by $\cup_{\mathcal{E}' \in \mathscr{E}_{\mathcal{E}}} \mathrm{I}(\overline{\mathcal{E}'})$ we mean $\lim_{\xi \to 0} \cup_{\mathcal{E}' \in \mathscr{E}_{\mathcal{E}}, \mathrm{diam}\mathcal{E}' > \xi} \mathrm{I}(\overline{\mathcal{E}'})$, and the limit exists because it is the limit of an increasing sequence of closed sets.

For any $\delta > 0$, $\mathscr{B}^0_D(\delta)$ is the collection of all sets $\mathcal{B}(\mathcal{E})$ defined above, for all clockwise excursions $\mathcal{E}$ in D of diameter at least δ.

For any $\eta > 0$, the collection $\mathscr{B}_D^\eta(\delta)$ contains all clusters completely contained in D plus all the connected components of the intersections of clusters in Λ_k with D. $\mathscr{B}_D^\eta(\delta)$ can be obtained with the following construction which mimics the continuum construction given earlier. We first give an orientation to the loops contained in Λ_k in such a way that loops that have red in their immediate interior are oriented clockwise and loops that have blue in their immediate interior are oriented counterclockwise. For each clockwise loop ℓ^η intersecting ∂D^η, we consider all excursions $\mathcal{E}^\eta$ inside D^η of diameter at least δ. Each excursion $\mathcal{E}^\eta$ runs from a point s_{in}^η on ∂D^η to a point s_{out}^η on ∂D^η. We call the counterclockwise segment of ∂D^η from s_{in}^η to s_{out}^η the base of $\mathcal{E}^\eta$. We call $\overline{\mathcal{E}^\eta}$ the concatenation of $\mathcal{E}^\eta$ with its base. We define the interior $\mathrm{I}\left(\overline{\mathcal{E}^\eta}\right)$ of $\overline{\mathcal{E}^\eta}$ to be the set of hexagons contained inside $\overline{\mathcal{E}^\eta}$.

We call $\mathscr{E}_{\mathcal{E}^\eta}^\eta$ the collection of all clockwise excursions in D^η of the same loop ℓ^η with base contained inside the base of $\mathcal{E}^\eta$. If $\mathcal{C}^\eta$ is the cluster whose outer boundary is the loop ℓ^η, we define $\mathcal{B}^\eta(\mathcal{E}^\eta)$ as follows:

$$\mathcal{B}^\eta(\mathcal{E}^\eta) := \overline{\mathrm{I}\left(\overline{\mathcal{E}^\eta}\right) \setminus \left\{\cup_{(\mathcal{E}^\eta)' \in \mathscr{E}_{\mathcal{E}^\eta}^\eta} \mathrm{I}\left(\overline{(\mathcal{E}^\eta)'}\right)\right\}} \cap \mathcal{C}^\eta.$$

We now note that the almost sure convergence $(\omega_\eta, L_\eta) \to (\omega_0, L_0)$, combined with Lemma 5.3.10, implies the same for the excursions in D. (Lemma 5.3.10 insures, via standard arguments, that an excursion cannot come close to the boundary of D without touching it, so that large lattice and continuum excursions will match exactly for η sufficiently small. For more details on how to use Lemma 5.3.10, the interested reader is referred to Lemma 6.1 of [22].) Together with the convergence of the clusters, this implies that $(\omega_\eta, L_\eta, \underline{\mathcal{B}}_D^\eta(\delta))$ converges in distribution to $(\omega_0, L_0, \underline{\mathcal{B}}_D^0(\delta))$ as $\eta \to 0$. The above result is valid for any $\delta > 0$, so letting $\delta \to 0$ gives the second part of the theorem. □

5.7 Limits of counting measures of clusters

Herein we state and prove Theorem 5.7.2, a precise and slightly stronger version of Theorem 5.1.2. We do this for the more general case of (portions of) clusters $\mathscr{B}_D^\eta(\delta)$ in a domain with piecewise smooth boundary D. The convergence of measures of the clusters which are completely contained in Λ_k follows immediately. For ease of notation we assume D to be Λ_1.

Let $\mathfrak{M}$ denote the set of finite Borel measures on Λ_1 endowed with the Prokhorov metric. Recall that $\mathfrak{M}$ is a separable metric space.

For $\delta, \varepsilon > 0$, $\eta \geq 0$ and $S \subseteq \Lambda_1$ we define

$$\mu_{S,n}^\eta := \sum_{z \in \mathbb{Z}[\mathbf{i}] : \Lambda_{3 \cdot 3^{-n}/2}(3^{-n} z) \cap S \neq \emptyset} \mu_{1, A(3^{-n} z; 3^{-n}/2, \delta/2 - 3^{-n})}^\eta. \tag{5.27}$$

That is the sum of counting measures $\mu_{1,A(z;3^{-n}/2,b)}^\eta$ where the inner box $\Lambda_{3^{-n}/2}(z)$ self or one of its neighbours has nonempty intersection with S.

Simple arguments show the following:

Observation 5.7.1. *Let B be a Borel set of $\mathbb{C}$ and $S \subseteq \Lambda_1$. Then $\mu_{S,n}^\eta(B) \geq \mu_{S,n'}^\eta(B)$ for $n' \geq n$ with probability 1 for fixed $\eta > 0$.*

It is easy to check that for all fixed $\eta > 0$ and $\mathcal{B} \in \mathscr{B}^{\eta}_{\Lambda_1}(\delta)$ the following limit exists

$$\lim_{n\to\infty} \mu^{\eta}_{\mathcal{B},n} \tag{5.28}$$

and is actually equal to $\mu^{\eta}_{\mathcal{B}}$ as defined in (5.7).

This motivates us to define, for any cluster $\mathcal{B} \in \mathscr{B}^{0}_{\Lambda_1}(\delta)$, $\mu^{0}_{\mathcal{B}}$ by (5.28) with $\eta = 0$ if the limit exists, and set $\mu^{0}_{\mathcal{B}} = 0$ when it does not.

Let $l(\mathfrak{M})$ denote the set of infinite sequences in $\mathfrak{M}$ with bounded distance from the empty measure. Similarly to (5.23), we set

$$d_l(\underline{\nu}, \underline{\phi}) := \sum_{j=1}^{\infty} \frac{d_P(\nu_j, \phi_j)}{1 + d_P(\nu_j, \phi_j)} 2^{-j}$$

for $\underline{\nu}, \underline{\phi} \in l(\mathfrak{M})$. It is easy to check that $l(\mathfrak{M})$ is separable, but not compact. Let $h^{\eta}(\delta) := |\mathscr{B}^{\eta}_{\Lambda_1}(\delta)|$, for $\eta \geq 0$. It follows from Lemma 5.5.3, together with the tightness of the number of excursions in Λ_1, of diameter at least δ, that $h^0(\delta)$ is a.s. finite. For $\eta \geq 0$ we define $\underline{\mu}^{\eta} = (\mu_j)_{j\geq 1}$, the vector of these measures, where $\mu^{\eta}_j := \mu^{\eta}_{\mathcal{B}_j}$ is as above for $j = 1, 2, \ldots, h^{\eta}(\delta)$, and we set $\mu^{\eta}_j = 0$ for $j > h^{\eta}(\delta)$. We define $\underline{\mu}^{\eta}$ similarly to $\underline{\mathcal{C}}^{\eta}$.

Now we are ready to state the main result from this section.

Theorem 5.7.2. *Suppose that Assumptions I-IV hold. Let D be a simply connected bounded domain with piece-wise smooth boundary. Let $\mathbb{P}$ be a coupling where $(\omega_\eta, L_\eta) \to (\omega_0, L_0)$ a.s. as $\eta \to 0$. Then $\underline{\mu}^{\eta}_D(\delta) \to \underline{\mu}^{0}_D(\delta)$ in probability as $\eta \to 0$, where $\underline{\mu}^{0}_D(\delta)$ is a measurable function of the pair (ω_0, L_0). In particular, the triple $(\omega_\eta, L_\eta, \underline{\mu}^{\eta}_D(\delta))$ converges in distribution to $(\omega_0, L_0, \underline{\mu}^{0}_D(\delta))$ as $\eta \to 0$. The same convergence result holds when $\underline{\mu}^{\eta}_D(\delta)$ is replaced by $\underline{\mu}^{\eta}_D$.*

The same conclusion holds for the measures of the clusters in $\hat{\mathbb{C}}$ which intersect a bounded domain D, that is, we keep the information of connections outside D.

Remark 5.7.3. Lemma 5.8.2 shows that clusters whose diameter is at least $\delta > 0$ have nonzero mass. Thus the convergence in Theorem 5.7.2 implies convergence in the metric (5.5) and so Theorem 5.1.2 is proved.

Let us first show that Theorem 5.1.3 follows easily from Theorems 5.5.1 and 5.7.2.

Proof of Theorem 5.1.3. The proof is analogous to the proof of Theorem 6 of [22], so we only give a sketch. Let D be any bounded subset of $\mathbb{C}$ and $k_1 > k_2$ be such that $D \subset \Lambda_{k_2}$. The measures $\mathbb{P}_{k_1}$ and $\mathbb{P}_{k_2}$ can be coupled in such a way that they coincide inside D, in the sense that they induced the same marginal distribution on $(\mathscr{C}^0_D, \mathscr{M}^0_D)$. This is because they are obtained from the scaling limit of the same full-plane lattice measure $\mathbb{P}_\eta$. The consistency relations needed to apply Kolmogorov's extension theorem are then satisfied, which insures the existence of a limit $\mathbb{P}$. □

The following lemma plays an important role in the proof of Theorem 5.7.2. Let $||\nu||_{TV}$ denote the total variation of a signed measure ν.

Lemma 5.7.4. *Suppose that Assumptions I-III hold. Let $\delta > 0$. Then there are positive constants $C = C(\delta), \varphi$ such that, for $n \in \mathbb{N}$ and $\eta > 0$ with $0 < 10\eta < 3^{-n} < \delta/10$*

$$\mathbb{P}_\eta(\exists \mathcal{B} \in \mathscr{B}^\eta_{\Lambda_1}(\delta), S \subseteq \Lambda_1 \text{ s.t.: } d_H(\mathcal{B}, S) < \varepsilon/2, ||\mu^\eta_{\mathcal{B}} - \mu^\eta_{S,n}||_{TV} \geq \varepsilon^\varphi) \leq C \cdot \varepsilon^\varphi$$

where $\varepsilon = 3^{-n}$.

Proof of Theorem 5.7.2 given Lemma 5.7.4. Let $\mathbb{P}$ as in Theorem 5.7.2, $\delta > 0$. It follows from Theorem 5.6.1 that the clusters in $\mathscr{B}^\eta_{\Lambda_1}(\delta)$ converge in probability as $\eta \to 0$.

Moreover, Theorem 5.3.19 shows that each of the measures

$$\mu^\eta_{1,A(3^{-n}z;3^{-n}/2,\delta/2-3^{-n})} \quad \text{for } n \geq 1 \text{ and } z \in \mathbb{Z}[\mathbf{i}] \text{ with } 3^{-n}z \in \Lambda_1.$$

converge in the Prokhorov metric in probability as $\eta \to 0$ to the version of measures where η is replaced by 0.

This implies that, for all fixed n and $S \subset \Lambda_1$, $\mu^\eta_{S,n} \to \mu^0_{S,n}$ weakly in probability as $\eta \to 0$. The monotonicity of the measures $\mu^\eta_{S,n}$ in n for a fixed subset S and fixed η of Observation 5.7.1 carries through the limit as $\eta \to 0$, thus the weak limit $\mu^0_S = \lim_{n\to\infty} \mu^0_{S,n}$ a.s. exists. Furthermore, since each of the measures $\mu^0_{S,n}$ is a function of (ω_0, L_0) and a.s. finite, we derive that μ^0_S is a.s. finite and is a function of (ω_0, L_0).

Recall the sequence $\underline{\mathcal{B}}^0_{\Lambda_1}(\delta)$ of clusters. Let $\mathcal{B}$ be the j-th element of this sequence and let $\mathcal{B}^\eta_j$ be the j-th element of $\underline{\mathcal{B}}^\eta_{\Lambda_1}(\delta)$. Let $\kappa > 0$ fixed. Lemma 5.7.4 implies that for some constants $\varphi, C = C(\delta)$ for $\kappa > \varepsilon^\varphi$, $\eta < \varepsilon/10$ and $3^{-n} = \varepsilon$ we have

$$\begin{aligned}
&\mathbb{P}(d_P(\mu^0_{\mathcal{B}}, \mu^\eta_{\mathcal{B}^\eta_j}) > 3\kappa) \\
&\quad \leq \mathbb{P}(d_P(\mu^0_{\mathcal{B}}, \mu^0_{\mathcal{B},n}) > \kappa) + \mathbb{P}(d_P(\mu^0_{\mathcal{B},n}, \mu^\eta_{\mathcal{B},n}) > \kappa) \\
&\qquad + \mathbb{P}(||\mu^\eta_{\mathcal{B},n} - \mu^\eta_{\mathcal{B}^\eta_j}||_{TV} > \kappa, d_H(\mathcal{B}, \mathcal{B}^\eta_j) < \varepsilon/2) + \mathbb{P}(d_H(\mathcal{B}, \mathcal{B}^\eta_j) \geq \varepsilon/2) \\
&\quad \leq \mathbb{P}(d_P(\mu^0_{\mathcal{B}}, \mu^0_{\mathcal{B},n}) > \kappa) + \mathbb{P}(d_P(\mu^0_{\mathcal{B},n}, \mu^\eta_{\mathcal{B},n}) > \kappa) \\
&\qquad + C\kappa + \mathbb{P}(d_H(\mathcal{B}, \mathcal{B}^\eta_j) \geq \varepsilon/2)
\end{aligned} \tag{5.29}$$

where d_P denotes the Prokhorov distance of Borel measures.

Now we take the limit first as $\eta \to 0$ then as $n \to \infty$ in (5.29). From the arguments above and Theorem 5.6.1 we deduce that

$$\lim_{\eta\to 0} \mathbb{P}(d_P(\mu^0_{\mathcal{B}}, \mu^\eta_{\mathcal{B}^\eta_j}) > 3\kappa) \leq C\kappa$$

for all $\kappa > 0$. Thus the measures $\mu^\eta_{\mathcal{B}^\eta_j}$ tend to $\mu^0_{\mathcal{B}}$ weakly in probability as $\eta \to 0$.

Recall that the convergence in $l^\infty(\mathfrak{M})$ is equivalent with coordinate-wise convergence. Thus $\underline{\mu}^\eta(\delta) \to \underline{\mu}^0(\delta)$ in probability as $\eta \to 0$. We have already proved in the lines above that $\underline{\mu}^0(\delta)$ is a measurable function of (ω_0, L_0), thus we deduced the results in Theorem 5.7.2 for $\underline{\mu}^\eta(\delta)$.

The results for $\underline{\mu}^\eta$ follow from the lines above by arguments similar to those at the end of the proof of Theorem 5.5.1. This concludes the proof of Theorem 5.7.2. ☐

We finish this section by proving Lemma 5.7.4 above. Its proof relies on Lemma 5.3.14.

Proof of Lemma 5.7.4. Let η, n, δ as in Lemma 5.7.4. To ease the notation, we set $\varepsilon := 3^{-n}$, $\delta' := \delta/2 - 3\varepsilon$ and $\beta := \frac{\lambda}{2(\lambda+\lambda_1)}$, with λ_1 as in Lemma 5.3.9 while λ as in Lemma 5.3.12.

Let $\nu^{\eta}_{\varepsilon^\beta}$ denote the normalized counting measure of the vertices close to the boundary of Λ_1 which have an open arm to distance $5\varepsilon^\beta$. That is,

$$\nu^{\eta}_{\varepsilon^\beta} := \frac{\eta^2}{\pi_1^\eta(\eta,1)} \sum_{v\in A(0;1-\varepsilon^\beta,1)\cap\eta V} \delta_v \mathbf{1}\{v \overset{(1)}{\longleftrightarrow} \partial\Lambda_{5\varepsilon^\beta}(v)\}. \tag{5.30}$$

Furthermore, we define the following collection of 'pivotal' boxes:

$$\mathrm{Piv}^\eta(\varepsilon,\varepsilon^\beta) := \{\Lambda_{\varepsilon/2}(\varepsilon z) \,|\, z\in\mathbb{Z}[\mathbf{i}]\cap\Lambda_{\varepsilon^{-1}+1}; \omega_\eta \in \mathcal{A}_{(1010),\emptyset}(\varepsilon z; 3\varepsilon/2, \varepsilon^\beta)\}.$$

Let $\mathcal{B}\in\mathscr{B}^\eta_{\Lambda_1}(\delta)$ and $S\subseteq\Lambda_1$ such that $d_H(S,\mathcal{B})<\varepsilon/2$. Note that $d_H(S,\mathcal{B})<\varepsilon/2$ implies that the counting measure $\mu^\eta_{S,n}$ is larger or equal the counting measure $\mu^\eta_{\mathcal{B}}$. As a consequence it is easy to check that, for these $\mathcal{B}$ and S, we have

$$\begin{aligned} &||\mu^\eta_{S,n} - \mu^\eta_{\mathcal{B}}||_{TV} \\ &\qquad \le ||\nu^\eta_{\varepsilon^\beta}||_{TV} + \sum_{z\in\mathbb{Z}[\mathbf{i}]\,:\,\Lambda_{\varepsilon/2}(\varepsilon z)\in\mathrm{Piv}^\eta(\varepsilon,\varepsilon^\beta)} ||\mu^\eta_{1,A(\varepsilon z;3\varepsilon/2,\delta')}||_{TV} \\ &\qquad \le ||\nu^\eta_{\varepsilon^\beta}||_{TV} + |\mathrm{Piv}^\eta(\varepsilon,\varepsilon^\beta)| \sup_{z\in\mathbb{Z}[\mathbf{i}]\cap\Lambda_{\varepsilon^{-1}+1}} ||\mu^\eta_{1,A(\varepsilon z;3\varepsilon/2,3\varepsilon)}||_{TV}. \end{aligned} \tag{5.31}$$

Let $\varphi>0$ to be fixed later and $a^\eta_\varepsilon := \varepsilon^{-(2+\varphi)}\pi^\eta_4(3\varepsilon/2,\varepsilon^\beta)$. From (5.31) we deduce that

$$\begin{aligned} &\mathbb{P}_\eta(\exists\mathcal{B}\in\mathscr{B}^\eta_{\Lambda_1}(\delta), S\subseteq\Lambda_1 \text{ s.t. } : d_H(S,\mathcal{B})<\varepsilon/2, ||\mu^\eta_{\mathcal{B}} - \mu^\eta_{S,n}||_{TV} \ge \varepsilon^\varphi) \\ &\qquad \le \mathbb{P}_\eta(||\nu^\eta_{\varepsilon^\beta}||_{TV} \ge \frac{1}{2}\varepsilon^\varphi) + \mathbb{P}_\eta(|\mathrm{Piv}^\eta(\varepsilon,\varepsilon^\beta)| \ge a^\eta_\varepsilon) \\ &\qquad + \quad \mathbb{P}_\eta(\sup_{z\in\Lambda_{\varepsilon^{-1}+1}\cap\mathbb{Z}[\mathbf{i}]} ||\mu^\eta_{1,A(\varepsilon z;3\varepsilon/2,3\varepsilon)}||_{TV} > \varepsilon^\varphi/2a^\eta_\varepsilon). \end{aligned} \tag{5.32}$$

By the Markov inequality, we have

$$\mathbb{P}_\eta(|\mathrm{Piv}^\eta(\varepsilon,\varepsilon^\beta)| \ge a^\eta_\varepsilon) \le C_1\varepsilon^\varphi \tag{5.33}$$

for some positive constant $C_1 = C_1(\delta)$ for all $\varphi>0$.

