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On the Extension Complexity of Stable Set Polytopes for Perfect Graphs

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Abstract

In linear programming one can formulate many combinatorial optimization problems as optimizing a linear function over a feasible region that is a polytope. Given a polytope P, any non-redundant description of P contains precisely one inequality for each facet. A polytope Q is called an extension of P if $\pi(Q) = P$ under some affine mapping π. Notice that Q could be in higher dimensional space than P. The extension complexity $\mathrm{xc}(P)$ of P is defined as the minimum number of facets among all polytopes Q that are extensions of P. If P has small extension complexity, one can optimize over P by means of a small linear program. This motivates the study of upper and lower bounds on extension complexity. In fact, Yannakakis [25] showed that the extension complexity of P is equal to the non-negative rank of the slack matrix of P. Moreover, there is a link between extension complexity and communication complexity, and one can often obtain lower bounds on extension complexity from lower bounds on nondeterministic communication complexity. Yannakakis [25] proved an upper bound $n^{\mathcal{O}(\log n)}$ for the extension complexities of the stable set polytopes of perfect graphs. Chudnovsky et al. [6] showed that every perfect graph either forms one of the basic perfect graphs or it admits one of the structural decompositions. These results motivate our study of extension complexity and graph operations.

The thesis starts with a global overview of extension complexity and its connection with communication theory. We then study the extension complexity of stable set polytopes, denoted by $\mathrm{xc}(\mathrm{STAB}(G))$, for some graphs G. One can apply graph operations to graphs G_1 and G_2 to obtain another graph G_0 and ask what is the relationship between $\mathrm{xc}(\mathrm{STAB}(G_0))$ and $\mathrm{xc}(\mathrm{STAB}(G_1))$, $\mathrm{xc}(\mathrm{STAB}(G_2))$. We also consider some special graphs. It is not clear whether the extension complexity of the stable set polytope of perfect graphs is polynomial or not. Nevertheless, we are able to show that the extension complexities of the stable set polytopes of double-split graphs are polynomial. Furthermore, we provide an upper bound on the extension complexity of any perfect graph G depending on the depth of a decomposition tree of G. We also study the extension complexity of other subclasses of perfect graphs as well, for example, Meyniel graphs.

Contents

Chapter 1

Introduction

1.1 Background

In combinatorial optimization, many algorithms are linear programming based. The most common approach is as follows: given a combinatorial problem, one identifies its feasible solutions with some vectors in such a way that optimizing a linear function over the convex hull of these vectors is equivalent to solving the original problem. The convex hull of a finite set of vectors is called a polytope. Minkowski [21] showed that every polytope can be written as the set of solutions of a system of linear equalities and inequalities. In other words, if one can find a linear system to describe the polytope of the underlying problem, then this provides a linear programming based algorithm.

One of the main obstacles behind the development of an efficient linear programming based algorithm is the well-known fact that there are polytopes which require exponentially many linear inequalities to describe it. For instance, Edmonds [12] showed that the spanning tree polytope, i.e., the convex hull of all characteristic vectors of spanning trees of the complete graph K_n, requires exponentially many inequalities to define it. Nevertheless, this obstacle may be resolved if one is allowed to introduce additional variables. Martin [19] used extra variables to provide a polynomial sized linear system whose underlying polyhedron has a projection to the spanning tree polytope. It turns out that one can often largely reduce the number of inequalities required to describe a polytope by introducing a few extra variables. This motivates the study of the so-called extension complexity, which for a polytope P denotes the smallest number of inequalities required to describe a higher dimensional polytope Q whose (affine) linear projection is P.

The extension complexity of the spanning tree polytope turns out to be polynomial, however it is clear that there are many polytopes whose extension complexities are exponential. Indeed, one should not expect an efficient algorithm for NP-hard problems, and the extension complexities of some of NP-hard problems are already shown to be exponential. The first result of this type can be found in Fiorini et al. [14] where the authors showed that the extension complexity of the traveling salesman polytope is $2^{\Omega(\sqrt{n})}$, where n is the number of vertices of the underlying graph. This answers a long standing open question by Yannakakis [25]. A more interesting question is whether the extension complexities of the problems in the class P are always at most polynomial. Rothvoß [22] answered this question negatively by showing that the extension complexity of the perfect matching polytope of a complete graph K_n is $2^{\Omega(n)}$.

Yannakakis [25] built a link between extension complexity and communication complexity. He showed that the minimum number of rectangles needed to cover the 1-entries of the slack matrix of the polytope P is a lower bound (known as the rectangle covering bound) on the extension complexity of P. Furthermore, he observed that the logarithm of the rectangle covering bound of a polytope P is equal to the nondeterministic communication complexity of the function f whose communication matrix is the slack matrix of P.

In this thesis, we will study the extension complexity of the stable set polytope $\mathrm{STAB}(G)$ for some graphs G. The maximum stable set problem is a well-known NP-hard problem in combinatorial optimization, which can be formulated as the maximization of a linear function over the stable set polytope. Grötschel et al. [16] showed that one can find the maximum stable set in polynomial time for perfect graphs using semi-definite programming. This is so far the only known method to find the maximum stable set in perfect graphs. Yannakakis [25] showed that the extension complexity of the stable set polytope of perfect graphs is at most $n^{\mathcal{O}(\log n)}$. In fact, it is still an open question whether the extension complexity of $\mathrm{STAB}(G)$ is polynomial, when G is a perfect graph.

Chudnovsky et al. [6] showed that every perfect graph either forms one of five types of basic perfect graphs or it admits one of two different types of structural decomposition into simpler graphs. This motivates our study of the relationship between $\mathrm{xc}(\mathrm{STAB}(G_0))$ and $\mathrm{xc}(\mathrm{STAB}(G_1))$, $\mathrm{xc}(\mathrm{STAB}(G_2))$, when the graph G_0 is composed from the graphs G_1 and G_2.

The thesis is organized as follows. In Chapter 1 we introduce the basic notation and terminology which will be used throughout this thesis. Furthermore we also collect some of the basic facts from polyhedral theory and graph theory.

In Chapter 2 we review the basics of extension complexity. We first define the extended formulation of a polytope (Section 2.1), and then introduce the non-negative rank and the slack matrix (Section 2.2 and 2.3). The main result of this chapter is Yannakakis' Factorization Theorem (Section 2.4).

In Chapter 3 we give a short introduction to communication complexity theory. Section 3.1 contains some basics and definitions in communication complexity. In Section 3.2, the deterministic communication complexity and several important quantities like the rectangle covering number are studied, and in Section 3.3 we introduce the nondeterministic communication complexity and its link with the extension complexity.

In Chapter 4 we look at the extension complexity of the stable set polytope of a graph G_0 obtained from two graphs G_1 and G_2 using some graph operation. In Section 4.1, we motivate the study of the relation between extension complexity and graph operations. In Section 4.2, 4.3 and 4.4, we study the following graph operations: graph substitutions, graph amalgamation and clique sum.

In Chapter 5 we study the extension complexity of perfect graphs. In Section 5.1 and 5.2 we give some basic facts about the extension complexity of general perfect graphs. In Section 5.3, the extension complexity of a subclass of perfect graphs known as double-split graph is shown to be polynomial in the size of the graph. In Section 5.4 and 5.5, we show an upper bound on the extension complexity of a graph G admitting some proper 2-join decompositions or skew-partitions, in terms of the extension complexities of some subgraphs of G. Finally, in Section 5.6, the extension complexity of Meyniel graphs, which are a subclass of perfect graphs, is shown to be polynomial under certain conditions.

1.2 Polyhedral Theory

We start this section by giving some basic definitions in polyhedral theory. The material is based on Bertsimas and Tsitsiklis [3].

Definition 1.2.1. *A* polyhedron *is a set of the form* $\{x \in \mathbb{R}^n : Ax \leq b\}$*, where* A *is an* $m \times n$ *matrix and* b *is an* m*-dimensional vector. The inequalities in the linear system* $Ax \leq b$ *are referred to as constraints.*

Definition 1.2.2. *Let $v_1, \ldots, v_k$ be vectors in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_k$ be non-negative scalars whose sum is one.*

(i) The vector $\sum_{i=1}^{k} \lambda_i v_i$ is called a convex combination *of $v_1, \ldots, v_k$;*

(ii) The convex hull *of $v_1, \ldots, v_k$, denoted by $\text{conv. hull}\{v_1, \ldots, v_k\}$, is the set of all convex combinations of these vectors.*

If the non-negative scalars $\lambda_1, \ldots, \lambda_k$ are not required to sum to one, then the vector $\sum_{i=1}^{k} \lambda_i v_i$ is called a conic combination *of $v_1, \ldots, v_k$.*

Definition 1.2.3. *A* polytope *P is the convex hull of a finite number of vectors. That is $P = \text{conv. hull}\{v_1, \ldots, v_k\}$ for some vectors $v_1, \ldots, v_k$.*

Definition 1.2.4. *Let P be a polyhedron. An element $x \in P$ is called a* vertex *of P if there exists a vector c such that $c^T x < c^T y$ for all $y \in P$ and $y \neq x$.*

Definition 1.2.5. *Let P be a polyhedron. A vector $x \in P$ is called an* extreme point *of P if there do not exist two vectors $y, z \in P$ and a scalar $\lambda \in (0, 1)$ such that $x = \lambda y + (1 - \lambda) z$.*

Theorem 1.2.6. *Let P be a polyhedron. A vector $x \in P$ is an extreme point if and only if it is a vertex of P.*

Let $A \in \mathbb{R}^{m \times n}$ be any matrix. For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, we denote the ith row of A and jth column of A by $A_i \in \mathbb{R}^{n \times 1}$ and $A^j \in \mathbb{R}^{m \times 1}$, respectively.

Definition 1.2.7. *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron. If an element $x^* \in P$ satisfies $A_i^T x^* = b_i$, we say that the corresponding ith constraint is* active *or* binding *at x^*. If a constraint $A_i^T x = b_i$ is active at some elements of P, then this constraint is called a* tight *constraint.*

Minkowski [21] showed that the vertices of bounded polyhedron and polytope are equivalent.

Theorem 1.2.8. [21] *A subset $P \subseteq \mathbb{R}^n$ is a bounded polyhedron if and only if it is a polytope.*

We conclude this section with the famous Farkas' Lemma, which is at the core of linear optimization.

Theorem 1.2.9. [13] *Given $A \in \mathbb{R}^{m \times d}, b \in \mathbb{R}^m$, precisely one of the following two statements is true:*
1. there exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$.
2. there exists $y \in \mathbb{R}^m_+$ such that $y^T b < 0$ and $y^T A \geq 0$.

The following is an equivalent statement of Farkas' Lemma.

Theorem 1.2.10. [13] *If the linear system $Ax \leq b$ implies the linear inequality $c^T x \leq d$, then there exists a non-negative vector $y \geq 0$ such that $y^T A = c^T$ and $y^T b \leq d$. Furthermore, if it also holds that $\{x : Ax \leq b,\ c^T x = d\}$ is non-empty, then there exists a non-negative vector $y \geq 0$ such that $y^T A = c^T$ and $y^T b = d$.*

1.3 Graph Theory

A graph G consists of a set of vertices, where some pairs of vertices are connected by edges. More formally, the set of vertices of G is $V(G) = \{v_1, \ldots, v_k\}$. The set of edges of G, denoted by $E(G)$, is a collection of unordered pairs of vertices. Let us introduce some basic definitions first.

Definition 1.3.1. *Let G be a graph and u be a vertex of G. The* neighborhood *of u, denoted by $N_G(u)$, is defined as $N_G(u) := \{v \in V(G) : \{u, v\} \in E(G)\}$. We simply write $N(u)$ when there is no ambiguity.*

Definition 1.3.2. *Let G be a graph and A be a subset of the vertices of G. The* subgraph *induced by A, denoted by $G(A)$, is the graph with vertex set $A \subseteq V(G)$ and edge set consisting of all edges of G whose both endpoints are in A. A graph G' is said to be an* induced subgraph *of G if $G' = G(A)$ for some $A \subseteq V(G)$.*

Definition 1.3.3. *Let G be a graph. A* path *of G is an ordered set of vertices $v_1, \ldots, v_k$ in $V(G)$ such that $\{v_i, v_{i+1}\} \in E(G)$ for every $i = 1, \ldots, k-1$. And we say this is a path from v_1 to v_k.*

Definition 1.3.4. *Let G be a graph. A* matching *is a subset of edges such that no pair of edges shares common endpoints.*

Definition 1.3.5. *Let G be a graph and u, v be the vertices of G. We say u is* reachable *from v if there is a path from u to v in G.*

Definition 1.3.6. *A graph G is called* connected *if for each pair of vertices u and v of G, u is reachable from v.*

Definition 1.3.7. *Let G be a graph and A, B be two disjoint subsets of the vertices of G. We define*

$$E(A) := \{\{u, v\} \in E(G) : u, v \in A\}$$

and

$$E(A,B) := \{\{u,v\} \in E(G) : u \in A,\ v \in B\}.$$

So $E(A)$ is the set of edges whose endpoints are both in A, and $E(A,B)$ is the set of edges crossing the vertex sets A and B.

Given a graph G, we ask whether G admits any interesting 'structure'. The so-called stable sets are the 'structure' that will be studied in this thesis.

Definition 1.3.8. *A* stable set *(also called* independent set*) is a subset of the vertices of G no two of which are adjacent. The cardinality of a largest stable set in G is denoted by $\alpha(G)$.*

Definition 1.3.9. *A* clique *is a set of pairwise adjacent vertices in G. The cardinality of a largest clique in G is denoted by $\omega(G)$.*

In what follows, the set of stable sets and the set of cliques in G are denoted by $\mathcal{I}(G)$ and $\mathcal{C}(G)$, respectively. Now we provide the definition of a very important class of graphs, called perfect graphs. These have been studied extensively in the past. The definitions of complete graphs, bipartite graphs, the line graph of a graph and the complement of a graph are also given.

Definition 1.3.10. *A graph G is said to be* perfect *if $\chi(G') = \omega(G')$ for every induced subgraph G' of G.*

Definition 1.3.11. *A* complete graph *with r vertices, denoted by K_r, is a graph whose edge set consists of all possible edges.*

Definition 1.3.12. *A* bipartite graph *is a graph whose vertices can be partitioned into two independent sets.*

Definition 1.3.13. *A graph L is the* line graph *of a graph G if $V(L) = E(G)$ and two vertices of L are adjacent if and only if their corresponding edges share a common endpoint in G.*

Definition 1.3.14. *The* complement of a graph G*, denoted by $\bar{G}$, is the graph which has the same vertex set as G, and $\{u,v\} \in E(\bar{G})$ if and only if $\{u,v\} \notin E(G)$ for all $u, v \in V(G)$.*

We may view a subset of vertices of some graph G as a binary vector of size $|V(G)|$. This enables us to analyze many graph problems from a different perspective.

