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Applications of Polyhedral Combinatorics to Multicommodity Flows and Compact Surfaces

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ABSTRACT. We survey results on graphs and curves on compact surfaces and outline the proof methods, which make essential use of polyhedral combinatorics. We derive results on planar multicommodity flows.

1. Introduction

In this paper we give a survey of some recent results on multicommodity flows and compact surfaces, derived with the help of methods from polyhedral combinatorics. For several of the results obtained we know, at this moment, no other proof method than polyhedral methods.

In fact, these polyhedral methods are none other than two well-known variants of Farkas' lemma. Let $a_1, \dots, a_k, b_1, \dots, b_m$ be vectors in $\mathbf{R}^n$. The first variant is the "blocking polyhedron theorem" of Fulkerson [**2**]:

(1) if the polyhedron $\{x \in \mathbf{R}^n | x \geq 0; a_i^{\mathrm{T}} x \geq 1 \ (i = 1, \dots, k)\}$ has vertices $b_1, \dots, b_m$, then the polyhedron $\{x \in \mathbf{R}^n | x \geq 0; b_j^{\mathrm{T}} x \geq 1 \ (j = 1, \dots, m)\}$ has all its vertices among $a_1, \dots, a_k$

(assuming $a_1, \dots, a_k \geq 0$).

The second variant is the "cone-form" of Farkas' lemma:

(2) if the convex cone $\{x \in \mathbf{R}^n | a_i^{\mathrm{T}} x \geq 0 \ \ (i = 1, \dots, k)\}$ is generated by $b_1, \dots, b_m$, then the convex cone $\{x \in \mathbf{R}^n | b_j^{\mathrm{T}} x \geq 0 \ (j = 1, \dots, m)\}$ is generated by $a_1, \dots, a_k$.

The first variant is applied to graphs embedded on the *Klein bottle* (Section 2), and the second variant is applied to graphs embedded on compact *orientable* surfaces (Section 3).

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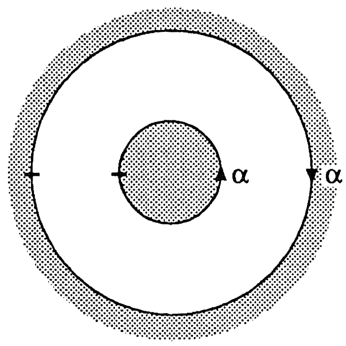
2. The Klein bottle

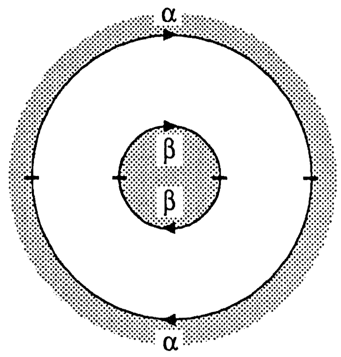
We first focus on the Klein bottle and its relations to planar multicommodity flows. The *Klein bottle* is a compact surface usually represented as follows. Consider an annulus (= cylinder) and identify the inner and outer boundaries, in *opposite* orientation. Schematically:

(3)



There is an alternative way of obtaining the Klein bottle from the annulus: identify opposite points on the outer boundary, and similarly, identify opposite points on the inner boundary. Schematically:

(4)



This corresponds to representing the Klein bottle as a 2-dimensional sphere with two "cross-caps" (one made by the outer boundary in (4), the other by the inner boundary).

The Klein bottle is a nonorientable surface. Hence there are two types of closed curves on the Klein bottle:

(5)
- *orientation-preserving* closed curves: those where the meaning of left and right is unchanged after one turn;
- *orientation-reversing* closed curves: those where the meaning of left and right is flipped after one turn.

It is not difficult to see that a closed curve is orientation-preserving (orientation-reversing, respectively) if it traverses the cross-caps an even (odd, respectively) number of times.

Now let $G = (V, E)$ be a graph embedded on the Klein bottle S. (By a graph we mean an *undirected* graph. *Embedding* assumes nonintersecting edges except for their end vertices. *Cellularly embedding* means that all faces are homeomorphic to open disks. We identify an embedded graph with its image.)

We will be interested in the orientation-reversing circuits in G. (A *circuit* is a simple closed curve in G. We identify a circuit with the set of edges traversed by it.)

Call a set B of edges a *blocker* if it intersects each orientation-reversing circuit. In **[10]** we proved the following min-max relations:

(6) (i) If G is bipartite, then the minimum size of an orientation-reversing circuit is equal to the maximum of pairwise disjoint blockers.
(ii) If G is Eulerian, then the minimum size of a blocker is equal to the maximum number of pairwise edge-disjoint orientation-reversing circuits.

(Here a graph is *Eulerian* if all degrees are even.)

We here sketch a proof of these equalities. In fact, we first show (6)(i), and next derive (ii) from (i) with the blocking polyhedron variant (1) of Farkas' lemma.

The starting point in the proof method is the following result proved in [7]:

THEOREM 1. *Let $G = (V, E)$ be a planar bipartite graph embedded in the plane. Let I_1 and I_2 be two of its faces. Then there exist pairwise edge-disjoint cuts $\delta(X_1), \ldots, \delta(X_t)$ so that for each two vertices v, w with $v, w \in \mathrm{bd}(I_1)$ or $v, w \in \mathrm{bd}(I_2)$, the distance in G from v to w is equal to the number of cuts $\delta(X_j)$ separating v and w.*

Here $\delta(X)$ denotes the set of nonloop edges of G with exactly one end point in X. The cut $\delta(X)$ is said to *separate* v and w if $v \neq w$ and $|\{v, w\} \cap X| = 1$. By bd(..) we denote the boundary of .. . Faces are considered as *open* regions.

From Theorem 1 we derive (6)(i):

THEOREM 2. *Let $G = (V, E)$ be a bipartite graph embedded on the Klein bottle S. Then the minimum length of any orientation-reversing circuit in G is equal to the maximum number of pairwise disjoint blockers.*

PROOF. Clearly, the maximum is not larger than the minimum. To show equality, we may assume that each face of G is orientable, i.e., contains no cross-cap. Indeed, if a face contains a cross-cap, we can add a path to G over this cross-cap in such a way that the graph remains bipartite and such that the minimum length of any orientation-reversing circuit remains unchanged (by taking the path with length large enough and of appropriate parity).

