On the strong connectivity problem

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We show that the strong connectivity problem is solvable in polynomial time in case each value α in the distance matrix with $0<\alpha<\infty$ is contained in a submatrix of form $\begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}$ (up to permuting rows or columns), thus extending a result of Lucchesi.

The *strong connectivity augmentation problem* is:

(1) *given*: a directed graph $G = (V,A)$, a length function $l: V \times V \to \mathbb{Z}_+$ and an integer B,
find: a set $A' \subseteq V \times V$ so that the graph $(V, A \cup A')$ is strongly connected and so that $\Sigma_{a \in A'} l(a) < B$.

(cf. Garey and Johnson [3]). This problem is easily seen to be NP-complete, since the problem of finding a Hamiltonian cycle in a directed graph (V, A'') is reducible to (1): just take $A := \emptyset$, $l: V \times V \to \mathbb{Z}_+$ defined by:

(2) $$l(u,v) := 0 \text{ if } (u,v) \in A'', \quad := 1 \text{ if } (u,v) \notin A'',$$

and $B := 1$ (cf. Eswaran and Tarjan [1]).

In fact the *traveling salesman problem*:

(3) *given*: a length function $l': V \times V \to \mathbb{Z}_+$ and an integer B',
find: a Hamiltonian cycle of length less than B'

is a direct special case of (1) (take $A := \emptyset$, $l(u,v) := l'(u,v) + B'$ and $B := B'|V| + B'$).

Another application of the strong connectivity augmentation problem is the planar feedback arc set problem (see below).

The strong connectivity augmentation problem is trivially equivalent to the *strong connectivity problem*:

(4) *given*: a length function $l: V \times V \to \mathbb{Z}_+ \cup \{\infty\}$ and an integer B,
find: a subset $A' \subseteq V \times V$ so that (V, A') is strongly connected and so that $\Sigma_{a \in A'} l(a) < B$.

Indeed, (4) is just the case $A = \emptyset$ in (1). Conversely, (1) can be reduced to (4) by resetting $l(a) := 0$ whenever $a \in A$. Allowing $l(a) = \infty$ in (4) is irrelevant: we could replace any ∞ by the value B.

We may assume in (4) without loss of generality that for all $i,j,k \in V$:

(5) (i) $l(i,i) = 0$
(ii) if $l(i,j) = 0$ and $l(j,k) = 0$ then $l(i,k) = 0$.

It was shown by Lucchesi [5] (cf. Frank [2] and Lucchesi and Younger [6]) that the strong connectivity problem (4) is solvable in polynomial time if the following condition on the length function holds:

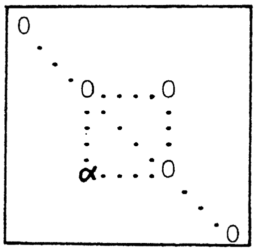
(6) for each $i,j \in V$: if $0 < l(i,j) < \infty$ then $l(j,i) = 0$.

Equivalently, the strong connectivity augmentation problem is solvable in polynomial time if:

(7) for each $i,j \in V$: if $0 < l(i,j) < B$ then $(j,i) \in A$.

So problem (1) is solvable in polynomial time if in the distance table we have that for each value α with $0 < \alpha < \infty$, the symmetric value is equal to 0:

(8)



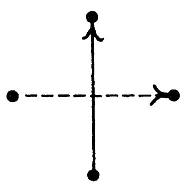
Lucchesi showed that this implies a polynomial-time algorithm for the following *feedback arc set problem*, in case G is planar:

(9) *given*: a directed graph $G = (V,A)$, a length function $l: A \rightarrow \mathbb{Z}_+$ and an integer B,
find: a subset $A' \subseteq A$ so that (V,A') is acyclic and so that $\Sigma_{a \in A'} l(a) > B$.

In general, this problem is NP-complete (Karp [4]).

