Uniqueness and Nonexistence of Some Graphs
Related to $M_{22}$

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Abstract. There is a unique distance regular graph with intersection array $(7, 6, 4; 1, 1, 1, 6)$; it has 330 vertices, and its automorphism group $M_{22} : 2$ acts distance transitively. It does not have an antipodal 2-cover, but it has a unique antipodal 3-cover, and this latter graph has automorphism group $3.M_{22} : 2$ acting distance transitively. As a side result we show uniqueness of the strongly regular graph with parameters $(n, k, \lambda, \mu) = (231, 30, 9, 3)$ under the assumption that it is a gammu space with lines of size 3.

1. Uniqueness of the Cameron Graph

There exists a strongly regular graph (sometimes called the Cameron graph) on 231 vertices with full automorphism group $M_{22} : 2$ constructed by taking as vertices the unordered pairs from a 22-set and joining two pairs whenever they are disjoint and their union is contained in a block of a (fixed) Steiner system $S(3, 6, 22)$ on this 22-set. (For undefined terminology, see e.g. Cameron & van Lint [3].) This graph becomes the collinearity graph of a partial linear space with lines of size 3 if one takes as lines the triples of pairwise disjoint pairs whose union is a block of the Steiner system. This partial linear space is a gammu space, that is, given a line $L$ and a point $x$ outside, then $x$ is collinear with zero, one or all points of $L$. The next theorem shows that this property characterizes our graph.

Theorem 1. Let $\mathcal{G}(X, \mathcal{L})$ be a gammu space with lines of size 3 such that its collinearity graph $\Gamma$ is strongly regular with parameters $(n, k, \lambda, \mu) = (231, 30, 9, 3)$. Then $\Gamma$ is isomorphic to the Cameron graph described above.

(Here, following common practice but unlike [3], $\nu$ denotes the number of vertices, $k$ the valency, $\lambda$ the number of common neighbours of two adjacent vertices and $\mu$ the number of common neighbours of two nonadjacent vertices of a graph $\Gamma$.)

Proof. Write $\Gamma(x)$ for the set of neighbours of a vertex $x$; $\mu(x, y) = \Gamma(x) \cap \Gamma(y)$ for the set of common neighbours of two nonadjacent vertices $x$ and $y$. The graph induced by $\Gamma$ on $\mu(x, y)$ is called a $\mu$-graph.

(1) Each $\mu$-graph is either a 3-coclique or a line.
(For: each $\{x\} \cup \Gamma(x)$ is a subspace of $\langle X, \mathcal{L} \rangle$, and the intersection of subspaces is again a subspace.)
(2) Each vertex is in ten 7-cliques and each line in two 7-cliques; each 7-clique is a subspace isomorphic to the Fano plane.  
(For: maximal cliques are subspaces and have at most 11 vertices (e.g. because $\lambda = 9$), and at least 7 vertices; but no STS (11) exists, so there are two maximal cliques on each line and each has 7 vertices.)

The $m$-clique extension of a graph is obtained by replacing each vertex $x$ by an $m$-clique $C_m$, and joining each vertex of $C_m$ to each vertex of $C_m$ whenever $x \sim y$.

(3) For each vertex $x$, $\Gamma(x)$ is isomorphic to the 2-clique extension of the line graph of the Petersen graph.  
(For: form a graph $\Delta$ with the Fano planes on $x$ as vertices and the lines on $x$ as edges and (reverse) inclusion as incidence. Then $\Delta$ has 10 vertices, valency 3, no triangles and no quadrangles (otherwise $\Gamma$ would have $\mu \geq 5$), so $\Delta$ is the Petersen graph.)

Note that the line graph of the Petersen graph is an antipodal 3-cover of the complete graph $K_5$ so that we have a concept of antipodal concurrent lines.