Definition 1.3.15. *Given a subset $S \subseteq V(G)$ of vertices of G, the characteristic vector of S, denoted by χ^S, is a $|V(G)|$-dimensional binary vector such that $\chi^S(u) = 1$ if and only if $u \in S$. So χ^S is indexed by the vertices of G.*

For a subset of vertices $K \subseteq V(G)$, $\chi^S(K) \in \{0,1\}^{|K|}$ is the 'subvector' of χ^S corresponding to the vertices of K.

Recall that a polytope is the convex hull of a finite number of vectors. The stable set polytope defined below is the central object of study in this thesis.

Definition 1.3.16. *The* stable set polytope $\mathrm{STAB}(G)$ *of a graph G is the convex hull of the characteristic vectors of the stable sets of G. It is defined as*

$$\mathrm{STAB}(G) := \mathrm{conv.\,hull}\{\chi^I : I \textit{ is stable in } G\} \subseteq \mathbb{R}^{|V(G)|}.$$

From Theorem 1.2.8, we know that the stable set polytope can be described by some linear system. Nevertheless, the proof of Theorem 1.2.8 only shows the existence and there is, in fact, so far no recipe to find such a linear system in general. Although the description of the stable set polytope in general is not known, an explicit description of $\mathrm{STAB}(G)$ can be given for some special class of graphs. Chvátal [7] provided an explicit description of the stable set polytope of perfect graphs.

Theorem 1.3.17. [7] *A graph is perfect if and only if* $\mathrm{STAB}(G)$ *is characterized by the following linear system, in the vector of variables $x \in \mathbb{R}^{|V(G)|}$,*

$$\sum_{v \in C} x_v \leq 1 \ \ \forall\, C \textit{ clique of } G, \tag{1.1}$$

$$x \geq 0. \tag{1.2}$$

Thus G is perfect if and only if $\mathrm{STAB}(G) = \{x \in \mathbb{R}^{|V|} : x \textit{ satisfies } (1.1) \textit{ and } (1.2)\}$. *In what follows, the set of inequalities in (1.1) are called* the clique constraints *and the set of inequalities in (1.2) are called* the non-negativity constraints. *Notice that it suffices to include all the maximal clique constraints in* (1.1).

Chapter 2

Extension Complexity

2.1 Extended Formulation

An extended formulation of a polytope is a linear system describing this polytope possibly using additional variables. The interest of extended formulations is due to the fact that one can often reduce the number of inequalities needed to define this polytope when additional variables are allowed.

Definition 2.1.1 (Extended Formulation)**.** *Let $P \subseteq \mathbb{R}^d$ be a polytope. Consider a linear system of the form:*

$$Ex + Ft = g,\ \hat{E}x + \hat{F}t \leq \hat{g} \tag{2.1}$$

in variables $(x, t) \in \mathbb{R}^{d+q}$, where $E \in \mathbb{R}^{p \times d}, F \in \mathbb{R}^{p \times q}, \hat{E} \in \mathbb{R}^{r \times d}, \hat{F} \in \mathbb{R}^{r \times q}, g \in \mathbb{R}^p$ and $\hat{g} \in \mathbb{R}^r$ for some $p, q, r \in \mathbb{Z}_+$. The linear system (2.1) is called an extended formulation *of P when $x \in P$ if and only if there exists a $t \in \mathbb{R}^q$ such that the vector (x, t) satisfies the system (2.1). The additional variable t is called the* lifting variable*.*

The size *of the extended formulation is defined as the number r of inequalities in the system (2.1). The extended formulation is said to be in* slack form *if the only inequalities are non-negativity conditions on the lifting variable t, i.e., if it is of the form:*

$$Ex + Ft = g,\ t \geq 0 \tag{2.2}$$

and then its size is the dimension of the variable t.

In order to prove that a linear system, say the linear system (2.1), is an extended formulation of some polytope P, one of the things we need to show is that for each

$x \in P$, there exists a $t \in \mathbb{R}^q$ such that the vector (x, t) satisfies the system (2.1). In fact, it suffices to show this statement for all vertices v of P.

Lemma 2.1.2. *Let $P \subseteq \mathbb{R}^d$ be a polytope. If $v \in V(P)$ implies that there exists a $t_v \in \mathbb{R}^q$ such that the vector (v, t_v) satisfies the linear system (2.1), then for each $x \in P$, there exists a $t \in \mathbb{R}^q$ such that the vector (x, t) satisfies the system (2.1).*

Proof. Let $x \in P$. Then x can be written as a convex combination of the vertices of P, i.e., $x = \sum_{v \in V(P)} \lambda_v \cdot v$, where λ_v's are non-negative scalars. By assumption, for each vertex v of P, there exists a $t_v \in \mathbb{R}^q$ such that the vector (v, t_v) satisfies the linear system (2.1). Then $\sum_{v \in V(P)} \lambda_v \cdot (v, t_v)$ also satisfies the linear system (2.1). Let $t = \sum_{v \in V(P)} \lambda_v \cdot t_v$. It holds that (x, t) satisfies the linear system (2.1). □

Definition 2.1.3 (Extension). *Let $P \subseteq \mathbb{R}^d$ be a polytope. A polytope $Q \subseteq \mathbb{R}^k$ is called an* extension *of P if there exists a linear mapping $\pi : \mathbb{R}^k \to \mathbb{R}^d$ such that $P = \pi(Q)$. The* size *of the extension Q, denoted by size(Q), is defined as the number of facets of Q.*

Definition 2.1.4 (Extension Complexity). *Let $P \subseteq \mathbb{R}^d$ be a polytope. The* extension complexity *of P is defined as*

$$\mathrm{xc}(P) = \min\{\textit{size}(Q) : Q \textit{ is an extension of } P\}.$$

2.2 Non-negative Rank

Given a non-negative matrix S, the non-negative rank of S is the smallest number of rank-one matrices with non-negative entries into which S can be decomposed additively. The non-negative rank of S is denoted by $\mathrm{rank}_+(S)$. Formally it is defined as follows.

Definition 2.2.1 (Non-negative rank). *The* non-negative rank *of a matrix $S \in \mathbb{R}_+^{m \times n}$ is defined as*

$$\mathrm{rank}_+(S) = \min\{r : S = TU,\;\; T \in \mathbb{R}_+^{m \times r},\;\; U \in \mathbb{R}_+^{r \times n}\}.$$

In what follows, $S = TU$ above is called a non-negative decomposition with intermediate dimension r.

While computing the (usual) rank of a matrix is an easy task, computing the non-negative rank is a non-trivial matter. Vavasis [24] showed that determining if $\mathrm{rank}_+(S) = \mathrm{rank}(S)$ is NP-hard. Some procedures to implement such computation have been investigated, see e.g. the work of Cohen and Rothblum [9].

In the rest of this subsection, we provide some basic and well-known properties of the non-negative rank.

Lemma 2.2.2. *For any non-negative matrix $S \in \mathbb{R}_+^{m\times n}$, we have*

(i) $\mathrm{rank}_+(S) = \mathrm{rank}_+(S^T)$*;*

(ii) $\mathrm{rank}_+(S) \leq \min\{m, n\}$*;*

(iii) $\mathrm{rank}(S) \leq \mathrm{rank}_+(S)$*.*

Proof.

(i) If $S = UV$, then $S^T = (UV)^T = V^TU^T$.

(ii) Consider the non-negative decompositions $S = SI_n = I_mS$.

(iii) The rank of S can be defined as $\mathrm{rank}(S) = \min\{r : S = TU,\ \ T \in \mathbb{R}^{m\times r},\ \ U \in \mathbb{R}^{r\times n}\}$.

□

Lemma 2.2.3. *Given matrices $S_1 \in \mathbb{R}^{m_1\times n}$, $S_2 \in \mathbb{R}_+^{m_2\times n}$, $S_1' \in \mathbb{R}^{m\times n_1}$, $S_2' \in \mathbb{R}_+^{m\times n_2}$, it holds that*

(i) $\mathrm{rank}_+(\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}) \leq \mathrm{rank}_+(S_1) + \mathrm{rank}_+(S_2)$*;*

(ii) $\mathrm{rank}_+(\begin{bmatrix} S_1' & S_2' \end{bmatrix}) \leq \mathrm{rank}_+(S_1') + \mathrm{rank}_+(S_2')$*.*

Proof. We shall prove the first statement; the second statement follows from the first one using Lemma 2.2.2 (i). Set $r_i := \mathrm{rank}_+(S_i)$ $(i = 1, 2)$ and consider some non-negative decompositions of S_1 and S_2 with intermediate dimensions r_1 and r_2, respectively. Namely, $S_1 = T_1U_1$ and $S_2 = T_2U_2$ for some matrices $T_1 \in \mathbb{R}_+^{m_1\times r_1}, U_1 \in \mathbb{R}_+^{r_1\times n}, T_2 \in \mathbb{R}_+^{m_2\times r_2}, U_2 \in \mathbb{R}_+^{r_2\times n}$. Then

$$S := \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix},$$

which gives a non-negative decomposition of S with intermediate dimension $r_1 + r_2$, and shows $\mathrm{rank}_+(S) \leq \mathrm{rank}_+(S_1) + \mathrm{rank}_+(S_2)$.

□

2.3 Slack Matrix

Definition 2.3.1 (Slack matrix)**.** *Let $P \subseteq \mathbb{R}^d$ be a polytope. Consider a set $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$ containing all vertices of P, i.e., $V(P) \subseteq V$, and a linear system $Ax \leq b$ describing P, i.e., $P = \{x \in \mathbb{R}^d : Ax \leq b\}$, where $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^{m \times 1}$. Then the $m \times n$ non-negative matrix $S = (S_{i,j})$ with entries*

$$S_{i,j} = b_i - A_i^T v_j$$

is called a slack matrix *of P and we say it is induced by V and the linear system $Ax \leq b$.*

Notice that the slack matrix of a polytope depends on the underlying linear system and the point set V. As every slack matrix is non-negative, the non-negative rank of any slack matrix is well-defined. In the next section, we will see that $\text{rank}_+(S)$ is equal to the extension complexity of the underlying polytope P and thus $\text{rank}_+(S)$ does not depend on the choice of the set $V \supseteq V(P)$ and the linear system defining P.

Lemma 2.3.2. [25] *Let S be a slack matrix of some polytope $P \subseteq \mathbb{R}^d$, induced by the linear system $Ax \leq b$ defining P and the point set $\{v_1, \ldots, v_n\} \supseteq V(P)$. It holds that* $\text{rank}(S) \leq d + 1$.

Proof. The slack matrix can be rewritten as $S = b\mathbf{1}^T - AV$ where $V = \left[v_1, \ldots, v_n\right]$ is the $d \times n$ matrix using the vectors of the vertices as columns. This implies the rank of any slack matrix is no greater than $d + 1$. □

2.4 Yannakakis' Factorization Theorem

Given a polytope P, Yannakakis' factorization theorem (see Yannakakis [25]) states that the smallest size of an extended formulation of P equals the nonnegative rank of its slack matrix S. This leads to a surprising connection between extension complexity and communication complexity, and we shall discuss them in Section 3. Before we provide the factorization theorem, we need the following lemmas.

Definition 2.4.1. *A polyhedron $P \subseteq \mathbb{R}^d$ is called* regular*, if there exists a direction $u \in \mathbb{R}^d$ such that $-\infty < \min\{u^T x : x \in P\} < \max\{u^T x : x \in P\} < \infty$.*

If P is regular, then P has dimension at least 1. Moreover if P is a polytope of dimension at least 1 or more generally if some linear image $\pi(P)$ is a polytope of dimension at least 1, then P is regular.

Lemma 2.4.2. *Consider a regular polyhedron $P = \{x \in \mathbb{R}^d : Ax \leq b\}$. Assume that the inequality $c^T x \leq d$ is implied by the system $Ax \leq b$. Then there exist non-negative scalars $\lambda \in \mathbb{R}^d_+$ such that $\lambda^T A = c^T$ and $\lambda^T b = d$, i.e., $c^T x \leq d$ is a conic combination of inequalities in $Ax \leq b$.*

Proof. First of all, we show that the inequality $0^T x \leq 1$ can be written as a conic combination of inequalities in $Ax \leq b$. As P is regular, we can find a direction u such that $l_1 = \min\{u^T x : x \in P\}$ and $l_2 = \max\{u^T x : x \in P\}$ are well-defined and $l_1 < l_2$. Moreover the inequalities $-u^T x \leq -l_1$ and $u^T x \leq l_2$ are valid and tight for P. By Farkas' lemma (Theorem 1.2.10), they are both conic combination of the inequalities in $Ax \leq b$; moreover the sum of them is $0^T x \leq 1$ after some positive scaling.

Now consider the valid inequality $c^T x \leq d$. If it is tight, the result follows again from Farkas' lemma (Theorem 1.2.10). If it is not tight, there exists a scalar $\tilde{d}$ strictly smaller than d such that $c^T x \leq \tilde{d}$ is tight. Now $c^T x \leq d$ is a non-negative combination of $c^T x \leq \tilde{d}$ and $0^T x \leq 1$, which are both conic combinations of inequalities of $Ax \leq b$, and thus this proves the claim. □

Lemma 2.4.3. *Let P be a polytope. Consider the slack matrix induced by the set of points $V = \{v_1, \ldots, v_n\} \supseteq V(P)$ and the linear system $Ax \leq b$ defining P, where $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^{m \times 1}$. The following holds:*

(i) Let $c^T x \leq d$ be a valid inequality for P. Let S' be the slack matrix induced by the linear system $Ax \leq b$, $c^T x \leq d$, and the set of points V. If P is regular, then

$$\text{rank}_+(S') = \text{rank}_+(S).$$

(ii) Let $v_{n+1} \in P$ be any point in P. Let S'' be the slack matrix induced by the linear system $Ax \leq b$ and the set of points $V \cup \{v_{n+1}\}$. Then

$$\text{rank}_+(S'') = \text{rank}_+(S).$$

Proof. By Lemma 2.2.3, we have $\text{rank}_+(S') \leq \text{rank}_+(S)$ and $\text{rank}_+(S'') \leq \text{rank}_+(S)$, as S' and S'' are sub-matrices of S. Let $S = TU$ be a non-negative factorization of S, for some $T \in \mathbb{R}^{m \times r}_+, U \in \mathbb{R}^{r \times n}_+$. We shall extend this factorization to a non-negative factorization of S' and S'' with the same intermediate dimension, which will prove the claim.

(i) As P is regular and the inequality $c^T x \leq d$ is implied by the linear system $Ax \leq b$, applying Lemma 2.4.2, there exist non-negative scalars $\lambda \in \mathbb{R}_+^m$ such that $\sum_{i=1}^m \mu_i A_i^T = c^T$ and $\sum_{i=1}^m \mu_i b_i = d$. Define $T_{m+1} := \sum_{i=1}^m \mu_i T_i \in \mathbb{R}^{1\times n}$. For every $j = 1, \dots, n$, we have

$$T_{m+1}U_j = (\sum_{i=1}^m \mu_i T_i)U_j = \sum_{i=1}^m \mu_i (b_i - A_i^T v_j) = d - c^T v_j.$$

This implies $S' = \begin{bmatrix} T \\ T_{m+1} \end{bmatrix} U$.