To see that (9) is solvable in polynomial time if G is planar, we consider the planar dual graph $G^* = (F,A^*)$ of G, directed in such a way that each arc of G crosses its dual arc in G^* 'from left to right':

(10)



(where the uninterrupted arrow is an arc of G, and the interrupted arc is the dual arc in G^*). Define for each pair $(f,g) \in F \times F$:

(11) $\left.\begin{array}{l} l^*(f,g) := 0 \\ l^*(g,f) := l(a) \end{array}\right\}$ $\left\{\begin{array}{l}\text{if } (f,g) = a^* \in A^*\text{, where} \\ a^* \text{ is the dual arc of } a \in A,\end{array}\right.$
$l^*(f,g) := \infty$ for all other pairs (f,g).

Let $B^* := \Sigma_{a \in A} l(a) - B$. Then for each subset A' of A one has:

(12) (V,A') is acyclic $\Leftrightarrow (F, A^* \cup [(A \setminus A')^*]^{-1})$ is strongly connected

(here C^{-1} denotes the set of inverse arcs of C). Moreover,

(13) $\sum_{a \in A'} l(a) > B \Leftrightarrow \sum_{(g,f) \in [(A \setminus A')^*]^{-1}} l^*(g,f) < B^*.$

This reduces the planar feedback arc set problem to the strong connectivity problem satisfying (6). Hence it is solvable in polynomial time.

Lucchesi's algorithm can also be used in a branch and bound method to solve the general strong connectivity problem. Typically, during the branching process, a node of the tree is labeled by a set R of 'required' arcs and a set F of 'forbidden' arcs. That is, the node only considers those subsets A' of $V \times V$ for which $R \subseteq A' \subseteq (V \times V) \setminus F$ and for which (V,A') is strongly connected. So the bound corresponding to the node should be a lower bound on the minimum length of these subsets A'.

In order to find such a bound, we can assume that R is reflexive (i.e., $(i,i) \in R$ for all i) and transitive (i.e., if (i,j) and (j,k) belong to R, then $(i,k) \in R$). Moreover, we can reset

(14) $l(a) := 0$ if $a \in R$,
$l(a) := \infty$ if $a \in F$.

If after this resetting, Lucchesi's condition:

(15) for all $i,j \in V$: if $0 < l(i,j) < \infty$ then $l(j,i) = 0$

is satisfied, Lucchesi's algorithm gives us the exact minimum value (instead of a lower bound) in polynomial time. This suggests that in our branching strategy, we should strive for a situation where (15) holds. That is, for choices of R and F satisfying:

(16) for all $i,j \in V$: $(i,j) \in R$, or $(j,i) \in R$, or both $(i,j) \in F$ and $(j,i) \in F$.

We show that the strong connectivity problem can also be solved in polynomial time if we weaken Lucchesi's condition (15) to:

(17) for all $i,j \in V$: if $0 < l(i,j) < \infty$ then $\exists i', j' \in V$ with
$l(i,i') = l(j',i') = l(j',j) = 0$.

This is indeed weaker than Lucchesi's condition, since if (15) holds we can take $i' = i$ and $j' = j$ in (17).

Condition (17) means that in the distance table we have that any value with $0 < \alpha < \infty$ is part of a 2×2-matrix $\begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}$ as in:

(18) $$\begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}$$

So the difference with Lucchesi's condition is that the diagonal elements of $\begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}$ need not be diagonal elements of the distance matrix.

THEOREM. *The strong connectivity problem is solvable in polynomial time if* (17) *is satisfied.*

PROOF. Let l satisfy (18). We may assume furthermore that $l(i,i) = 0$ for all $i \in V$, and if $l(i,j) = l(j,k) = 0$ then $l(i,k) = 0$ for all $i,j,k \in V$.