(4) Each line $L$ is contained in a unique subspace isomorphic to the $GQ(2,2)$ generalised quadrangle (For: let $L = \{x, y, z\}$ and let $M, N$ be the two lines on $x$ antipodal to $L$, say $M = \{x, u, v\}$. Let $p$ be a common neighbour of $u$ and $v$ distinct from $x$. Since $M$ is antipodal to $L$ we have that $\mu(u, y)$ is a 3-coclique, so $p \sim x$ and $u(p, x)$ is a 3-coclique, so $pu$ is antipodal to $L$ and $pu$ is antipodal to $M$. It follows that the 8 common neighbours of a point of $L \setminus \{x\}$ and a point of $M \setminus \{x\}$ lie on the 8 lines not on $x$ antipodal to $L$ or $M$, and we find two $3 \times 3$ grids having $L \cup M$ in common. But these same 8 points are also joined to $N \setminus \{x\}$ by the 4 lines not on $x$ antipodal to $N$, and the 15 points and 15 lines we have found form a $GQ(2,2)$. Uniqueness follows since in a $GQ(2,2)$ all $\mu$-graphs are 3-cocliques so that any two intersecting lines are antipodal and the whole construction was forced.)

Let us call a $GQ(2,2)$ subgraph (subgeometry) a quad.

(5) There are 77 quads, 5 on each vertex, 1 on each line, and any two have at most one vertex in common. Two nonadjacent vertices $x, y$ are in a quad if and only if $\mu(x, y)$ is a 3-coclique. Quads are geodetically closed.  
(For: if $x \sim y$ and $\mu(x, y)$ is a 3-coclique and $p$ is a common neighbour of $x$ and $y$ then the lines $px$ and $py$ are antipodes, and $y$ is in the unique quad containing $px$.)

We shall write $Q(L)$ and $Q(x, y)$ for the unique quad on the line $L$ or on the nonadjacent vertices $x, y$ (this notation implying that $\mu(x, y)$ is a 3-coclique).

(6) Let $Q$ be a quad and $x \not\in Q$. Then $\Gamma(x) \cap Q$ is either empty or a line. If we write $\Gamma_1(Q) = \{y' | d(y', Q) = 1\}$ then $|\Gamma_1(Q)| = 15$, $|\Gamma_2(Q)| = 120$, $|\Gamma_3(Q)| = 96$.  
(For: let $L$ be a line on $x$ meeting $Q$ in $y$, then $L$ is in a Fano plane together with one of the three lines on $y$ in $Q$.)

(7) If $Q, Q'$ are two quads, and $Q \cap Q' = \{z\}$ then the 8 nonneighbours of $z$ in $Q'$ are in $\Gamma_1(Q)$. There are 60 quads meeting $Q$ in a single point, 5 on each point of $\Gamma_2(Q)$, so there are 16 quads disjoint from $Q$ and these are entirely contained within $\Gamma_1(Q)$.  
(For: let $x \in Q, y \in Q, x \sim y, x \sim z$ then $\Gamma(x) \cap Q$ is a line $L$ on $y$. This line does not contain $z$, so $z$ has a neighbour on it and we may assume $z \sim y$. But now $x \sim y \sim z$ and $Q'$ is geodetically closed, so $y \in Q'$, contradiction.)

(8) There are no three pairwise disjoint quads.
(For: suppose $Q_1$, $Q_2$, $Q_3$ are pairwise disjoint, and define $\gamma_x: Q_1 \rightarrow Q^*$ by $\gamma(x) = f(x) \cap Q$, where $Q^*$ denotes the generalized quadrangle dual to $Q$.
Then $\psi = \gamma_x \circ \gamma_y$ is an isomorphism from $Q_2$ onto $Q_3$. If $x \in Q_2$ and $x \sim \psi(x)$ then $\mu(x, \psi(x))$ is the line $\gamma_y(x)$, but $\psi(x)$ also has a neighbour on the line $\gamma_y(x)$, contradiction. Thus $x \sim \psi(x)$ for each $x \in Q$, and $\gamma_y \circ \psi^{-1}$ is a polarity of $Q_3$, where all points are absolute. But $GQ(2,2)$ has no such polarity, contradiction.)

(9) For a graph $d$ with the quads as vertices, two quads being adjacent whenever they are disjoint. Then $d$ is the unique strongly regular graph with parameters $(n, k, \lambda, \mu) = (77, 16, 0, 4)$ and is isomorphic with the graph that has the blocks of $S(3,6,22)$ as vertices and pairs of disjoint blocks as edges.
(For: we have seen $u, k, \lambda$ and $\mu = 4$ is easily checked. Now the result follows from Brouwer [21].)