(ii) As v_{n+1} is a convex combination of the points in V, there exist non-negative coefficients $\lambda_1, \dots, \lambda_n$ such that $v_{n+1} = \sum_{i=1}^n \lambda_i v_i$ and $\sum_{i=1}^n \lambda_i = 1$. Define $U^{n+1} := \sum_{i=1}^n \lambda_i U^i \in \mathbb{R}^{m\times 1}$. For every $i = 1, \dots, m$, we have

$$T_i U^{n+1} = \sum_{j=1}^n \lambda_j T_i U^{n+1} = \sum_{j=1}^n \lambda_j (b_i - A_i v_{n+1}) = b_i - A_i v_{n+1}.$$

Thus $S'' = T \begin{bmatrix} U & U_{n+1} \end{bmatrix}$. □

By Minkowski-Weyl theorem (see Minkowski [21]), any polytope P can be alternatively described as the set of solutions of a linear system or as the convex hull of a finite set of points. These descriptions are not unique and thus the slack matrix of P is also not unique. However if P is regular, as a direct corollary of Lemma 2.4.3, then the non-negative rank of the slack matrix of P does not depend on the choice of the linear system describing P or the set of points $V \supseteq V(P)$. Hence the non-negative rank of the slack matrix of P characterizes a geometric property of P, it does not depend on the algebraic description of P.

The following theorem from Yannakakis [25], builds a link between the non-negative rank of the slack matrix of P and the extension complexity of P. We provide another version of the proof due to Fiorini et al. [14].

Theorem 2.4.4. [25] *Let $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ be a polytope of dimension at least 1, where $A \in \mathbb{R}^{m\times d}$, $b \in \mathbb{R}^{m\times 1}$, and $V = \{v_1, \dots, v_n\} \supseteq V(P)$. Let $S \in \mathbb{R}^{m\times n}$ be the induced slack matrix. Let r be a positive integer. The following assertions are equivalent:*

(i) $\operatorname{rank}_+(S) \leq r$*;*

(ii) P has an extension of size at most r;

(iii) P has an extended formulation in slack form of size at most r.

(iv) P has an extended formulation of size at most r.

Proof. $(i) \Rightarrow (ii)$: By assumption, $S = TU$ for some non-negative matrices $T \in \mathbb{R}_+^{m\times r}$ and $U \in \mathbb{R}_+^{r\times n}$. Moreover we may assume T contains no zero-column, otherwise r can be decreased. Let Q be a polyhedron defined by the following linear system

$$Ax + Tt = b,\ t \geq 0. \tag{2.3}$$

So $Q = \{(x,t) \in \mathbb{R}^{d+r} : (x,t) \text{ satisifes linear system (2.3)}\}$. Since T is non-negative and it contains no zero-columns, one can check that Q is a bounded polyhedron and thus it is a polytope. Moreover Q has at most r facets as it is defined by r inequalities. So Q is an extension of P of size at most r if we can show $\pi(Q) = P$.

$\pi(Q) \subseteq P$: Take any $(x,t) \in Q$, we have $Tt \geq 0$ as both T and t are non-negative, this implies $Ax \leq b$ and thus $x \in P$.

$\pi(Q) \supseteq P$: It suffices to show any vertex of P is contained in $\pi(Q)$. Take any vertex $v_j \in V(P)$, then $v_j \in V$ and take the corresponding column $U^j \in \mathbb{R}_+^r$ in the matrix U. Using the definition of slack matrix, we have

$$Av_j + TU^j = Av_j + S^j = Av_j + (b - Av_j) = b.$$

This implies that the vector $(v_j, U^j) \in \mathbb{R}^{d+r}$ is in Q and thus $v_j \in \pi(Q)$.

$(ii) \Rightarrow (iv)$: Let $Q \subseteq \mathbb{R}^k$ be an extension of P of size at most r. By definition, Q is a polytope with at most r facets, thus it can be defined by some linear system with at most r inequalities, say

$$Q := \{y \in \mathbb{R}^k : A'y = b', By \leq d\}.$$

So B has at most r rows. Let π be the affine mapping such that $P = \pi(Q)$, say $\pi(y) = Cy + c$ for some matrix C and vector c. Now consider the following linear system

$$A'y = b', By \leq d, x = Cy + c$$

where $x \in \mathbb{R}^d$. This linear system is an extended formulation of P, i.e., $x \in P$ if and only if there exists $y \in \mathbb{R}^k$ such that the pair (x,y) satisfies the above system. The size of an extended formulation is the number of inequalities in the linear system, which is at most r in this case.

$(iv) \Rightarrow (i)$: Assume we have an extended formulation of P of size r given by the following linear system

$$Ex + Ft = g,\ \hat{E}x + \hat{F}t \leq \hat{g} \tag{2.4}$$

in variables $(x, t) \in \mathbb{R}^{d+q}$, where $E \in \mathbb{R}^{p\times d}, F \in \mathbb{R}^{p\times q}, \hat{E} \in \mathbb{R}^{r\times d}, \hat{F} \in \mathbb{R}^{r\times q}, g \in \mathbb{R}^p$ and $\hat{g} \in \mathbb{R}^r$ for some $p, q \in \mathbb{Z}_+$. Furthermore let $Q \subseteq \mathbb{R}^{d+r}$ be the polyhedron defined by the above linear system (2.4). Clearly Q is regular, as it has a linear image which is a polytope of dimension at least 1.

Let $S' \in \mathbb{R}_+^{(r+m)\times|V|}$ be the slack matrix whose first r rows correspond to the inequalities in the linear system (2.4) and whose last m rows correspond to the inequalities in the linear system $Ax \le b$ and whose columns corresponds to the set of points $V' = \{(v, 0) \in \mathbb{R}^{d+q} : v \in V\}$. Now the slack matrix S is a submatrix of S', since S consists of the last m rows of S'. This implies $\text{rank}_+(S) \le \text{rank}_+(S')$.

As Q is regular and each inequality in the linear system $Ax \le b$ is implied by the linear system (2.4), applying Lemma 2.4.3, we conclude that $\text{rank}_+(S') = \text{rank}_+(S'')$, where S'' is the submatrix of S' corresponding to its first r rows. It remains to notice that $\text{rank}_+(S'') \le r$ as S'' has r rows, using Lemma 2.2.2. Thus $\text{rank}_+(S') \le r$, which in turn implies $\text{rank}_+(S) \le r$.

$(iii) \Rightarrow (iv)$: An extended formulation in slack form is an extended formulation.

$(iv) \Rightarrow (iii)$: If P has an extended formulation of size at most r, then $\text{rank}_+(S) \le r$ by implication $(iv) \Rightarrow (i)$. We claim that the linear system (2.3) is an extended formulation of P in slack form of size r. Indeed, in $(i) \Rightarrow (ii)$, we already prove that $x \in P$ if and only if there exists a $t \in \mathbb{R}^r$ such that (x, t) satisfies the linear system (2.3). □

The central question in this field is finding good upper and lower bounds for the extension complexity of a given polytope P. For example, a trivial lower bound based on the dimension of P is provided below.

Lemma 2.4.5. *If P is a d-dimensional polytope, then* $\text{xc}(P) \ge d + 1$.

Proof. Assume $\text{xc}(P) < d + 1$. From Theorem 2.4.4, we know P has an extension $Q \subseteq \mathbb{R}^k$ with at most d facets and the dimension of Q satisfies $k \ge d$. Since a k-dimensional polytope has at least $k + 1$ facets, Q contains at least $k + 1 \ge d + 1$ facets. This is a contradiction. □

This trivial lower bound is tight in the following case.

Lemma 2.4.6. $\text{xc}(\text{STAB}(K_p)) = p + 1$.

Proof. From Lemma 2.4.5, we know $\text{xc}(\text{STAB}(K_p)) \ge p+1$. As the complete graph is perfect, it follows from Theorem 1.3.17 that the stable set polytope of K_p can be characterized by the following linear system

$$\sum_{i=1}^{p} x_i \leq 1 \text{ and } x_i \geq 0 \;\; \forall i = 1, \ldots, p.$$

in variables $x \in \mathbb{R}^p$. This is an extended formulation of size $p + 1$ and thus $\mathrm{xc}(\mathrm{STAB}(K_p)) \leq p + 1$. □

Chapter 3

Communication Complexity

3.1 Introduction

In this chapter we introduce communication compexity and establish its connection with extension complexity via non-negative rank of slack matrices. Part of the material is based on Kushilevitz and Nisan [17].

Let $f : X \times Y \to Z$ be a function for some sets X, Y, Z. The communication problem involves two separated parties (Alice and Bob), and each of them receives part of the input, i.e. Alice gets $x \in X$ and Bob gets $y \in Y$. The goal is to design a protocol so that they can compute $f(x, y)$ with the least amount of communication between them. Here we assume that Alice and Bob have unlimited computational power.

A communication protocol computing f is a distributed algorithm that, at each stage, must determine whether the run terminates; if the run terminates, both parties should know the outcome; and if the run has not terminated, the protocol must specify a player who should send a bit next. If Alice is specified, the protocol also specifies what she sends. The whole process depends solely on the bits communicated so far. The *cost of a protocol* $\mathcal{P}$ on input (x, y) is the total number of bits communicated by $\mathcal{P}$ on this input (x, y). The *cost of a protocol* $\mathcal{P}$ is the maximum cost of $\mathcal{P}$ over all inputs (x, y). The *complexity of* f is defined as the minimum cost among all protocols computing f.

We shall see that the complexity of f depends on the structure of the underlying *communication matrix*, which is defined as follows.

Definition 3.1.1. *Given a function* $f : X \times Y \to Z$*, the* communication matrix *of* f *is a matrix* $M_f \in \mathbb{R}^{|X| \times |Y|}$ *defined as* $(M_f)_{x,y} = f(x, y)$ *for every* $(x, y) \in X \times Y$*.*

3.2 Deterministic Communication

We would like to analyze the protocol from a combinatorial point of view. To this end, the protocol is defined in terms of binary trees.

Definition 3.2.1. *Let* $f : X \times Y \to Z$ *be a function. A* protocol $\mathcal{P}$ *is a binary tree such that*

(i) Each leaf is associated with an element $z \in Z$*;*

(ii) Each internal node v *is associated with a function* $a_v : X \to \{0, 1\}$ *or* $b_v : Y \to \{0, 1\}$*.*

The output of the protocol $\mathcal{P}$ *on input* (x, y) *is the element* $z \in Z$ *associated to the leaf node reached as follows: starting from the root; at each node associated with* a_v *walk left if and only if* $a_v(x) = 0$*, and at each node associated with* b_v *walk left if and only if* $b_v(y) = 0$*. The* cost of the protocol *is the depth of the tree.*

Definition 3.2.2. *A* deterministic protocol $\mathcal{P}$ *computing* f *is one whose output on input* (x, y) *always equals to* $f(x, y)$ *for every* $x \in X$ *and* $y \in Y$*.*

Definition 3.2.3. *Let* $f : X \times Y \to Z$ *be a function. The deterministic communication complexity of* f*, denoted by* $D(f)$*, is the minimum cost of* $\mathcal{P}$ *among all deterministic protocols computing* f*.*

There is a trivial protocol to compute function f: Alice sends her input to Bob, then Bob calculates $f(x, y)$ locally and sends the answer back to Alice. This yields the following upper bound on $D(f)$.

Proposition 3.2.4. *For every function* $f : X \times Y \to Z$*, the deterministic communication complexity of* f *satisfies:*

$$D(f) \leq \log_2 |X| + \log_2 |Z|.$$

Proof. It costs at most $\log_2 |X|$ bits to send Alice's input to Bob. Now Bob can calculate $f(x, y)$ locally and sending the answer back to Alice costs at most $\log_2 |Z|$ bits. This is a deterministic protocols computing f with costs $\log_2 |X| + \log_2 |Z|$. □

There is also a simple a lower bound on $D(f)$. Namely, $D(f)$ is at least $\log_2 |\text{range}(f)|$, where $\text{range}(f) = \{z \in Z : f(x,y) = z \text{ for some } (x,y) \in X \times Y\}$, as either Alice or Bob sends the answer to another party in the last step. This lower bound can be useful if the range of f is large. However we are concerned with boolean functions $f : X \times Y \to \{0,1\}$ in the remaining discussion, and this lower bound does not help. We will find a better bound by exploiting the tree structure of protocol $\mathcal{P}$.

Definition 3.2.5. *Given a protocol $\mathcal{P}$ and a node v from the underlying binary tree, we define R_v to be the set of inputs (x,y) that reach node v.*

As each input (x,y) results in a path from root to some leaf node, $\{R_l\}_{l \in L}$ is a partition of the input $X \times Y$, where L is the set of leaves. It turns out that such a partition possesses special structure. Let us define the so-called monochromatic rectangles first.

Definition 3.2.6. *A* rectangle *R of $X \times Y$ is a subset $R \subseteq X \times Y$ such that $R = A \times B$ for some $A \subseteq X$ and $B \subseteq Y$.*

Definition 3.2.7. *A subset $R \subseteq X \times Y$ is called* f-monochromatic *if the value of f is constant on R.*

We will see that deterministic communication complexity and rectangles are connected via the following proposition:

Proposition 3.2.8. *Given a protocol $\mathcal{P}$ and a node v from the underlying protocol tree, R_v is a rectangle.*

Proof. We prove this by induction on the level k of the node.
The base step: When $k = 0$, v is the root node and every $(x,y) \in X \times Y$ reaches v. Clearly $X \times Y$ is a rectangle.
Induction step: Assume the case for $k-1$ is true. Pick any node v at level k whose parent is u. By induction hypothesis, $R_u = A_u \times B_u$ is a rectangle. Assume without loss of generality that it is Alice's turn to send the bit at u and the associated function is $a_u : X \to \{0,1\}$. Assume that node v is the left child of u. Then we have

$$\begin{aligned} R_v &= R_u \cap \{(x,y) : a_u(x) = 0\} \\ &= (A_u \times B_u) \cap \{(x,y) : a_u(x) = 0\} \\ &= (A_u \cap \{(x,y) : a_u(x) = 0\}) \times B_u. \end{aligned}$$

This shows that R_v is a rectangle. The reasoning is analogous in the case when v is the right child of u. □

Finally we are ready to state the connection explicitly.

Proposition 3.2.9. *Let $f : X \times Y \to Z$ be a function and $\mathcal{P}$ be a protocol computing f. If L is the set of leaves of the protocol $\mathcal{P}$, then $\{R_l\}_{l \in L}$ forms a partition of $X \times Y$ into f-monochromatic rectangles. Furthermore, the number of rectangles is precisely the number of leaves of $\mathcal{P}$.*

Proof. The statement follows immediately from the fact that each input (x, y) reaches precisely one leaf l and all inputs reaching the same leaf l have the same value z (corresponding to the leaf l). □

This connection yields the following lower bound on deterministic communication complexity.