Now we bound the third term in (5.32). With some positive constants C_2, C_3, C_4

depending on δ we have

$$\begin{aligned}
\mathbb{P}_\eta(\sup_{z\in\Lambda_{\varepsilon^{-1}+1}\cap\mathbb{Z}[\mathbf{i}]} ||\mu^\eta_{1,A(\varepsilon z;3\varepsilon/2,3\varepsilon)}||_{TV} > \varepsilon^\varphi/2a^\eta_\varepsilon) \\
&\leq C_2\varepsilon^{-2}\mathbb{P}_\eta(||\mu^\eta_{1,A(3\varepsilon/2,3\varepsilon)}||_{TV} > \varepsilon^\varphi/2a^\eta_\varepsilon) \\
&= C_2\varepsilon^{-2}\mathbb{P}_\eta(|\mathcal{V}^\eta_{3\varepsilon}| \geq \varepsilon^\varphi\eta^{-2}\pi^\eta_1(\eta,1)/2a^\eta_\varepsilon) \\
&\leq C_2\varepsilon^{-2}\exp\left(-C_3\varepsilon^{2\varphi}\frac{\pi^\eta_1(\eta,1)}{\pi^\eta_1(\eta,3\varepsilon)\pi^\eta_4(3\varepsilon/2,\varepsilon^\beta)}\right) \qquad (5.34)\\
&\leq C_2\varepsilon^{-2}\exp\left(-C_4\varepsilon^{2\varphi}\frac{\pi^\eta_1(3\varepsilon,\varepsilon^\beta)}{\pi^\eta_4(3\varepsilon/2,\varepsilon^\beta)}\pi^\eta_1(\varepsilon^\beta,1)\right),
\end{aligned}$$

where in the second inequality we used Lemma 5.3.14 and in the last line we used Lemma 5.3.8 twice. Lemmas 5.3.9 and 5.3.12, (5.34) and the choice of β give that

$$\begin{aligned}
\mathbb{P}_\eta(\sup_{z\in\Lambda_{\varepsilon^{-1}+1}\cap\mathbb{Z}[\mathbf{i}]} ||\mu^\eta_{1,A(\varepsilon z;3\varepsilon/2,3\varepsilon)}||_{TV} > \varepsilon^\varphi/2a^\eta_\varepsilon) &\leq C_2\varepsilon^{-2}\exp(-C_5\varepsilon^{2\varphi+\lambda(\beta-1)+\lambda_1\beta}) \\
&= C_2\varepsilon^{-2}\exp(-C_5\varepsilon^{2\varphi-\lambda/2}) \qquad (5.35)
\end{aligned}$$

with $C_5 > 0$. Computations similar to those above give the following upper bound for the second term in (5.32):

$$\begin{aligned}
\mathbb{P}_\eta(||\nu^\eta_{\varepsilon^\beta}||_{TV} \geq \frac{1}{2}\varepsilon^\varphi) &\leq C_6\varepsilon^{-\beta}\exp\left(-C_7\varepsilon^{\varphi-\beta}\frac{\pi^\eta_1(\eta,1)}{\pi^\eta_1(\eta,\varepsilon^\beta)}\right) \\
&\leq C_6\varepsilon^{-\beta}\exp\left(-C_8\varepsilon^{\varphi-\beta+\beta\lambda_1}\right) \qquad (5.36)
\end{aligned}$$

for suitable constants C_6, C_7, C_8. We set $\varphi = \frac{\lambda\wedge(\beta(1-\lambda_1))}{4} > 0$. A combination of (5.32), (5.33), (5.35) and (5.36) finishes the proof of Lemma 5.7.4. □

5.8 Properties of the continuum clusters and their normalized counting measures

We start with the connections between the clusters and their counting measures. The first result of the section shows, roughly speaking, that the scaling limit of the clusters as closed sets contains the same information as their normalized counting measures. Then we show conformal invariance of the clusters and conformal covariance of their normalized counting measures.

5.8.1 Basic properties

Recall the notation $\mathscr{C}^\eta(\delta)$ from (5.2). We set $\mathscr{C}^0 = \bigcup_{n=1}^\infty \mathscr{C}^0(3^{-n})$. For $\mathcal{C} \in \mathscr{C}^0$ and $\psi > 0$ we write

$$\tilde{\mu}^0_{\mathcal{C},\psi} := \frac{4\psi^2}{\pi^0_1(2\psi,1)} \sum_{z\in\mathbb{Z}[\mathbf{i}]:\Lambda_{\psi/2}(\psi z)\cap\mathcal{C}\neq\emptyset} \delta_{\psi z}. \qquad (5.37)$$

Theorem 5.8.1. *Suppose that Assumptions I-IV hold. Then* $\mathrm{supp}(\mu^0_{\mathcal{C}}) = \mathcal{C}$ *for all* $\mathcal{C} \in \mathscr{C}^0$*. Moreover,*

$$\tilde{\mu}^0_{\mathcal{C},\psi} \to \mu^0_{\mathcal{C}} \text{ weakly in probability as } \psi \to 0 \tag{5.38}$$

for all $\mathcal{C} \in \mathscr{C}^0$*.*

The proof of the theorem above relies on the following two lemmas.

Lemma 5.8.2. *Assume that Assumptions I - III hold. Let* $k, \delta > 0$*. Then for all* $\varphi > 0$ *there is* $x_\varphi = x_\varphi(k,\delta) > 0$ *so that*

$$\mathbb{P}_\eta(\exists \mathcal{C} \in \mathscr{B}^\eta_k(\delta) \text{ with } ||\mu^\eta_{\mathcal{C}}||_{TV} < x_\varphi) < \varphi \tag{5.39}$$

for all $\eta \in (0,\delta)$*.*

Proof of Lemma 5.8.2. For critical percolation the proof of Lemma 5.8.2 follows from the proof of Theorem 3.1.2: (3.34) with $x = 0$ can be shown in the same manner as for $x > 0$. Alternatively, Lemma 5.8.2 can be deduced from a combination of [17, Lemma 4.4 and part i) of Theorem 3.1 and 3.3].

It is easy to verify that actually all these arguments just need Assumptions I - III. □

The second is essentially [38, Proposition 4.13] see also [38, Eqn. (4.39)]. Let A be the annulus $A = A(a,b)$ with $0 < a < b$ and $\mathcal{C} \in \mathscr{C}^0$. For $\eta \geq 0$ and $\psi > 0$ we set

$$\tilde{\mu}^\eta_{A,\psi} := \frac{4\psi^2}{\pi^\eta_1(2\psi,1)} \sum_{z \in \mathbb{Z}[\mathbf{i}] \cap \Lambda_{\psi^{-1}a}} \mathbf{1}\{\Lambda_{\psi/2}(\psi z) \overset{1}{\longleftrightarrow} \partial\Lambda_b\}\delta_{\psi z}.$$

Lemma 5.8.3 (Proposition 4.13 of [38])**.** *Suppose that Assumptions I-IV hold. Let* $f : \mathbb{C} \to \mathbb{R}$ *be a continuous function with compact support, and* $A = A(a,b)$ *an annulus with* $0 < a < b$*. Then*

$$\tilde{\mu}^0_{A,\psi}(f) \to \mu^0_A(f) \text{ in } L^2 \text{ as } \psi \to 0. \tag{5.40}$$

Remark 5.8.4. For the proof of Theorem 5.8.1 convergence in probability is enough in (5.40).

Proof of Theorem 5.8.1. Since $\mathscr{C}^0 = \bigcup_{n=1}^\infty \mathscr{C}^0(3^{-n})$ and $\mathscr{C}^0(3^{-n}) = \bigcup_{k\in\mathbb{N}} \mathscr{C}^0_k(3^{-n})$, it is enough to show the required equalities hold with probability 1 for all $\mathcal{C} \in \mathscr{C}^0_k(\delta)$ for any fixed $\delta > 0$ and $k \in \mathbb{N}$. We will work under a coupling $\mathbb{P}$ such that $\omega_\eta \to \omega_0$ a.s.

The proofs of Theorems 5.5.1 and 5.7.2 show that $\mathrm{supp}(\mu^0_{\mathcal{C}}) \subseteq \mathcal{C}$ for all $\mathcal{C} \in \mathscr{C}^0(\delta)$ with probability 1. We turn to the proof of $\mathrm{supp}(\mu^0_{\mathcal{C}}) \supseteq \mathcal{C}$. Let $\varphi > 0$ and x_φ as in Lemma 5.8.2. By covering Λ_k with at most $4(k/\varepsilon)^2$ squares with side length ε we get

$$\begin{aligned}\mathbb{P}_\eta(\exists z \in \mathbb{Z}[\mathbf{i}], \exists \mathcal{C} \in \mathscr{C}^\eta(\delta) \text{ s.t. } \Lambda_{\varepsilon/2}(\varepsilon z) \cap \mathcal{C} \neq \emptyset \text{ and } \mu^\eta_{\mathcal{C}}(\Lambda_\varepsilon(\varepsilon z)) < x_\varphi)& \\ \leq 4(k/\varepsilon)^2 \mathbb{P}_\eta(\exists \mathcal{B} \in \mathscr{B}^\eta_{\Lambda_\varepsilon}(\varepsilon/2) \text{ with } ||\mu^\eta_{\mathcal{B}}||_{TV} < x_\varphi)& \\ \leq 4(k/\varepsilon)^2\varphi.&\end{aligned} \tag{5.41}$$

By Theorem 5.7.2 we have that $\underline{\mu}^\eta(\delta) \xrightarrow{p} \underline{\mu}^0(\delta)$ in the metric d_l for all $\delta > 0$ as $\eta \to 0$. This combined with the tightness of $|\mathscr{C}_k^0(\delta)|$, (5.41) and the Portmanteau theorem gives that

$$\mathbb{P}_0(\exists z \in \mathbb{Z}[\mathbf{i}], \exists \mathcal{C} \in \mathscr{C}_k^0(\delta) \text{ s.t. } \Lambda_{\varepsilon/2}(\varepsilon z) \cap \mathcal{C} \neq \emptyset \text{ and } \mu_{\mathcal{C}}^0(\Lambda_\varepsilon(\varepsilon z)) < x_\varphi) \\ \leq 4(k/\varepsilon)^2 \varphi \quad (5.42)$$

for all $\varepsilon \in (0, \delta/10)$. We take the limit $\varphi \to 0$ in (5.42) and get

$$\mathbb{P}_0(\exists z \in \mathbb{Z}[\mathbf{i}], \exists \mathcal{C} \in \mathscr{C}_k^0(\delta) \text{ s.t. } \Lambda_{\varepsilon/2}(\varepsilon z) \cap \mathcal{C} \neq \emptyset \text{ and } \mu_{\mathcal{C}}^0(\Lambda_\varepsilon(\varepsilon z)) = 0) = 0, \quad (5.43)$$

which shows that $\text{supp}(\mu_{\mathcal{C}}^\eta) + \Lambda_\varepsilon \supseteq \mathcal{C}$ for all $\mathcal{C} \in \mathscr{C}_k^0(\delta)$ with probability 1 for each fixed $\varepsilon > 0$. Thus $\text{supp}(\mu_{\mathcal{C}}^0) \supseteq \mathcal{C}$ for all $\mathcal{C} \in \mathscr{C}^0$ with probability 1, and finishes the proof of the first statement of Theorem 5.8.1.

Since the proof of (5.38) is analogous to that of Lemma 5.7.4, we only give a sketch. Let $\delta, \varepsilon > 0$, $\mathcal{C} \in \mathscr{C}^0(\delta)$ and $f : \mathbb{C} \to \mathbb{R}$ be a continuous function with compact support. Recall the definition of $\mu_{\mathcal{C},\varepsilon}^0$ from the lines above Lemma 5.7.4. We set

$$\bar{\mu}_{\mathcal{C},\varepsilon,\psi}^0 := \sum_{z \in \mathbb{Z}[\mathbf{i}] : \Lambda_{3\varepsilon/2}(\varepsilon z) \cap \mathcal{C} \neq \emptyset} \tilde{\mu}_{A(\varepsilon z, \varepsilon/2, \delta/2 - \varepsilon), \psi}^0.$$

Note that when we replace $\mu_{A(\varepsilon z, \varepsilon/2, \delta/2-\varepsilon)}^0$ by $\tilde{\mu}_{A(\varepsilon z, \varepsilon/2, \delta/2-\varepsilon),\psi}^0$ in the definition of $\mu_{\mathcal{C},\varepsilon}^0$, we arrive to the measure $\bar{\mu}_{\mathcal{C},\varepsilon,\psi}^0$. Thus for any fixed $\varepsilon > 0$ Lemma 5.8.3 shows that $\bar{\mu}_{\mathcal{C},\varepsilon,\psi}^0(f)$ and $\mu_{\mathcal{C},\varepsilon}^0(f)$ are close to each other in L^2 when ψ is small. In particular, $\bar{\mu}_{\mathcal{C},\varepsilon,\psi}^0 \to \mu_{\mathcal{C},\varepsilon}^0$ weakly in probability as $\psi \to 0$.

Arguments similar to those in the proof of Lemma 5.7.4 give that $\tilde{\mu}_{\mathcal{C},\psi}^0$ and $\bar{\mu}_{\mathcal{C},\varepsilon,\psi}^0$ are close to each other in total variation distance (hence in Prokhorov distance as well) with high probability when ψ and ε are both small.

By the proof of Theorem 5.7.2, $\mu_{\mathcal{C},\varepsilon}^0$ is close to $\mu_{\mathcal{C}}^0$ in Prokhorov distance when ε is small with high probability. Thus

$$\tilde{\mu}_{\mathcal{C},\psi}^0 \approx \bar{\mu}_{\mathcal{C},\varepsilon,\psi}^0 \xrightarrow{\psi \to 0} \mu_{\mathcal{C},\varepsilon}^0 \xrightarrow{\varepsilon \to 0} \mu_{\mathcal{C}}^0,$$

where the limits are in Prokhorov metric in probability, and $\tilde{\mu}_{\mathcal{C},\psi}^0 \approx \bar{\mu}_{\mathcal{C},\varepsilon,\psi}^0$ means that the Prokhorov distance of these measures is small with high probability when ε and ψ are both small. Thus (5.38) follows, and Theorem 5.8.1 is proved. □

5.8.2 Conformal invariance and covariance

In this section we prove Theorem 5.1.4 and the stronger conformal covariance of Bernoulli percolation clusters as stated in Theorem 5.2.2.

Let us first restrict ourselves to critical site percolation on the triangular lattice. At the end of this section we will show how to obtain the weaker invariance of Theorem 5.1.4 from our general assumptions.

Recall Definition 5.3.4 of the restriction of a configuration to a bounded domain D.

Theorem 5.8.5. *Let, for $\eta \geq 0$, $\mathbb{P}_\eta$ denote the measure for critical site percolation on the triangular lattice. Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ be a conformal map. The laws of $(f(\omega_{0,D}), f(L_{0,D}))$ and $(\omega_{0,f(D)}, L_{0,f(D)})$ coincide.*

The conformal invariance of the continuum loop process was proved in [22, Theorem 3, item 4]. The conformal invariance of the quad crossings, follows immediately because of the measurability with respect to the loop process.

The construction of the continuum clusters and their measures was obtained in Sections 5.4 - 5.7 by approximating the cluster by boxes $\Lambda_{\varepsilon/2}(z)$. In order to prove conformal invariance / covariance we would like to approximate the clusters by conformally transformed boxes $f(\Lambda_{\varepsilon/2}(z))$. More precisely, let $\phi > 0$ and $f : \Lambda_{1+\phi} \to \hat{\mathbb{C}}$ be a conformal map. We set $D = f(\Lambda_1)$ and $D' := f(\Lambda_{1+\phi})$. Let d_f denote the push-forward of the L^∞ metric on $\Lambda_{1+\phi}$. That is,

$$d_f(x, y) := ||f^{-1}(x) - f^{-1}(y)||_\infty$$

for $x, y \in D'$. Note that f is defined in an open neighbourhood of Λ_1 because when we approximate the cluster measures using one arm measures, we need to consider annuli whose inner square is contained in Λ_1 but which are not completely contained in Λ_1.

Clearly, $(\Lambda_{1+\phi}, d_\infty)$ and (D', d_f) are isomorphic as metric spaces. Thus all the geometric constructions in Section 5.4 can be repeated for the clusters in D just by applying the map f. We denote these analogues of the objects by an additional 'f' subscript. Thus all the statements apart from those in Section 5.4.1 remain valid if we keep the constants such as ε, δ unchanged, but add an additional subscript f in the objects appearing in the claims. Moreover, the bounds in Section 5.4.1 remain valid asymptotically, as $\eta \to 0$, if we use the transformed boxes $f(\Lambda_{\varepsilon/2}(z))$ to define the relevant events because of the conformal invariance of the scaling limit.