Now we might continue describing $\Gamma$ in terms of $d$, exploiting detailed knowledge of $d$. Instead I'll choose another way, showing the rank 4 structure of $\Gamma$.

(10) $\Gamma$ carries a 3-class association scheme with $(x, y) \in R_0$iff $x = y$, $(x, y) \in R_1$, iff $x \sim y$, $(x, y) \in R_1$, iff $x \sim y$ and $\mu(x, y)$ is a line $(x, y) \in R_3$ iff $x \sim y$ and $\mu(x, y)$ is a 3-coclique. The parameters are $(p_{ij}) = I$.

$$
(p_{ij})_0 = \begin{pmatrix}
0 & 30 & 0 & 0 \\
1 & 9 & 16 & 4 \\
0 & 3 & 21 & 6 \\
0 & 3 & 24 & 3
\end{pmatrix},
(p_{ij})_1 = \begin{pmatrix}
0 & 16 & 112 & 32 \\
0 & 16 & 112 & 32 \\
1 & 21 & 108 & 30 \\
0 & 24 & 120 & 16
\end{pmatrix}.

(p_{ij})_2 = \begin{pmatrix}
0 & 0 & 0 & 40 \\
0 & 4 & 32 & 4 \\
0 & 6 & 30 & 4 \\
0 & 3 & 24 & 3
\end{pmatrix}.

(For:

a) $p_{ij} = 4$: If $(x, y) \in R_1$, $(x, z) \in R_3$, then by (7) $x, y, z$ are all in one quad, the unique quad on $xy$, and in this quad there are 4 points nonadjacent to $x$ and $y$.

b) $p_{ij} = 4$: Suppose $\mu(x, y) = L$. There are two quads on $x$ disjoint from $L$, and if $Q$ is such a quad then $y \notin Q$ and $\eta(x) \cap Q = \emptyset$ (otherwise $\eta(y) \cap Q$ would be a line and $\mu(x, y)$ would contain at least 4 points); $y$ is in 5 quads and each meets $Q$ in a single point. These 5 points form an ovoid, as follows from a) and hence 3 of them are adjacent to $x$. Remain 2 possibilities for $z$ with $(x, z) \in R_3$ on each $Q$, so $p_{ij} = 4$.

c) $p_{ij} = 4$: Suppose $x \sim y \sim z$, $(x, z) \in R_3$. Then $Q(x, z)$ is the unique quad on $xy$ and $z$ is one of the 4 neighbours of $y$ not on $xy$ in this quad.

d) $p_{ij} = 3$: Suppose $x \sim z$, $(x, y) \in R_3$. Then $x, y, z$ are all in one quad $Q(x, y)$ and $z$ is one of the 3 neighbours of $x$ nonadjacent to $y$ in this quad.

e) $p_{ij} = 6$: Suppose $x \sim z$, $(x, y) \in R_3$, $(y, z) \in R_3$. Let $Q = \eta(y, z)$. By b) the line $L = \mu(x, y)$ meets $Q$ in a point $p$, say. Now $\eta(x) \cap Q$ is the line $p_x$, and $Q$ is the unique quad containing the line $p_x$. For $p$ there are 3 choices on $L$ and in each case we find two possibilities for $z$.

f) $p_{ij} = 20$: Suppose $(x, y) \in R_3$, $Q = Q(x, y)$. Inside $Q$ there are 4 points nonadjacent to $x$ and $y$. Any other point $z$ with $(x, z), (y, z) \in R_3$ must be in $\eta(y, z)$.}
If \( z \in \Gamma_2(Q) \) then the five quads on \( z \) meet \( Q \) in five points forming an oval \( O \) in \( Q \). Now \( Q \) has 6 ovals, and if \( O, O' \) are any two ovals then there are precisely 32 points \( z \) determining either \( O \) or \( O' \) (for: \( O \cap O' = \{ p \} \) and there are 32 points nonadjacent to \( p \) in the four quads distinct from \( Q \) on \( p \)); it follows that any given oval (and in particular the one containing \( x, y \)) is determined by 16 points \( z \).

Thus \( p_{34}^2 = 4 + 16 = 20 \).

All other \( p_{4k} \) are determined by these (and the parameters of \( \Gamma \) as a strongly regular graph).