Corollary 3.2.10. *Let $f : X \times Y \to Z$ be a function. If d is a lower bound on the number of rectangles to partition $X \times Y$ into f-monochromatic rectangles, then $D(f) \geq \log_2 d$.*

Proof. Assume $D(f) < \log_2 d$, there exists a deterministic protocol $\mathcal{P}$ of f whose cost is less than $\log_2 d$, i.e., the depth of the underlying binary tree is less than $\log_2 d$. Then the number of leaves in the binary tree is strictly less than $2^{\log_2 d} = d$. By Proposition 3.2.9, it induces a partition of $X \times Y$ using strictly less than d f-monochromatic rectangles. This is a contradiction. □

3.3 Nondeterministic Communication

We start this section with the definition of the rectangle covering number and nondeterministic communication complexity.

Definition 3.3.1. *The* rectangle covering number $C^1(f)$ *of some function $f : X \times Y \to \{0, 1\}$ is the smallest number of monochromatic rectangles to cover the* 1*-entries of its communication matrix M_f.*

Definition 3.3.2. *The* nondeterministic communication complexity *of a boolean function $f : X \times Y \to \{0, 1\}$ is $N^1(f) := \lceil \log_2 C^1(f) \rceil$.*

Consider an all-powerful prover who knows both inputs of Alice and Bob, and this all-powerful prover is trying to convince them that $f(x, y) = 1$ by sending a certificate. Upon receiving this certificate, Alice and Bob verify it using a deterministic verification protocol. The cost of a proof system is the sum of the size (in bits) of the certificate and the cost of the deterministic verification protocol. In addition, this proof system must satisfy the following two conditions:

(i) If $f(x, y) = 1$, then there exists a certificate such that Alice and Bob conclude $f(x, y) = 1$;

(ii) If $f(x, y) = 0$, then Alice and Bob always declare $f(x, y) = 0$ for every certificate.

We will show the following relations between the minimum cost of a proof system for f, denoted by c_f, and the rectangle covering number $C^1(f)$.

Lemma 3.3.3. $c_f \leq \log_2 C^1(f) + 2$.

Proof. Let C be a covering of the 1-entries of M_f by $C^1(f)$ 1-monochromatic rectangles. Consider the following proof system:

(i) If $f(x, y) = 1$, the all-powerful prover sends the index of a 1-monochromatic rectangle $R = A \times B$ containing the input (x, y); otherwise the all-powerful prover sends any rectangle index. ($\log_2 C^1(f)$ bits);

(ii) Alice verifies if $x \in A$, and sends the result to Bob. (1 bit);

(iii) Bob verifies if $y \in B$, and sends the result to Alice. (1 bit)

This is a valid proof system and thus $c_f \leq \log_2 C^1(f) + 2$. □

Lemma 3.3.4. $c_f \geq \log_2 C^1(f)$.

Proof. Consider any proof system for f of cost c_f which contains 2^{k_1} certificates and each of the 2^{k_1} certificates c induces a k_2-cost deterministic protocol P_c, so $k_1 + k_2 = c_f$. Let $\mathcal{R}_c$ be the set of (at most 2^{k_2}) 1-monochromatic rectangles corresponding to the 1-leaves of P_c. Then the union $\cup_c \mathcal{R}_c$ is a set of at most $2^{k_1+k_2}$ 1-monochromatic rectangles which covers exactly the 1s of M_f. Hence $2^{k_1+k_2} \geq C^1(f)$. □

Now we investigate the link between nondeterministic communication complexity and extension complexity. First of all, the definitions of the support of a matrix and the support matrix of a matrix are provided.

Definition 3.3.5. *Let M be a matrix. The* support *of the matrix M, denoted by* supp(M)*, is the set of entries where the value of the matrix is not zero. It is formally defined as*

$$\text{supp}(M) := \{(a, b) : M_{a,b} \neq 0\}.$$

Definition 3.3.6. *The* support matrix *of some matrix S, denoted by* $\text{suppmat}(S)$, *is the matrix defined as*

$$(\text{suppmat}(S))_{xy} = \begin{cases} 1 & \text{if } S_{xy} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.3.7. [25] *Let S be any non-negative matrix. Let $f : X \times Y \to \{0,1\}$ be a boolean function whose communication matrix M_f coincides with* $\text{suppmat}(S)$. *It holds that $C^1(f) \leq \text{rank}_+(S)$.*

Proof. Let $r = \text{rank}_+(S)$. Then there exists a non-negative decomposition of S with intermediate dimension r, namely $S = TU$ for some non-negative matrices $T \in \mathbb{R}_+^{m\times r}$ and $U \in \mathbb{R}_+^{r\times n}$. We have

$$\begin{aligned} \text{supp}(\text{suppmat}(S)) &= \text{supp}(S) \\ &= \text{supp}(\sum_{i=1}^{r} T^i U_i^T) \\ &= \cup_{i=1}^{r} \text{supp}(T^i U_i^T) \\ &= \cup_{i=1}^{r} \text{supp}(T^i) \times \text{supp}(U_i^T). \end{aligned}$$

This yields a monochromatic rectangle covering with r rectangles, namely taking rectangles $R_i = \text{supp}(T^i) \times \text{supp}(U_i^T)$ for $1 \leq i \leq r$. □

From Theorem 2.4.4, the extension complexity of a polytope P equals the non-negative rank of its slack matrix S induced by a certain linear system and set of points. One may define a function f whose induced communication matrix is $\text{suppmat}(S)$. The above theorem asserts that the rectangle covering number $C^1(f)$ lower bounds the non-negative rank of $\text{suppmat}(S)$. Thus Theorem 2.4.4 and Theorem 3.3.7 imply the following result.

Corollary 3.3.8. [25] *Let $P \subseteq \mathbb{R}^d$ be a polytope and S be its induced slack matrix. Let f be the boolean function whose communication matrix is* $\text{suppmat}(S)$. *Then*

$$\text{xc}(P) \geq C^1(f).$$

The above corollary asserts that the nondeterministic communication complexity of f is a lower bound on the logarithm of the extension complexity of P. This enables us to analyze the extension complexity of polytopes using the nondeterministic communication complexity of the corresponding function.

In the rest of this section, we consider a lower bound technique for the rectangle covering number, called the "fooling set" technique.

Definition 3.3.9. [1] *Let* $f : X \times Y \to \{0, 1\}$*. A* fooling set S *is a subset of* $X \times Y$ *such that*

(i) $f(x, y) = 1$ *for every* $(x, y) \in S$*;*

(ii) if (x_1, y_1) *and* (x_2, y_2) *are two distinct pairs in* S*, then* $f(x_1, y_2) = 0$ *or* $f(x_2, y_1) = 0$*.*

Given a fooling set S and two distinct pairs (x_1, y_1) and (x_2, y_2) in S, a f-monochromatic rectangle cannot contain both (x_1, y_1) and (x_2, y_2). Indeed, if this is the case, then $f(x_1, y_2) = 0$ or $f(x_2, y_1) = 0$, and this is a contradiction. This gives the following well-known lower bound for the rectangle covering number.

Lemma 3.3.10. [1] *If* S *is a fooling set of* f*, then* $C^1(f) \geq |S|$*.*

Now we provide an application of the "fooling set" technique, which allows us to derive a tight lower bound for the extension complexity of the stable set polytope of a disjoint union of some edges.

Lemma 3.3.11. *If* $G = (V, E)$ *is the disjoint union of* p *edges, then* $\mathrm{xc}(\mathrm{STAB}(G)) = 3p$*.*

Proof. Notice that every induced subgraph of G is a bipartite graph and thus G is perfect. We know that $\mathrm{xc}(\mathrm{STAB}(G)) \leq 3p$ by Theorem 1.3.17.

To see $\mathrm{xc}(\mathrm{STAB}(G)) \geq 3p$, we consider the slack matrix S of $\mathrm{STAB}(G)$, induced by the non-negativity constraints, clique constraints and the stable sets of G. As G is perfect, these constraints characterize $\mathrm{STAB}(G)$ and thus $\mathrm{xc}(\mathrm{STAB}(G)) = \mathrm{rank}_+(S)$. Let $f : X \times Y \to \{0, 1\}$ be the function whose communication matrix M_f coincides with S. So X is indexed by the non-negativity constraints and clique constraints, and Y is indexed by the stable sets of G. Define $[p] := \{1, \ldots, p\}$. Let $V = \{a_i, b_i : i \in [p]\}$ and $E = \{(a_i, b_i) : i \in [p]\}$. For every $i \in [p]$, we consider the stable sets

$$I_i = \{a_i, \ldots, a_p, b_1, \ldots, b_{i-1}\},$$
$$J_i = \{a_1, \ldots, a_{i-1}, b_i, \ldots, b_p\},$$
$$K_i = \{a_1, \ldots, a_p\} \backslash \{a_i\}.$$

For every $i \in [p]$, define the following subsets of $X \times Y$,

$$(a_i \geq 0, I_i),$$
$$(b_i \geq 0, J_i),$$
$$(a_i + b_i \leq 1, K_i).$$

We shall prove that the above subsets of $X \times Y$ form a fooling set of size $3p$ for f. This will prove the lemma. To this end, we check the condition (i) and (ii) in Definition 3.3.9. It is clear that each listed subset satisfies (i), thus we will check (ii) now.

Consider two distinct subsets $(a_i \geq 0, I_i)$ and $(a_j \geq 0, I_j)$ for some $i, j \in [p]$ and $i \neq j$. Assume $f(a_i \geq 0, I_j) = 1$ and $f(a_j \geq 0, I_i) = 1$. This implies that $i \geq j$ and $j \leq i$, which is a contradiction. Similarly we can check the subsets $(b_i \geq 0, J_i)$ and $(b_j \geq 0, J_j)$ for some $i, j \in [p]$ and $i \neq j$. For two distinct subsets $(a_i + b_i \leq 1, K_i)$ and $(a_j + b_j \leq 1, K_j)$ for some $i, j \in [p]$ and $i \neq j$, it holds that $f(a_i + b_i \leq 1, K_j) = 0$.

Now we consider two distinct subsets $(a_i \geq 0, I_i)$ and $(b_j \geq 0, J_j)$ for some $i, j \in [p]$ and $i \neq j$. Assume $f(a_i \geq 0, J_i) = 1$ and $f(b_j \geq 0, I_i) = 1$. This implies that $i \leq j - 1$ and $j \leq i - 1$, which is a contradiction.

Finally let us consider two distinct subsets $(a_i \geq 0, I_i)$ and $(a_j + b_j \leq 1, K_j)$ for some $i, j \in [p]$. It holds that $f(a_j + b_j \leq 1, I_i) = 0$, as the clique $\{a_j, b_j\}$ must intersects the stable set I_i. The same result holds for subsets $(b_i \geq 0, J_i)$ and $(a_j + b_j \leq 1, K_j)$, where $i, j \in [p]$. □

Finally, we conclude this section with an application of communication complexity to upper bound extension complexity. At the beginning of this section, we have shown the nondeterministic communication complexity of a function is linearly related to the minimum cost of a proof system for this function. A proof system is called *unambiguous*, if it is additionally required that for each input (x, y), there exists precisely one certificate such that Alice and Bob conclude $f(x, y) = 1$.

First of all, the following result from communication complexity is needed for us.

Lemma 3.3.12. [25] *If the minimum cost of an unambiguous proof system for a function f is k, then the deterministic complexity of f is at most $\mathcal{O}(k^2)$.*

Now we are ready to prove the following upper bound for the extension complexities of the stable set polytopes of perfect graphs.

Theorem 3.3.13. [25] *The extension complexities of the stable set polytopes of n-vertex perfect graphs are $n^{\mathcal{O}(\log n)}$.*

Proof. Let G be a perfect graph with n vertices. Let S be a submatrix of the slack matrix of STAB(G), whose rows correspond to the clique constraints and whose columns correspond to the stable sets of G. So the rows corresponding to the non-negative constraints are discarded. Since there are only n non-negative constraints, the non-negative rank of S satisfies $\mathrm{xc}(\mathrm{STAB}(G)) \leq \mathrm{rank}_+(S) + n$ by Lemma 2.2.3. The complement of S, denoted by $\bar{S}$, is a matrix of the same size, and whose entries are defined by $\bar{S}(C, I) = 1 - S(C, I)$ for every clique C and independent set I. Notice that S and $\bar{S}$ are binary matrices.

Let f be a boolean function whose communication matrix M_f coincides with $\bar{S}$. Now we construct an unambiguous proof system computing f. Assume Alice and Bob receive part of the input C^* and I^*, respectively. Now the all-powerful prover sends a vertex v as a certificate. Then Alice verifies if $v \in C^*$ and sends the result to Bob, and Bob verifies whether $v \in I^*$ and sends the result to Alice. Finally they conclude $\bar{S}(C^*, I^*) = 1$ if and only if $v \in C^*$ and $v \in I^*$.

If $\bar{S}(C^*, I^*) = 1$, then $|C^* \cap I^*| = 1$. Thus there exists precisely one certificate such that Alice and Bob conclude $\bar{S}(C^*, I^*) = 1$. If $\bar{S}(C^*, I^*) = 0$, then $C^* \cap I^* = \emptyset$ and Alice and Bob always conclude $\bar{S}(C^*, I^*) = 0$. By definition, this is an unambiguous proof system for computing the function f, whose cost is $\mathcal{O}(\log n)$.

Let f' be a function whose communication matrix $M_{f'}$ coincides with S. Applying Lemma 3.3.12, we know that the deterministic communication complexity of f' is at most $\mathcal{O}((\log n)^2)$. This induces a monochromatic rectangle partitioning of all the entries of S with at most $2^{\mathcal{O}(\log^2 n)} = n^{\mathcal{O}(\log n)}$ rectangles. As S is a binary matrix, the monochromatic rectangle partitioning yields a non-negative decomposition of size $n^{\mathcal{O}(\log n)}$. By Theorem 2.4.4, we conclude that the extension complexity of $\mathrm{STAB}(G)$ is $n^{\mathcal{O}(\log n)}$. □

Chapter 4

Graph Operations

4.1 Introduction

In this chapter we consider several graph operations and investigate the behavior of the extension complexity of the stable set polytope under these graph operations. Roughly speaking, given two graphs G_1 and G_2 (possibly satisfying certain conditions), a graph operation produces a new graph G_0 obtained by 'composing' the two graphs G_1 and G_2. We will consider the following operations:

(i) the graph substitution operation;

(ii) the amalgam operation;

(iii) the clique sum operation.

These operations share the property that they preserve perfection: if G_1 and G_2 are perfect then their composition G_0 too is perfect. For all these graph operations we will upper bound the extension complexity of the stable set polytope of G_0 in terms of the extension complexities of the stable set polytopes of G_1 and G_2. In this chapter, we will show that if G_1 and G_2 compose G_0 via operation (i), (ii), or (iii) listed above, then it holds that $\mathrm{xc}(\mathrm{STAB}(G_0)) \leq \mathrm{xc}(\mathrm{STAB}(G_1)) + \mathrm{xc}(\mathrm{STAB}(G_2))$. For the amalgam operation operation, it will be used in the next section to show that the extension complexity of Meyniel graphs is polynomial in the size (number of vertices) of the graph.