Next note that there is a positive constant $K = K(f)$ such that $|f'(u)| \in [1/K, K]$ for $u \in \Lambda_{1+\phi/2}$. Thus d_f and the L^∞-metric are equivalent on D. As above, we add a subscript 'f' for the metrics built from d_f. Thus $d_{H,f}$ and $d_{P,f}$ are equivalent to d_H and d_P respectively, where $d_{H,f}$ and $d_{P,f}$ are built on d_f.

We can obtain the clusters in D in two ways: via the square boxes $\Lambda_{\varepsilon/2}(z)$, that is, using the metric L^∞ in D, or via the transformed boxes $f(\Lambda_{\varepsilon/2}(z))$, that is, using the metric d_f. The equivalence of the metrics implies that these two approximations provide the same continuum clusters in the scaling limit.

Now notice that the scaling limit in D in terms of quad crossings is distributed like the image under f of the scaling limit in Λ_1, because of the conformal invariance of quad crossing configurations. This implies that the construction in D, using the transformed boxes $f(\Lambda_{\varepsilon/2}(z))$, gives clusters that have the same distribution as the images of the continuum clusters in Λ_1. This proves the following theorem.

Theorem 5.8.6. *For $\eta \geq 0$, let $\mathbb{P}_\eta$ denote the measure for critical site percolation on the triangular lattice. Let $\phi > 0$, $f : \Lambda_{1+\phi} \to \hat{\mathbb{C}}$ be a conformal map, and $D := f(\Lambda_1)$.*

Then the laws of $\mathscr{B}^0_D$ and $f(\mathscr{B}^0_{\Lambda_1})$ are identical, where

$$f(\mathscr{B}^0_{\Lambda_1}) := \{f(\mathcal{B}) : \mathcal{B} \in \mathscr{B}^0_{\Lambda_1}\}.$$

In addition to the convergence of arm measures, Garban, Pete and Schramm also proved in [38] the conformal covariance of these measures. They prove the following theorem, which is Theorem 6.7 in their paper.

Theorem 5.8.7. *For $\eta \geq 0$, let $\mathbb{P}_\eta$ denote the measure for critical site percolation on the triangular lattice. Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ be a conformal map. Let $A \subset \mathbb{C}$ be a proper annulus with piece-wise smooth boundary with $\overline{A} \subset D$. For a Borel set $B \subseteq f(D)$, let*

$$\mu_{1,A}^{0*}(B) := \int_{f^{-1}(B)} |f'(z)|^{2-\alpha_1} d\mu_{1,A}^0(z).$$

Then the law of $\mu_{1,f(A)}^0 = \mu_{1,f(A)}^0(\omega_{0,f(D)})$ and $\mu_{1,A}^{0*} = \mu_{1,A}^{0*}(\omega_{0,D})$ coincide.

The arguments in the proof of Lemma 5.7.4 imply that approximating the cluster measures by one-arm measures of annuli of the form $f(\Lambda_{\delta/2} \setminus \Lambda_{\varepsilon/2})$ provides the same limit as approximating the cluster measures, in D, by one-arm measures of annuli of the form $\Lambda_{\delta/2} \setminus \Lambda_{\varepsilon/2}$. This observation and Theorem 5.8.7 imply the following result, where $\tilde{\mathscr{M}}_D^0$ denotes the collection of measures of all clusters in $\mathscr{B}_D^0$.

Theorem 5.8.8. *For $\eta \geq 0$, let $\mathbb{P}_\eta$ denote the measure for critical site percolation on the triangular lattice. Let $\phi > 0$, $f : \Lambda_{1+\phi} \to \hat{\mathbb{C}}$ be a conformal map, and $D := f(\Lambda_1)$. Then the laws of $\tilde{\mathscr{M}}_D^0$ and $f(\tilde{\mathscr{M}}_{\Lambda_1}^0)$ are identical, where, with the notation of Theorem 5.8.7,*

$$f(\tilde{\mathscr{M}}_{\Lambda_1}^0) := \{\mu^{0*} \,:\, \mu^0 \in \tilde{\mathscr{M}}_{\Lambda_1}^0\}.$$

We are now ready to give the proofs of two of our main results, Theorems 5.2.2 and 5.1.4.

Proof of Theorem 5.2.2. This is a combination of Theorems 5.8.6 and 5.8.8. □

Proof of Theorem 5.1.4. Note that it is sufficient to prove that the pairs $(f(\mathscr{C}^0), f(\mathscr{M}^0))$ and $(\mathscr{C}^0, \mathscr{M}^0)$ have the same distribution. This follows from a straightforward modification of the arguments above. Namely, the rotation and translation invariance and scaling covariance of the 1-arm measures under the general Assumptions I - IV follows easily from the proof of Theorem 5.3.19. See also [38, Equation (6.1) and Proposition 6.4]. □

5.9 Proof of the convergence of the largest Bernoulli percolation clusters

Now we turn to the precise version and to the proof of Theorem 5.2.1.

Theorem 5.9.1. *Let $\mathbb{P}$ be a coupling where $(\omega_\eta, L_\eta) \to (\omega_0, L_0)$ a.s. as $\eta \to 0$. Then for all $i \in \mathbb{N}$ the i-th largest cluster $\mathcal{M}_{(i)}^\eta$ converges in $\mathbb{P}$-probability to $\mathcal{M}_{(i)}^0$ as $\eta \to 0$, where $\mathcal{M}_{(i)}^0$ is a measurable function of (ω_0, L_0). In particular, $(\omega_\eta, L_\eta, \mathcal{M}_{(i)}^\eta) \to (\omega_0, L_0, \mathcal{M}_{(i)}^0)$ in distribution. The same convergence holds for the measures $\mu_{\mathcal{M}_{(i)}^\eta}^\eta$.*

Let us start with some preliminary results. Recall the definition of collections of (portions of) clusters $\mathscr{B}^{\eta}_{\Lambda_1}(\delta)$ in Section 5.6.

Proposition 5.9.2. *Let $\delta \in (0,1)$. For all $\varphi > 0$ there exist $\eta_0, \alpha > 0$ such that for all $\eta < \eta_0$:*

$$\mathbb{P}_\eta(\exists \mathcal{B}, \mathcal{B}' \in \mathscr{B}^{\eta}_{\Lambda_1}(\delta) : \mathcal{B} \neq \mathcal{B}' : |\mu^{\eta}_{\mathcal{B}}(\Lambda_1) - \mu^{\eta}_{\mathcal{B}'}(\Lambda_1)| < \alpha) < \varphi.$$

Proof of Proposition 5.9.2. In Chapter 3 a proof for Proposition 5.9.2 was given for bond percolation on the square lattice, however the proof also works for other models, like site percolation on the triangular lattice as noted in remark (i) after Theorem 3.1.1. □

The following lemma is a complement of Lemma 5.3.14.

Lemma 5.9.3 (Lemma 4.4 of [17])**.** *There are positive constants c, C such that for all $x, y > 0$*

$$\mathbb{P}_\eta(\exists \mathcal{B} \in \mathscr{B}^{\eta}_{\Lambda_1} : |\mu^{\eta}_{\mathcal{B}}(\Lambda_1)| > x \textit{ and } \operatorname{diam}(\mathcal{B}) < y) < C y^{-1} \exp(-cx/\sqrt{y})$$

for all $\eta < \eta_0 = \eta_0(x, y)$.

The next proposition follows easily from a combination of Lemma 5.9.3 and [17, Theorem 3.1, 3.3 and 3.6]. See also Corollary 3.2.4.

Proposition 5.9.4. *Let $i \in \mathbb{N}$. For all $\varphi > 0$ there exist $\delta > 0, \eta_0 > 0$ such that for all $\eta < \eta_0$:*

$$\mathbb{P}_\eta(\exists j \leq i : \mathcal{M}^{\eta}_{(j)} \notin \mathscr{B}^{\eta}_{\Lambda_1}(\delta)) < \varphi.$$

Proof of Theorem 5.9.1. Let $i \in \mathbb{N}$ be fixed and $\mathbb{P}$ be a coupling such that $(\omega_\eta, L_\eta) \to (\omega_0, L_0)$ a.s. as $\eta \to 0$. First we show that the i largest clusters in the scaling limit can almost surely be defined as a function of the pair (ω_0, L_0). Then we show that the i-th largest cluster $\mathcal{M}^{\eta}_{(i)}$ in the discrete configuration ω_η converges to the i-th largest continuum cluster.

Let $m \in \mathbb{N}$. Theorems 5.6.1 and 5.7.2 show that the sequence of clusters $\underline{\mathcal{B}}^{0}_{\Lambda_1}(3^{-m})$ and their corresponding measures $\underline{\mu}^{0}(3^{-m})$ are a.s. well defined.

We define the *volume* of a continuum cluster $\mathcal{B} \in \mathscr{B}^{0}_{\Lambda_1}$ as $\mu^{0}_{\mathcal{B}}(\Lambda_1)$. Lemma 5.3.14 shows that the volumes of the clusters $\mathcal{B} \in \mathscr{B}^{0}_{\Lambda_1}(3^{-m})$ are a.s. finite. Moreover, Lemma 5.5.3, together with the tightness of the number of excursions in Λ_1, of diameter at least 3^{-m}, gives that $h^0(3^{-m}) := |\mathscr{B}^{0}_{\Lambda_1}(3^{-m})|$ is a.s. finite. Thus we can reorder the sequence of clusters $\underline{\mathcal{B}}^{0}(3^{-m})$ in decreasing order by their volume. We break ties in some deterministic way. However, we will see below that ties occur with probability 0. Let $\mathcal{M}^{0}_{(j)}(3^{-m})$ denote the j-th cluster in this new ordering.

Let $\varphi > 0$ arbitrary. Take α and η_0 as in Proposition 5.9.2. Then, for $\eta < \eta_0$

$$\begin{aligned}
&\mathbb{P}(\exists \mathcal{B}, \mathcal{B}' \in \mathscr{B}^{0}_{\Lambda_1}(3^{-m}) : \mathcal{B} \neq \mathcal{B}', |\mu^{0}_{\mathcal{B}}(\Lambda_1) - \mu^{0}_{\mathcal{B}'}(\Lambda_1)| < \alpha/2) \\
&\qquad \leq \mathbb{P}(\exists \mathcal{B}, \mathcal{B}' \in \mathscr{B}^{\eta}_{\Lambda_1}(3^{-m}) : \mathcal{B} \neq \mathcal{B}' : |\mu^{\eta}_{\mathcal{B}}(\Lambda_1) - \mu^{\eta}_{\mathcal{B}'}(\Lambda_1)| < \alpha) \\
&\qquad\quad + \mathbb{P}(\exists j \leq h^0(3^{-m}) : |\mu^{\eta}_{\mathcal{B}^{\eta}_j}(\Lambda_1) - \mu^{0}_{\mathcal{B}^{0}_j}(\Lambda_1)| > \alpha/4) \\
&\qquad \leq \varphi + \mathbb{P}(\exists j \leq h^0(3^{-m}) : |\mu^{\eta}_{\mathcal{B}^{\eta}_j}(\Lambda_1) - \mu^{0}_{\mathcal{B}^{0}_j}(\Lambda_1)| > \alpha/4).
\end{aligned} \tag{5.44}$$

The second term in the right hand side of (5.44) tends to 0 as $\eta \to 0$, since h^0 is a.s. finite and $\underline{\mu}^\eta(3^{-m}) \to \underline{\mu}^0(3^{-m})$ in probability by Theorem 5.7.2. Since $\varphi > 0$ was arbitrary, this shows that

$$\mathbb{P}(\exists \mathcal{B}, \mathcal{B}' \in \mathscr{B}^0_{\Lambda_1}(3^{-m}) : \mathcal{B} \neq \mathcal{B}', |\mu^0_{\mathcal{B}}(\Lambda_1) - \mu^0_{\mathcal{B}'}(\Lambda_1)| = 0) = 0,$$

that is, there are no ties in the ordering above with probability 1.

Now we show that, for all $j \leq i$,

$$\mathbb{P}(\exists m_0 \in \mathbb{N} \text{ s.t. } \mathcal{M}^0_{(j)}(3^{-m_0}) = \mathcal{M}^0_{(j)}(3^{-m}) \text{ for all } m \geq m_0) = 1. \tag{5.45}$$

Suppose the contrary, and let j_0 be the smallest $j \leq i$ so that (5.45) fails. Let

$$E = \{\nexists m_0 \in \mathbb{N} \text{ s.t. } \mathcal{M}^0_{(j_0)}(3_0^{-m}) = \mathcal{M}^0_{(j_0)}(3^{-m}) \text{ for all } m \geq m_0\},$$

and $\varphi = \mathbb{P}(E) > 0$.

The definition of j_0 implies that, on the event E, there is a sequence of clusters $(\tilde{\mathcal{B}}_n)_{n\geq 1} \subseteq \mathscr{B}^0_{\Lambda_1}$ so that $\operatorname{diam}(\tilde{\mathcal{B}}_n) \to 0$ as $n \to 0$, and $\mu_{\tilde{\mathcal{B}}_n}(\Lambda_1)$ is increasing. Take $\delta > 0$ so small that

$$\mathbb{P}(\exists n \in \mathbb{N} \text{ s.t. } \tilde{\mathcal{B}}_n \in \mathscr{B}^0_{\Lambda_1}(\delta), E) > \varphi/2.$$

Since $\mu_{\tilde{\mathcal{B}}_n}(\Lambda_1)$ is increasing, the equation above combined with Lemma 5.8.2 shows that there is $x > 0$ so that

$$\mathbb{P}(\lim_{n\to\infty} \mu_{\tilde{\mathcal{B}}_n}(\Lambda_1) > x, E) > \varphi/4.$$

Since $\operatorname{diam}(\tilde{\mathcal{B}}_n) \to 0$ as $n \to 0$, the above implies that there are deterministic sequences δ_n, δ'_n tending to 0 as $n \to 0$ so that

$$\mathbb{P}(\exists \mathcal{B} \in \mathscr{B}^0_{\Lambda_1} \text{ with } \mu^0_{\mathcal{B}}(\Lambda_1) > x/2 \text{ and } \operatorname{diam}(\mathcal{B}) \in (\delta_n, \delta'_n)) \geq \varphi/8.$$

Theorem 5.5.1 and 5.7.2 implies that for all $n \geq 0$ there is $\eta_0(n)$ so that

$$\mathbb{P}(\exists \mathcal{B} \in \mathscr{B}^\eta_{\Lambda_1} \text{ with } \mu^\eta_{\mathcal{B}}(\Lambda_1) > x/4 \text{ and } \operatorname{diam}(\mathcal{B}) \in (\delta_n/2, 2\delta'_n)) \geq \varphi/16,$$

for all $\eta \leq \eta_0(n)$. Since $\delta_n \to 0$ as $n \to 0$, taking n large enough, we get a contradiction with Lemma 5.3.14. Hence (5.45) is proved, and for each $j \leq i$ we set $\mathcal{M}^0_{(j)} := \mathcal{M}^0_{(j)}(3^{-m_0})$ where m_0 as in the event on the left hand side of (5.45).

It remains to show that $\mathcal{M}^\eta_{(i)}$ converges in probability to $\mathcal{M}^0_{(i)}$ as well as their measures. Let $\varepsilon, \alpha > 0$ and $m > 0$, first we check that

$$\begin{aligned}
\mathbb{P}(d_H(\mathcal{M}^\eta_{(i)}, \mathcal{M}^0_{(i)}) > \varepsilon)& \\
\leq \mathbb{P}(\exists j \leq i : \mathcal{M}^0_{(j)} \neq \mathcal{M}^0_{(j)}(3^{-m}))& \\
+ \mathbb{P}(\exists j \leq i : \mathcal{M}^\eta_{(j)} \neq \mathcal{M}^\eta_{(j)}(3^{-m}))& \\
+ \mathbb{P}(\exists \mathcal{B}, \mathcal{B}' \in \mathscr{B}^0_{\Lambda_1}(3^{-m}) : \mathcal{B} \neq \mathcal{B}', |\mu^0_{\mathcal{B}}(\Lambda_1) - \mu^0_{\mathcal{B}'}(\Lambda_1)| < \alpha)& \\
+ \mathbb{P}(\exists \mathcal{B}, \mathcal{B}' \in \mathscr{B}^\eta_{\Lambda_1}(3^{-m}) : \mathcal{B} \neq \mathcal{B}', |\mu^\eta_{\mathcal{B}}(\Lambda_1) - \mu^\eta_{\mathcal{B}'}(\Lambda_1)| < \alpha)& \\
+ \mathbb{P}(\exists k \leq h^0(3^{-m}) : |\mu^\eta_{\mathcal{B}^\eta_k}(\Lambda_1) - \mu^0_{\mathcal{B}^0_k}(\Lambda_1)| > \alpha/3)& \\
+ \mathbb{P}(\exists k \leq h^0(3^{-m}) : d_H(\mathcal{B}^\eta_k, \mathcal{B}^0_k) > \varepsilon),&
\end{aligned} \tag{5.46}$$

where $\mathcal{B}_k^\eta$ and $\mathcal{B}_k^0$ are the k-th cluster in the ordering used in the proofs of Theorem 5.5.1 and 5.7.2 of the clusters in $\mathscr{B}_{\Lambda_1}^\eta(3^{-m})$ and in $\mathscr{B}_{\Lambda_1}^0(3^{-m})$, respectively.

We justify (5.46) as follows. On the complement of the first two events on the right hand side of (5.46), all of the i largest clusters at scale η and 0 (i.e in the scaling limit) have diameter at least 3^{-m}. On the complement of the third and fourth event on the right hand side of (5.46), the normalized volumes of the different clusters with diameter at least 3^{-m} are at least α apart at both scales η and 0. Thus on the complement of the first five events on the right hand side of (5.46) the ordering according to their volume of the k largest clusters at scale η and 0 coincide, that is for all $j \leq i$, there is a unique $k_j \leq h^0(3^{-m})$ so that $\mathcal{M}_{(j)}^\eta = \mathcal{B}_{k_j}^\eta$ and $\mathcal{M}_{(j)}^0 = \mathcal{B}_{k_j}^0$. This together with the last term in the right hand side of (5.46) proves (5.46).