(11) \( (x, R_3) \) is isomorphic with the triangular graph \( \binom{22}{2} \).

(For: it has the right parameters by (10), and uniqueness follows by Connor [4].)

Let us identify the quads in this triangular graph.

**Lemma.** Let \( \Delta \) be a triangular graph \( \binom{n}{2} \) and \( T \) a noncomplete subgraph isomorphic to \( \binom{m}{2} \). If \( \Delta \) is labelled with \( \binom{Y}{2} \) for some \( n \)-set \( Y \) then this labelling induces a labelling with \( \binom{Z}{2} \) on \( T \), where \( Z \) is some \( m \)-subset of \( Y \). (In other words, there are only canonical ways to embed noncomplete triangular subgraphs.)

**Proof.** Let \( x, y \) be vertices of \( T \) labelled with \( ab \) and \( ac \) respectively. We prove that some vertex of \( T \) is labelled with \( bc \). Choose \( z \in T, z \approx y, z \approx x \). Then \( z \) is labelled with \( cd \), say. Now \( \mu(x, z) \) is a 4-circuit, so there are two vertices, \( u, v \in T \) adjacent to each of \( x, y, z \). But these must be labelled \( ad \) and \( bc \).

(This Lemma reminds me of recent work by J.I. Hall on Kneser graphs – probably it is a special case of some of his results.)

(12) \( \Gamma \) is the graph with as vertices the pairs from a set of 22 symbols, where two pairs are adjacent whenever they are disjoint and their union is contained in a block of a \( S(3, 6, 22) \) design on the set of symbols.

(For: the collinearity graph of a quad \( GQ(2, 2) \) is the complement of the triangular graph \( \binom{6}{2} \), so by the Lemma and (11) we can label \( X \) with the pairs from a set \( \Sigma \) of 22 symbols and the quads correspond to certain 6-subsets of \( \Sigma \). Each triple in \( \Sigma \) determines a unique quad, so these 6-sets form a Steiner system \( S(3, 6, 22) \) on \( \Sigma \). If two pairs are adjacent in \( \Gamma \) then they are nonadjacent in \( (X, R_3) \), i.e. disjoint, and they are contained in a quad.)

**Remark.** The association scheme described in (10) corresponds to the group action, i.e., \( M_{22} \) acts rank 4 on \( X \) with suborbits \( 1 + 30 + 160 + 40 \).

2. **Uniqueness of a Graph on 330 Vertices**

Define a graph \( \Gamma \) with as vertices the 330 blocks of the Steiner system \( S(5, 8, 24) \) missing two fixed symbols, where two blocks are adjacent whenever they are disjoint.
We have the following correspondence between graph distance and size of intersection:

\[
\begin{array}{c|c}
|B \cap B'| & d(B, B') \\
8 & 0 \\
0 & 1 \\
2 & 3 \\
4 & \begin{cases} 
2 & \text{if the sextet determined by } B \text{ and } B' \text{ has both} \\
4 & \text{fixed symbols in the same tetrad,} \\
& \text{otherwise.}
\end{cases}
\end{array}
\]

It is easy to check that \( \Gamma \) is distance regular (in fact, distance transitive) with intersection array \( i(7, 6, 4; 4; 1, 1, 1, 6) \). \( \Gamma \) has full group of automorphisms \( M_{22}.2 \). The vertices, edges and Petersen subgraphs of \( \Gamma \) form a geometry with Bu
ckenhout diagram

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\[
\begin{array}{c}
5, 6, 5, 6 \quad 330 \\
5, 5, 2, 2 \quad 1155 \\
1, 2, 1, 2 \quad 231
\end{array}
\]
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points edges Petersen subgraphs

Our aim here is to prove uniqueness of \( \Gamma \) from its parameters. Let us start with a lemma producing the Petersen subgraphs.

**Lemma.** Let \( \Gamma \) be a graph with \( \mu = c_3 = 1 \) and \( \lambda = 0, a_2 = 2 \). Then any two vertices at distance two in \( \Gamma \) determine a unique induced Petersen graph.

(For notation, see Biggs [1]; we do not suppose that \( \Gamma \) is distance regular.)