Our study is motivated by the fact that it is still an open problem whether the extension complexity of the stable set polytope of every perfect graph is polynomially bounded in terms of the size of the graph. For many classes of graphs, the graphs in a given class admit certain structural decompositions which roughly states that every graph in this class can be constructed via some graph operations from some predefined basic graphs. As we will see in the next chapter, the proof of the strong perfect graph theorem provides such a structural decomposition for perfect graphs. In other words, given a perfect graph G, it can be decomposed into basic graphs. Here the decomposition refers to proper 2-join decomposition or skew partitions, so they are not the operations considered in this chapter.

Suppose we have a class of graphs that can be decomposed into some predefined basic graphs via some graph operation. Then the decomposition process induces a tree whose leaves represent some 'basic' graphs. If the decomposition used in each step, say G is decomposed into G_1 and G_2, has the property that $\text{xc}(\text{STAB}(G)) \leq \text{xc}(\text{STAB}(G_1)) + \text{xc}(\text{STAB}(G_2))$, then we obtain an upper bound on the extension complexity of the stable set polytope of G as a function of the extension complexity of the stable set polytope of the basic graphs and the depth of the tree. We will use this approach in the next chapter to upper bound the extension complexity for some Meyniel graphs

4.2 Graph Substitution

Definition 4.2.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two vertex-disjoint graphs and u be a vertex of G_1. The* graph-substitution operation *of graphs G_1 and G_2, denoted by $G_0 = \mathcal{S}(G_1, u, G_2)$, is the graph $G_0 = (V_0, E_0)$ with*

$$V_0 = (V_1 \backslash \{u\}) \cup V_2,$$

$$E_0 = (\cup_{v \in V_2} \{(v, w) : (u, w) \in E_1\}) \cup E(G_1(V_1 \backslash \{u\})) \cup E_2.$$

In other words, the graph-substitution operation constructs a new graph G_0 by replacing a vertex u of G_1 by G_2 and connecting every vertex of G_2 to all neighbors of u in G_1. Lovász [18] showed that if G_1 and G_2 are perfect graphs, then the constructed new graph $G_0 = \mathcal{S}(G_1, u, G_2)$ is also a perfect graph.

Theorem 4.2.2. [18] *The graph-substitution operation preserves perfection.*

In the rest of this section, we show that the extension complexity of $\text{STAB}(G_0)$ is no greater than the sum of the extension complexities of $\text{STAB}(G_1)$ and $\text{STAB}(G_2)$. First of all, we need the following lemma.

Lemma 4.2.3. *Let P be a non-empty polytope. Let the linear system*

$$Ex + Fs = g,\ s \geq 0. \tag{4.1}$$

be an extended formulation of P. If the pair (x_0, s_0) satisfies $Ex_0 + Fs_0 = 0$ and $s_0 \geq 0$, then $x_0 = 0$.

Proof. Let (x, s) be a feasible solution of the linear system (4.1), whose existence is guaranteed by the assumption $P \neq \emptyset$. Then $(x, s) + \lambda \cdot (x_0, s_0)$ also satisfies this linear system for every $\lambda \geq 0$. This implies $x + \lambda \cdot x_0 \in P$ for every $\lambda \geq 0$. As P is a polytope, it is bounded by Definition 1.2.8. This implies $x_0 = 0$. □

Let G be a graph. If $x \in \mathbb{R}^{|V(G)|}$ be the vector of variables corresponding to the vertices of a graph G and $S \subseteq V(G)$ be a subset of vertices of G, then $x(S) \in \mathbb{R}^{|S|}$ denotes the 'subvector' of x containing only entries for the vertices in S. Now we are ready to show the main result of this section.

Theorem 4.2.4. *Let G_1 and G_2 be two vertex-disjoint graphs and u be a vertex of G_1. Let $G_0 = \mathcal{S}(G_1, u, G_2)$ be the graph obtained by replacing u by G_2 in G_1. It holds that*

$$\mathrm{xc}(\mathrm{STAB}(G_0)) \leq \mathrm{xc}(\mathrm{STAB}(G_1)) + \mathrm{xc}(\mathrm{STAB}(G_2)).$$

Proof. For $i = 1, 2$, let $x_i \in \mathbb{R}^{|V(G_i)|}$ be the vector of variables corresponding to the vertices of G_i and let the linear system

$$E_i x_i + F_i s_i = g_i,\ s_i \geq 0, \tag{4.2}$$

be an extended formulation (in slack form) of $\mathrm{STAB}(G_i)$ of size r_i, with the lifting variables $s_i \in \mathbb{R}^{r_i}$. These extended formulations can be assumed to be in slack form by Theorem 2.4.4.

We claim that the following linear system

$$\begin{cases} E_1 y_1 + F_1 t_1 = g_1, & t_1 \geq 0, \\ E_2 y_2 + F_2 t_2 - g_2 \cdot y_1(u) = 0, & t_2 \geq 0. \end{cases} \tag{4.3}$$

in the vector of variables $y_i \in \mathbb{R}^{|V(G_i)|}$ and $t_i \in \mathbb{R}^{r_i}$ $(i = 1, 2)$, is an extended formulation (not in slack form) of $\mathrm{STAB}(G_0)$ with the lifting variables $(t_1, t_2, y_1(u)) \in \mathbb{R}^{r_1 + r_2 + 1}$. In what follows, $y_1(\bar{u})$ denotes $y_1(V_1 \backslash \{u\})$, so $y_1 = (y_1(\bar{u}), y_1(u))$ and the vector $(y_1(\bar{u}), y_2)$ is indexed by the vertices of G_0.

By definition, we need to show that $(y_1(\bar{u}), y_2) \in \text{STAB}(G_0)$ if and only if there exists a vector $(t_1, t_2, y_1(u)) \in \mathbb{R}^{r_1+r_2+1}$ such that $(y_1, y_2, t_1, t_2) \in \mathbb{R}^{|V_1|+|V_2|+r_1+r_2}$ satisfies the linear system (4.3).

The "only if" part ($\Longrightarrow$): Let $(y_1(\bar{u}), y_2) \in \text{STAB}(G_0)$. From Lemma 2.1.2, it suffices to consider the case when $(y_1(\bar{u}), y_2) \in \mathbb{R}^{|V_0|}$ is a vertex of $\text{STAB}(G_0)$. Then $(y_1(\bar{u}), y_2) \in \{0,1\}^{|V_0|}$ is the characteristic vector of some stable set I of G_0. Let $J_i = I \cap V(G_i)$ $(i = 1, 2)$. Clearly J_i is stable in G_i. Recall that χ^S denotes the characteristic vector of some subset S of the vertices. Consider the following two cases depending on whether the set $J_1 \cup \{u\}$ is stable in G_1.

(i) If $J_1 \cup \{u\}$ is stable in G_1, then there exists a vector $s_1 \in \mathbb{R}^{r_1}$ such that $(\chi^{J_1\cup\{u\}}, s_1)$ satisfies the linear system (4.2) for $i = 1$, as this linear system is an extended formulation of $\text{STAB}(G_1)$. Similarly, since J_2 is stable in G_2, there exists a vector $s_2 \in \mathbb{R}^{r_2}$ such that (χ^{J_2}, s_2) satisfies the linear system (4.2) for $i = 2$.

    Let $t_1 = s_1$, $t_2 = s_2$ and $y_1(u) = 1$. Then $y_1 = \chi^{J_1\cup\{u\}}$ and $y_2 = \chi^{J_2}$. It holds that (y_1, y_2, t_1, t_2) satisfies the linear system (4.3), due to the choice of t_1, t_2 and $y_1(u) = 1$.

(ii) If $J_1 \cup \{u\}$ is not stable in G_1, then $J_2 = \emptyset$ as $N(u) \cap J_1 \neq \emptyset$. Furthermore $\chi^{J_1}(u) = 0$ as $u \notin J_1$. We know that there exists a vector $s_1 \in \mathbb{R}^{r_1}$ such that (χ^{J_1}, s_1) satisfies the linear system (4.2) for $i = 1$, as this linear system is an extended formulation of $\text{STAB}(G_1)$.

    Let $t_1 = s_1$, $t_2 = 0$ and $y_1(u) = 0$. Then $y_1 = \chi^{J_1}$ and $y_2 = 0$. We have $E_1y_1 + F_1t_1 = g_1$, $t_1 \geq 0$ by the choice of t_1. Furthermore $E_2y_2 = F_2t_2 = g_2 \cdot y_1(u) = 0$ as $y_2 = 0$, $t_2 = 0$ and $y_1(u) = 0$. Thus (y_1, y_2, t_1, t_2) satisfies the linear system (4.3).

In both cases, we have constructed lifting variables $(t_1, t_2, y_1(u)) \in \mathbb{R}^{r_1+r_2+1}$ such that the vector $(y_1, y_2, t_1, t_2) \in \mathbb{R}^{|V_1|+|V_2|+r_1+r_2}$ satisfies the linear system (4.3).

The "if" part ($\Longleftarrow$): Let $(y_1, y_2, t_1, t_2) \in \mathbb{R}^{|V_1|+|V_2|+r_1+r_2}$ be a vector satisfying the linear system (4.3). We distinguish the following two cases depending on the value of $y_1(u)$:

(i) If $y_1(u) = 0$, then $y_2 = 0$ by Lemma 4.2.3. As the linear system (4.2) is an extended formulation of $\text{STAB}(G_1)$, y_1 is in $\text{STAB}(G_1)$. If $I \subseteq V(G_1)$ is stable in G_1 and $u \notin I$, then I is also stable in G_0. Thus $(y_1(\bar{u}), y_2) \in \text{STAB}(G_0)$.

(ii) If $y_1(u) > 0$, then the linear system (4.3) implies $y_1 \in \mathrm{STAB}(G_1)$ and $\frac{y_2}{y_1(u)} \in \mathrm{STAB}(G_2)$. Thus we can write y_1 and $\frac{y_2}{y_1(u)}$ as convex combination of the vertices of $\mathrm{STAB}(G_1)$ and $\mathrm{STAB}(G_2)$, respectively. By definition, there exist non-negative coefficients $\lambda_I \geq 0$ summing to 1, such that $y_1 = \sum_{I \in \mathcal{I}(G_1)} \lambda_I \chi^I$, where χ^I is the vertex of $\mathrm{STAB}(G_1)$ associated to the stable set I. Similarly, $\frac{y_2}{y_1(u)} = \sum_{J \in \mathcal{I}(G_2)} \mu_J \chi^J$ for some non-negative coefficients $\mu_J \geq 0$ summing to 1 and χ^J is the vertex of $\mathrm{STAB}(G_2)$ associated to the stable set J.

As $y_1(u) = \sum_{I \in \mathcal{I}(G_1), I \ni u} \lambda_I$, we have

$$\sum_{I \in \mathcal{I}(G_1), I \not\ni u} \lambda_I \begin{pmatrix} \chi^I \\ 0 \end{pmatrix} + \sum_{J \in \mathcal{I}(G_2)} \mu_J \sum_{I \in \mathcal{I}(G_1), I \ni u} \lambda_I \begin{pmatrix} \chi^I \\ \chi^J \end{pmatrix} = \begin{pmatrix} y_1 \\ y_1(u) \cdot \sum_{J \in \mathcal{I}(G_2)} \mu_J \chi^J \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

by the choice of λ_I's and μ_J's. It remains to notice that

$$\sum_{I \in \mathcal{I}(G_1), I \not\ni u} \lambda_I + \sum_{J \in \mathcal{I}(G_2)} \mu_J \sum_{I \in \mathcal{I}(G_1), I \ni u} \lambda_I = 1$$

and each term in the summation is non-negative. Furthermore consider the set of the vectors appearing in the convex combination above,

$$\{\begin{pmatrix} \chi^I \\ 0 \end{pmatrix} : I \in \mathcal{I}(G_1), I \not\ni u\} \cup \{\begin{pmatrix} \chi^I \\ \chi^J \end{pmatrix} : I \in \mathcal{I}(G_1), I \ni u, J \in \mathcal{I}(G_2)\}.$$

Each vector in this set corresponds to a unique vertex of $\mathrm{STAB}(G_0)$ when the entry for the vertex u is ignored. Thus $(y_1(\bar{u}), y_2) \in \mathrm{STAB}(G_0)$.

This proves that the linear system (4.3) is an extended formulation of $\mathrm{STAB}(G_0)$. The size of an extended formulation of $\mathrm{STAB}(G_0)$ is the number of inequalities in the linear system, which is precisely $r_1 + r_2$ in this case. If we take $r_1 = \mathrm{xc}(\mathrm{STAB}(G_1))$ and $r_2 = \mathrm{xc}(\mathrm{STAB}(G_2))$, the theorem follows immediately. □

For any graphs G_1 and G_2, the above theorem gives an upper bound on the extension complexity of the graph G_0 obtained by replacing some vertex $u \in V(G_1)$ by G_2. For some special cases, a slightly tighter upper bound can be obtained. We look at the cases when G_2 is K_p or $\overline{K_2}$. From Lemma 2.4.5 and Theorem 1.3.17, we know that $\mathrm{xc}(\mathrm{STAB}(K_p)) = p + 1$. Fiorini et al. [15] showed that $\mathrm{xc}(\mathrm{STAB}(\overline{K_p})) = 2p$. We will show that $\mathrm{xc}(\mathrm{STAB}(G_0)) \leq \mathrm{xc}(\mathrm{STAB}(G_1)) + p$ when $G_2 = K_p$ (See Proposition 4.2.5), and $\mathrm{xc}(\mathrm{STAB}(G_0)) \leq \mathrm{xc}(\mathrm{STAB}(G_1)) + 3$ when $G_2 = \overline{K_2}$ (See Proposition 4.2.6).

In the remaining part of this section, we use the linear system

$$Ex_1 + Fs_1 = g,\ s \geq 0 \tag{4.4}$$

as an extended formulation (in slack form) of $\mathrm{STAB}(G_1)$ of size r with the lifting variables $s_1 \in \mathbb{R}^r$. Rearrange the columns of the matrix E if necessary, so that $E = [E_{\bar{u}}\ E_u]$ where E_u is the column associated with the vertex u. The entries of the variable x_1 are rearranged correspondingly such that $x_1 = (x_1(\bar{u}), x_1(u))$.