Let C_1 be a minimum-length orientation-reversing circuit in G, say with length t_1. We "cut open" the Klein bottle S along C_1. In this way we obtain a bordered surface S', with a circle B_1 as border, so that S arises from S' by identifying opposite points on B_1. So S' is a Möbius strip. Let $i: S' \to S$ denote the identification map. The graph $G' := i^{-1}[G]$ is a bipartite graph on S', and $B_1 = i^{-1}[C_1]$.

Let C_2 be a minimum-length orientation-reversing circuit in G' (on S'), say with length t_2. We may assume that C_2 is edge-disjoint from B_1 (by adding parallel edges). Next we "cut open" the Möbius strip S' along C_2. We now obtain an annulus S'', with two circles B_1 and B_2 as boundaries (in the ideal case where C_2 is vertex-disjoint from B_1—the general case is similar).

The Klein bottle S arises from the annulus S'' by identifying opposite points on B_1 and by identifying opposite points on B_2. Let $i': S'' \to S$ be the identification map, and let $G'' := (i')^{-1}[G]$. So G'' is a planar bipartite graph, embeddable in the plane $\mathbf{R}^2$, in such a way that two of its faces I_1 (= unbounded face) and I_2 have the following properties:

(7)
(i) the boundary of I_1 is a circuit B_1 of length $2t_1$, and the boundary of I_2 is a circuit B_2 of length $2t_2$;
(ii) S arises from $\mathbf{R}^2 \setminus (I_1 \cup I_2)$ by identifying pairs of opposite points on B_1 and by identifying pairs of opposite points on B_2.

In fact, we identify S'' and $\mathbf{R}^2 \setminus (I_1 \cup I_2)$.

Since t_1 is the minimum length of an orientation-reversing circuit in G, each pair of opposite vertices on B_1 has distance exactly t_1. Similarly, since t_2 is the minimum length of an orientation-reversing circuit in G', each pair of opposite vertices on B_2 has distance exactly t_2.

By Theorem 1, there exist pairwise disjoint cuts $\delta(X_1), \ldots, \delta(X_t)$ so that for each two vertices v and w of G'' with $v, w \in \mathrm{bd}(I_1)$ or $v, w \in \mathrm{bd}(I_2)$, the distance in G'' from v to w is equal to the number of cuts $\delta(X_j)$ separating v and w. We may assume that each $\delta(X_j)$ separates at least one such pair v, w (all other cuts can be deleted).

Each cut $\delta(X_j)$ intersects any subpath P of B_1 of length t_1 at most once (as P is intersected by t_1 of the $\delta(X_j)$, as P is a shortest path between its end points). So if $\delta(X_j)$ intersects B_1, it intersects B_1 exactly twice, in two opposite edges. Similarly, if $\delta(X_j)$ intersects B_2, it intersects B_2 exactly twice, in two opposite edges.

We classify $\delta(X_1), \ldots, \delta(X_t)$ into three classes:

(8) (i) those intersecting both B_1 and B_2, say $\delta(X_1), \ldots, \delta(X_s)$;
(ii) those intersecting B_1 but not B_2, say $\delta(X_{s+1}), \ldots, \delta(X_{t_1})$;
(iii) those intersecting B_2 but not B_1, say $\delta(X_{t_1+1}), \ldots, \delta(X_t)$.

Note that B_2 is intersected by exactly t_2 of the $\delta(X_j)$, and hence $t_2 = s + (t - t_1)$, i.e., $s = t_1 + t_2 - t$.

Now it is not difficult to see that the images of the $\delta(X_j)$, properly composed, give blockers in G as required. In fact, we can take:

(9)
$$i'[\delta(X_1)], \ldots, i'[\delta(X_s)], i'[\delta(X_{s+1}) \cup \delta(X_{t_1+1})], \ldots, i'[\delta(X_{t_1}) \cup \delta(X_{2t_1-s})]. \quad \square$$

A standard corollary in polyhedral combinatorics now is:

THEOREM 3. *Let $G = (V, E)$ be a graph embedded on the Klein bottle S. Then each vertex of the polytope in $\mathbf{R}^E$ determined by*

(10)
$$\text{(i) } x(e) \geq 0 \quad (e \in E),$$
$$\text{(ii) } \sum_{e \in C} x(e) \geq 1 \quad (C \textit{ orientation-reversing circuit})$$

is the incidence vector of some blocker.

PROOF. Let x be a positive rational vector satisfying (10). We show that there exist blockers $B_1, \ldots, B_t$ and rationals $\lambda_1, \ldots, \lambda_t > 0$ so that $\lambda_1 + \cdots + \lambda_t = 1$ and so that

$$x \geq \lambda_1 \chi^{B_1} + \cdots + \lambda_t \chi^{B_t} \tag{11}$$

(where χ^B denotes the incidence vector of B). This suffices to prove the theorem.

Let N be a natural number so that $Nx(e)$ is an even integer for each edge e. Replace each edge of G by a path of length $Nx(e)$ (that is, put $Nx(e) - 1$ new vertices on e). We obtain a bipartite graph G'. Let C' be a minimum-length orientation-reversing circuit in G', of length t, say. As x satisfies (10), we know $t \geq N$. By Theorem 2 there exist t pairwise edge-disjoint blockers $B'_1, \ldots, B'_t$ in G'. Their "projections" to G give t blockers $B_1, \ldots, B_t$ in G with the property that each edge e of G is contained in at most $Nx(e)$ of the B_j. Hence

$$tx \geq Nx \geq \chi^{B_1} + \cdots + \chi^{B_t}. \tag{12}$$

Taking $\lambda_j := 1/t$ for each j gives (11). $\square$

Lehman's theorem (1) now implies the dual statement of Theorem 3:

THEOREM 4. *Let $G = (V, E)$ be a graph embedded on the Klein bottle S. Then the vertices of the polytope in $\mathbf{R}^E$ determined by*

$$(13) \qquad \begin{aligned} &\text{(i) } x(e) \geq 0 \quad (e \in E), \\ &\text{(ii) } \sum_{e \in B} x(e) \geq 1 \quad (B \subseteq E,\ B \text{ blocker}) \end{aligned}$$

are exactly the characteristic vectors of orientation-reversing circuits.