Let $\varphi > 0$ arbitrary. By (5.45) and Proposition 5.9.4, we find m and $\eta_0 > 0$ such that the first and second term on the right hand side of (5.46) are less than $\varphi/6$ for all $\eta < \eta_0$. Then we use the bounds in (5.44) and Proposition 5.9.2 and find $\alpha, \eta_1 > 0$ so that the third and fourth term on the right hand side of (5.46) are less than $\varphi/6$ for all $\eta < \eta_1$. Finally, we apply Theorem 5.5.1 for the fifth term and Theorem 5.7.2 for the sixth term and deduce that $\limsup_{\eta\to 0} \mathbb{P}(d_H(\mathcal{M}_{(i)}^\eta, \mathcal{M}_{(i)}^0) > \varepsilon) < \varphi$. Since φ and ε were arbitrary, this shows that $\mathcal{M}_{(i)}^\eta \to \mathcal{M}_{(i)}^0$ in probability as $\eta \to 0$.

The proof for the convergence of normalized counting measures goes in a similar way: notice that if we replace the fifth term on the right hand side of (5.46) with

$$\mathbb{P}(\exists j \leq h^0(3^{-m}) : d_p(\mu^\eta_{\mathcal{B}_j^\eta}, \mu^0_{\mathcal{B}_j^0}) > \alpha/3),$$

then we get an upper bound for the probability $\mathbb{P}(\exists j \leq i : d_P(\mu^\eta_{\mathcal{M}_{(j)}^\eta}, \mu^0_{\mathcal{M}_{(j)}^0}) > \alpha/3)$. This completes the proof of Theorem 5.9.1. □

6 Factorization formulas for percolation

This chapter is based on [31].

We consider critical site percolation on the triangular lattice in the upper half-plane. Let u_1, u_2 be two sites on the boundary and w a site in the interior. It was predicted by Simmons, Kleban and Ziff [74] that the ratio $\mathbb{P}(nu_1 \leftrightarrow nu_2 \leftrightarrow nw)^2 \,/\, \mathbb{P}(nu_1 \leftrightarrow nu_2) \cdot \mathbb{P}(nu_1 \leftrightarrow nw) \cdot \mathbb{P}(nu_2 \leftrightarrow nw)$ converges to K_F as $n \to \infty$, where $x \leftrightarrow y$ denotes that x and y are in the same cluster, and K_F is a constant. Beliaev and Izyurov [9] proved an analog of this in the scaling limit. We prove, using their result and a generalized coupling argument, the earlier mentioned prediction. Furthermore we prove a factorization formula for $\mathbb{P}(nu_2 \leftrightarrow [nu_1, nu_1 + s]; nw \leftrightarrow [nu_1, nu_1 + s])$, where $s > 0$.

6.1 Introduction and Main results.

We consider critical site percolation on the triangular lattice. See [41] for a general introduction and [80, 82] for more recent progress in two-dimensional percolation. A lot of attention has been given to crossing probabilities and critical exponents, which are believed to be universal. In particular it is believed that in the continuum limit of many two-dimensional critical percolation models, crossing probabilities are conformally invariant. However this has only been proved for site percolation on the triangular lattice by Smirnov [77]. Another interesting question is whether it is possible to examine the higher order correlation functions. These are the functions $\mathbb{E}[X_{v_1} X_{v_2} \cdots X_{v_n}]$, where v_i is a vertex and $X_{v_i} = \mathbf{1}\{0 \leftrightarrow v_i\}$ is the indicator function of the event that v_i is in the open cluster of the origin. A possible approach to compute these correlation functions might be via factorization formulas.

To state our main results we consider the hexagonal lattice, where every center of a hexagon is a site of the triangular lattice $\mathbb{T}$ in the closure of the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$. In this lattice two neighbouring sites $x, y \in \mathbb{T}$ have $|x - y| = 1$. By $\mathbb{P}_\eta$ we denote the probability measure of critical percolation on $\eta\mathbb{T}$, for $\eta > 0$. Let $\eta > 0$ and let the random set $Q \subset \overline{\mathbb{H}}$ be the union of all hexagons for which the center is open. The points $u, v \in \overline{\mathbb{H}}$ are connected if u, v are in the same connected component of Q. We denote this by $u \leftrightarrow v$. Let, for $u \in \eta\mathbb{T}$, $\mathcal{C}(u)$ denote the open cluster containing u. Let, for $A \subset \overline{\mathbb{H}}$,

$$\mathcal{C}(A) := \bigcup_{u \in A \cap \eta \mathbb{T}} \mathcal{C}(u).$$

Further we will denote the hypergeometric function by ${}_2F_1(a,b;c;d)$ (see for example [1]). We denote by $\mathbb{S} := \{z \in \mathbb{C} : \Im(z) \in (0,1), \Re(z) > 0\}$ the semi-infinite strip.

Our first main result is a factorization formula for the probability that three given vertices are in the same cluster, where two of the vertices are on the boundary of the half-plane.

Theorem 6.1.1. *Let $u_1, u_2 \in \mathbb{R}$ and $w \in \mathbb{H}$ and $u_1 \neq u_2$, then*

$$\lim_{\eta \to 0} \frac{\mathbb{P}_\eta(u_1 \leftrightarrow u_2 \leftrightarrow w)^2}{\mathbb{P}_\eta(u_1 \leftrightarrow u_2)\mathbb{P}_\eta(u_1 \leftrightarrow w)\mathbb{P}_\eta(u_2 \leftrightarrow w)} = K_F, \tag{6.1}$$

where

$$K_F = \frac{2^7 \pi^5}{3^{3/2}\Gamma(1/3)^9}.$$

This factorization formula was heuristically derived, using Conformal Field Theory arguments, by Simmons, Kleban and Ziff in [74]. Using the convergence of percolation exploration interfaces to SLE_6 (See e.g. [71, 77]), a mathematical rigorous proof of an analog of this formula in the continuum scaling limit was given by Beliaev and Izyurov in [9]. See Theorem 6.2.1 for their result. That result is the starting point in the proof of Theorem 6.1.1. To obtain Theorem 6.1.1 from it we state and prove a quite general and robust form of a coupling result for one-arm like events (see Proposition 6.3.4 in Section 6.3.1).

Our second main result involves the limiting behaviour of the probability $\mathbb{P}(\{u_2, w\} \subset \mathcal{C}([u_1, u_1 + s]))$, where u_1, u_2 are on the boundary of the half-plane and w is in the half-plane. We have the following theorem.

Theorem 6.1.2. *Let $u_1 \in \mathbb{R}, w \in \mathbb{H}, s > 0$ and $u_2 > u_1 + s$, then*

$$\lim_{\eta \to 0} \frac{\mathbb{P}_\eta(\{u_2, w\} \subset \mathcal{C}([u_1, u_1+s]))}{\mathbb{P}_\eta(w \in \mathcal{C}([u_1, u_1+s]))\ \mathbb{P}_\eta(u_2 \in \mathcal{C}([u_1, u_1+s]))} = \psi(u_1, s, u_2, w), \tag{6.2}$$

where ψ is the function

$$\psi(u_1, s, u_2, w) = e^{\pi x/3} \cdot \frac{{}_2F_1\left(-\frac{1}{2}, -\frac{1}{3}; \frac{7}{6}; e^{-2\pi x}\right)}{{}_2F_1\left(-\frac{1}{2}, -\frac{1}{3}; \frac{7}{6}; 1\right)},$$

with $x = \Re(\Psi_{u_1,s,u_2}(w))$ where Ψ_{u_1,s,u_2} is the conformal map that transforms $\{\mathbb{H}, u_1, u_1 + s, u_2\}$ to $\{\mathbb{S}, \mathbf{i}, 0, \infty\}$.

Simmons, Ziff and Kleban studied in [76] the probability in the numerator in (6.2). They used Conformal Field Theory arguments to find several predictions for formulas of the probabilities in (6.2). Theorem 6.1.2 is a discrete analog of one of their predictions (Equation (29) in Section III B of [76]).

Our interest in these factorization formulas came from the paper [9] by Beliaev and Izyurov. They rigorously proved an analog of the formula (6.2) above in the scaling limit, but with the probability $\mathbb{P}(w \in \mathcal{C}([u_1, u_1 + s]))$ replaced by $s_3^{5/48}$, see Theorem 6.2.2. However their theorem involves probabilities where the cluster does not necessarily touch w, but comes within a certain distance from it. More precisely, their formula is about the limits where first the mesh size, and secondly the above mentioned distance tends to zero.

Remark: We believe that our coupling argument, Proposition 6.3.4, is more generally applicable. For example Simmons, Ziff and Kleban also predicted in [76] a factorization formula for the probability $\mathbb{P}_\eta(u_2 \leftrightarrow w \leftrightarrow [u_1, u_1 + s])$. We hope that as soon as an analog of this result in the scaling limit has been proved, our Proposition 6.3.4 can be used to prove this factorization formula in a discrete setting. More recently Delfino and Viti heuristically derived in [33] (see also [86]) a factorization formula for the probability $\mathbb{P}(x \leftrightarrow y \leftrightarrow w)$, where all three points are in the interior of the half-plane. We also believe that Proposition 6.3.4 might be an ingredient for a rigorous proof of a discrete analog of this factorization formula, again after the scaling limit analog has been proved.

The rest of this chapter is organized as follows. In Section 6.2 we introduce some notation and sum up some preliminary results, which are crucial for our proofs. In Section 6.3.1 we state and proof a quite general and abstract ratio limit result, Proposition 6.3.4, which is based on a coupling argument. This proposition forms a key ingredient for the proofs of both main theorems. In the last Sections 6.3.2 and 6.3.3 we give the proofs of our main results.

6.2 Notation and Preliminaries.

We begin with some notation. Let $\Omega^\eta := \{0,1\}^{\eta\mathbb{T}}$. Elements of Ω^η will typically be denoted by ω, ν and called *configurations*. We call a vertex $v \in \eta\mathbb{T}$ *open* if $\omega_v = 1$, otherwise we say that v is *closed*. For two configurations $\omega, \nu \in \Omega^\eta$ we write $\omega \leq \nu$ if and only if $\omega_v \leq \nu_v$ for all $v \in \eta\mathbb{T}$. Let $P \subset \mathbb{H}$, we write $\omega_P \in \{0,1\}^{\eta\mathbb{T}\cap P}$ for the restriction of ω to the vertices which are contained in P. For two disjoint sets $P, Q \subset \mathbb{H}$, and configurations ω_P, ω_Q we define $\omega_P \times \omega_Q$ to be the configuration $\tilde{\omega}_{P\cup Q} \in \{0,1\}^{\eta\mathbb{T}\cap(P\cup Q)}$ such that $\tilde{\omega}_P = \omega_P$ and $\tilde{\omega}_Q = \omega_Q$. Let $V \subset \Omega^\eta$ be an event and $A \subset \mathbb{H}$. We define the event

$$V_A := \{\omega \,|\, \exists\, \tilde{\omega}_{\mathbb{H}\setminus A} : \omega_A \times \tilde{\omega}_{\mathbb{H}\setminus A} \in V\}. \tag{6.3}$$

Further, with some abuse of notation, for $A \subset \mathbb{H}, \omega_A \in \{0,1\}^{A\cap\eta\mathbb{T}}$ and $V \subset \Omega^\eta$ we write $\mathbb{P}_\eta(V \,|\, \omega_A)$ for the conditional probability of V given that the configuration on A equals ω_A. Similarly we write $\{\omega_A\}$ for the event that the configuration on A equals ω_A.

For $z = z_1 + z_2\mathbf{i} \in \mathbb{H}$ and $a > 0$, we write $B_a(z)$ for the intersection of the half-plane with the $2a \times 2a$-box centered at z. We denote annuli by $A(z; a, b) := B_b(z) \setminus B_a(z)$. A *circuit* in an annulus $A(z; a, b)$ is a sequence of neighbouring vertices in $\eta\mathbb{T}$, such that every vertex appears at most once in the sequence, the last vertex is a neighbour of the first and it surrounds $B_a(z)$. We will often encounter annuli which intersect the boundary of $\mathbb{H}$, in that case we will also consider *semi-circuits*. A semi-circuit

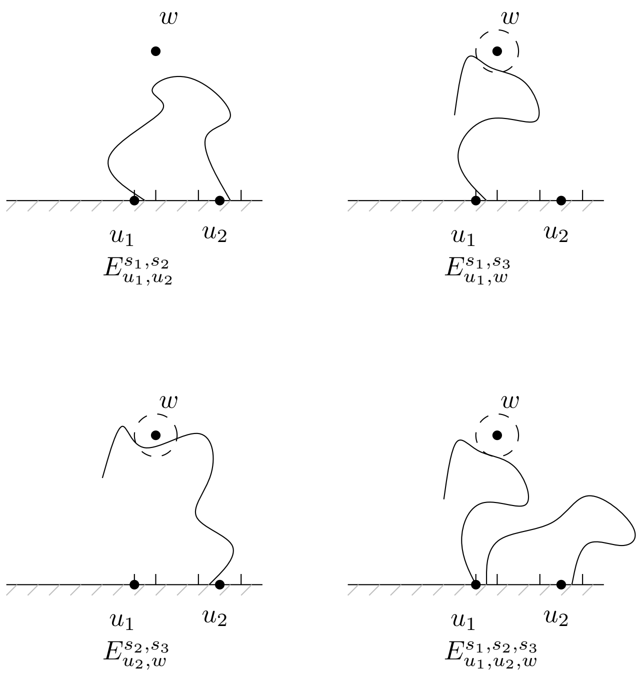

Figure 6.1: The events $E^{s_1,s_2}_{u_1,u_2}, E^{s_1,s_3}_{u_1,w}, E^{s_2,s_3}_{u_2,w}$ and $E^{s_1,s_2,s_3}_{u_1,u_2,w}$. Note that the clusters in $E^{s_1,s_2,s_3}_{u_1,u_2,w}$ might be disjoint.

in an annulus $A(z;a,b)$ is a sequence of neighbouring vertices such that every vertex appears at most once in the sequence, the first and the last vertex are both on the boundary $\partial\mathbb{H}$ and the semi-circuit 'surrounds' $B_a(z)$. In other words a semi-circuit is a path in $\mathbb{H}$ from the boundary of $\mathbb{H}$ to the boundary of $\mathbb{H}$ which disconnects $B_a(z)$ from infinity. A (semi-)circuit is called open if all its vertices are open. For a (semi-)circuit γ we denote by $int(\gamma)$ the bounded connected component of $\mathbb{H} \setminus \bar{\gamma}$ containing $B_a(z)$, where $\bar{\gamma}$ is the curve in the plane described by γ. Further $ext(\gamma)$ is the unbounded connected component of $\mathbb{H} \setminus \bar{\gamma}$.

Let $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ be the open ball of radius one. For $w \in \mathbb{H}$ and a closed connected set $A \subset \mathbb{H}$ we denote by $\rho(w, A)$ the conformal radius of the component of w in $\mathbb{H} \setminus A$ seen from w. It is defined as follows. If $w \notin A$, let V be the connected component of w in $\mathbb{H} \setminus A$. Let $\phi : V \to \mathbb{U}$ be the unique conformal map with $\phi(w) = 0$ and $\phi'(w) > 0$. Then we set $\rho(w, A) := 1/\phi'(w)$. Otherwise, if $w \in A$ we set $\rho(w, A) := 0$. We can compare the conformal radius with the euclidean distance from the point to the set, namely it follows from Koebe's 1/4-Theorem and Schwarz' Lemma that

$$\frac{1}{4}\rho(w, A) \leq \min_{x\in A} |w - x| \leq \rho(w, A). \tag{6.4}$$

(See e.g. [2])

We introduce the following events, which all represent the existence of clusters which come close to certain vertices. See Figure 6.1. For $u_1, u_2 \in \mathbb{R}$, $w \in \mathbb{H}$ and

$s_1, s_2, s_3 > 0$,

$$\begin{aligned} E^{s_1,s_2}_{u_1,u_2} &:= \{\mathcal{C}([u_1, u_1+s_1]) \cap [u_2 - s_2, u_2 + s_2] \neq \emptyset\}; \\ E^{s_1,s_3}_{u_1,w} &:= \{\rho(w, \mathcal{C}([u_1, u_1+s_1])) < s_3\}; \\ E^{s_2,s_3}_{u_2,w} &:= \{\rho(w, \mathcal{C}([u_2 - s_2, u_2+s_2])) < s_3\}; \\ E^{s_1,s_2,s_3}_{u_1,u_2,w} &:= E^{s_1,s_2}_{u_1,u_2} \cap E^{s_1,s_3}_{u_1,w}. \end{aligned} \tag{6.5}$$

Although all these events depend on η, we omit this from the notation. They represent the discrete versions of the events used by Beliaev and Izyurov in [9]. Note the difference between the events $E^{s_1,s_3}_{u_1,w}$ and $E^{s_2,s_3}_{u_2,w}$. This is to stay as close as possible to the events defined in that paper. As mentioned before Beliaev and Izyurov considered the limits, as $\eta \to 0$, of the probabilities of the events above. That is

$$\begin{aligned} f^{s_1,s_2}_{u_1,u_2} &:= \lim_{\eta\to 0} \mathbb{P}_\eta(E^{s_1,s_2}_{u_1,u_2}); \\ f^{s_1,s_3}_{u_1,w} &:= \lim_{\eta\to 0} \mathbb{P}_\eta(E^{s_1,s_3}_{u_1,w}); \\ f^{s_2,s_3}_{u_2,w} &:= \lim_{\eta\to 0} \mathbb{P}_\eta(E^{s_2,s_3}_{u_2,w}); \\ f^{s_1,s_2,s_3}_{u_1,u_2,w} &:= \lim_{\eta\to 0} \mathbb{P}_\eta(E^{s_1,s_2,s_3}_{u_1,u_2,w}). \end{aligned}$$

The existence of these limits follow from the results in [61, 77]. Namely the existence of the first one (which is actually given by Cardy's formula) was proved by Smirnov in [77]. The second and third are described in the article on the one-arm exponent for critical $2D$ percolation [61], using the so called exploration path, started at, respectively u_1+s_1 and u_2+s_2. The fourth one can also be described in terms of exploration path. It is the intersection of the events: (1) the exploration path starting at u_1+s_1 swallows u_2-s_2 before it swallows u_1 or u_2+s_2 and (2) the exploration path, or union of nested exploration paths, comes s_3 close to w in conformal radius. See [61] for the definition of the exploration path and more details.