Proposition 4.2.5. *Let $G_1 = (V_1, E_1)$ be a graph and $u \in V_1$ be a vertex of G_1. Let $G_0 = \mathcal{S}(G_1, u, K_p)$ be the graph obtained by replacing u by K_p in G_1. It holds that*

$$\mathrm{xc}(\mathrm{STAB}(G_0)) \leq \mathrm{xc}(\mathrm{STAB}(G_1)) + p.$$

Proof. Let $y_1 \in \mathbb{R}^{|V_1|-1}$ and $y_2 \in \mathbb{R}^p$ be the vectors of variables corresponding to the vertices of G_0 from $V(G_1)\backslash\{u\}$ and $V(K_p)$, respectively. So y_1 is only indexed by the vertices in $V(G_1)\backslash\{u\}$ now. We will show that the following linear system (4.5) is an extended formulation of $\mathrm{STAB}(G_0)$ with the lifting variables $t \in \mathbb{R}^r$:

$$\begin{bmatrix} E_{\bar{u}} & E_u & \cdots & E_u \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + Ft = g,\ t \geq 0,\ y_2 \geq 0. \tag{4.5}$$

As the size of this linear system (4.5) is $r + p$, this suffices to prove the claim. By definition, we need to show that $(y_1, y_2) \in \mathrm{STAB}(G_0)$ if and only if there exists a vector $t \in \mathbb{R}^r$ such that $(y_1, y_2, t) \in \mathbb{R}^{|V_0|+r}$ satisfies the linear system (4.5).

The "only if" part ($\Longrightarrow$): Let $(y_1, y_2) \in \mathrm{STAB}(G_0)$. From Lemma 2.1.2, it suffices to consider the case when $(y_1, y_2) \in \mathbb{R}^{|V_0|}$ is a vertex of $\mathrm{STAB}(G_0)$. In other words, $(y_1, y_2) \in \{0,1\}^{|V_0|}$ is the characteristic vector of some stable set I of G_0. Since at most one of the vertices from K_p can be included in the stable set I, we have that $\sum_{v \in V(K_p)} y_2(v) = |I \cap V(K_p)| \in \{0,1\}$.

We now define a vector $x_1 \in \mathbb{R}^{|V_1|}$ indexed by $V(G_1)$ as follows: Take $x_1(u) = \sum_{v \in V(K_p)} y_2(v) \in \{0,1\}$, $x_1(\bar{u}) = y_1 \in \mathbb{R}^{|V_1|-1}$, so $x_1 = (x_1(\bar{u}), x_1(u)) \in \{0,1\}^{|V_1|}$. Now x_1 can be considered as the characteristic vector of a subset of vertices J of G_1 when the last entry is indexed by the vertex u. We claim that J is a stable set of G_1. Indeed, if $x_1(u) = |I \cap V(K_p)| = 1$, then $I \cap N(u) = \emptyset$ and $J = (I \backslash V(K_p)) \cup \{u\}$ is stable in G_1; and if $x_1(u) = |I \cap V(K_p)| = 0$, then $I \subseteq V(G_1)$ and $J = I$ is stable in G_1. Thus $x_1 \in \mathrm{STAB}(G_1)$ and there exists a vector $s_1 \in \mathbb{R}^r$ such that $(x_1, s_1) \in \mathbb{R}^{|V_1|+r}$ satisfies the linear system (4.4), as this linear system (4.4) is an extended formulation of $\mathrm{STAB}(G_1)$. Let $t = s_1 \in \mathbb{R}^r$. It can be easily shown that

$(y_1, y_2, t) \in \mathbb{R}^{|V_0|+r}$ satisfies the linear system (4.5) using the feasibility of (x_1, s_1) for the linear system (4.4).

The "if" part ($\Longleftarrow$): Let $(y_1, y_2, t) \in \mathbb{R}^{|V_0|+r}$ be a feasible solution of the linear system (4.5). We need to show $(y_1, y_2) \in \mathrm{STAB}(G_0)$. It is not hard to see that the following linear system

$$\sum_{v \in V(K_p)} x_2(v) \leq 1,\ x_2 \geq 0, \tag{4.6}$$

in variable $x_2 \in \mathbb{R}^p$, provides the linear inequality description of $\mathrm{STAB}(K_p)$. Applying the argument used in the proof of Theorem 4.2.4 to the extended formulation (4.4) of $\mathrm{STAB}(G_1)$ and the extended formulation (4.6) of $\mathrm{STAB}(K_p)$, we can conclude that the following linear system is an extended formulation of $\mathrm{STAB}(G_0)$ with lifting variables $(x_1'(u), s')$:

$$\begin{cases} Ex_1' + Fs' = g,\ s' \geq 0, \\ \sum_{v \in V(K_p)} x_2'(v) \leq x_1'(u), \\ x_2' \geq 0. \end{cases} \tag{4.7}$$

Hence x_1', x_2' and s' are variables living in $\mathbb{R}^{|V(G_1)|}$, $\mathbb{R}^p$ and r, respectively.

As the vector (y_1, y_2, t) satisfies the linear system (4.5), it also satisfies the following relation,

$$E \begin{bmatrix} y_1 \\ \sum_{v \in V(K_p)} y_2(v) \end{bmatrix} + Ft = g,\ t \geq 0,\ y_2 \geq 0.$$

This implies that the vector $(y_1, \sum_{v \in V(K_p)} y_2(v))$ satisfies the linear system (4.4) which is an extended formulation of $\mathrm{STAB}(G_1)$. Thus $(y_1, \sum_{v \in V(K_p)} y_2(v)) \in \mathrm{STAB}(G_1)$ and thus $\sum_{v \in V(K_p)} y_2(v) \leq 1$. Let $x_1'(u) = \sum_{v \in V(K_p)} y_2(v) \in \mathbb{R}$, $x_1'(\bar{u}) = y_1 \in \mathbb{R}^{|V_1|-1}$, $x_1' = (x_1'(\bar{u}), x_1'(u)) \in \mathbb{R}^{|V_1|}$, $x_2' = y_2 \in \mathbb{R}^p$ and $s' = t \in \mathbb{R}^p$. Then $(x_1', x_2', s') \in \mathbb{R}^{|V_1|+p+r}$ satisfies the linear system (4.7). As the linear system (4.7) is an extended formulation of $\mathrm{STAB}(G_0)$, this implies $(y_1, y_2) = (x_1'(\bar{u}), x_2') \in \mathrm{STAB}(G_0)$.

□

Proposition 4.2.6. *Let $G_1 = (V_1, E_1)$ be a graph and $u \in V_1$ be a vertex. Let $G_0 = \mathcal{S}(G_1, u, \overline{K_2})$ be the graph obtained by replacing u by $\overline{K_2}$ in G_1. It holds that*

$$\mathrm{xc}(\mathrm{STAB}(G_0)) \leq \mathrm{xc}(\mathrm{STAB}(G_1)) + 3.$$

Proof. Let $y_1 \in \mathbb{R}^{|V_1|-1}$ and $y_2 \in \mathbb{R}^2$ be the vectors of variables corresponding to the vertices of G_0 from $V(G_1)\backslash\{u\}$ and $V(\overline{K_2})$, respectively. Denote the two vertices of G_2 by u_1 and u_2, respectively. So $y_2 = (y_2(u_1), y_2(u_2))$. We will show that the following linear system (4.8) is an extended formulation of STAB(G_0) with lifting variables $t_1 \in \mathbb{R}^r$ and $t_2 \in \mathbb{R}$:

$$\begin{bmatrix} E_{\bar{u}} & E_u & E_u \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} F & -E_u \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = g,\ t_1 \geq 0,\ t_2 \geq 0,\ y_2(u_1) \geq t_2,\ y_2(u_1) \geq t_2. \tag{4.8}$$

As the size of this extended formulation (4.8) is $r+3$, this will prove the claim. By definition, we need to show that $(y_1, y_2) \in$ STAB(G_0) if and only if there exists a vector $(t_1, t_2) \in \mathbb{R}^{r+1}$ such that $(y_1, y_2, t_1, t_2) \in \mathbb{R}^{|V_0|+r_1+r_2}$ satisfies the linear system (4.8).

The "only if" part ($\Longrightarrow$): Let $(y_1, y_2) \in$ STAB(G_0). From Lemma 2.1.2, it suffices to consider the case when $(y_1, y_2) \in \mathbb{R}^{|V_0|}$ is a vertex of STAB(G_0). Then $(y_1, y_2) \in \{0,1\}^{|V_0|}$ is the characteristic vector of some stable set I of G_0. Consider the following two cases depending on the cardinality of $|I \cap V(\overline{K_2})|$ which is equal to $y_2(u_1) + y_2(u_2)$:

(i) If $|I \cap V(\overline{K_2})| \leq 1$, then $(y_1, y_2(u_1) + y_2(u_2)) \in \{0,1\}^{|V_1|}$ is the characteristic vector of some stable set of G_1. Indeed, if $|I \cap V(\overline{K_2})| = 1$, then $N(u) \cap I = \emptyset$ and $(I \cap V(G_1)) \cup \{u\}$ is stable in G_1; and if $|I \cap V(\overline{K_2})| = 0$, then $y_2(u_1) + y_2(u_2) = 0$ and we know that y_1 corresponds to some stable set in G_1. We know that there exists a vector $s_1 \in \mathbb{R}^r$ such that $(y_1, y_2(u_1) + y_2(u_2), s_1) \in \mathbb{R}^{|V_1|+r}$ satisfies the linear system (4.4), as the linear system (4.4) is an extended formulation of STAB(G_1).

Let $t_1 = s_1$ and $t_2 = 0$. Using the feasibility of $(y_1, y_2(u_1) + y_2(u_2), s_1)$ for the linear system (4.4), we get that (y_1, y_2, t_1, t_2) satisfies the linear system (4.8);

(ii) If $|I \cap V(\overline{K_2})| = 2$, then $y_2(u_1) + y_2(u_2) = 2$ and $(y_1, 1) \in \mathbb{R}^{|V_1|}$ is the characteristic vector of some stable set of G_1. As the linear system (4.4) is an extended formulation of STAB(G_1), there exists a vector $s_1 \in \mathbb{R}^r$ such that $(y_1, 1, s_1) \in \mathbb{R}^{|V_1|+r}$ satisfies the linear system (4.4).

Let $t_1 = s_1$ and $t_2 = 1$. Using the feasibility of $(y_1, 1, s_1)$ for the linear system (4.4) and the relation $y_2(u_1) + y_2(u_2) = 2$, one can show that (y_1, y_2, t_1, t_2) satisfies the linear system (4.8);

In both cases, we have constructed the lifting variables $(t_1, t_2) \in \mathbb{R}^{r+1}$ such that the vector $(y_1, y_2, t_1, t_2) \in \mathbb{R}^{|V_0|+r_1+r_2}$ satisfies the linear system (4.8).

The "if" part ($\Longleftarrow$): Consider any feasible solution $(y_1, y_2, t_1, t_2) \in \mathbb{R}^{|V_0|+r_1+r_2}$ of (4.8). We need to show that $(y_1, y_2) \in \mathrm{STAB}(G_0)$. The stable set polytope of $\overline{K_2}$ can be characterized by $\mathrm{STAB}(\overline{K_2}) = \{x_2 \in \mathbb{R}^2 : 0 \leq x_2 \leq 1\}$. From the proof of Theorem 4.2.4, the following linear system

$$\begin{cases} Ex_1' + Fs' = g, \ s' \geq 0 \\ 0 \leq x_2'(v) \leq x_1'(u) \ \ \forall\, v \in V(\overline{K_2}) \end{cases} \tag{4.9}$$

is an extended formulation of $\mathrm{STAB}(G_0)$ with lifting variables $(x_1'(u), s') \in \mathbb{R}^{r+1}$.

As the vector (y_1, y_2, t_1, t_2) satisfies the linear system (4.8), it also satisfies the following relation,

$$E \begin{bmatrix} y_1 \\ y_2(u_1) + y_2(u_2) - t_2 \end{bmatrix} + Ft_1 = g, \ t_1 \geq 0, \ t_2 \geq 0, \ y_2(u_1) \geq t_2, \ y_2(u_1) \geq t_2.$$

This implies the vector $(y_1, y_2(u_1) + y_2(u_2) - t_2, t_1)$ satisfies the linear system (4.4) which is an extended formulation of $\mathrm{STAB}(G_1)$, thus $(y_1, y_2(u_1) + y_2(u_2) - t_2, t_1) \in \mathrm{STAB}(G_1)$ and $y_2(u_1)+y_2(u_2)-t_2 \leq 1$. Take $x_1(u)' = y_2(u_1)+y_2(u_2)-t_2$, $x_1(\bar{u})' = y_1$, $x_1' = (x_1(\bar{u})', x_1(u)')$, $x_2' = y_2$ and $s' = t$. Then (x_1', x_2', s') satisfies the linear system (4.9). As this linear system (4.9) is an extended formulation of $\mathrm{STAB}(G_0)$ and this implies that $(y_1, y_2) = (x_1'(\bar{u}), x_2') \in \mathrm{STAB}(G_0)$. □

Finally, we provide an application of the graph substitution operation. Given a graph G, if one recognizes that G can be composed from some 'basic' graphs via graph substitution operation, then the result in this section immediately yields an upper bound on $\mathrm{xc}(\mathrm{STAB}(G))$.

Lemma 4.2.7. *Let $G = (V, E)$ be the complement of a disjoint union of q edges. It holds that* $\mathrm{xc}(\mathrm{STAB}(G)) \leq 4q + 1$.

Proof. By Lemma 2.4.6, the extension complexity of a complete graph K_q satisfies $\mathrm{xc}(K_q) = q + 1$. Now for each vertex $u \in V(K_q)$, we substitute u by an independent set $\overline{K_2}$ of size 2. The resulting graph is G. From Proposition 4.2.6, each substitution increases the extension complexity by at most 3 and there are q substitutions, thus $\mathrm{xc}(\mathrm{STAB}(G)) \leq 4q + 1$. □

4.3 Amalgam Operation

Definition 4.3.1. [4] *Let G_1 and G_2 be graphs. Let $v_k \in V(G_k)$ and $C_k \subseteq N(v_k)$, for $k = 1, 2$, such that* *(i)* C_1 *and* C_2 *are cliques of the same cardinality;*

(ii) $N(v_1)\backslash C_1 = \emptyset$ *if and only if* $N(v_2)\backslash C_2 = \emptyset$*;*

(iii) $\{(u, w) : u \in C_k,\ w \in N(v_k)\backslash C_k\} \subseteq E(G_k)$ $(k = 1, 2)$.