This is in fact the fractional packing version of (6)(ii). We derive the integer packing result (6)(ii):

THEOREM 5. *Let $G = (V, E)$ be an Eulerian graph embedded on the Klein bottle S. Then the minimum size of a blocker is equal to the maximum number of pairwise edge-disjoint orientation-reversing circuits.*

PROOF. Clearly, the maximum is not more than the minimum. Suppose equality does not hold, and let G form a counterexample with

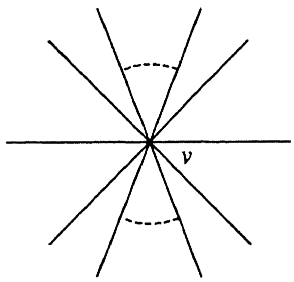
$$(14) \qquad \sum_{v \in V} 2^{\deg(v)}$$

as small as possible (where $\deg(v)$ denotes the degree of v). Then

(15) each vertex of G has degree at most 4.

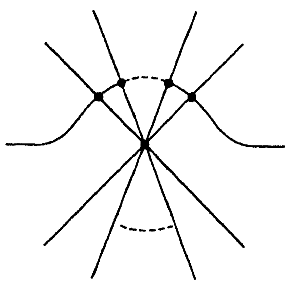
For suppose v has degree at least 6:

(16)



Replace this (on the Klein bottle) by:

(17)

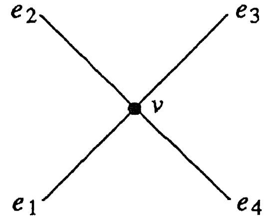


This modification does not change the minimum size, t say, of a blocker, as one may check. However, it reduces the sum (14), so in the new graph there exist t pairwise edge-disjoint orientation-reversing circuits. This gives also in the original graph t pairwise edge-disjoint orientation-reversing circuits, contradicting our assumption.

This shows (15). Let t be the minimum size of a blocker in G. Hence the vector x with all entries equal to $1/t$ satisfies (13). So by Theorem 4 there exist orientation-reversing circuits $C_1, \ldots, C_k$ (pairwise different) and reals $\lambda_1, \ldots, \lambda_k > 0$ so that

$$\text{(18)} \qquad \begin{aligned} &\text{(i) } \lambda_1 + \cdots + \lambda_k = 1, \\ &\text{(ii) } \lambda_1 \chi^{C_1} + \cdots + \lambda_k \chi^{C_k} \leq x. \end{aligned}$$

Consider a vertex v of G of degree 4, and the edges e_1, e_2, e_3, e_4 incident to v in cyclic order:

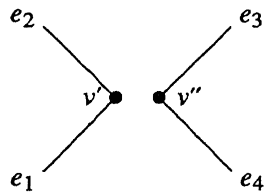
(19) 

Thus e_1 and e_3 are "opposite" in v, and similarly, e_2 and e_4 are opposite in v. We show that for each circuit C_i

$$\text{(20)} \qquad \begin{aligned} &\text{(i) } C_i \text{ traverses } e_1 \Longleftrightarrow C_i \text{ traverses } e_3, \\ &\text{(ii) } C_i \text{ traverses } e_2 \Longleftrightarrow C_i \text{ traverses } e_4. \end{aligned}$$

Having shown this for each vertex v and each C_i, it follows that $C_1, \ldots, C_k$ are pairwise edge-disjoint. Since $k \geq t$ (since $\lambda_i \leq 1/t$ for each i), this proves the theorem.

If (20) does not hold, we may assume without loss of generality that C_1 traverses e_1 and e_2. Replace (19) by

(21) 

Let G' be the new graph obtained. So G arises from G' by identifying v' and v''. Graph G' is Eulerian again, with sum (14) smaller than for G. So by the minimality assumption, the theorem to be proved holds for G'.

Let t' be the minimum size of a blocker in G'. If $t' \geq t$, there exist t pairwise edge-disjoint orientation-reversing circuits in G', and hence also in G, contradicting our assumption. So $t' < t$. By the Euler condition, $t' \leq t - 2$. Let B' be a blocker of size t' in G'. Then $B := B' \cup \{e_1, e_2\}$ is a blocker of size at most $t' + 2$ in G. Since $|B| \geq t$, we know $|B| = t$.

Since $|C_i \cap B| \geq 1$ while $|C_1 \cap B| > 1$, this gives the contradiction:

$$(22)\qquad \begin{aligned} 1 &= \sum_{e \in B} 1/t \geq \sum_{e \in B} (\lambda_1 \chi^{C_1}(e) + \cdots + \lambda_k \chi^{C_k}(e)) \\ &= \lambda_1 |C_1 \cap B| + \lambda_2 |C_2 \cap B| + \cdots + \lambda_k |C_k \cap B| \\ &> \lambda_1 + \cdots + \lambda_k = 1 . \quad \Box \end{aligned}$$

Theorem 5 has a number of corollaries. First, a theorem of Lins [**3**] follows, which is in fact the analogue of Theorem 5 for the projective plane. Note that the orientation-reversing circuits in the projective plane are exactly the non-null-homotopic circuits, and exactly the nonseparating circuits.

THEOREM 6 (Lins' theorem). *Let $G = (V, E)$ be an Eulerian graph embedded in the projective plane. Then the maximum number of pairwise edge-disjoint non-null-homotopic circuits in G is equal to the minimum number of edges intersected by any non-null-homotopic closed curve not intersecting V.*

PROOF. This follows directly from Theorem 5 by putting a cross-cap in one of the faces of G, thus transforming the projective plane to a Klein bottle. Note that the minimal blockers in G are exactly the minimal sets of edges intersected by some non-null-homotopic closed curve not intersecting V. $\Box$

Theorem 5 also implies two results on planar multicommodity flows. Let $G = (V, E)$ be a graph, and let $r_1, \ldots, r_k, s_1, \ldots, s_k$ be vertices of G (so that $r_i \neq s_i$ for all i). Clearly, the following *cut condition* is a necessary condition for the existence of pairwise edge-disjoint paths $P_1, \ldots, P_k$ where P_i connects r_i and s_i $(i = 1, \ldots, k)$:

(23) (cut condition): for each $X \subseteq V$: $|\delta(X)| \geq$ number of pairs r_i, s_i separated by $\delta(X)$.