As Beliaev and Izyurov already mentioned in [9, Remark 4], the factorization formula they proved, Proposition 4.1 in their paper, implies the following Theorem.

Theorem 6.2.1 (Remark 4 in [9]). *Let u_1, u_2, w and K_F be as in Theorem 6.1.1. For every $\varepsilon, s_0 > 0$ there exist $s_1, s_2, s_3 < s_0$ such that*

$$\left| \frac{(f^{s_1,s_2,s_3}_{u_1,u_2,w})^2}{f^{s_1,s_2}_{u_1,u_2} \cdot f^{s_1,s_3}_{u_1,w} \cdot f^{s_2,s_3}_{u_2,w}} - K_F \right| < \varepsilon. \tag{6.6}$$

The following Theorem is the main result in [9], and will be used in the proof of Theorem 6.1.2.

Theorem 6.2.2 (Theorem 1.1 in [9]). *Let u_1, u_2, w, s be as in Theorem 6.1.2. One has*

$$\lim_{s_3\to 0} \lim_{s_2\to 0} s_3^{-5/48} \cdot \frac{f^{s,s_2,s_3}_{u_1,u_2,w}}{f^{s,s_2}_{u_1,u_2}} = K_1 |\Psi'_{u_1,s,u_2}(w)|^{5/48} G\left(\Re(\Psi_{u_1,s,u_2}(w)), \Im(\Psi_{u_1,s,u_2}(w))\right), \tag{6.7}$$

where Ψ_{u_1,s,u_2} *is the conformal map that transforms* $\{\mathbb{H}, u_1, u_1+s, u_2\}$ *to* $\{\mathbb{S}, \mathbf{i}, 0, \infty\}$ *and*

$$\begin{aligned} K_1 &= \frac{18\pi^{5/48}}{5\pi \cdot 2^{5/48}} H(0)^{-1} \\ G(x,y) &= e^{\pi x/3} H(x) \sinh(\pi x)^{-1/3} \left(\frac{\sinh(\pi x)^2 \sin(\pi y)^2}{\sinh(\pi x)^2 + \sin(\pi y)^2} \right)^{11/96}, \end{aligned} \tag{6.8}$$

with

$$H(x) = {}_2F_1\left(-\frac{1}{2}, -\frac{1}{3}; \frac{7}{6}; e^{-2\pi x}\right). \tag{6.9}$$

The lemma below, proved by Beliaev and Izyurov, is an improvement of a result by Lawler, Schramm and Werner in [61].

Lemma 6.2.3 (Lemma 2.2 in [9]). *Let* u_1, w *be as in Theorem 6.1.1 and let* $s > 0$. *One has*

$$\lim_{s_3 \to 0} s_3^{-5/48} \cdot f^{s,s_3}_{u_1,w} = K_2 |\phi'(w)|^{5/48} (\sin(\pi\omega/2))^{1/3}, \tag{6.10}$$

where ω *is the harmonic measure of* (u_1, u_1+s) *seen from* w; ϕ *is a conformal map from* $\mathbb{H}$ *to the unit disc such that* $\phi(w) = 0$, *and*

$$K_2 = \frac{18}{5\pi}. \tag{6.11}$$

We end this section with a lemma which is a simple generalization of the FKG inequality.

Lemma 6.2.4. *Let* $A \subset \mathbb{H}$ *and let* B, E *be increasing events. Let* $\nu_A \in \{0,1\}^{\eta\mathbb{T}\cap A}$. *If* B *is completely determined by the vertices in* $\mathbb{H} \setminus A$, *that is* $B = B_{\mathbb{H}\setminus A}$, *then*

$$\mathbb{P}_\eta(B \cap E \cap \{\nu_A\}) \geq \mathbb{P}_\eta(B)\mathbb{P}_\eta(E \cap \{\nu_A\}).$$

Proof of Lemma 6.2.4: The proof of this lemma is straightforward and we omit it. □

6.3 Proofs of the main results.

6.3.1 Coupling of one-arm like events.

The proof of our first main result, Theorem 6.1.1, has two ingredients. The first is Theorem 6.2.1. The second ingredient for our proof is a coupling argument for one-arm like events which appeared in somewhat different forms in [54] and more recently in [38]. However our coupling result is developed in a more general framework of one-arm like events; see Definitions 6.3.1-6.3.3 below.

Our second main result, Theorem 6.1.2, also has this coupling argument as one of the main ingredients. The other main ingredients for the proof of Theorem 6.1.2 are Theorem 6.2.2 and Lemma 6.2.3.

The proof of our coupling argument is along the lines of the sketch in [38]. In that paper, among other very interesting results, a ratio limit theorem was proved. They proved that, for every $a > 0$

$$\lim_{\eta \to 0} \frac{\mathbb{P}_\eta(0 \leftrightarrow \mathbb{C} \setminus [-a,a]^2)}{\mathbb{P}_\eta(0 \leftrightarrow \mathbb{C} \setminus [-1,1]^2)} = a^{-5/48},$$

see section 5.1 in that paper. Here we show that their arguments can be modified, which makes them more generally applicable. In the arguments of [38], when a cluster comes s close to a point z it means that the cluster touches the boundary of $B_s(z)$. Hence the configuration in $B_s(z)$ is independent of the event that the cluster comes close. However, in our situation, when a cluster comes close to a vertex z it means in some occasions that the conformal radius is small and in other occasions it means that the cluster touches the interval $[z-s, z+s]$, as we saw in Section 6.2. Hence in our situation the configuration in $B_s(z)$ is not independent from the event that the cluster comes s close to z. This difference in measuring the distance of a cluster to a point makes the arguments more complicated. Our way to solve these complications is to grasp the essence which makes things work. This led us to the following formal definition of a class of events which intuitively describe the occurrence of a cluster coming within a distance s from z.

Definition 6.3.1. *Let $s, C > 0$. Let $z \in \mathbb{H}$ and $V \subset \Omega^\eta$ be an increasing event. We say that V is an (s,C)-*one-arm like event *around z if, for every (semi-)circuit γ in $A(z; s, C)$,*

$$V \begin{cases} \subset & \{B_s(z) \leftrightarrow \mathbb{H} \setminus B_C(z)\} \\ \supset & \{\gamma \textit{ open}\} \cap V_{ext(\gamma)} \cap V_{int(\gamma)} \end{cases} \tag{6.12}$$

and

$$\{I(z,s) \leftrightarrow \gamma\} \subset V_{int(\gamma)},$$

where $I(z,s)$ is the horizontal line segment $[z, z+s/8] \subset \overline{\mathbb{H}}$ and $V_{int(\gamma)}, V_{ext(\gamma)}$ as in (6.3).

For example, for every $x, s, C \in \mathbb{R}$ and $a \in [1/8, 1]$, the events $\{B_{as}(x\mathbf{i}) \leftrightarrow (x\mathbf{i} + 2C(1+\mathbf{i}))\}$ and $\{I(x,s) \leftrightarrow \mathbb{H} \setminus B_{2C}(x)\}$ are (s,C)-one-arm like events around $x\mathbf{i}$, respectively x. In the proof of Theorem 6.1.1 we will see that also certain events concerning a small conformal radius from z to a certain cluster are (s,C)-one-arm like events.

Observe that the definition above implies that for every (semi-)circuit γ in $A(z; s, C)$,

$$V \cap \{\gamma \text{ open}\} = V_{ext(\gamma)} \cap V_{int(\gamma)} \cap \{\gamma \text{ open}\}, \tag{6.13}$$

where V is an (s,C)-one-arm like event around z.

If V is an (s,C)-one-arm like event around z, there is a certain open cluster which comes within a distance s from z. For any such event V we will also consider a related event where this cluster hits z. Intuitively a good candidate for such an event would be $V \cap \{z \leftrightarrow \mathbb{H} \setminus B_C(z)\}$, but this is not appropriate: under this event the cluster $\mathcal{C}(z)$ and the earlier mentioned cluster, could be disjoint. In other words, this event is too large. It turns out that the following definition is suitable for our purposes.

Definition 6.3.2. *Let V be an (s,C)-one-arm like event around z. Let $V^\bullet$ be an increasing event. We call $V^\bullet$ a* point version *of V if, for every (semi-)circuit γ in $A(z;s,C)$,*

$$V^\bullet \begin{cases} \subset & V \cap \{z \leftrightarrow \mathbb{H} \setminus B_C(z)\} \\ \supset & \{\gamma \text{ open}\} \cap V_{ext(\gamma)} \cap \{z \leftrightarrow \gamma\}. \end{cases} \tag{6.14}$$

For example, for every $x, s, C \in \mathbb{R}$ and $a \in [1/8, 1]$, the event $\{x\mathbf{i} \leftrightarrow (x\mathbf{i} + 2C(1+\mathbf{i}))\}$ is a point version of $\{B_{as}(x\mathbf{i}) \leftrightarrow (x\mathbf{i} + 2C(1+\mathbf{i}))\}$ and $\{x \leftrightarrow \mathbb{H} \setminus B_{2C}(x)\}$ is a point version of $\{I(x,s) \leftrightarrow \mathbb{H} \setminus B_{2C}(x)\}$. To state the coupling proposition we need one more definition.

Definition 6.3.3. *Let $z \in \mathbb{H}$ and $s, C > 0$. Let V and W be (s,C)-one-arm like events around z. We say that V, W are (s,C)-*comparable *around z if the events $V_{B_C(z)}$ and $W_{B_C(z)}$ are equal.*

It follows easily from this definition, that equality also holds for any subset of $B_C(z)$. In other words, let V, W be (s,C)-comparable around z, then $V_A = W_A$ for every $A \subset B_C(z)$.

Our coupling argument is contained in the following proposition.

Proposition 6.3.4. *Let $C > 0$ and $z \in \mathbb{H}$. There exist increasing functions $\varepsilon(s), m(s) : \mathbb{R}_+ \to (0,1)$, with $\varepsilon(s) \to 0$ and $m(s) \to 0$ as $s \to 0$ such that the following holds. For all $s > 0$, for all $\eta < m(s)$ and for every pair $V, W \subset \Omega^\eta$ of (s,C)-comparable events around z and point versions $V^\bullet$ of V and $W^\bullet$ of W we have*

$$\left| \frac{\mathbb{P}_\eta(V^\bullet \,|\, V)}{\mathbb{P}_\eta(W^\bullet \,|\, W)} - 1 \right| < \varepsilon(s). \tag{6.15}$$

Before we give a proof of this proposition, we introduce some notation and state a lemma which is crucial in the proof of Proposition 6.3.4.

Let $C, s > 0$ and $z \in \mathbb{H}$. Let $l(i) := 4^{-i}C$. Let $N(s,C) = \lfloor \log_4(C/s) \rfloor - 2$ and let $P_i := \mathbb{H} \setminus B_{l(i)}(z)$. We define for every $i \in \{0, 1, 2, \cdots, N(s,C)\}$ the annuli $AI_i := A(z; \frac{1}{4}l(i), \frac{1}{2}l(i))$, $AO_i := A(z; \frac{1}{2}l(i), l(i))$ and $A_i := AI_i \cup AO_i$. We denote by ΓI_i the outermost open (semi-)circuit in AI_i and by ΓO_i the innermost open (semi-)circuit in AO_i, if they exist. Otherwise, if there is no (semi-)circuit in AI_i (resp. AO_i) we set $\Gamma I_i = \emptyset$ (resp. $\Gamma O_i = \emptyset$). Let γ_I be a fixed (semi-)circuit in AI_i and γ_O be a fixed (semi-)circuit in AO_i. The following observation is quite standard. Conditioned on $\{\Gamma I_i = \gamma_I; \Gamma O_i = \gamma_O\}$, the configuration in $int(\gamma_I) \cup ext(\gamma_O)$ is a fresh independent copy of a percolation configuration.

Lemma 6.3.5. *There exists a universal constant $C_1 \in (0,1)$ such that the following holds. Let $z \in \mathbb{H}, s, C > 0, i \leq N(s,C)$ and let γ_I be a deterministic (semi-)circuit. Let V be an (s,C)-one-arm like event around z. Then, for every $\nu \in V_{P_i}$ we have*

$$\mathbb{P}_\eta(\Gamma I_i = \gamma_I \,|\, V \cap \{\nu_{P_i}\}) \geq C_1 \, \mathbb{P}_\eta(\{\Gamma I_i = \gamma_I\} \cap \{\Gamma O_i \text{ exists}\} \cap \{\gamma_I \leftrightarrow \Gamma O_i\}). \tag{6.16}$$

Proof of Lemma 6.3.5: It is sufficient to prove that, for every (semi-)circuit γ_O,

$$\begin{aligned} &\mathbb{P}_\eta(\{\Gamma I_i = \gamma_I\} \cap \{\Gamma O_i = \gamma_O\} \cap \{\gamma_I \leftrightarrow \gamma_O\} \,|\, V \cap \{\nu_{P_i}\}) \\ \geq \;& C_1 \, \mathbb{P}_\eta(\{\Gamma I_i = \gamma_I\} \cap \{\Gamma O_i = \gamma_O\} \cap \{\gamma_I \leftrightarrow \gamma_O\}). \end{aligned} \tag{6.17}$$

Namely (6.16) immediately follows from (6.17) after summing over the possible (semi-)circuits γ_O.

Let γ_O be an arbitrary (semi-)circuit and

$$D = \{\Gamma I_i = \gamma_I\} \cap \{\Gamma O_i = \gamma_O\} \cap \{\gamma_I \leftrightarrow \gamma_O\}.$$

Then the left hand side of (6.17) is equal to

$$\frac{\mathbb{P}_\eta(D \cap V \cap \{\nu_{P_i}\})}{\mathbb{P}_\eta(V \cap \{\nu_{P_i}\})}. \tag{6.18}$$

It follows from (6.13) and Definition 6.3.1 that

$$\begin{aligned}\mathbb{P}_\eta(D \cap V \cap \{\nu_{P_i}\}) &= \mathbb{P}_\eta(D \cap V_{ext(\gamma_O)} \cap V_{int(\gamma_O)} \cap \{\nu_{P_i}\}) \\ &\geq \mathbb{P}_\eta(D \cap V_{ext(\gamma_O)} \cap \{I(z,s) \leftrightarrow \gamma_I\} \cap \{\nu_{P_i}\}).\end{aligned}$$

The last probability is, by the observation about inner- and outermost (semi-)circuits, equal to

$$\mathbb{P}_\eta(D)\mathbb{P}_\eta(I(z,s) \leftrightarrow \gamma_I)\mathbb{P}_\eta(V_{ext(\gamma_O)} \cap \{\nu_{P_i}\}). \tag{6.19}$$

On the other hand the denominator in (6.18) is, again by Definition 6.3.1, less than or equal to

$$\begin{aligned}\mathbb{P}_\eta(V_{ext(\gamma_O)} \cap \{\nu_{P_i}\} \cap \{B_s(z) \leftrightarrow \gamma_I\}) &= \mathbb{P}_\eta(V_{ext(\gamma_O)} \cap \{\nu_{P_i}\})\mathbb{P}_\eta(B_s(z) \leftrightarrow \gamma_I) \quad (6.20) \\ &\leq \mathbb{P}_\eta(V_{ext(\gamma_O)} \cap \{\nu_{P_i}\}) \cdot \frac{1}{C_1}\mathbb{P}_\eta(I(z,s) \leftrightarrow \gamma_I),\end{aligned}$$

where the constant C_1 comes from standard RSW and FKG arguments. A combination of (6.18), (6.19) and (6.20) gives (6.17). This finishes the proof of Lemma 6.3.5. □

Proof of Proposition 6.3.4: We will describe a coupling of the conditional distributions given V and given W, denoted by $\tilde{\mathbb{P}}$. More precisely we construct $\tilde{\mathbb{P}}$ such that, for $\nu, \omega \in \Omega^\eta$,

$$\tilde{\mathbb{P}}(\nu \times \Omega^\eta) = \mathbb{P}_\eta(\nu \,|\, V), \qquad \tilde{\mathbb{P}}(\Omega^\eta \times \omega) = \mathbb{P}_\eta(\omega \,|\, W). \tag{6.21}$$

Furthermore $\tilde{\mathbb{P}}$ will be such that the probability that the two distributions are successfully coupled (in a sense defined precisely below) goes to 1 as s tends to zero, uniformly in η. We will finish the proof by showing how this coupling can be used to prove the proposition.

Let us first describe the coupling procedure. First we draw, independently of each other, ν_{P_0} and ω_{P_0} according to, respectively $\mathbb{P}_\eta(\cdot \,|\, V)$ and $\mathbb{P}_\eta(\cdot \,|\, W)$. Next we draw, step by step, the random elements ν_{A_i}, ω_{A_i}, starting from $i = 0$.

Every step goes as follows. The outermost (semi-)circuits $\Gamma I_i(\nu)$, $\Gamma I_i(\omega)$ are drawn from the optimal coupling of $\mathbb{P}_\eta(\Gamma I_i(\nu) = \cdot \,|\, V; \nu_{P_i})$ and $\mathbb{P}_\eta(\Gamma I_i(\omega) = \cdot \,|\, W; \omega_{P_i})$. That is, the coupling is such that $\tilde{\mathbb{P}}(\Gamma I_i(\nu) = \Gamma I_i(\omega) \neq \emptyset \,|\, \nu_{P_i}; \omega_{P_i})$ is as large as possible.