Suppose now that $0 < l(i,j) < \infty$ for some $i,j \in V$ while $l(j,i) \neq 0$. By (17) there exist $i',j' \in V$ so that $l(j',i) = l(j',i') = l(j,i') = 0$. We introduce two new points, i'' and j'' say. Let $\bar{V} := V \cup \{i'',j''\}$, and

$$\text{(19)} \quad \begin{aligned} &\bar{l}(a,b) := l(a,b) \quad \text{if } a,b \in V, (a,b) \neq (i,j), \\ &\bar{l}(i,j) := \infty \\ &\bar{l}(i,i'') := \bar{l}(i'',i') := \bar{l}(j',j'') := \bar{l}(j',j'') := \bar{l}(j'',j) := 0, \\ &\bar{l}(a,b) := \infty \quad \text{for all other } a,b \in \bar{V}. \end{aligned}$$

We show that the strong connectivity problem for $\bar{V},\bar{l}$ is equivalent to that for V,l. First, let A be a minimum length subset of $V \times V$ with (V,A) strongly connected. Let:

$$\text{(20)} \quad \begin{aligned} &\bar{A} := A \cup \{(i,i''),(i'',i'),(j'',i''),(j',j''),(j'',j)\} && \text{if } (i,j) \notin A, \\ &\bar{A} := (A \setminus \{(i,j)\}) \cup \{(i,i''),(i'',i'),(j'',i''),(j',j''),(j'',j),(i'',j'')\} && \text{if } (i,j) \in A. \end{aligned}$$

Clearly,

$$\text{(21)} \quad \sum_{a \in A} l(a) = \sum_{a \in \bar{A}} \bar{l}(a).$$

Moreover, $(\bar{V},\bar{A})$ is strongly connected. This follows directly from (20) if $(i,j) \notin A$. If $(i,j) \in A$, then $(i,i''),(i'',j''),(j'',j)$ form a path in $\bar{A}$ from i to j. Hence also in this case, $(\bar{V},\bar{A})$ is strongly connected.

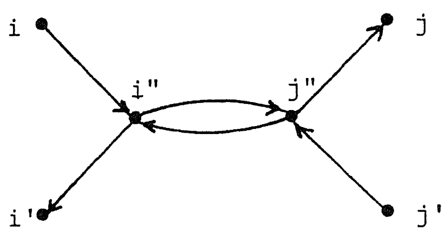
Conversely, let $\bar{A}$ be a minimum length subset of $\bar{V} \times \bar{V}$ with $(\bar{V},\bar{A})$ strongly connected. Without loss of generality, if $l(a,b) = 0$ then $(a,b) \in \bar{A}$. Let:

$$\text{(22)} \quad \begin{aligned} &A := \bar{A} \cap (V \times V) && \text{if } (i'',j'') \notin \bar{A}, \\ &A := (\bar{A} \cap (V \times V)) \cup \{(i,j)\} && \text{if } (i'',j'') \in \bar{A}. \end{aligned}$$

Again (21) holds. Moreover (V,A) is strongly connected. To see this, take

$a,b \in V$. We show that A contains a path from a to b. Since $(\bar{V},\bar{A})$ is strongly connected, $\bar{A}$ contains a path P from a to b. Assume that P passes i'' and j'' as few as possible. If P does not traverse i'' nor j'', it is also a path in A. So supposes P traverses i'' or j''. Consider all arcs incident to i'' or j'' with finite length:

(23)



Since $(i,i'),(j',i'),(j',j) \in A$, and since $(i,j) \in A$ if $(i'',j'') \in \bar{A}$, it follows that P does not intersect $\{i'',j''\}$.

So replacing V,l by $\bar{V},\bar{l}$ gives an equivalent problem, and decreases the number of pairs (i,j) with $0 < l(i,j) < \infty$ and $l(j,i) \neq 0$. Therefore, after at most $|V|^2$ such replacements, we attain an equivalent strong connectivity problem satisfying Lucchesi's condition. This is solvable in polynomial time by Lucchesi's algorithm. □

This theorem suggests that in a branch and bound process, our branching strategy should strive for a situation where the following holds:

(24) for all $i,j \in V$: $(i,j) \in F$, or $(i,i'),(j',i'),(j',j) \in R$ for some $i',j' \in V$.

(The second alternative includes the case $(i,j) \in R$, by taking $i' = j$, $j' = i$.)

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