Simple examples show that this cut condition is not sufficient in general.

However, an *Euler condition* turns out to be quite helpful:

(24) (Euler condition) the graph $(V, E \cup \{\{r_1, s_1\}, \ldots, \{r_k, s_k\}\})$ is Eulerian.

First we sketch how to derive

THEOREM 7. *Let $G = (V, E)$ be a planar graph embedded in the plane $\mathbf{R}^2$ and let $r_1, \ldots, r_k, s_1, \ldots, s_k$ be vertices of G so that the Euler condition holds. Let $r_1, \ldots, r_k$ be incident to the unbounded face I_1 in clockwise order. Let $s_1, \ldots, s_k$ be incident to some other face I_2 in counterclockwise order. Then there exist pairwise edge-disjoint paths $P_1, \ldots, P_k$ where P_i connects r_i and s_i $(i = 1, \ldots, k)$ if and only if the cut condition is satisfied.*

PROOF. Let the cut condition be satisfied. Extend $\mathbf{R}^2 \setminus (I_1 \cup I_2)$ to the Klein bottle, by adding a cylinder between the boundaries of I_1 and I_2. Extend G to a graph G' on the Klein bottle by adding edges $e_1, \ldots, e_k$

over this cylinder, in such a way that e_i connects r_i and s_i $(i = 1, \dots, k)$. Then a circuit in G' is orientation-reversing if and only if it contains an odd number of edges from $e_1, \dots, e_k$. So it suffices to show that G' contains k pairwise edge-disjoint orientation-reversing circuits.

Since G' is Eulerian, we can apply Theorem 5. That is, we must show that each blocker in G' has size at least k. It is not difficult to derive this from the cut condition. □

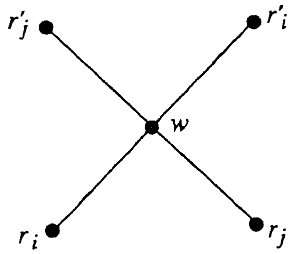
Also a theorem of Okamura [4] can be derived:

THEOREM 8 (Okamura's theorem). *Let $G = (V, E)$ be a planar graph embedded in the plane $\mathbf{R}^2$ and let $r_1, \dots, r_k, s_1, \dots, s_k$ be vertices of G so that the Euler condition holds. Let there exist two faces I_1 and I_2 of G so that for each $i = 1, \dots, k$, $r_i, s_i \in \mathrm{bd}(I_1)$ or $r_i, s_i \in \mathrm{bd}(I_2)$. Then there exist pairwise edge-disjoint paths $P_1, \dots, P_k$ where P_i connects r_i and s_i $(i = 1, \dots, k)$ if and only if the cut condition is satisfied.*

PROOF. Without loss of generality, I_1 is the unbounded face, and $r_1, \dots, r_t, s_1, \dots, s_t \in \mathrm{bd}(I_1)$ and $r_{t+1}, \dots, r_k, s_{t+1}, \dots, s_k \in \mathrm{bd}(I_2)$. By an argument due to S. Lins we may assume that $r_1, \dots, r_t, s_1, \dots, s_t$ occur in this order cyclically around $\mathrm{bd}(I_1)$. To see this, first note that we may assume that the vertices $r_1, \dots, r_t, s_1, \dots, s_t$ are distinct and have degree 1 (as we can add a new vertex of degree 1 to any r_i or s_i and replace this r_i or s_i by the new vertex). Call two pairs r_i, s_i and r_j, s_j on $\mathrm{bd}(I_1)$ *crossing* if $i \neq j$ and r_i, r_j, s_i, s_j occur in this order cyclically around $\mathrm{bd}(I_1)$, clockwise or counterclockwise. Suppose not all pairs r_i, s_i are crossing. Then there exist i, j so that r_i, s_i and r_j, s_j are noncrossing and so that there is no pair r_h, s_h on the part of the boundary of I_1 that connects r_i and s_i and that does not contain s_i or s_j (maybe after exchanging r_i and s_i).

Now we can add in I_1 three new vertices, w, r_i', and r_j', and four new edges as follows:

(25)



Replacing r_i and r_j by r_i' and r_j' does not violate the cut condition. Moreover, any pair of edge-disjoint paths P_i', P_j' in the extended graph, where P_i' connects r_i' and s_i and P_j' connects r_j' and s_j, contains edge-disjoint paths P_i and P_j in the original graph, where P_i connects r_i and s_i and P_j connects r_j and s_j.

Repeating this construction, we end up with $r_1, \dots, r_t, s_1, \dots, s_t$ occurring in this order cyclically around $\mathrm{bd}(I_1)$ (possibly after reordering indices and exchanging r_i and s_i). Similarly, we can assume that $r_{t+1}, \dots, r_k, s_{t+1}, \dots, s_k$ occur in this order cyclically around $\mathrm{bd}(I_2)$.

Now extend $\mathbf{R}^2 \setminus (I_1 \cup I_2)$ to the Klein bottle, by adding two cross-caps (in fact, two Möbius strips) along the boundaries of I_1 and of I_2. Extend G to a graph G' on the Klein bottle by adding edges $e_1, \dots, e_k$ over the cross-caps, in such a way that e_i connects r_i and s_i ($i = 1, \dots, k$). The remainder of the proof is similar to that of Theorem 7. □

Okamura's theorem has as special case the theorem of Okamura and Seymour [5], where $r_1, \dots, r_k, s_1, \dots, s_k$ are all on the boundary of one face.

3. Compact orientable surfaces

We next show how some results on curves and graphs on compact orientable surfaces can be derived with the help of polyhedral combinatorics. Recall that a *compact orientable surface* is a 2-dimensional sphere with a finite number of "handles" added.

Let S be a compact orientable surface. A *closed curve* on S is a continuous function $C: S_1 \to S$, where S_1 is the unit circle. We call two closed curves C and C' *homotopic*, in notation: $C \sim C'$, if C can be shifted continuously to C', *without fixing a base point*; in other words, there exists a continuous function $\Phi: [0, 1] \times S_1 \to S$ so that

(26) $\Phi(0, z) = C(z)$ and $\Phi(1, z) = C'(z)$ for all $z \in S_1$.