We say that this step of the coupling is successful if $\Gamma I_i(\nu) \neq \emptyset$ and $\Gamma I_i(\nu) = \Gamma I_i(\omega) =: \gamma$. In that case we can finish the coupling procedure as follows. First we draw $\nu_{ext(\Gamma I_i(\nu)) \cap A_i}$ and $\omega_{ext(\Gamma I_i(\omega)) \cap A_i}$ from the appropriate conditional probability measures, independently of each other. So $\nu_{ext(\Gamma I_i(\nu))\cap A_i}$ is drawn from the probability measure $\mathbb{P}_\eta(\cdot \,|\, \Gamma I_i(\nu) = \gamma; V; \nu_{P_i})$. Since V is an (s, C)-one-arm like event we have for every $\nu_{int(\gamma)} \in \{0,1\}^{\eta\mathbb{T}\cap int(\gamma)}$

$$\begin{aligned}\mathbb{P}_\eta(\nu_{int(\gamma)} \,|\, \Gamma I_i(\nu) = \gamma; V; \nu_{ext(\gamma)}) &= \mathbb{P}_\eta(\nu_{int(\gamma)} \,|\, V_{int(\gamma)};\, V_{ext(\gamma)};\, \Gamma I_i(\nu) = \gamma;\, \nu_{ext(\gamma)}) \\ &= \mathbb{P}_\eta(\nu_{int(\gamma)} \,|\, V_{int(\gamma)}),\end{aligned}$$

where we used (6.13) in the first equality and independence of $\nu_{int(\gamma)}$ and $V_{int(\gamma)}$ from the rest in the second. The same holds for W. Now we use that V and W are (s, C)-comparable around z. As we saw immediately after Definition 6.3.3 this implies that $V_{int(\gamma)} = W_{int(\gamma)}$, hence the two conditional distributions of the interior of γ are equal. Thus we can draw $\nu_{int(\gamma)}$ according to $\mathbb{P}_\eta(\cdot \,|\, V_{int(\gamma)})$ and take $\omega_{int(\gamma)} := \nu_{int(\gamma)}$.

If this step of the coupling was not successful, let γ_ν and γ_ω be the outcome of $\Gamma I_i(\nu)$ and $\Gamma I_i(\omega)$ respectively, we draw the random elements ν_{A_i}, ω_{A_i} according to $\mathbb{P}_\eta(\cdot \,|\, \Gamma I_i(\nu) = \gamma_\nu; V; \nu_{P_i})$ and $\mathbb{P}_\eta(\cdot \,|\, \Gamma I_i(\omega) = \gamma_\omega; W; \omega_{P_i})$ independently of each other and continue to the next step with $i + 1$.

If all steps, $i = 0, \cdots, N(s, C)$, of the coupling were not successful, we draw ν_{RM} and ω_{RM} according to the appropriate conditional probabilities, independently of each other, where

$$RM := B_{l(N(s,C)+1)}(z) \supset B_{2s}(z). \tag{6.22}$$

That this procedure defines a coupling for the measures in (6.21) follows from standard arguments.

Let S denote the event that the coupling is successful (i.e. that some step in the above described procedure is succesful). The crucial property of this coupling is that

$$(\Omega^\eta \times W^\bullet) \cap S = (V^\bullet \times \Omega^\eta) \cap S, \tag{6.23}$$

which follows easily from Definition 6.3.2. To see that $\tilde{\mathbb{P}}(S) \to 1$ as $s \to 0$, note that it follows easily from Lemma 6.3.5 together with RSW, FKG arguments that there exists a constant $C_2 > 0$ such that for every i

$$\sum_{\gamma_I} \min_{\substack{E\in\{V,W\} \\ \omega_{P_i}\in\{0,1\}^{P_i}}} (\mathbb{P}_\eta(\Gamma I_i = \gamma_I \,|\, E;\, \omega_{P_i})) \geq C_2.$$

Hence, for every step in the procedure described above, the probability that the coupling is successful is at least C_2. Thus

$$\tilde{\mathbb{P}}(S) \geq 1 - (1 - C_2)^{N(s,C)+1} \tag{6.24}$$

if η is small enough.

Now we show how this coupling can be used to prove the proposition. First rewrite the quotient in (6.15)

$$\frac{\mathbb{P}_\eta(V^\bullet \,|\, V)}{\mathbb{P}_\eta(W^\bullet \,|\, W)} = \frac{\tilde{\mathbb{P}}((V^\bullet \times \Omega^\eta) \cap S) + \tilde{\mathbb{P}}(V^\bullet \times \Omega^\eta \,|\, S^c)\tilde{\mathbb{P}}(S^c)}{\tilde{\mathbb{P}}((\Omega^\eta \times W^\bullet) \cap S) + \tilde{\mathbb{P}}(\Omega^\eta \times W^\bullet \,|\, S^c)\tilde{\mathbb{P}}(S^c)}. \tag{6.25}$$

We claim that

$$\begin{aligned}\tilde{\mathbb{P}}(V^\bullet \times \Omega^\eta \,|\, S^c) &\asymp \mathbb{P}_\eta(z \leftrightarrow \mathbb{H} \setminus B_{2s}(z)); &(6.26)\\ \tilde{\mathbb{P}}(V^\bullet \times \Omega^\eta \,|\, S) &\asymp \mathbb{P}_\eta(z \leftrightarrow \mathbb{H} \setminus B_{2s}(z)); &(6.27)\end{aligned}$$

for η small enough. Similarly for $\Omega^\eta \times W^\bullet$. Applying these claims together with (6.23) and the fact that $\tilde{\mathbb{P}}(S^c)$ converges to zero as s tends to zero, uniformly in η as follows from (6.24), proves the proposition.

It remains to prove the claims (6.26) and (6.27). At first sight one might think that these bounds are easy consequences of RSW, FKG arguments. This is not completely true since we have to deal with the condition that the coupling was not successful, respectively successful, which are neither increasing nor decreasing events. Recall the definition of RM in (6.22). Let $PN := \mathbb{H} \setminus RM$. It is sufficient to show that, for all suitable $\nu_{PN} \times \omega_{PN}$,

$$\tilde{\mathbb{P}}(V^\bullet \times \Omega^\eta \,|\, \nu_{PN} \times \omega_{PN}) \asymp \mathbb{P}_\eta(z \leftrightarrow \mathbb{H} \setminus B_{2s}(z)). \tag{6.28}$$

First note that it follows from the coupling procedure that

$$\tilde{\mathbb{P}}(V^\bullet \times \Omega^\eta \,|\, \nu_{PN} \times \omega_{PN}) = \mathbb{P}_\eta(V^\bullet \,|\, V \cap \{\nu_{PN}\}).$$

First we prove that in (6.28), the left hand side is less than or equal to a constant times the right hand side. To do this we introduce the event B, that there is an open (semi-)circuit in $A(z; s, 2s)$. We will prove this upper bound by showing that there exist universal constants $C_3, C_4 > 0$ such that, for all suitable ν_{PN}

$$\begin{aligned} \mathbb{P}_\eta(V^\bullet \cap B \,|\, V \cap \{\nu_{PN}\}) &\geq C_3\, \mathbb{P}_\eta(V^\bullet \,|\, V \cap \{\nu_{PN}\}); & (6.29)\\ \mathbb{P}_\eta(V^\bullet \cap B \,|\, V \cap \{\nu_{PN}\}) &\leq C_4\, \mathbb{P}_\eta(z \leftrightarrow \mathbb{H} \setminus B_{2s}(z)). & (6.30)\end{aligned}$$

First we consider the lower bound (6.29). Let ν_{PN} be arbitrary. Using Lemma 6.2.4 and standard RSW, FKG arguments we get that

$$\begin{aligned} \mathbb{P}_\eta(V^\bullet \cap B \,|\, V \cap \{\nu_{PN}\}) &\geq \mathbb{P}_\eta(B)\mathbb{P}_\eta(V^\bullet \,|\, V \cap \{\nu_{PN}\}). \\ &\geq C_3\, \mathbb{P}_\eta(V^\bullet \,|\, V \cap \{\nu_{PN}\}) \end{aligned}$$

This proves (6.29).

Next we prove the upper bound (6.30). Therefore let Γ denote the outermost open (semi-)circuit in $A(z; s, 2s)$. Since V is an (s, C)-one-arm like event, we have by Definition 6.3.1,

$$\bigcup_\gamma V_{ext(\gamma)} \cap \{\Gamma = \gamma\} \cap \{I(z,s) \leftrightarrow \gamma\} \;\subset\; V. \tag{6.31}$$

This, together with standard RSW, FKG arguments, implies that there exists a constant $C_5 > 0$ such that

$$\begin{aligned} \mathbb{P}_\eta(B \cap V \,|\, \nu_{PN}) &\geq \mathbb{P}_\eta(B \cap V_{ext(\Gamma)} \cap \{I(z,s) \leftrightarrow \Gamma\} \,|\, \nu_{PN}) \\ &\geq C_5\, \mathbb{P}_\eta(B \cap V_{ext(\Gamma)} \,|\, \nu_{PN}), \qquad (6.32)\end{aligned}$$

since $\mathbb{P}_\eta(I(z,s) \leftrightarrow \Gamma \,|\, B;\, V_{ext(\Gamma)};\, \nu_{PN}) \geq C_5$. Hence

$$\begin{aligned} \mathbb{P}_\eta(V^\bullet \cap B \,|\, V \cap \{\nu_{PN}\}) &\leq \mathbb{P}_\eta(\{z \leftrightarrow \Gamma\} \cap B \,|\, V \cap \{\nu_{PN}\}) \\ &\leq \mathbb{P}_\eta(z \leftrightarrow \mathbb{H} \setminus B_s(z)) \cdot \frac{\mathbb{P}_\eta(B \cap V_{ext(\Gamma)} \,|\, \nu_{PN})}{\mathbb{P}_\eta(V \,|\, \nu_{PN})} \\ &\leq \frac{1}{C_5 C_6} \mathbb{P}_\eta(z \leftrightarrow \mathbb{H} \setminus B_{2s}(z)) \cdot \frac{\mathbb{P}_\eta(B \cap V \,|\, \nu_{PN})}{\mathbb{P}_\eta(V \,|\, \nu_{PN})} \\ &\leq \frac{1}{C_5 C_6} \mathbb{P}_\eta(z \leftrightarrow \mathbb{H} \setminus B_{2s}(z)), \qquad (6.33)\end{aligned}$$

where we used in the first inequality Definition 6.3.2. In the second inequality we used the fact that $V \subset V_{ext(\Gamma)}$ together with the fact that $\{z \leftrightarrow \Gamma\}$ is independent of everything outside Γ (which exists because of B). The third inequality follows from (6.32) and the existence of a universal constant $C_6 > 0$ such that $\mathbb{P}_\eta(z \leftrightarrow \mathbb{H} \setminus B_{2s}(z)) \geq C_6 \, \mathbb{P}_\eta(z \leftrightarrow \mathbb{H} \setminus B_s(z))$. This gives the desired inequality (6.30) and completes the proof of the upper bound in (6.28).

Next we consider the lower bound in (6.28). We prove that

$$\mathbb{P}_\eta(V^\bullet \,|\, V \cap \{\nu_{PN}\}) \geq C_3 \, \mathbb{P}_\eta(z \leftrightarrow \mathbb{H} \setminus B_{2s}(z)). \tag{6.34}$$

To prove this, we again use the event B. The inequality (6.34) follows immediately from the following inequality

$$\mathbb{P}_\eta(V^\bullet \cap B \,|\, V \cap \{\nu_{PN}\}) \geq C_3 \, \mathbb{P}_\eta(z \leftrightarrow \mathbb{H} \setminus B_{2s}(z)), \tag{6.35}$$

where $C_3 > 0$ is the same as in (6.29). Similarly to (6.31), but now using Definition 6.3.2, we have

$$\bigcup_\gamma \{\Gamma = \gamma\} \cap V_{ext(\gamma)} \cap \{z \leftrightarrow \gamma\} \;\subset\; V^\bullet, \tag{6.36}$$

where Γ is the outermost circuit in $A(z; s, 2s)$. Hence

$$\begin{aligned}
&\mathbb{P}_\eta(V^\bullet \cap B \,|\, V \cap \{\nu_{PN}\}) \\
&\overset{(6.36)}{\geq} \quad \sum_\gamma \frac{\mathbb{P}_\eta(\{\Gamma = \gamma\} \cap V_{ext(\gamma)} \cap \{z \leftrightarrow \gamma\} \cap \{\nu_{PN}\})}{\mathbb{P}_\eta(V \cap \{\nu_{PN}\})}, \\
&\geq \quad \mathbb{P}_\eta(z \leftrightarrow \mathbb{H} \setminus B_{2s}(z)) \sum_\gamma \frac{\mathbb{P}_\eta(\{\Gamma = \gamma\} \cap V_{ext(\gamma)} \cap \{\nu_{PN}\})}{\mathbb{P}_\eta(V \cap \{\nu_{PN}\})}, \\
&\geq \quad \mathbb{P}_\eta(z \leftrightarrow \mathbb{H} \setminus B_{2s}(z)) \frac{\mathbb{P}_\eta(B \cap V \cap \{\nu_{PN}\})}{\mathbb{P}_\eta(V \cap \{\nu_{PN}\})}.
\end{aligned} \tag{6.37}$$

It follows from Lemma 6.2.4 together with the fact that $\mathbb{P}_\eta(B) \geq C_3$ that

$$\mathbb{P}_\eta(B \cap V \cap \{\nu_{PN}\}) \geq C_3 \cdot \mathbb{P}_\eta(V \cap \{\nu_{PN}\}). \tag{6.38}$$

This completes the proof of (6.35)
and finishes the proof of Proposition 6.3.4 □

6.3.2 Proof of Theorem 6.1.1.

Let u_1, u_2, w be fixed. Because of Theorem 6.2.1 it is sufficient to show that for every $\varepsilon > 0$, there exists $s > 0$, such that $\forall s_1, s_2, s_3 < s : \exists \eta_0 > 0$ with the property that

$$\left| \frac{\mathbb{P}_\eta(u_1 \leftrightarrow u_2 \leftrightarrow w \,|\, E^{s_1,s_2,s_3}_{u_1,u_2,w})^2}{\mathbb{P}_\eta(u_1 \leftrightarrow u_2 \,|\, E^{s_1,s_2}_{u_1,u_2}) \, \mathbb{P}_\eta(u_1 \leftrightarrow w \,|\, E^{s_1,s_3}_{u_1,w}) \, \mathbb{P}_\eta(u_2 \leftrightarrow w \,|\, E^{s_2,s_3}_{u_2,w})} - 1 \right| < \varepsilon, \tag{6.39}$$

for all $\eta < \eta_0$.

In order to prove (6.39) we define the following events:

$$\begin{aligned} E^{s_1,\bullet}_{u_1,u_2} &:= \{[u_1,u_1+s_1]\cap\mathcal{C}(u_2)\neq\emptyset\}; \\ E^{\bullet,s_3}_{u_1,w} &:= \{\rho(w,\mathcal{C}(u_1))<s_3\}; \\ E^{\bullet,s_3}_{u_2,w} &:= \{\rho(w,\mathcal{C}(u_2))<s_3\}; \\ E^{s_1,\bullet,s_3}_{u_1,u_2,w} &:= \{[u_1,u_1+s_1]\cap\mathcal{C}(u_2)\neq\emptyset\}\cap\{\rho(w,\mathcal{C}([u_1,u_1+s_1]))<s_3\}; \\ E^{\bullet,\bullet,s_3}_{u_1,u_2,w} &:= \{u_1\leftrightarrow u_2\}\cap\{\rho(w,\mathcal{C}(u_1))<s_3\}. \end{aligned} \tag{6.40}$$

Let $C:=(\min\{|u_1-u_2|,|u_1-w|,|u_2-w|\})/(2\sqrt{2})$. We claim the following about the events defined in (6.5) and (6.40).

1. Every event of the form $E^{s_1,s_2,s_3}_{a_1,a_2,a_3}$ or $E^{s_1,s_2}_{a_1,a_2}$ where the a_i's are in $\{u_1,u_2,w\}$ and each s_i is in $\mathbb{R}_+$ or $s_i=\bullet$, defined in (6.5) and (6.40), is, for each $s_j\neq\bullet$ an (s_j,C)-one-arm like event around a_j. For example $E^{s_1,\bullet,s_3}_{u_1,u_2,w}$ is an (s_1,C)-one-arm like event around u_1, and an (s_3,C)-one-arm like event around w.

2. The events $\{u_1\leftrightarrow u_2\}$, $\{u_1\leftrightarrow w\}$, $\{u_2\leftrightarrow w\}$, $\{u_1\leftrightarrow u_2\leftrightarrow w\}$ are point versions of respectively $E^{s_1,\bullet}_{u_1,u_2}$, $E^{\bullet,s_3}_{u_1,w}$, $E^{\bullet,s_3}_{u_2,w}$ and $E^{\bullet,\bullet,s_3}_{u_1,u_2,w}$.

3. Each event in (6.40) is a point version of the corresponding event $E^{s_1,s_2,s_3}_{a_1,a_2,a_3}$ or $E^{s_1,s_2}_{a_1,a_2}$, where the "$\bullet$" is replaced by a positive number s_j. E.g. $E^{\bullet,\bullet,s_3}_{u_1,u_2,w}$ is a point version of $E^{s_1,\bullet,s_3}_{u_1,u_2,w}$ and $E^{s_1,\bullet}_{u_1,u_2}$ is a point version of $E^{s_1,s_2}_{u_1,u_2}$.

4. Each pair of events of the form $E^{s_1,s_2,s_3}_{a_1,a_2,a_3}$ and $E^{s_1,s_2}_{a_1,a_2}$ where the a_i's are in $\{u_1,u_2,w\}$ and each s_i is in $\mathbb{R}_+$ or $s_i=\bullet$, defined in (6.5) and (6.40), are, for each j where both events have $s_j\neq\bullet$, (s_j,C)-comparable around a_j. For example the events $E^{s_1,s_2}_{u_1,u_2}$, $E^{s_1,s_3}_{u_1,w}$, $E^{s_1,s_2,s_3}_{u_1,u_2,w}$, $E^{s_1,\bullet}_{u_1,u_2}$, $E^{s_1,\bullet,s_3}_{u_1,u_2,w}$ are pairwise (s_1,C)-comparable around u_1.