The amalgam composition *of the graphs* G_1 *and* G_2 *generates a new graph, denoted by* $G_0 = (G_1, v_1, C_1)\ \Phi\ (G_2, v_2, C_2)$*, using the following procedure:*

(a) identify the cliques C_1 *and* C_2*, and the new clique is denoted by* C*;*

(b) add all possible edges between the vertex sets $N(v_1)\backslash C_1$ *and* $N(v_2)\backslash C_2$*;*

(c) delete the vertices v_1 *and* v_2*.*

Conversely, if for a graph G_0*, there exist two triples* (G_k, v_k, C_k) $(k = 1, 2)$ *satisfying* $(i), (ii), (iii)$ *such that* $G_0 = (G_1, v_1, C_1)\ \Phi\ (G_2, v_2, C_2)$*, then* G_0 *is said to be* amalgam decomposed *into the graphs* G_1 *and* G_2*.*

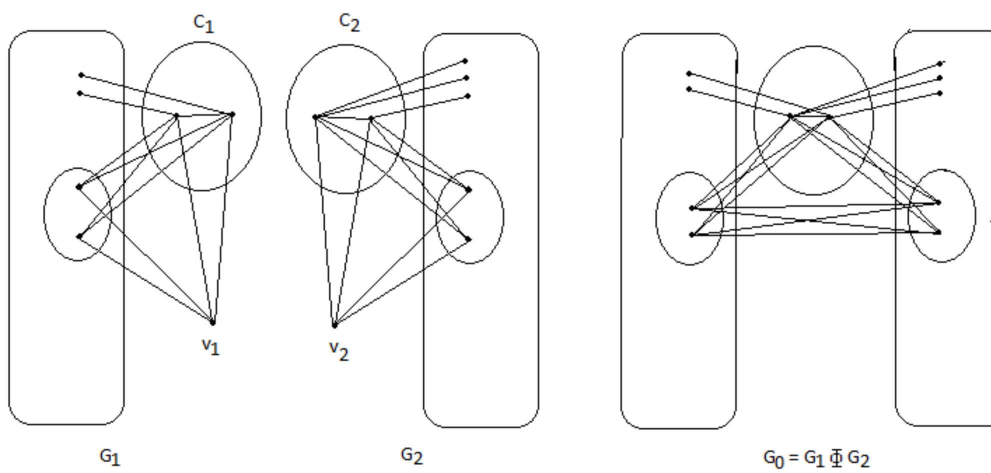

Figure 4.1: Amalgamation of G_1 and G_2

Burlet and Fonlupt [4] showed that the amalgam composition preserves perfection.

Theorem 4.3.2. [4] *The amalgam composition preserves perfection.*

Burlet and Fonlupt [5] constructed a linear system defining STAB(G_0) by using the linear systems defining STAB(G_1) and STAB(G_2). We follow the same line of reasoning as in [5] to show that the extension complexity of STAB(G_0) is at most the sum of the extension complexities of STAB(G_1) and STAB(G_2). After the completion of this section, we found that a recent result in Conforti et al. [10] already gives the same upper bound in a more general setting.

Theorem 4.3.3. *Let G_0 be a graph such that G_0 can be amalgam decomposed into some graphs G_1 and G_2. It holds that*

$$\mathrm{xc}(\mathrm{STAB}(G_0)) \leq \mathrm{xc}(\mathrm{STAB}(G_1)) + \mathrm{xc}(\mathrm{STAB}(G_2)).$$

Proof. By definition, there exist two triples (G_k, v_k, C_k) $(k = 1, 2)$ satisfying $(i), (ii), (iii)$ in Definition 4.3.1, such that $G_0 = (G_1, v_1, C_1) \ \Phi \ (G_2, v_2, C_2)$. Let $V_k := V(G_k)$ and $E_k := E(G_k)$ $(k = 0, 1, 2)$. Let $x_k \in \mathbb{R}^{|V_k|}$ be the vector of variables corresponding to the vertices of G_k and let the linear system

$$A_k x_k \leq b_k \tag{4.10}$$

define the stable set polytope STAB(G_k) $(k = 1, 2)$.

Let $y = (y(V_0), y(v_1), y(v_2)) \in \mathbb{R}^{|V_0|+2}$ be the vector of variables corresponding to the vertices of G_0 and the vertices v_1 and v_2. So $y(V_0) \in \mathbb{R}^{|V_0|}$ is indexed by the vertices of G_0. The entries $y(v_1) \in \mathbb{R}$ and $y(v_2) \in \mathbb{R}$ represent the vertices v_1 and v_2, respectively. Notice that $V_0 \cup \{v_1, v_2\} = (V_1 \backslash C_1) \cup (V_2 \backslash C_2) \cup C$. Define $V_{0k} := (V_k \backslash C_k) \cup C$ for $k = 1, 2$. Thus V_{0k} contains the vertex v_k and the vertices of G_0 generated from G_k and hence the vector $y(V_{0k}) \in \mathbb{R}^{|V_k|}$ is indexed by the vertex v_k and by the vertices of G_0 generated from G_k for $k = 1, 2$.

Consider the following linear system obtained from the linear systems (4.10),

$$\begin{cases} A_k y(V_{0k}) \leq b_k, & k = 1, 2, \\ y(v_1) + y(v_2) + \sum_{v \in C} y(v) = 1. \end{cases} \tag{4.11}$$

The linear systems $A_1 y(V_{01}) \leq b_1$ and $A_2 y(V_{02}) \leq b_2$ in (4.11) are called the first set of constraints and the second set of constraints, respectively. The last equation in (4.11) is denoted by the auxiliary constraint.

Let $Q := \{y \in \mathbb{R}^{|V_0|+2} : y \text{ satisfies the linear system (4.11)}\}$ be the polyhedron defined by the linear system (4.11). From the linear systems (4.10), we know that the linear system $A_k y(V_{0k}) \leq b_k$ defines the stable set polytope of G_k $(k = 1, 2)$, and thus Q is bounded. From Theorem 1.2.8, Q is a polytope. We claim that this polytope Q is an extension of P under the linear mapping π defined by $\pi(y) = \pi(y(V_0), y(v_1), y(v_2)) = y(V_0)$. By definition, we need to show that STAB(G_0) =

$\pi(Q)$. As both STAB(G_0) and $\pi(Q)$ are convex sets, it suffices to show that each vertex of STAB(G_0) is contained in $\pi(Q)$ and vice versa.

To show STAB(G_0) $\subseteq \pi(Q)$: Let $y(V_0) \in \mathbb{R}^{|V_0|}$ be a vertex of STAB(G_0). Then $y(V_0) \in \{0,1\}^{|V_0|}$ is the characteristic vector of some stable set I of G_0. We need to find the lifting variables $(y(v_1), y(v_2)) \in \mathbb{R}^2$ such that $y = (y(V_0), y(v_1), y(v_2)) \in \mathbb{R}^{|V_0|+2}$ satisfies the linear system (4.11). This shows that $y \in Q$. As $\pi(y) = y(V_0)$ and $y(V_0)$ was an arbitrarily picked vertex of STAB(G_0), this proves that STAB(G_0) $\subseteq \pi(Q)$.

For $k = 1, 2$, define $J_k \subseteq V_k$ as follows: If $v \in |I \cap C|$, then $v' \in J_k$, where $v' \in C_k \subseteq V_k$ is the vertex corresponding to $v \in C$ before the identification of C_1 and C_2; If $v \in |I \cap (V_k \backslash C_k)|$, then $v \in J_k$; and these are the only vertices included in J_k. It is clear that J_k is stable in G_k, and $v_k \notin J_k$. Recall that χ^{J_k} is the characteristic vector of J_k. Consider the following cases depending on the cardinality of $I \cap C$:

(i) $|I \cap C| = 1$: Let $y(v_1) = y(v_2) = 0$. Then $y(V_{0k}) = \chi^{J_k} \in$ STAB(G_k) ($k = 1, 2$). Thus the first and second sets of constraints in the linear system (4.11) are satisfied. As $|I \cap C| = 1$, we have $\sum_{v \in C} y(v) = 1$ and $y(v_1) + y(v_2) + \sum_{v \in C} y(v) = 1$. Thus the auxiliary constraint in the linear system (4.11) is also satisfied.

(ii) $|I \cap C| = 0$ and $J_1 \cup \{v_1\}$ is stable in G_1: Let $y(v_1) = 1$ and $y(v_2) = 0$. Then $y(V_{01}) = \chi^{J_1 \cup \{v_1\}} \in$ STAB(G_1) and $y(V_{02}) = \chi^{J_2} \in$ STAB(G_2). As $|I \cap C| = 0$, we have $\sum_{v \in C} y(v) = 0$ and thus $y(v_1) + y(v_2) + \sum_{v \in C} y(v) = 1$. Thus all the constraints in the linear system (4.11) are satisfied.

(iii) $|I \cap C| = 0$ and $J_1 \cup \{v_1\}$ is not stable in G_1: In this case, we must have $J_1 \cap (N_{G_1}(v_1) \backslash C_1) \neq \emptyset$. This implies $J_2 \cap (N_{G_2}(v_2) \backslash C_2) = \emptyset$ and thus $J_2 \cup \{v_2\}$ is stable in G_2. Then the result follows in the same way as in the second case.

In all cases, we have constructed the lifting variables $(y(v_1), y(v_2)) \in \mathbb{R}^2$ such that $y = (y(V_0), y(v_1), y(v_2)) \in \mathbb{R}^{|V_0|+2}$ satisfies the linear system (4.11).

To show STAB(G_0) $\supseteq \pi(Q)$: Let $y = (y(V_0), y(v_1), y(v_2)) \in \mathbb{R}^{|V_0|+2}$ be a vertex of Q. So y satisfies the linear system (4.11). We need to show that $\pi(y) = y(V_0)$ is contained in STAB(G_0). From the first two sets of constraints in the linear system (4.11), we know that $y(V_{0k}) \in$ STAB(G_k) ($k = 1, 2$), and thus $0 \leq y \leq 1$. We will show that y is actually a 0-1 vector. Consider the following two cases depending on the value of y on $C \cup \{v_1, v_2\}$:

(i) $0 < y(u) < 1$ for some vertex $u \in C$: For $k = 1, 2$, as $y(V_{0k}) \in$ STAB(G_k), there exists a vertex y_k^* of STAB(G_k) such that

(a) $y_k^*(u) = 1$;

(b) If a constraint from the kth set of constraint is active at $y(V_{0k})$, then this constraint is also active at y_k^*.

So $y_k^* \in \{0,1\}^{|V_k|}$ is the characteristic vector of some independent set in G_k $(k = 1, 2)$. Since y_1^* and y_2^* agree on C, they uniquely determine the 0-1 vector $y^* \in \mathbb{R}^{|V_0|+2}$. As $y^*(V_{0k}) = y_k^* \in \text{STAB}(G_k)$ $(k = 1, 2)$ by construction, y^* satisfies the first two sets of constraints in the linear system (4.11) and this implies that $y^*(v) = 0$ for every vertex $v \in (C\backslash\{u\}) \cup \{v_1, v_2\}$ and $y^*(u) = 1$. Hence it holds that $y^*(v_1) + y^*(v_2) + \sum_{v\in C} y^*(v) = y^*(u) = 1$. Thus the auxiliary constraint is also satisfied at y^*. This implies $y^* \in Q$.

(ii) $y(v) = 0$ for every $v \in C$ and $y(v_1) \in (0, 1)$: In this case, it holds that $\sum_{v\in C} y(v) = 0$ and thus the auxiliary constraint gives that $y(v_1) + y(v_2) = 1$. So $y(v_2) = 1 - y(v_1) \in (0, 1)$. For $k = 1, 2$, as $y(V_{0k}) \in \text{STAB}(G_k)$, there exists a vertex y_k^* of $\text{STAB}(G_k)$ such that

(a) $y_k^*(v_k) = k - 1$;

(b) If a constraint from the kth set of constraint is active at $y(V_{0k})$, then this constraint is also active at y_k^*.

Similar to case (i), the vectors y_1^* and y_2^* uniquely determine the 0-1 vector $y^* \in \mathbb{R}^{|V(G_0)|+2}$ and y^* satisfies the first two sets of constraint in the linear system (4.11). Since $y(v) = 0$ for every $v \in C$, from (b) above, we have that $y^*(v) = 0$ for every $v \in C$, and thus $\sum_{v\in C} y^*(v) = 0$. This implies that $y^*(v_1) + y^*(v_2) + \sum_{v\in C} y^*(v) = y^*(v_2) = 1$. Therefore the auxiliary constraint is also satisfied at y^*. This implies $y^* \in Q$.

(iii) $y(v) = 0$ for every $v \in C$ and $y(v_1) \in \{0, 1\}$: In this case, it holds that $\sum_{v\in C} y(v) = 0$, and thus the auxiliary constraint gives that $y(v_1) + y(v_2) = 1$. If $y(v_1) = 1$, then $y(v) = 0$ for every $v \in N(v_1)\backslash C_1$. Using the fact that the kth set of constraints defines the stable set polytope $\text{STAB}(G_k)$ $(k = 1, 2)$, and $y(v) = 0$ for every $v \in C \cup (N_{G_1}(v_1)\backslash C_1)$, one can show that y has to be 0-1, otherwise it contradicts that y is a vertex. If $y(v_1) = 0$, then $y(v_2) = 1$ and the same argument as above can be applied.

In the first two cases, we have constructed a feasible solution $y^* \in Q$ such that if a constraint in the linear system (4.11) is active at y, then this constraint is also active at y^* and $y \neq y^*$. This contradicts the assumption that y is a vertex of Q. Thus y is a 0-1 vector in all of the cases.

Now we can prove that $\pi(y) \in \mathrm{STAB}(G_0)$ using the fact that y is a 0-1 vector. Assume $\pi(y) \notin \mathrm{STAB}(G_0)$. As y is a 0-1 vector, there must be some adjacent vertices $u, v \in V_0$ such that $y(u) = y(v) = 1$. Since $y(V_{0k}) \in \mathrm{STAB}(G_k)$ $(k = 1, 2)$, from the definition of amalgamation, we must have $y(u) = y(v) = 1$ for some vertices $u \in N_{G_1}(v_1) \backslash C_1$ and $v \in N_{G_2}(v_2) \backslash C_2$. But this implies that $y(v_1) = y(v_2) = \sum_{v \in C} y(v) = 0$ and thus the auxiliary constraint is not satisfied at y. This is a contradiction. Thus $\pi(y) \in \mathrm{STAB}(G_0)$. This finishes the proof the claim that the polytope Q defined by the linear system (4.11) is an extension of P.

Now we are going to construct an extended formulation of $\mathrm{STAB}(G_0)$ using the extended formulation of $\mathrm{STAB}(G_1)$ and $\mathrm{STAB}(G_2)$. Let the vectors $x_k \in \mathbb{R}^{|V_k|}$ and $y = (y(V_0), y(v_1), y(v_2)) \in \mathbb{R}^{|V_0|+2}$ be defined as before. Let the linear system

$$E_k x_k + F_k s_k = g_k, \ s_k \geq 0 \tag{4.12}$$

be an extended formulation (in slack form) of $\mathrm{STAB}(G_k)$ of size r_k with lifting variables $s_k \in \mathbb{R}^{r_k}$, for $k = 1, 2$. We claim that the following linear system

$$\begin{cases} E_k y(V_{0k}) + F_k t_k = g_k, \ t_k \geq 0, \ k = 1, 2, \\ y(v_1) + y(v_2) + \sum_{v \in C} y(v) = 1. \end{cases} \tag{4.13}$$

is an extended formulation of $\mathrm{STAB}(G_0)$ of size $r_1 + r_2$ with lifting variables $t_1 \in \mathbb{R}^{r_1}$, $t_2 \in \mathbb{R}^{r_2}$, $y(v_1) \in \mathbb{R}$ and $y(v_2) \in \mathbb{R}$. By definition, we need to show that $y(V_0) \in \mathrm{STAB}(G_0)$ if and only if there exists a vector $(y(v_1), y(v_1), t_1, t_2) \in \mathbb{R}^{r_1+r_2+2}$ such that $(y(V_0), y(v_1), y(v_2), t_1, t_2) \in \mathbb{R}^{|V_0|+r_1+r_2+2}$ satisfies the linear system (4.13).