We call a closed curve *primitive* if there do not exist a closed curve D and an integer $n \geq 2$ so that $C \sim D^n$.

By $\mathrm{cr}(C, D)$ we denote the number of intersections of C and D (counting multiplicities):

(27) $$\mathrm{cr}(C, D) := |\{(y, z) \in S_1 \times S_1 | C(y) = D(z)\}|.$$

By $\min\mathrm{cr}(C, D)$ we denote the minimum number of intersections of C' and D', ranging over all $C' \sim C$ and $D' \sim D$:

(28) $$\min\mathrm{cr}(C, D) := \min\{\mathrm{cr}(C', D') | C' \sim C, D' \sim D\}.$$

One objective in this section is to derive the following result in combinatorial topology [8]. It describes under which conditions two systems of primitive closed curves are homotopically the same:

THEOREM 17. *Let $C_1, \dots, C_k$ and $C'_1, \dots, C'_{k'}$ be primitive closed curves on S. Then the following are equivalent:*

(i) *$k = k'$ and there exists a permutation π of $\{1, \dots, k\}$ so that $C'_{\pi(i)} \sim C_i$ or $C'_{\pi(i)} \sim C_i^{-1}$ for each $i = 1, \dots, k$;*

(ii) *for each closed curve D on S:*

$$\sum_{i=1}^{k} \min\mathrm{cr}(C_i, D) = \sum_{i=1}^{k'} \min\mathrm{cr}(C'_i, D).$$

The implication (ii) $\Rightarrow$ (i) is the essence of the theorem. It asserts that if two systems of primitive closed curves cannot be shifted to each other, then there exists a closed curve D distinguishing between them. Note that we cannot skip the primitiveness condition.

A second objective is a result in topological graph theory **[11]**. We need some further terminology and notation. If G is a graph embedded on S and D is a closed curve on S, we denote by $\mathrm{cr}(G, D)$ the number of intersections of G and D (counting multiplicities):

(29) $$\mathrm{cr}(G, D) := |\{z \in S_1 | D(z) \in G\}|.$$

By $\mu_G(D)$ we denote the minimum number of intersections of G and D', ranging over all $D' \sim D$:

(30) $$\mu_G(D) := \min\{\mathrm{cr}(G, D') | D' \sim D\}.$$

If G' arises from G by deleting edges and isolated vertices and by contracting nonloop edges, we say that G' is a *minor* of G. It is called a *proper* minor if at least one edge is deleted or contracted. Note that if G' is a minor of G then $\mu_{G'} \le \mu_G$. We call G a *kernel* (on S) if for each proper minor G' of G one has $\mu_{G'} \ne \mu_G$ (i.e., $\mu_{G'}(D) < \mu_G(D)$ for at least one D).

The theorem states that a kernel G is in a sense determined by μ_G:

THEOREM 18. *Let G and G' be cellularly embedded kernels on S with $\mu_G = \mu_{G'}$. Then G' can be obtained from G by a series of the following operations*:

(i) *shifting the graph homotopically over S*;
(ii) *taking the (surface) dual graph*;
(iii) *ΔY-exchange.*

Here we take the dual graph only if the graph is cellularly embedded on S (i.e., every face is a disk). *ΔY-exchange* means replacing a triangular face by a new vertex of degree three, connected by edges to the three vertices of the triangle:

(31)


or conversely.

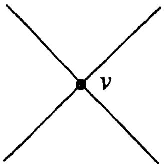
Note that each of the operations (i), (ii), (iii) keeps the function μ_G invariant. For the projective plane the analogue of Theorem 18 was proved by Scott Roundby **[6]**.

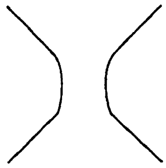
We sketch how Theorems 17 and 18 are proved with the help of polyhedral results. The basic notion is that of a tight graph on S. For any graph

$G = (V, E)$ on S and any closed curve D on S, let $\overline{\mu}_G(D)$ denote the minimum number of intersections of G and D', ranging over all $D' \sim D$ not intersecting V:

(32) $$\overline{\mu}_G(D) := \min\{\mathrm{cr}(G, D') | D' \sim D, D' \text{ does not intersect } V\}.$$

If G is 4-regular and v is a vertex of G, we call replacing

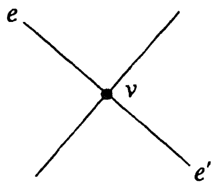
(33) 

by 

opening of G at v (there are two possible openings at v). If G' arises from G by a series of openings, we call G' an *opening* of G. If there is at least one opening, it is called a *proper* opening.

Note that if G' is an opening of G then $\overline{\mu}_{G'} \leq \overline{\mu}_G$. We call G *tight* (*on* S) if for each proper opening G' of G one has $\overline{\mu}_{G'} \neq \overline{\mu}_G$ (i.e., $\overline{\mu}_{G'}(D) < \overline{\mu}_G(D)$ for at least one D).

If G is a 4-regular graph on S, the *straight decomposition* of G is the partition of the edges of G into closed curves obtained as follows. Follow an edge, e say, until one of its end points, v say. Next continue along the edge, e' say, opposite in v to e:

(34) 

Similarly, if we arrive in the other end point of e', v' say, we continue along the edge opposite to e' in v'. Repeating this, we finally will return in e. Thus we have obtained a closed curve.

Repeating this for the edges left, we obtain a system of closed curves $C_1, \ldots, C_k$ traversing each edge exactly once. Clearly, this system is unique up to the choice of the starting points of the curves and up to reversing any of the closed curves. We call $C_1, \ldots, C_k$ the *straight decomposition* of G.

In **[9]** we proved the following theorem:

THEOREM 9. *Let G be a 4-regular graph embedded on the compact orientable surface S. Then G is tight if and only if the straight decomposition $C_1, \ldots, C_k$ forms a minimally crossing system of primitive closed curves.*

Here $C_1, \ldots, C_k$ is called *minimally crossing* if any C_i has a minimum number of self-crossings (over all $C_i' \sim C_i$), and any two C_i and C_j have a minimum number of mutual crossings (over all $C_i' \sim C_i$ and $C_j' \sim C_j$).