Before we give proofs of these claims we show how Theorem 6.1.1 follows from them. We factorize the numerator in (6.39) as follows

$$\begin{aligned} &\mathbb{P}_\eta(u_1\leftrightarrow u_2\leftrightarrow w\,|\,E^{s_1,s_2,s_3}_{u_1,u_2,w})^2 \\ &= \mathbb{P}_\eta(u_1\leftrightarrow u_2\leftrightarrow w\,|\,E^{\bullet,\bullet,s_3}_{u_1,u_2,w})^2\cdot\mathbb{P}_\eta(E^{\bullet,\bullet,s_3}_{u_1,u_2,w}\,|\,E^{s_1,\bullet,s_3}_{u_1,u_2,w})^2\cdot\mathbb{P}_\eta(E^{s_1,\bullet,s_3}_{u_1,u_2,w}\,|\,E^{s_1,s_2,s_3}_{u_1,u_2,w})^2. \end{aligned} \tag{6.41}$$

The probabilities in the denominator in (6.39) can be factorized as follows

$$\mathbb{P}_\eta(u_1\leftrightarrow u_2\,|\,E^{s_1,s_2}_{u_1,u_2}) = \mathbb{P}_\eta(u_1\leftrightarrow u_2\,|\,E^{s_1,\bullet}_{u_1,u_2})\mathbb{P}_\eta(E^{s_1,\bullet}_{u_1,u_2}\,|\,E^{s_1,s_2}_{u_1,u_2}) \tag{6.42}$$

$$\mathbb{P}_\eta(u_1\leftrightarrow w\,|\,E^{s_1,s_3}_{u_1,w}) = \mathbb{P}_\eta(u_1\leftrightarrow w\,|\,E^{\bullet,s_3}_{u_1,w})\mathbb{P}_\eta(E^{\bullet,s_3}_{u_1,w}\,|\,E^{s_1,s_3}_{u_1,w}) \tag{6.43}$$

$$\mathbb{P}_\eta(u_2\leftrightarrow w\,|\,E^{s_2,s_3}_{u_2,w}) = \mathbb{P}_\eta(u_2\leftrightarrow w\,|\,E^{\bullet,s_3}_{u_2,w})\mathbb{P}_\eta(E^{\bullet,s_3}_{u_2,w}\,|\,E^{s_2,s_3}_{u_2,w}). \tag{6.44}$$

Plugging this into the quotient in (6.39) and applying Proposition 6.3.4 to the 6 pairs of (s_i,C)-comparable events completes the proof.

It remains to prove claims 1-4 above. Some of these claims follow immediately, for the others we use two standard properties of conformal radius. The first is (6.4). The second property is *monotonicity*: the conformal radius is non-decreasing as the domain A decreases, (as is well known and follows easily from Schwarz' Lemma. See for example [2]).

We prove claim 1 for a particular event, namely $E^{\bullet,s_3}_{u_1,w}$.
(a) It is increasing: Let $\omega \in E^{\bullet,s_3}_{u_1,w}$ and $\nu \geq \omega$, then $\mathcal{C}(u_1)(\omega) \subset \mathcal{C}(u_1)(\nu)$. Here $\mathcal{C}(u_1)(\omega)$ means the cluster of u_1 under the configuration ω. Thus by monotonicity of the conformal radius $\rho(w, \mathcal{C}(u_1)(\nu)) \leq \rho(w, \mathcal{C}(u_1)(\omega)) < s_3$ and $\nu \in E^{\bullet,s_3}_{u_1,w}$.
(b) $E^{\bullet,s_3}_{u_1,w} \subset \{B_{s_3}(w) \leftrightarrow \mathbb{H} \setminus B_C(w)\}$: Suppose that $\omega \in E^{\bullet,s_3}_{u_1,w}$. It follows from (6.4) that $\min_{x\in\mathcal{C}(u_1)} |w - x| < s_3$. Further $\sqrt{2}C \leq |u_1 - w|/2$, which implies that $\omega \in \{B_{s_3}(w) \leftrightarrow \mathbb{H} \setminus B_C(w)\}$.
Let γ be an arbitrary (semi-)circuit in $A(w; s_3, C)$. Let $D := E^{\bullet,s_3}_{u_1,w}$
(c) $\{\gamma \text{ open}\} \cap D_{ext(\gamma)} \cap D_{int(\gamma)} \subset D$: Let $\omega \in D_{int(\gamma)}$ and $\nu \in D_{ext(\gamma)}$. By definition there exists $\tilde{\nu}$ such that $\nu_{ext(\gamma)} \times \tilde{\nu} \in D$. With the second inequality in (6.4) this implies that $u_1 \leftrightarrow \gamma$ in $ext(\gamma)$. Next let $\tilde{\omega}$ be such that $\omega_{int(\gamma)} \times \tilde{\omega} \in D$. Then it is easy to see that $\mathcal{C}(u_1)(\omega_{int(\gamma)} \times \tilde{\omega}) \cap int(\gamma) \subset \mathcal{C}(\gamma)(\omega) \cap int(\gamma)$. Monotonicity of the conformal radius implies now that

$$\rho\left(w, \mathcal{C}(\gamma)(\omega)\right) \leq \rho\left(w, \mathcal{C}(u_1)(\omega_{int(\gamma)} \times \tilde{\omega})\right) < s_3$$

Let $\upsilon := \omega_{int(\gamma)} \times \{1\}^\gamma \times \nu_{ext(\gamma)}$. Note that $\mathcal{C}(u_1)(\upsilon) \cap int(\gamma) = \mathcal{C}(\gamma)(\omega) \cap int(\gamma)$. Thus $\rho(w, \mathcal{C}(u_1)(\upsilon)) = \rho(w, \mathcal{C}(\gamma)(\omega))$, and hence $\upsilon \in D$.
(d) $\{I(w, s_3) \leftrightarrow \gamma\} \subset D_{int(\gamma)}$: Let $\omega \in \{I(w, s_3) \leftrightarrow \gamma\}$ and $\nu \in \{u_1 \leftrightarrow \gamma\}$. Then the first inequality in (6.4) implies that $\omega_{int(\gamma)} \times \{1\}^\gamma \times \nu_{ext(\gamma)} \in D$, hence $\omega \in D_{int(\gamma)}$.
This completes the proof of claim 1 for this particular event. The proofs for the other events and claims are very similar and we omit them. □

6.3.3 Proof of Theorem 6.1.2.

We will use the notation

$$E^{s_1,\bullet,\bullet}_{u_1,u_2,w} := \{\{u_2, w\} \subset \mathcal{C}([u_1, u_1 + s])\}. \tag{6.45}$$

With this notation we can write the quotient in (6.2) as

$$\frac{\mathbb{P}(\{u_2, w\} \subset \mathcal{C}([u_1, u_1 + s]))}{\mathbb{P}(w \in \mathcal{C}([u_1, u_1 + s]))\ \mathbb{P}(u_2 \in \mathcal{C}([u_1, u_1 + s]))} = \frac{\mathbb{P}_\eta(E^{s,\bullet,\bullet}_{u_1,u_2,w})}{\mathbb{P}_\eta(E^{s,\bullet}_{u_1,w})\mathbb{P}_\eta(E^{s,\bullet}_{u_1,u_2})}. \tag{6.46}$$

Similarly to the proof of Theorem 6.1.1 we factorize this as follows

$$\begin{aligned} &\frac{\mathbb{P}_\eta(E^{s,\bullet,\bullet}_{u_1,u_2,w})}{\mathbb{P}_\eta(E^{s,\bullet}_{u_1,w})\mathbb{P}_\eta(E^{s,\bullet}_{u_1,u_2})} \\ &= \frac{\mathbb{P}_\eta(E^{s,\bullet,\bullet}_{u_1,u_2,w} \mid E^{s,\bullet,s_3}_{u_1,u_2,w})}{\mathbb{P}_\eta(E^{s,\bullet}_{u_1,w} \mid E^{s,s_3}_{u_1,w})} \cdot \frac{\mathbb{P}_\eta(E^{s,\bullet,s_3}_{u_1,u_2,w} \mid E^{s,s_2,s_3}_{u_1,u_2,w})}{\mathbb{P}_\eta(E^{s,\bullet}_{u_1,u_2} \mid E^{s,s_2}_{u_1,u_2})} \cdot \frac{\mathbb{P}_\eta(E^{s,s_2,s_3}_{u_1,u_2,w})}{\mathbb{P}_\eta(E^{s,s_3}_{u_1,w})\mathbb{P}_\eta(E^{s,s_2}_{u_1,u_2})}. \end{aligned} \tag{6.47}$$

The first two ratio's converge to 1 by Proposition 6.3.4, uniformly in η. Namely the involved events are point versions and (s, C)-comparable, by similar arguments as in the proof of Theorem 6.1.1. We claim that the ratio

$$\frac{\mathbb{P}_\eta(E^{s,s_2,s_3}_{u_1,u_2,w})}{\mathbb{P}_\eta(E^{s,s_3}_{u_1,w})\mathbb{P}_\eta(E^{s,s_2}_{u_1,u_2})} \tag{6.48}$$

converges to the function $\psi(u_1, s, u_2, w)$, as η, s_2, s_3 tend to zero. To prove this claim we note that

$$\frac{\mathbb{P}_\eta(E^{s,s_2,s_3}_{u_1,u_2,w})}{\mathbb{P}_\eta(E^{s,s_3}_{u_1,w})\mathbb{P}_\eta(E^{s,s_2}_{u_1,u_2})} = \frac{s_3^{-5/48} \cdot \mathbb{P}_\eta(E^{s,s_2,s_3}_{u_1,u_2,w} \mid E^{s,s_2}_{u_1,u_2})}{s_3^{-5/48} \cdot \mathbb{P}_\eta(E^{s,s_3}_{u_1,w})}. \tag{6.49}$$

Theorem 6.2.2 and Lemma 6.2.3 imply that the following limit of (6.49) exists: First send η to zero, after that send s_2 to zero and finally let s_3 go to zero. This, together with the uniform convergence in η of the first two ratio's in (6.47), implies that the limit in (6.2) exists and is equal to

$$\frac{\pi^{5/48}|\Psi'_{u_1,s,u_2}(w)|^{5/48} G\left(\Re(\Psi_{u_1,s,u_2}(w)), \Im(\Psi_{u_1,s,u_2}(w))\right)}{2^{5/48}H(0) \cdot |\phi'(w)|^{5/48}(\sin(\pi\omega/2))^{1/3}}, \tag{6.50}$$

where Ψ, G, ϕ, H, ω are as in Theorem 6.2.2 and Lemma 6.2.3.

To finish the proof of Theorem 6.1.2 we have to simplify (6.50) and show that it is equal to the function $\psi(u_1, s, u_2, w)$ given in that Theorem. Hereto let $\Pi : \mathbb{H} \to \mathbb{H}$ be a conformal map such that the points $u_1, u_1 + s, u_2$ are mapped to $-1, 1, \infty$ respectively. Let $\tilde{w} = \Pi(w)$. Let $\tilde{\Psi} : \mathbb{H} \to \mathbb{S}$ be the conformal map, such that $\Psi = \tilde{\Psi} \circ \Pi$, thus

$$\tilde{\Psi}(z) = \frac{-\mathbf{i}}{\pi} \arcsin(z) + \frac{1}{2}\mathbf{i}.$$

Further let $\tilde{\phi}$ be the conformal map such that $\phi = \tilde{\phi} \circ \Pi$. We have that

$$|\phi'(w)| = \frac{|\Pi'(w)|}{2\Im(\tilde{w})}, \qquad |\Psi'(w)| = \frac{|\Pi'(w)|}{\pi\sqrt{|1 - \tilde{w}^2|}}. \tag{6.51}$$

Recall that $x = \Re(\Psi_{u_1,s,u_2}(w))$, $y = \Im(\Psi_{u_1,s,u_2}(w))$ and $\Psi_{u_1,s,u_2}(w) = \tilde{\Psi}(\tilde{w})$, thus

$$\begin{aligned} \sinh(\pi x) &= \sinh(\Im(\arcsin(\tilde{w}))), \\ \sin(\pi y) &= \cos(\Re(\arcsin(\tilde{w}))). \end{aligned}$$

It follows from standard formulas for hyperbolic functions that

$$\sinh(\pi x)^2 \sin(\pi y)^2 = \Im(\tilde{w})^2, \tag{6.52}$$

$$\sinh(\pi x)^2 + \sin(\pi y)^2 = |1 - \tilde{w}^2|. \tag{6.53}$$

Further note that

$$\begin{aligned} &\left(\frac{1}{\sinh(\pi x)}\right)^{1/3} \left(\frac{\sinh(\pi x)^2 \sin(\pi y)^2}{\sinh(\pi x)^2 + \sin(\pi y)^2}\right)^{11/96} \\ = &\left(\frac{\sin(\pi y)^2}{\sinh(\pi x)^2 + \sin(\pi y)^2}\right)^{1/6} \left(\frac{\sinh(\pi x)^2 + \sin(\pi y)^2}{\sinh(\pi x)^2 \sin(\pi y)^2}\right)^{5/96}. \end{aligned} \tag{6.54}$$

Putting together the definition of G in (6.8) and equations (6.51) - (6.54) gives that (6.50) is equal to

$$\frac{e^{\pi x/3}H(x)}{H(0)} \cdot \left(\frac{\cos(\Re(\arcsin(\tilde{w})))}{\sqrt{|1 - \tilde{w}^2|}\sin(\pi\omega/2)}\right)^{1/3}. \tag{6.55}$$

Recall that $\omega\pi$ is equal to the angle at $\tilde{w}$ in the triangle with corners $-1, 1, \tilde{w}$. It follows easily that

$$\sin(\pi\omega/2) = \sqrt{\frac{1}{2} - \frac{|\tilde{w}|^2 - 1}{2|1 - \tilde{w}^2|}},$$

and from formulas for hyperbolic functions, including (6.53), that

$$2\cos(\Re(\arcsin(\tilde{w})))^2 = |1 - \tilde{w}^2| + 1 - |\tilde{w}|^2,$$

which together imply that the last factor in (6.55) equals 1. This completes the proof of Theorem 6.1.2. □

Summary

Planar critical percolation can stand for various models. Let us give three examples. First critical bond percolation on the square lattice $\mathbb{Z}^2$. We equip this lattice with edges between neighbouring vertices. We keep edges with probability 1/2 and remove them otherwise, independently of the other edges. The second model is site percolation on the triangular lattice and is very similar to the previous model. Here the lattice consists of the vertices $\{x + y\mathbf{j} : x, y \in \mathbb{Z}\}$, where $\mathbf{j} = e^{\frac{1}{3}\pi\mathbf{i}}$. Similarly to bond percolation on $\mathbb{Z}^2$, we equip the triangular lattice with edges between the neighbouring vertices. In this model we keep vertices with probability 1/2 and otherwise remove them together with their incident edges, independently of the other vertices. The third example is critical FK-percolation with parameter $q = 2$, which is again a model on the square lattice $\mathbb{Z}^2$. It is defined as a limit of measures defined on $\Lambda_n = [-n, n]^2$. Let $E(\Lambda_n \cap \mathbb{Z}^2)$ be the set of edges of the graph with vertex set $\Lambda_n \cap \mathbb{Z}^2$. Let ϕ_{Λ_n} be the measure on subsets of $E(\Lambda_n \cap \mathbb{Z}^2)$, defined by

$$\phi_{\Lambda_n}(\omega) := \frac{1}{Z_{\Lambda_n}} \left(\frac{\sqrt{2}}{1+\sqrt{2}}\right)^{|\omega|} \cdot \left(\frac{1}{1+\sqrt{2}}\right)^{|E(\Lambda_n \cap \mathbb{Z}^2)\setminus\omega|} \cdot 2^{k(\omega)},$$

where Z_{Λ_n} is a normalizing constant, $\omega \subset E(\Lambda_n \cap \mathbb{Z}^2)$ denotes the set of edges which are retained and $k(\omega)$ denotes the number of connected components in the graph with vertex set $\Lambda_n \cap \mathbb{Z}^2$ and edge set ω. FK-percolation on $\mathbb{Z}^2$ with parameter $q = 2$ is defined as the limit of the measures ϕ_{Λ_n} as $n \to \infty$.

The main motivation to study these models is their simple definition on the one hand and interesting properties on the other hand. In the end one hopes to gain a better understanding of two dimensional stochastic systems defined on lattices, which appear in theoretical physics, mathematical epidemiology, etcetera.

In Chapters 2 and 3 we consider critical bond percolation in $\Lambda_n \cap \mathbb{Z}^2$. Let $\mathcal{M}_n^{(i)}$ denote the size of the i-th largest cluster (i.e. connected component in the remaining graph) in terms of the number of vertices it contains. Let $\pi(n)$ denote the probability that the origin is connected to the boundary of Λ_n. It is well known in the literature [17] that, for all $i \in \mathbb{N}$ and n large, $\mathcal{M}_n^{(i)}$ is of the order $n^2\pi(n)$. In Chapter 2 we show that, for any nonempty interval $(a, b) \subset [0, \infty)$, there exists a strictly positive constant δ depending on a and b such that, for all n sufficiently large,

$$\mathbb{P}\big(an^2\pi(n) \leq \mathcal{M}_n^{(1)} \leq bn^2\pi(n)\big) > \delta.$$

To prove this result we divide the $2n \times 2n$-box in smaller boxes. Then we construct, using RSW and FKG arguments, a cluster which contains circuits in all these boxes.

Using a suitable form of independence we show that, with a probability bounded away from zero, the cluster has the desired size. It is well known that there exist constants $C, \alpha > 0$, such that $\pi(n) \geq Cn^{-\alpha}$. The fact that α is strictly less than 1 is important in our proof.

The difference in size between the largest and the second largest cluster is considered in Chapter 3. We prove that the order of the difference is equal to the order of the sizes themselves, i.e. $n^2\pi(n)$. A Similar result holds for the i-th and $(i+1)$-th largest cluster. Járai [50] proved a weaker result, namely that the order of the difference is at least $\sqrt{n^2\pi(n)}$. The proof makes use of a concentration inequality for sums of independent random variables. The independent random variables are roughly speaking obtained from boxes with side length of order n, which are intersected by the i-th largest cluster. We also prove in Chapter 3 that, for any $x \geq 0$,

$$\limsup_{n\to\infty} \mathbb{P}\big(\exists u \in \Lambda_n : xn^2\pi(n) \leq \mathcal{C}(u) \leq (x+\varepsilon)n^2\pi(n)\big) \to 0 \quad \text{as } \varepsilon \to 0,$$

where $\mathcal{C}(u)$ is the cluster containing u.

The proofs in Chapters 2 and 3 are given for bond percolation on $\mathbb{Z}^2$. Essentially the same proofs work for site percolation on the triangular lattice.

In Chapters 4 and 6 we only consider site percolation on the triangular lattice. Chapter 5 also contains results on the scaling limit of FK-percolation with $q = 2$.