**Proof.**

Let \( d(a, x) = 2 \), with \( a \sim p \sim x \). Let \( x \) have neighbours \( w, y \) in \( \Gamma_2(a) \), with \( a \sim r \sim w, a \sim q \sim y \). Since \( \Gamma \) has girth 5, all points mentioned are distinct. Since \( w \) has two neighbours (\( r \) and \( x \)) in \( \Gamma_2(q) \) we have \( d(w, q) = 2 \), \( s \sim q \sim v \sim w \), and similarly \( r \sim z \sim y \). Again \( d(v, z) = 2 \) (not \( v \sim z \), since \( v \sim z \sim r \sim w \sim v \) would be a 4-circuit, and not \( d(v, z) = 3 \) since \( \{w, q\} < \Gamma(v) \cap \Gamma_2(z) \) so \( u \sim v \sim z \) for some vertex \( u \). We must have \( d(a, u) = 2 \) and \( d(p, v) = 2 \) so \( p \sim u \), completing our Petersen graph. \( \Box \)

(We find that there are \( k(k - 1)/6 \) Petersen graphs on a vertex and \( sk(k - 1)/60 \) Petersen graphs altogether, so these numbers must be integers for a graph \( \Gamma \) satisfying the hypotheses of the Lemma.)

This Lemma applies to our graph on 330 vertices as well as to its antipodal 2-covers and 3-covers (these pass all known existence criteria). The three distance distribution diagrams are
In these cases we have $k = 7$, each vertex $x$ is in 7 Petersen graphs and the triples induced by these on $\Gamma(x)$ must form the Fano plane; in particular, two Petersen graphs on a vertex $x$ have an edge in common.

**Proposition.** There is no distance regular graph on 660 vertices with intersection array $i(7, 6, 4, 4, 3, 1, 1, 1; 1, 1, 1, 3, 4, 4, 6, 7)$.

**Proof.** Let $\Gamma$ be such a graph. Choose vertices $x_0, a, b, x_8$ with $d(x_0, x_8) = 8$, $a \sim b$, \{a, b\} $\subseteq \Gamma_4(x_0) \cap \Gamma_4(x_8)$. Since $c_4 = 3$ is odd, there must be a Petersen graph $P$ on the edge $ab$ meeting both $\Gamma_3(x_0) \cap \Gamma(a)$ and $\Gamma_3(x_8) \cap \Gamma(a)$, say $x_3 \in P \cap \Gamma(a) \cap \Gamma_3(x_0)$, $x_3 \in P \cap \Gamma(a) \cap \Gamma_3(x_8)$. Let $x_0 \sim x_1 \sim x_2 \sim x_3 \sim a \sim x_3 \sim x_6 \sim x_7 \sim x_8$ be a geodesic. $P$ must have an edge in common with the Petersen graph determined by $x_1$ and $x_3$, and this edge lies in $\Gamma_3(x_0)$; similarly, $P$ must have an edge in $\Gamma_3(x_8)$—but one easily sees that this is impossible since $a_4 = 1$.

**Theorem 2.** There is a unique distance regular graph $\Gamma$ on 330 vertices with intersection array $i(7, 6, 4, 4, 1, 1, 1, 6)$.

**Proof.** We first show that the graph $\mathcal{D}$ with as vertices the Petersen subgraphs of $\Gamma$ where two Petersen graphs are adjacent when they meet, is isomorphic to the Cameron graph. First of all $\mathcal{D}$ has 231 vertices and valency 30.