Our proof in **[9]** is quite hard. From Theorem 9 one can derive quite easily (see [9]):

THEOREM 10. *Let G be a tight graph on the compact orientable surface S, with straight decomposition $C_1, \dots, C_k$. Then for each closed curve D on S:*

$$\overline{\mu}_G(D) = \sum_{i=1}^{k} \min \operatorname{cr}(C_i, D). \tag{35}$$

We show that this implies

THEOREM 11. *Let G be an Eulerian graph embedded on the compact orientable surface S. Then the edges of G can be partitioned into closed curves $C_1, \dots, C_k$ in such a way that for each closed curve D on S*

$$\overline{\mu}_G(D) = \sum_{i=1}^{k} \min \operatorname{cr}(C_i, D). \tag{36}$$

PROOF. By applying the same modification as given by (16) and (17), we may assume that G is 4-regular. Moreover, we may assume that G is tight, as we can open G at vertices as long as we do not change the function $\overline{\mu}_G$. Hence the theorem follows from Theorem 10. □

The analogue of Theorem 11 for the projective plane is Lins' theorem (Theorem 6 ot above). At the moment we do not know a similar result for arbitrary compact nonorientable surfaces.

By passing to the surface dual graph, Theorem 11 transforms to

THEOREM 12. *Let $G = (V, E)$ be a cellularly embedded bipartite graph on the compact orientable surface S. Then there exist closed curves $D_1, \dots, D_t$ not intersecting V and crossing (altogether) each edge exactly once, in such a way that for each closed curve C on S there exists a closed curve $C' \sim C$ in G with the property*

$$\operatorname{length}_G(C') = \sum_{j=1}^{t} \min \operatorname{cr}(C, D_j). \tag{37}$$

[Here $\operatorname{length}_G(C')$ is the number of edges of G traversed by C' (counting multiplicities).]

PROOF. The theorem follows directly by applying Theorem 11 to the surface dual graph of G. □

It should be noted here that the theorem is not true if we delete "cellularly embedded."

Observe the analogy of Theorem 12 with Theorem 2 on the Klein bottle. We can now derive theorems analogous to Theorems 3 and 4, using the cone version (2) of Farkas' lemma. In fact, we only give the analogue of Theorem 4.

Let $G = (V, E)$ be a graph embedded on the compact orientable surface and let $C_1, \dots, C_k$ be closed curves on S. Consider the convex cone K in

$\mathbf{R}^k \times \mathbf{R}^E$ generated by the vectors

$$\text{(38)} \quad \begin{aligned} &\text{(i) } (\varepsilon_i\,;\chi^C) && (i=1,\ldots,k;\ C \text{ closed curve in } G \text{ with } C \sim C_1);\\ &\text{(ii) } (0\,;\varepsilon_e) && (e \in E).\end{aligned}$$

Here χ^C is the vector in $\mathbf{R}^E$ defined by

$$\text{(39)} \qquad \chi^C(e) := \text{ number of times } C \text{ traverses } e$$

for $e \in E$. Moreover, ε_i and ε_e denote the ith and eth unit basis vectors in $\mathbf{R}^k$ and $\mathbf{R}^E$, respectively, while $\mathbf{0}$ is the all-zero vector in $\mathbf{R}^k$.

The cone K is a polyhedral cone, i.e., is generated by only finitely many vectors among (38). To see this, observe that for each fixed $i = 1, \ldots, k$, there exist only finitely many minimal vectors in the collection $\{\chi^C | C$ is a closed curve in G with $C \sim C_i\}$ (minimal with respect to componentwise comparison). This follows from the fact that these are vectors in $\mathbf{Z}^E_+$. We can restrict (38)(i) to those with χ^C such a minimal vector.

Now the analogue of Theorem 4 is

THEOREM 13. *K is exactly the set of vectors $(z\,;x)$ in $\mathbf{R}^k \times \mathbf{R}^E$ satisfying*

$$\text{(40)} \quad \begin{aligned} &\text{(i) } x(e) \geq 0 && (e \in E),\\ &\text{(ii) } \sum_{e\in E} \gamma^D(e)\cdot x(e) \geq \sum_{i=1}^{k} \min \operatorname{cr}(C_i, D)\cdot z_i && (D \text{ closed curve in } S \text{ not intersecting } V).\end{aligned}$$

[Here $\gamma^D(e)$ denotes the number of times D intersects e.]

PROOF. It is not difficult to check that each vector (38) satisfies (40). Suppose next that some vector $(z\,;x) \in \mathbf{R}^k \times \mathbf{R}^E$ satisfies (40) but does not belong to K. Then by Farkas' lemma (cone-form) there exists a vector $(p\,;\ell)$ in $\mathbf{R}^k \times \mathbf{R}^E$ so that $(p\,;\ell)$ has nonnegative inner product with all vectors (38) but not with $(z\,;x)$. That is,

$$\text{(41)} \quad \begin{aligned} &\text{(i) } p_i + \sum_{e\in E} \ell(e)\cdot\chi^C(e) \geq 0 && (i=1,\ldots,k;\ C \text{ closed curve in } G \text{ with } C\sim C_i);\\ &\text{(ii) } \ell(e) \geq 0 && (e\in E);\\ &\text{(iii) } \sum_{i=1}^{k} p_i z_i + \sum_{e\in E} \ell(e)x(e) < 0.\end{aligned}$$

We may assume (by increasing $\ell(e)$ slightly) that ℓ is rational and positive. Hence we may assume (by blowing up $(p\,;\ell)$) that each $\ell(e)$ is a positive even integer.

Now replace each edge e of G by a path of length $\ell(e)$ making the graph G'. So G' arises from G by putting $\ell(e) - 1$ new vertices on any edge e.

Moreover, we make G' cellularly embedded by adding paths over nondisk faces of length T, where T is even and $T \geq \max\{-p_1, \dots, -p_k\}$.