The expected number of clusters intersecting a line segment is considered in Chapter 4. For site percolation on the upper half-plane and a line segment, of length n, on the boundary of the half-plane, we prove that, asymptotically, the expected number of clusters which intersect the line segment is given by

$$\left(\mathbb{P}(1 \not\leftrightarrow (-\infty, 0]) - \frac{1}{2}\right) \cdot n + \frac{\sqrt{3}}{4\pi}\log(n) + o(\log(n)),$$

where $\mathbb{P}(1 \not\leftrightarrow (-\infty, 0])$ denotes the probability that there is no path (in the remaining graph on the upper half-plane) from the vertex at the point 1 to the half infinite line $(-\infty, 0]$.

For the case of percolation in the full plane we can only give the first term and an upper bound for the prefactor of the logarithmic term. The prefactors for the logarithmic term were heuristically derived by Cardy [25] for the half plane and Kovács, Iglói and Cardy [59] for the full plane.

In Chapter 5 we consider the convergence of the largest clusters for site percolation on the triangular lattice in the following sense. Replace the vertices by hexagons, such that the hexagons form a tiling of the plane. We keep a hexagon with probability 1/2 and remove it otherwise, independently of the other hexagons. The obtained connected components are closed subsets of the plane $\mathbb{R}^2$. Now we rescale the plane by a factor $\eta \in (0, 1)$, hence the side lengths of the hexagons become $\eta/\sqrt{3}$. Let us denote by σ_η the union of all remaining η-scaled hexagons in $\mathbb{R}^2$. Since we were originally considering the largest clusters in Λ_n we now restrict ourselves to the η-scaled hexagons in Λ_1, that is $\sigma_\eta \cap \Lambda_1$. The connected components of $\sigma_\eta \cap \Lambda_1$ form the clusters. We prove that, as η tends to zero, the largest clusters converge in the Hausdorff metric to continuum clusters. This is not the only sense in which we prove convergence of the clusters. We also show that the so called "counting measures" of the clusters converge after an appropriate scaling. An interesting consequence of this latter result is the proof of convergence of the size of the largest clusters in a $2n \times 2n$-box. More precisely, for every $i \in \mathbb{N}$, there exists a random variable $M^{(i)}$ such that

$$\frac{\tilde{\mathcal{M}}_n^{(i)}}{n^2\pi(n)} \xrightarrow{d} M^{(i)} \qquad \text{as } n \to \infty,$$

where $\tilde{\mathcal{M}}_n^{(i)}$ denotes the size of the i-th largest cluster in Λ_n for site percolation on the triangular lattice.

The proofs in Chapter 5 make use of the so-called full scaling limit result by Camia and Newman [22] and the convergence of "area" / "one-arm" measures by Garban, Pete and Schramm [38].

Assuming that FK-percolation with $q = 2$ also has a unique full scaling limit in the same spirit as the one by Camia and Newman, we obtain the convergence of the large clusters in that model as well. We use this to obtain a geometric representation of the scaling limit of the Ising magnetization field. (The existence of the scaling limit of the Ising magnetization field was already proved in [21].)

Additionally we prove in Chapter 5 conformal invariance / covariance properties of the clusters.

In Chapter 6 we consider the probability that three fixed points are in a single cluster. More precisely we consider percolation on the upper half-plane and take two distinct points $u, v \in \mathbb{Z}$ on the boundary of the upper half-plane and a third point $w = m + l\mathbf{i} \in \mathbb{H}$, with $m \in \mathbb{Z}$ and $l \in \mathbb{N}$, in the interior of the upper half-plane. We prove that, asymptotically in n, the probability that the vertices nu, nv and nw are in a single cluster can be factorized as follows

$$\frac{\mathbb{P}(nu \leftrightarrow nv \leftrightarrow nw)^2}{\mathbb{P}(nu \leftrightarrow nv) \cdot \mathbb{P}(nu \leftrightarrow nw) \cdot \mathbb{P}(nv \leftrightarrow nw)} \to \frac{2^7\pi^5}{3^{3/2}\Gamma(1/3)^9} \qquad \text{as } n \to \infty,$$

where $x \leftrightarrow y$ denotes the event that x and y are in the same cluster. Note that the constant does not depend on the precise choice of u, v, w. A similar result, but in terms of the scaling limit, was obtained by Beliaev and Izyurov in [9]. The difference is that informally, on the left hand-side, the vertices nu, nv, nw are replaced by neighbourhoods of the vertices, where the definition of neighbourhood is different for the different vertices. The proof in chapter 6 combines the result by Beliaev and Izyurov with a generalized coupling argument.

Furthermore, an asymptotic factorization for the probability that nu and nw are both in a cluster which hits a given interval is proved. The proof of the latter factorization result is based on the same coupling arguments and again on a similar result by Beliaev and Izyurov in the last mentioned paper.

Those factorization formulas, together with the constants were heuristically derived by Simmons, Kleban and Ziff [74, 76].

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Bibliography

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, Dover, 1965.

[2] L. V. Ahlfors, *Conformal invariants*, AMS Chelsea Publishing, Providence, RI, 1973. Topics in geometric function theory.

[3] M. Aizenman, *On the number of incipient spanning clusters*, Nuclear Physics B, 485 (1997), pp. 551—-582.

[4] M. Aizenman and D. J. Barsky, *Sharpness of the phase transition in percolation models*, Comm. Math. Phys., 108 (1987), p. 489–526.

[5] M. Aizenman and A. Burchard, *Hölder regularity and dimension bounds for random curves*, Duke mathematical journal, 99 (1999), pp. 419–453.

[6] V. Beffara, *Is critical 2d percolation universal?*, In and Out of Equilibrium 2, 60 (2008), p. 31.

[7] V. Beffara and H. Duminil-Copin, *The self-dual point of the two-dimensional random-cluster model is critical for $q \geq 1$*, Probab. Theory Rel. Fields, 153 (2012), pp. 511–542.

[8] V. Beffara and P. Nolin, *On monochromatic arm exponents for 2d critical percolation*, Ann. Probab., 39 (2011), pp. 1286–1304.

[9] D. Beliaev and K. Izyurov, *A proof of factorization formula for critical percolation*, Comm. Math. Phys., 310 (2012), pp. 611–623.

[10] J. van den Berg and R. P. Conijn, *On the size of the largest cluster in 2D critical percolation*, Electron. Commun. Probab., 17 (2012), pp. no. 58, 13.

[11] ——, *The gaps between the sizes of large clusters in 2D critical percolation*, Electron. Commun. Probab., 18 (2013), pp. no. 92, 1–9.

[12] ——, *The expected number of critical percolation clusters intersecting a line segment*, arXiv:1505.08046, (2015).

[13] J. van den Berg, B. N. B. de Lima, and P. Nolin, *A percolation process on the square lattice where large finite clusters are frozen*, Random Structures Algorithms, 40 (2012), pp. 220–226.

[14] J. van den Berg and H. Kesten, *Inequalities with applications to percolation and reliability*, Journal of applied probability, (1985), pp. 556–569.

[15] B. Bollobás and O. Riordan, *Percolation*, Cambridge University Press, New York, 2006.

[16] C. Borgs, J. T. Chayes, H. Kesten, and J. Spencer, *Uniform boundedness of critical crossing probabilities implies hyperscaling*, Random Structures Algorithms, 15 (1999), pp. 368–413.

[17] ———, *The birth of the infinite cluster: finite-size scaling in percolation*, Comm. Math. Phys., 224 (2001), pp. 153–204. Dedicated to Joel L. Lebowitz.

[18] S. R. Broadbent and J. M. Hammersley, *Percolation processes*, in Mathematical Proceedings of the Cambridge Philosophical Society, vol. 53, Cambridge Univ Press, 1957, pp. 629–641.

[19] F. Camia, *Towards conformal invariance and a geometric representation of the 2D Ising magnetization field*, Markov Processes and Related Fields, 18, (2012), pp. 89–110.

[20] F. Camia, R. P. Conijn, and D. Kiss, *Conformal measure ensembles for percolation and the FK-Ising model*, arXiv:1507.01371, (2015).

[21] F. Camia, C. Garban, and C. M. Newman, *Planar Ising magnetization field I. uniqueness of the critical scaling limit*, Ann. Probab., 43 (2015), pp. 528–571.

[22] F. Camia and C. M. Newman, *Two-dimensional critical percolation: the full scaling limit*, Comm. Math. Phys., 268 (2006), p. 1–38.

[23] ———, *Critical percolation exploration path and SLE 6: a proof of convergence*, Probab. Theory Rel. Fields, 139 (2007), pp. 473–519.

[24] ———, *Ising (conformal) fields and cluster area measures*, Proceedings of the National Academy of Sciences, 106 (2009), pp. 5457–5463.

[25] J. Cardy, *Lectures on conformal invariance and percolation*, Arxiv:0103018, (2001).

[26] J. L. Cardy, *Critical percolation in finite geometries*, Journal of Physics A: Mathematical and General, 25 (1992), p. L201.

[27] D. Chelkak, *Robust discrete complex analysis: a toolbox*, arXiv:1212.6205, (2012). To appear in the Annals of Probability.

[28] D. Chelkak, H. Duminil-Copin, and C. Hongler, *Crossing probabilities in topological rectangles for the critical planar FK-Ising model*, arXiv:1312.7785, (2013).

[29] D. Chelkak, H. Duminil-Copin, C. Hongler, A. Kemppainen, and S. Smirnov, *Convergence of Ising interfaces to Schramm's SLE curves*, Comptes Rendus Mathematique, 352 (2014), pp. 157–161.

[30] D. Chelkak, C. Hongler, and K. Izyurov, *Conformal invariance of spin correlations in the planar Ising model*, Annals of Mathematics, 181 (2015), pp. 1087–1138.

[31] R. P. Conijn, *Factorization formulas for 2D critical percolation, revisited*, arXiv:1502.04387, (2015). To appear in Stochastic Process. Appl.

[32] J. T. Cox and G. R. Grimmett, *Central limit theorems for associated random variables and the percolation model*, Ann. Probab., 12 (1984), pp. 514–528.

[33] G. Delfino and J. Viti, *On three-point connectivity in two-dimensional percolation*, Journal of Physics. A, Mathematical and Theoretical, 44 (2011).

[34] J. Dubédat, *Excursion decompositions for SLE and Watts' crossing formula*, Probab. Theory Rel. Fields, 134 (2006), pp. 453–488.

[35] H. Duminil-Copin, C. Hongler, and P. Nolin, *Connection probabilities and rsw-type bounds for the two-dimensional fk ising model*, Comm. Pure and Applied Math., 64 (2011), pp. 1165–1198.

[36] H. Duminil-Copin and V. Tassion, *A new proof of the sharpness of the phase transition for Bernoulli percolation on Z^d*, arXiv:1502.03051, (2015).

[37] C. G. Esseen, *On the concentration function of a sum of independent random variables*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 9 (1968), pp. 290–308.

[38] C. Garban, G. Pete, and O. Schramm, *Pivotal, cluster, and interface measures for critical planar percolation*, J. Amer. Math. Soc., 26 (2013), pp. 939–1024.

[39] ——, *The scaling limits of near-critical and dynamical percolation*, arXiv:1305.5526v2, (2013), pp. 1–86.

[40] G. R. Grimmett, *The stochastic random-cluster process and the uniqueness of random-cluster measures*, Ann. Probab., (1995), pp. 1461–1510.

[41] ——, *Percolation*, vol. 321 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, second ed., 1999.

[42] ——, *The Random-Cluster Model*, Springer-verlag, Berlin Heidelberg, 2006.

[43] ——, *Probability on Graphs*, Cambridge University Press, Cambridge, 2010.

[44] G. R. Grimmett and H. Kesten, *Percolation since Saint-Flour*, arXiv:1207.0373, (2012).

[45] J. M. Hammersley, *Percolation processes: Lower bounds for the critical probability*, The Annals of Mathematical Statistics, (1957), pp. 790–795.

[46] T. E. Harris, *A lower bound for the critical probability in a certain percolation process*, in Mathematical Proceedings of the Cambridge Philosophical Society, vol. 56, Cambridge Univ Press, 1960, pp. 13–20.

[47] R. van der Hofstad and W. Kager, *Pattern theorems, ratio limit theorems and Gumbel maximal clusters for random fields*, J. Stat. Phys., 130 (2008), pp. 503–522.

[48] R. van der Hofstad and F. Redig, *Maximal clusters in non-critical percolation and related models*, J. Stat. Phys., 122 (2006), pp. 671–703.

[49] C. Hongler and S. Smirnov, *Critical percolation: the expected number of clusters in a rectangle*, Probab. Theory Rel. Fields, 151 (2011), pp. 735–756.

[50] A. A. Járai, *Incipient infinite percolation clusters in 2D*, Ann. Probab., 31 (2003), pp. 444–485.

[51] A. Kemppainen, *Private communication to D. Kiss.*

[52] A. Kemppainen and S. Smirnov, *Random curves, scaling limits and Loewner evolutions*, arXiv:1212.6215v2, (2013).

[53] H. Kesten, *The critical probability of bond percolation on the square lattice equals 1/2*, Comm. Math. Phys., 74 (1980), pp. 41–59.

[54] ——, *The incipient infinite cluster in two-dimensional percolation*, Probab. Theory Rel. Fields, 73 (1986), pp. 369–394.

[55] ——, *Scaling relations for 2D-percolation*, Comm. Math. Phys., 109 (1987), pp. 109–156.

[56] D. Kiss, *Frozen percolation in two dimensions*, Probab. Theory Rel. Fields, (2013), pp. 1–56.

[57] ——, *Large deviation bounds for the volume of the largest cluster in 2D critical percolation*, Electron. Commun. Probab., 19 (2014), pp. no. 32, 1–11.

[58] P. Kleban, J. J. H. Simmons, and R. M. Ziff, *Anchored critical percolation clusters and 2D electrostatics*, Physical review letters, 97 (2006), p. 115702.

[59] I. A. Kovács, F. Iglói, and J. Cardy, *Corner contribution to percolation cluster numbers*, Phys. Rev. B, 86 (2012), pp. 1–6.

[60] G. F. Lawler, O. Schramm, and W. Werner, *Values of Brownian intersection exponents, I: Half-plane exponents*, Acta Math, 187 (2001), pp. 237–273.

[61] ——, *One-arm exponent for critical 2D percolation*, Electron. J. Probab., 7 (2002), pp. no. 2, 13 pp. (electronic).

[62] ——, *Conformal invariance of planar loop-erased random walks and uniform spanning trees*, Ann. Probab., 32 (2004), pp. 939–995.

[63] L. Le Cam, *On the distribution of sums of independent random variables*, in Proc. Internat. Res. Sem., Statist. Lab., Univ. California, Berkeley, Calif, Springer-Verlag, New York, 1965, pp. 179–202.

[64] B. G. Nguyen, *Typical cluster size for two-dimensional percolation processes*, J. Stat. Phys., 50 (1988), pp. 715–726.

[65] P. Nolin, *Near-critical percolation in two dimensions*, Electron. J. Probab., 13 (2008), pp. no. 55, 1562–1623.

[66] L. Onsager, *Crystal statistics. I. a two-dimensional model with an order-disorder transition*, Physical Review, 65 (1944), p. 117.

[67] M. D. Penrose, *A central limit theorem with applications to percolation, epidemics and Boolean models*, Ann. Probab., (2001), pp. 1515–1546.

[68] S. Rohde and O. Schramm, *Basic properties of SLE*, Annals of Mathematics, 161 (2005), pp. 883–924.

[69] L. Russo, *A note on percolation*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, 43 (1978), pp. 39–48.

[70] ———, *On the critical percolation probabilities*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, 56 (1981), pp. 229–237.

[71] O. Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math., 118 (2000), pp. 221–288.

[72] P. D. Seymour and D. J. A. Welsh, *Percolation probabilities on the square lattice*, Advances in graph theory, 3 (1978), pp. 227–245.

[73] S. Sheffield and D. B. Wilson, *Schramm's proof of Watts' formula*, Ann. Probab., 39 (2011), pp. 1844–1863.

[74] J. J. H. Simmons, P. Kleban, and R. M. Ziff, *Exact factorization of correlation functions in two-dimensional critical percolation*, Phys. Rev. E, 76 (2007), p. 041106.

[75] ———, *Percolation crossing formulae and conformal field theory*, Journal of Physics A: Mathematical and Theoretical, 40 (2007), p. F771.

[76] J. J. H. Simmons, R. M. Ziff, and P. Kleban, *Factorization of percolation density correlation functions for clusters touching the sides of a rectangle*, Journal of Statistical Mechanics: Theory and Experiment, 2009 (2009), p. P02067.

[77] S. Smirnov, *Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits*, C. R. Acad. Sci. Paris Sér. I Math., 333 (2001), pp. 239–244.

[78] ———, *Conformal invariance in random cluster models. I. holmorphic fermions in the Ising model*, Annals of Mathematics, 172 (2010), pp. 1435–1467.

[79] S. Smirnov and O. Schramm, *On the scaling limits of planar percolation*, Ann. Probab., 39 (2011), p. 1768–1814. (with an appendix by Garban, C.).

[80] N. Sun, *Conformally invariant scaling limits in planar critical percolation*, Probab. Surv., 8 (2011), pp. 155–209.

[81] W. Werner, *Random planar curves and Schramm-Loewner evolutions*, in Lectures on probability theory and statistics, Springer, 2004, pp. 107–195.

[82] ——, *Lectures on two-dimensional critical percolation*, in Statistical mechanics, vol. 16 of IAS/Park City Math. Ser., Amer. Math. Soc., Providence, RI, 2009, pp. 297–360.

[83] T. T. Wu, *Theory of Toeplitz determinants and the spin correlations of the two-dimensional Ising model. I*, Phys. Rev., 149 (1966), pp. 380–401.

[84] R. Yu, H. Saleur, and S. Haas, *Entanglement entropy in the two-dimensional random transverse field ising model*, Physical Review B, 77 (2008), p. 140402.

[85] Y. Zhang, *A martingale approach in the study of percolation clusters on the* $\mathbf{Z}^{\mathbf{d}}$ *lattice*, J. Theoret. Probab., 14 (2001), pp. 165–187.

[86] R. M. Ziff, J. J. H. Simmons, and P. Kleban, *Factorization of correlations in two-dimensional percolation on the plane and torus*, Journal of Physics A: Mathematical and Theoretical, 44 (2011), p. 065002.