**Claim.** The distance distribution around a Petersen graph $P$ is

$$
\begin{array}{c}
10 & 40 & 120 & 160 \\
3 & 4 & 1 & 6 \\
\end{array}
$$

Indeed: $P$ is geodetically closed so any point in $\Gamma_3(P)$ has a unique neighbour in $P$. Also, if $x, y \in P$ with $d(x, y) = 2$ then the two neighbours of $y$ at distance two to $x$ are in $P$ and it follows that $\Gamma_3(P)$ is a coclique. If $z \in \Gamma_2(P)$ then at most 7 points of $P$ can be in $\Gamma_2(z)$, so there are points of $P$ in $\Gamma_3(z)$. Now since $c_3 = 1$ we must have that $\Gamma_3(z) \cap P$ is geodetically closed and hence is an edge. If $P$ is the unique Petersen graph on $z$ meeting $P$ then $P'$ contains an edge on $z$ in $\Gamma_3(P)$, so $z$ has at most 4 neighbours in $\Gamma_3(P)$. The maximum possible distance to $P$ is 3 since $\Gamma$ has
diameter 4 and \( a_4 = 1 < 3 \). If some point \( u \in \mathcal{G}_2(P) \) had at most two neighbours in \( \mathcal{G}_2(P) \) then \( |P \cap \mathcal{G}_2(u)| \leq 4 \), and removing at most two edges from \( P \) we are left with a graph where each vertex has degree at most one – impossible. Thus the number of edges between \( \mathcal{G}_2(P) \) and \( \mathcal{G}_3(P) \) is both at most and at least 480 = 4.120 = 3.160 and we have equality everywhere, proving the claim.

Let us compute \( \lambda \). If \( P, P' \) and \( P'' \) are three Petersen graphs that have pairwise nonempty intersection then by the previous \( P \cap P' \cap P'' \) is nonempty. Let \( P \cap P' = \{u, v\} \), then there is one more Petersen graph on \( \{u, v\} \), and 4 others on \( u \) and on \( v \) so that \( \lambda = 1 + 4 + 4 = 9 \).

Next look at \( \mu \). There are two possibilities (as was to be expected, since the Cameron graph is rank 4, not rank 3): (i) \( P' \) meets \( \mathcal{G}_1(P) \), and (ii) \( d(P, P') > 1 \).

In the first case we see from \( \alpha_2(P) = 1 \) that any \( P'' \) meeting both \( P \) and \( P' \) must contain the (unique, since \( c_3 = 1 \)) edge joining \( P \) and \( P' \) so that \( P \) and \( P' \) have three common neighbours.

In the second case we see from the distance distribution diagram around \( P \) and the fact that any two Petersen graphs on a point have an edge in common, that \( P' \) contains 3 edges in \( \mathcal{G}_3(P) \).

(Indeed, if \( u \in \mathcal{G}_3(P) \) then \( u \) is in 6 Petersen graphs meeting \( \mathcal{G}_1(P) \), two on each edge \( uv \) with \( v \in \mathcal{G}_2(P) \), so \( u \) is in a unique Petersen graph \( P' \) not meeting \( \mathcal{G}_1(P) \), and \( P' \) contains the three neighbours of \( u \) in \( \mathcal{G}_3(P) \) so that \( \mathcal{G}_2(P) \cap \mathcal{G}_3(P) \) is a coclique. But the only way to split a Petersen graph into a coclique and a graph where each vertex has degree (at most) one is as \( K_3 + 3K_2 \).

Thus \( \mu = 3 \), and by Theorem 1 the graph \( \delta \) is isomorphic to the Cameron graph. (Clearly the 3-lines of \( \delta \) are the triples of Petersen graphs on a given edge, and the computation of \( \lambda \) also proved the Gamma space property.)

[This gives us a 22-set \( S \) and a Steiner system \( S(3, 6, 22) \) on \( S \) and a labelling of \( \delta \) with \( \frac{S}{2} \) such that Petersen graphs at distance 2 correspond to intersecting pairs and intersecting Petersen graphs correspond to disjoint pairs contained in a block of the Steiner system. We want to let the vertices of \( \Gamma \) correspond to 8-subsets of \( S \). This is done as follows:

Given a vertex \( x \), it is in 7 Petersen graphs labelled with 7 pairwise disjoint pairs of symbols. Label \( x \) with the set of \( 22 - 2 \cdot 7 = 8 \) remaining symbols. We shall however not use this labelling.)

Each vertex \( x \) determines a 7-clique in \( \delta \), and we find 330 7-cliques in \( \delta \) in this way; but \( \delta \) has only 330 7-cliques, 10 on each vertex of \( \delta \), so we indentify \( \Gamma \) as the graph with as vertices the Fano planes in \( \delta \), where two Fano planes are adjacent when they have a line in common. This shows that \( \Gamma \) is uniquely determined. \( \Box \)

Remarks. The Petersen subgraphs arise as follows:

Let \( \alpha, \beta \) be the two fixed symbols chosen in the symbol set \( S \cup \{\alpha, \beta\} \) in order to define \( \Gamma \). Any sextet such that \( \alpha \) and \( \beta \) lie in the same tetrad \( T \) of the sextet has 5 remaining tetrads, and the union of the any two of these is a block of \( S(5, 8, 24) \), giving 10 blocks altogether, and these 10 blocks induce a Petersen subgraph in \( \Gamma \). The pair this Petersen graph is labelled with is \( T \setminus \{\alpha, \beta\} \), showing that the labelling proposed above is the correct one.