Since C' is bipartite, by Theorem 12 there exist closed curves $D_1, \dots, D_t$ not intersecting the vertex set of G' and crossing each edge of G' exactly once, in such a way that for each $i = 1, \dots, k$ there exists a closed curve $C'_i \sim C_i$ in G' with the property that

$$\text{length}_{G'}(C'_i) = \sum_{j=1}^{t} \min \text{cr}(C_i, D_j). \tag{42}$$

Note that

$$\text{length}_{G'}(C'_i) = \sum_{e \in E} \ell(e)\chi^{C'_i}(e) \quad \text{and } \ell(e) = \sum_{j=1}^{t} \gamma^{D_j}(e). \tag{43}$$

In particular, by (41)(i)

$$p_i + \text{length}_{G'}(C'_i) \geq 0 \quad (i = 1, \dots, k). \tag{44}$$

This implies the following contradiction to (41)(iii):

$$\begin{aligned} -\sum_{i=1}^{k} p_i z_i \leq \sum_{i=1}^{k} \text{length}_{G'}(C'_i) z_i &= \sum_{i=1}^{k}\sum_{j=1}^{t} \min \text{cr}(C_i, D_j) z_i \\ &\leq \sum_{j=1}^{t}\sum_{e \in E} \gamma^{D_j}(e) x(e) = \sum_{e \in E} \ell(e) x(e). \quad \square \end{aligned} \tag{45}$$

Theorem 13 implies the following "homotopic circulation theorem":

THEOREM 14. *Let $G = (V, E)$ be a graph embedded on the compact orientable surface S and let $C_1, \dots, C_k$ be closed curves on S. Then there exist closed curves $C_{i1}, \dots, C_{ir_i} \sim C_i$ in G and rationals $\lambda_{i1}, \dots, \lambda_{ir_i} > 0$ for $i = 1, \dots, k$ such that*

$$\begin{aligned} &\text{(i) } \sum_{j=1}^{r_i} \lambda_{ij} = 1 \qquad (i = 1, \dots, k), \\ &\text{(ii) } \sum_{i=1}^{k}\sum_{j=1}^{r_i} \lambda_{ij}\chi^{C_{ij}}(e) \leq 1 \quad (e \in E) \end{aligned} \tag{46}$$

if and only if for each closed curve D on S not intersecting V

$$\text{cr}(G, D) \geq \sum_{i=1}^{k} \min \text{cr}(C_i, D). \tag{47}$$

PROOF. Directly from Theorem 13, since (46) is equivalent to the all-one vector $(\mathbf{1}; \mathbf{1})$ belonging to K, while (47) is equivalent to $(\mathbf{1}; \mathbf{1})$ satisfying (40). $\square$

In general, we cannot require the λ_{ij} in (46) to be integer, even if we require G to be Eulerian. That is, the analogue of the "integer-packing" theorem, Theorem 5, does not hold. However, if S is the torus the analogue does hold, as was shown in **[1]**:

THEOREM 15. *Let $G = (V, E)$ be an Eulerian graph embedded on the torus S and let $C_1, \dots, C_k$ be closed curves on S. Then there exist pairwise edge-disjoint closed curves $C_1' \sim C_1, \dots, C_k' \sim C_k$ in G (such that no C_i' traverses any edge more than once) if and only if for each closed curve D on S not intersecting V condition* (47) *is satisfied.*

This theorem can be derived from Theorem 14, in a way similar to the derivation of Theorem 5 from the fractional version of Theorem 5 (i.e., Theorem 4).

A consequence of Theorem 14 similar to Theorems 7 and 8 is the following "homotopic flow-cut theorem":

THEOREM 16. *Let $G = (V, E)$ be a planar graph embedded in the plane $\mathbf{R}^2$ and let $I_1, \dots, I_p$ be some of the faces of G, including the unbounded face. Let $P_1, \dots, P_k$ be curves in $\mathbf{R}^2 \setminus (I_1 \cup \dots \cup I_p)$ with end points on $\mathrm{bd}(I_1 \cup \dots \cup I_p)$. Then there exist paths $P_{i1}, \dots, P_{ir_i} \sim P_i$ in G and rationals $\lambda_{i1}, \dots, \lambda_{ir_i} > 0$ for $i = 1, \dots, k$ such that*

$$\text{(48)} \qquad \text{(i)} \sum_{j=1}^{r_i} \lambda_{ij} = 1 \qquad (i = 1, \dots, k),$$

$$\text{(ii)} \sum_{i=1}^{k} \sum_{j=1}^{r_i} \lambda_{ij} \chi^{P_{ij}}(e) \le 1 \quad (e \in E)$$

if and only if for each curve D in $\mathbf{R}^2 \setminus (I_1 \cup \dots \cup I_p)$ not intersecting V and connecting two points on $\mathrm{bd}(I_1 \cup \dots \cup I_p)$:

$$\text{(49)} \qquad \mathrm{cr}(G, D) \ge \sum_{i=1}^{k} \min \mathrm{cr}(P_i, D).$$

[Here we use similar terminology and notation as above. A *curve* is a continuous function $C: [0, 1] \to \mathbf{R}^2$, while *homotopic* requires fixing the end points.]

PROOF (SKETCH). We can reduce this theorem to Theorem 14 by adding for each $i = 1, \dots, k$ a handle connecting the end vertices of P_i, extending the graph by an edge over this handle (connecting the end points of P_i) and by extending P_i to a closed curve over the handle. □

We now finally come to showing Theorems 17 and 18.

THEOREM 17. *Let $C_1, \dots, C_k$ and $C_1', \dots, C_{k'}'$ be primitive closed curves on the compact orientable surface S. Then the following are equivalent:*

(i) *$k = k'$ and there exists a permutation π of $\{1, \dots, k\}$ so that $C_{\pi(i)}' \sim C_i$ or $C_{\pi(i)}' \sim C_i^{-1}$ for each $i = 1, \dots, k$;*

(ii) *for each closed curve* D *on* S

$$\sum_{i=1}^{k} \min \operatorname{cr}(C_i, D) = \sum_{i=1}^{k'} \min \operatorname{cr}(C_i', D).$$

PROOF (SKETCH). The implication (i) $\Rightarrow$ (ii) is trivial since $\min \operatorname{cr}(C_i^{-1}, D) = \min \operatorname{cr}(C, D)$. To see the implication (ii) $\Rightarrow$ (i) we may assume that both $C_1, \dots, C_k$ and $C_1', \dots, C_{k'}'$ form minimally crossing collections of closed curves, and that the system $C_1, \dots, C_k$ has at least as many crossings as $C_1', \dots, C_{k'}'$. Let $G = (V, E)$ be the graph made up by $C_1', \dots, C_{k'}'$. Without loss of generality, each vertex of G has degree 2 or 4.