The suborbit lengths \( \{1 + 7 + 42 + 168 + 112\} \) were given incorrectly by Fischer & McKay [5] but are stated correctly in Ivanov, Klin & Faradjev
3. Uniqueness of a Graph on 990 Vertices

Recently, existence of a distance transitive graph on 990 vertices with intersection array $i(7, 6, 4, 4, 4, 1, 1, 1, 1, 1, 2, 4, 6, 7)$ was shown by Ivanov, Ivanov & Faradjev [7]. Its full automorphism group is $3.M_{22}$; it is already distance transitive under $3.M_{22}$. This graph is interesting for several reasons; for instance, it provides an example of a distance regular graph where the sequence $(a_j)_{0 \leq j \leq d}$ is not unimodal.

**Theorem 3.** There is a unique distance regular graph $\Gamma$ on 990 vertices with intersection array $i(7, 6, 4, 4, 4, 1, 1, 1, 1, 1, 2, 4, 6, 7)$.

**Proof.** As before we find Petersen graphs; the distance distribution around a Petersen graph $P$ is

\[
\begin{array}{cccccccc}
  & 10 & 40 & 120 & 480 & 240 & 80 & 20 \\
 1 & \_ & \_ & 6 & \_ & 2 & \_ \_ & \_ \\
 2 & \_ & \_ & \_ & \_ & 5 & \_ & \_ \\
 3 & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\
 4 & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\
 5 & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\
 6 & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\
 7 & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\
\end{array}
\]

Let $\Gamma$ be an antipodal 3-cover of $\Gamma$. Pick a Petersen graph $P$ in $\Gamma$ and a vertex $x \in \Gamma_3(P)$. Then $\Gamma_3(x) \cap P \cong 3K_2$. Since $P$ has only five subgraphs isomorphic to $3K_2$ we see that $\Gamma_3(P)$ is a 32-cover of the complete graph $K_5$. Put $A = \Gamma_3(P)$ and consider the inverse images of $P$, $x$ and $\delta$ in $\Gamma$. Above $P$ we see three Petersen graphs $P_1$, $P_2$, $P_3$ at mutual distance 6. If $\hat{x}$ is one of the three vertices above $x$ then $\Gamma_3(\hat{x}) \cap P_j$ is a single edge $(j = 1, 2, 3)$ so that we find a labelling of the three edges in $\Gamma_3(x) \cap P$ with $\{1, 2, 3\}$. If $\hat{x} \sim \hat{y} \in \Gamma_3(P)$ then the labelling of the three edges in $\Gamma_3(y) \cap P$ determined by $\hat{y}$ is given by the requirement that each edge in $\Gamma_3(y) \cap P$ has the same label as the edge in $\Gamma_3(x) \cap P$ it meets.

If $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ are the three vertices above $x$ then these determine three labellings of the three edges in $\Gamma_3(x) \cap P$ that are cyclic shifts of each other (since for $i \neq j$ we have $d(\hat{x}_i, \hat{x}_j) = 8$ so $\hat{x}_i$ and $\hat{x}_j$ cannot both have distance 3 to the same vertex of some $P_j$).

Now $A$ is connected, so identifying the vertex set of $\tilde{A}$ with $A \times Z_2$ all adjacencies in $\tilde{A}$ are determined and clearly this determines $\Gamma$. Thus there is at most one possibility for $\tilde{A}$ and by the result of Ivanov, Ivanov & Faradjev there is a unique $\Gamma$.

\[\square\]

**Remark.** I have not determined whether $\tilde{A}$ is the union of three copies of $A$ or is connected, since that is unimportant for the above argument.

The distance distribution diagram for $A$ follows. (We have $v = 160$, $k = (1, 4, 12, 30, 60, 46, 7)$.)
Uniqueness and Nonexistence of Some Graphs Related to $M_{23}$

References