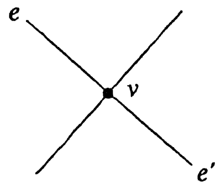
Now by (ii), for each closed curve D on S not intersecting V

$$\text{(50)}\quad \operatorname{cr}(G, D) = \sum_{i=1}^{k'} \operatorname{cr}(C_i', D) \geq \sum_{i=1}^{k'} \min \operatorname{cr}(C_i', D) = \sum_{i=1}^{k} \min \operatorname{cr}(C_i, D).$$

Hence by Theorem 14 there exist $C_{ij} \sim C_i$ and $\lambda_{ij} > 0$ satisfying (46).

Now it can be proved that each C_{ij}, if it enters a vertex v over an edge e, continues over the edge e' opposite to e:

(51)



The reason is that $C_1, \dots, C_k$ necessarily have at least as many crossings as $C_1', \dots, C_{k'}'$. Hence the C_{ij} should "use" all crossings of the C_i'—if any C_{ij} makes a turn in v, there is not enough room left for crossings of the C_{ij}. This intuitive argument can be made precise at the cost of several technicalities—see [**8**].

It follows that each C_{ij} in fact is one of $C_1', \dots, C_{k'}'$ and their inverses. As we may assume that the C_{ij} are different, the theorem now follows. □

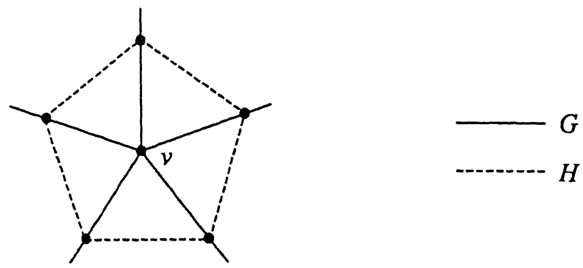
Finally

THEOREM 18. *Let* G *and* G' *be cellularly embedded kernels on* S *with* $\mu_G = \mu_{G'}$. *Then* G' *can be obtained from* G *by a series of the following operations:*

(i) *shifting the graph homotopically over* S;
(ii) *taking the (surface) dual graph*;
(iii) ΔY*-exchange.*

PROOF (SKETCH). From G we make an auxiliary graph H as follows. For each edge e of G, put a vertex w_e of H on the "middle" of e. For each vertex v of G, make a circuit connecting the vertices w_e on the edges e of G incident to v:

(52)



Thus we obtain a 4-regular graph H. Note that we can reconstruct G from H, up to shifting G and up to duality.

Now deletion and contraction of an edge e of G corresponds to the two ways of opening vertex w_e of H. Moreover, $\overline{\mu}_H = 2\mu_G$. Therefore, as G is a kernel, H is tight.

Similarly, we make a tight graph H' from G'. Then

$$\overline{\mu}_{H'} = 2\mu_{G'} = 2\mu_G = \overline{\mu}_H. \tag{53}$$

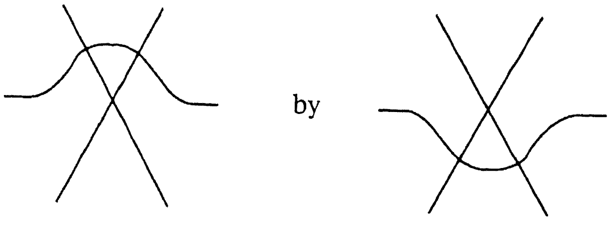
Let $C_1, \dots, C_k$ and $C'_1, \dots, C'_{k'}$ be the straight decompositions of H and H', respectively. By Theorem 10 we have for each closed curve D

$$\sum_{i=1}^{k} \min \operatorname{cr}(C_i, D) = \overline{\mu}_H(D) = \overline{\mu}_{H'}(D) = \sum_{i=1}^{k'} \min \operatorname{cr}(C'_i, D). \tag{54}$$

So by Theorem 17 we may assume that $k = k'$ and that $C_i \sim C'_i$ for $i = 1, \dots, k$.

By Theorem 9, both $C_1, \dots, C_k$ and $C'_1, \dots, C'_k$ are minimally crossing collections of primitive closed curves. It can be shown (using the hyperbolic plane representation of the universal covering surface of S) that $C_1, \dots, C_k$ can be shifted to $C'_1, \dots, C'_k$ keeping the collection minimally crossing throughout the shifting process. In fact $C_1, \dots, C_k$ can be transformed to $C'_1, \dots, C'_k$ by a number of "swappings", i.e., replacing

(55)



(and by shifting the whole graph $C_1 \cup \cdots \cup C_k$).

Any such swapping corresponds to transforming H, and hence to transforming G. One easily checks that it corresponds to the ΔY-exchange. Hence G' can be obtained from G by the operations (i), (ii), and (iii). □

REFERENCES

1. A. Frank and A. Schrijver, *Disjoint homotopic cycles on the torus*, preprint.
2. D. R. Fulkerson, *Blocking and anti-blocking pairs of polyhedra*, Math. Programming **1** (1971), 168–194.
3. S. Lins, *A minimax theorem on circuits in projective graphs*, J. Combin. Theory Ser. B **30** (1981), 253–262.
4. H. Okamura, *Multicommodity flows in graphs*, Discrete Appl. Math. **6** (1983), 55–62.
5. H. Okamura and P. D. Seymour, *Multicommodity flows in planar graphs*, J. Combin. Theory Ser. B **31** (1981), 75–81.
6. S. Randby, personal communication by N. Robertson (1989).
7. A. Schrijver, *Distances and cuts in planar graphs*, J. Combin. Theory Ser. B **46** (1989), 46–57.
8. ——, *Homotopy and crossings of systems of curves on a surface*, Linear Algebra Appl. **114** (1989), 157–167.
9. ——, *Decomposition of graphs on surfaces and a homotopic circulation theorem*, Report OS-R8719, Mathematical Centre, 1987 (to appear in Journal of Combinatorial Theory (B)).
10. ——, *The Klein bottle and multicommodity flows*, Combinatorica 9 (1989), 375–384.
11. ——, *On the uniqueness of kernels*, Report OS-R8924, Mathematical Centre, 1989.

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