The "only if" part ($\Longrightarrow$): Let $y(V_0) \in \mathrm{STAB}(G_0)$. Since the polytope Q defined by the linear system (4.11) is an extension of P, there exists a vector $(y(v_1), y(v_2)) \in \mathbb{R}^2$ such that $y = (y(V_0), y(v_1), y(v_2)) \in \mathbb{R}^{|V_0|+2}$ is in Q, i.e., y satisfies the linear system (4.11). From the auxiliary constraint of the linear system (4.11), we know that $y(v_1) + y(v_2) + \sum_{v \in C} y(v) = 1$ and thus the last equation in the linear system (4.13) is satisfied at y.

From the first two sets of constraints in (4.11), we know that $y(V_{0k}) \in \mathrm{STAB}(G_k)$ $(k = 1, 2)$. As the linear system (4.12) is an extended formulation of $\mathrm{STAB}(G_k)$ and $y(V_{0k}) \in \mathrm{STAB}(G_k)$, there exists a vector $s_k \in \mathbb{R}^{r_k}$ such that $(y(V_{0k}), s_k)$ satisfies the linear system (4.12) $(k = 1, 2)$. Let $t_1 = s_2$ and $t_2 = s_2$. Then $y = (y(V_0), y(v_1), y(v_2), t_1, t_2) \in \mathbb{R}^{|V_0|+r_1+r_2+2}$ satisfies the linear system (4.13) by construction.

The "if" part ($\Longleftarrow$): Let $(y(V_0), y(v_1), y(v_2), t_1, t_2) \in \mathbb{R}^{|V_0|+r_1+r_2+2}$ be a feasible solution of the linear system (4.13). Then $(y(V_0), y(v_1), y(v_2)) \in \mathbb{R}^{|V_0|+r+2}$ satisfies the linear system (4.11). This implies $\pi(y(V_0), y(v_1), y(v_2)) = y(V_0) \in \mathrm{STAB}(G_0)$ as the polytope Q defined by the linear system (4.11) is an extension of $\mathrm{STAB}(G_0)$ under π.

This shows that the linear system (4.13) is an extended formulation of $\mathrm{STAB}(G_0)$. The size of this extended formulation is clearly $r_1 + r_2$. The theorem follows if we take $r_1 = \mathrm{xc}(\mathrm{STAB}(G_1))$ and $r_2 = \mathrm{xc}(\mathrm{STAB}(G_2))$. □

4.4 Clique Sum

Definition 4.4.1. [2] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that the vertex-intersection $V_1 \cap V_2$ is a clique in both G_1 and G_2. The* clique sum *of G_1 and G_2, denoted by $G_1 \oplus G_2$, is a graph $G_0 = (V_0, E_0)$ with*

$$V_0 = V_1 \cup V_2 \text{ and } E_0 = E_1 \cup E_2.$$

It is well-known that clique-sum operation preserve perfection.

Proposition 4.4.2. *The clique sum operation preserves perfection.*

Proof. Let G_1 and G_2 be perfect graphs. Let G_0 be the clique-sum composed graph from G_1 and G_2. It is not hard to see that $\omega(G_0) = \max\{\omega(G_1), \omega(G_2)\}$ and $\chi(G_0) = \max\{\chi(G_1), \chi(G_2)\}$. As G_1 and G_2 are perfect, it holds that $\omega(G_1) = \chi(G_1)$ and $\omega(G_2) = \chi(G_2)$. Therefore $\omega(G_0) = \chi(G_0)$. The same arguments hold if we consider a subgraph of G_0 which can be considered as the clique sum of some subgraphs of G_1 and G_2. Thus G_0 is also perfect. □

Chvátal [7] showed the following polyhedral consequences of the clique-sum operation.

Theorem 4.4.3. [7] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that the vertex-intersection $V_1 \cap V_2$ is a clique in both G_1 and G_2. Let $G_0 = G_1 \oplus G_2$ be the clique sum of G_1 and G_2. If the linear system*

$$\sum_{v \in V_1} a_i^1(v) \cdot x_i(v) \leq b_i^1, \quad i = 1, \ldots, n_1,$$

defines the stable set polytope of G_1, and the linear system

$$\sum_{v \in V_2} a_i^2(v) \cdot y_i(v) \leq b_i^2, \quad i = 1, \ldots, n_2,$$

defines the stable set polytope of G_2, then the union of these linear systems, with the variables for the vertices in $V_1 \cap V_2$ identified, i.e., the following linear system

$$\begin{cases} \sum_{v\in V_1} a_i^1(v)\cdot z_i(v) \le b_i^1, & i=1,\ldots,n_1, \\ \sum_{v\in V_2} a_i^2(v)\cdot z_i(v) \le b_i^2, & i=1,\ldots,n_2. \end{cases}$$

defines the stable set polytope of G_0.

This result has a direct corollary that the extension complexity of $\mathrm{STAB}(G_0)$ is upper bounded by the sum of the extension complexities of $\mathrm{STAB}(G_1)$ and $\mathrm{STAB}(G_2)$.

Corollary 4.4.4. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that the vertex-intersection $V_1 \cap V_2$ is a clique in both G_1 and G_2. Let $G_0 = G_1 \oplus G_2$ be the clique sum of G_1 and G_2. It holds that*

$$\mathrm{xc}(\mathrm{STAB}(G_0)) \le \mathrm{xc}(\mathrm{STAB}(G_1)) + \mathrm{xc}(\mathrm{STAB}(G_2)).$$

If one does not assume that the vertex-intersection $V_1 \cap V_2$ is a clique in the definition of clique-sum composition, then one gets the definition of *the subgraph identification operation* which is a generalization of the clique sum operation. The subgraph identification does not preserve perfection in general.

Consider another graph operation called disjoint union.

Definition 4.4.5. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two vertex-disjoint graphs. The disjoint union of G_1 and G_2, denoted by $G_1 \cup G_2$, is a graph $G_0 = (V_0, E_0)$ with*

$$V_0 = V_1 \cup V_2 \text{ and } E_0 = E_1 \cup E_2.$$

Clearly disjoint union is a special case of clique sum when $K = \emptyset$. Thus we have the following result about the extension complexity of $\mathrm{STAB}(G_1 \cup G_2)$ as a direct consequence of Corollary 4.4.4.

Corollary 4.4.6. *Let G_1 and G_2 be two vertex-disjoint graphs. Let $G_0 = G_1 \cup G_2$ be the disjoint union of G_1 and G_2. It holds that*

$$\mathrm{xc}(\mathrm{STAB}(G_0)) \le \mathrm{xc}(\mathrm{STAB}(G_1)) + \mathrm{xc}(\mathrm{STAB}(G_2)).$$

This result can also be proven easily without using Theorem 4.4.4. The interesting question is whether the inequality $\mathrm{xc}(\mathrm{STAB}(G_0)) \ge \mathrm{xc}(\mathrm{STAB}(G_1)) + \mathrm{xc}(\mathrm{STAB}(G_2))$ also holds. Intuitively this is true, but this is not known currently.

Chapter 5

Perfect Graphs

5.1 Introduction

The maximum stable set problem is difficult to solve in general. However, it can be solved efficiently on some classes of graphs. Grötschel et al. [16] showed that the maximum stable set for perfect graphs may be found in polynomial time using semi-definite programming. This is so far the only known method to solve the maximum stable set problem in perfect graphs in polynomial time and it is still an open question whether this is also possible using *linear programming*. This motivates the study of the extension complexity of the stable set polytope of perfect graphs.

A graph G is a Berge graph if neither G nor $\bar{G}$ contains an odd-length induced cycle of length 5 or more. The strong perfect graph theorem states that a graph G is perfect if and only if G is a Berge graph. The recent proof of the strong perfect graph theorem in Chudnovsky et al. [6], which characterizes perfect graphs in terms of forbidden induced subgraphs, relies on a structural decomposition result for perfect graphs. This result says (roughly) that any perfect graph G either belongs to one of the following basic classes:

(i) bipartite graphs and their complements;

(ii) line graphs of bipartite graphs and their complements;

(iii) double-split graphs;

or one of $G, \bar{G}$ admits one of the following decompositions:

(a) proper 2-join;

(b) balanced skew partition.

It is well-known that the stable set polytopes of the basic perfect graphs (i) and (ii) above have polynomial-sized linear programs. Thus the extension complexities of the stable set polytopes of these graphs are polynomial. In Section 5.3, we will show that the extension complexity of the stable set polytope of (iii) double-split graphs is also polynomial in the size (number of vertices) of the graph. Furthermore if a graph G admits the decompositions (a) or (b), then we show that the extension complexity of STAB(G) is upper bounded by some linear functions in the extension complexities of the stable set polytope of some subgraphs of G. This is done in Section 5.4 for the 2-join operation and in Section 5.5 for the skew partition operation.

Although this does not settle the question whether the extension complexity of the stable set polytope of any perfect graph is polynomial in the size (number of vertices) of the graph, we do obtain a positive result for perfect graphs which has a decomposition tree whose depth is logarithmic in its number of vertices. Furthermore, we also consider a subclass of perfect graphs known as decomposable Meyniel graphs. We showed that the extension complexities of the stable set polytopes of decomposable Meyniel graphs are also polynomial. In Section 5.6 we show that the extension complexity of the stable set polytope of any decomposable Meyniel graph is polynomial. For this we use our result about the amalgam operation from Chapter 4 and we in fact show that the extension complexity is at most cubic in the number of vertices.

5.2 The Extension Complexity of Perfect Graphs

In this section, we provide some general results about the extension complexity of perfect graphs.

Let G be a perfect graph. Then we know that STAB(G) can be characterized by the non-negativity constraints and the clique constraints as stated in Theorem 1.3.17. In what follows, the slack matrix of STAB(G), where the rows are indexed by the non-negativity constraints and a set of clique constraints containing all maximal clique constraints, and the columns by all independent sets of G, is denoted by S_G. Notice that it suffices to include all the maximal clique constraints and the non-negative constraints to the linear system defining STAB(G), as each clique of G is contained in some maximal clique of G. From Lemma 2.4.3 and Theorem 2.4.4, we know that $\text{rank}_+(S_G) = \text{xc}(\text{STAB}(G))$, i.e., the non-negative rank of the slack matrix S_G coincides with the extension complexity of the stable set polytope of G.

We give an example to illustrate the structure of S_G when G is a cycle of length 4. Assume $G = (V, E)$ is C_4, where $V = \{a, b, c, d\}$ and $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}$. The maximal cliques of G are precisely the set of edges. The collection of the stable sets in G is given by $\mathcal{I}(G) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{b, d\}\}$. Thus S_G is an 8 by 7 matrix as follows,

$$S_G = \begin{array}{r} \\ x_a \geq 0 \\ x_b \geq 0 \\ x_c \geq 0 \\ x_d \geq 0 \\ x_a + x_b \leq 1 \\ x_b + x_c \leq 1 \\ x_c + x_d \leq 1 \\ x_a + x_d \leq 1 \end{array} \begin{array}{c} \begin{array}{ccccccc} \emptyset & \{a\} & \{b\} & \{c\} & \{d\} & \{a,c\} & \{b,d\} \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

In what follows, S'_G denotes the submatrix of S_G corresponding to the rows of non-negative constraints, and S''_G denotes the submatrix of S_G corresponding to the rows of clique constraints. Then each column of S'_G is precisely the characteristic vector of the stable set associated to that column. The rows and columns of the submatrix S''_G can be indexed by the cliques and the stable sets of G, respectively. Notice that $S''_G(C, I) = 1 - |C \cap I|$, where C is a clique and I is a stable set of G.

The number of rows of S_G is at most the number of vertices $|V(G)|$ plus the number of cliques in G. From Lemma 2.2.2, we immediately have that

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq |V(G)| + \text{ number of cliques in } G.$$

If one only includes the maximal clique constraints, then it yields the following sharper upper bound on $\mathrm{xc}(\mathrm{STAB}(G))$.

Lemma 5.2.1. [25] *Let G be a perfect graph. If G has at most k maximal cliques, then* $\mathrm{xc}(\mathrm{STAB}(G)) \leq |V(G)| + k$.

One can also show that the extension complexity of a perfect graph G is linearly related to that of its complement.

Lemma 5.2.2. [25] *Let G be a perfect graph. It holds that*

$$\mathrm{xc}(\mathrm{STAB}(\bar{G})) \leq \mathrm{xc}(\mathrm{STAB}(G)) + |V(G)|.$$

Proof. The slack matrix S_G of G has the form $S_G = \begin{bmatrix} S'_G \\ S''_G \end{bmatrix}$. Recall that S''_G has one row for every clique C and one column for every stable set I of G; and the corresponding entry is $S(C, I) = 1 - |C \cap I|$. Notice that a clique in G is a stable set in $\bar{G}$ and a stable set in G is a clique in $\bar{G}$. Thus the slack matrix $S_{\bar{G}} = \begin{bmatrix} S'_{\bar{G}} \\ S''_{\bar{G}} \end{bmatrix}$ of G satisfies that $S''_{\bar{G}} = (S''_G)^T$.

Applying Lemma 2.2.2, Lemma 2.2.3 and Theorem 2.4.4, we have

$$\begin{aligned} \text{xc}(\text{STAB}(\bar{G})) &= \text{rank}_+(S_{\bar{G}}) \\ &= \text{rank}_+(\begin{bmatrix} S'_{\bar{G}} \\ S''_{\bar{G}} \end{bmatrix}) \\ &\leq \text{rank}_+(S'_{\bar{G}}) + \text{rank}_+(S''_{\bar{G}}) \\ &\leq |V(G)| + \text{rank}_+(S''_G). \end{aligned}$$

as the number of rows of $S'_{\bar{G}}$ is $|V(G)|$ and $S''_{\bar{G}} = (S''_G)^T$. Now S''_G is a submatrix of S_G whose non-negative rank equals $\text{xc}(\text{STAB}(G))$, by Lemma 2.2.3, we have $\text{rank}_+(S''_G) \leq \text{rank}_+(S_G) = \text{xc}(\text{STAB}(G))$. This finishes the proof. □

Finally, we provide some well-known upper bounds on the extension complexities of the stable set polytopes of basic perfect graphs.

Proposition 5.2.3. [23] *If G is a bipartite graph, then* $\text{xc}(\text{STAB}(G)) \leq |V(G)| + |E(G)|$ *and* $\text{xc}(\text{STAB}(\bar{G})) \leq 2 \cdot |V(G)| + |E(G)|$.

Proof. Each edge in the bipartite graph G is a clique and there are no other cliques in G. From Theorem 5.2.1, we have that $\text{xc}(\text{STAB}(G)) \leq |V(G)| + |E(G)|$. Applying Lemma 5.2.2, we have that $\text{xc}(\text{STAB}(\bar{G})) \leq 2 \cdot |V(G)| + |E(G)|$. □

Proposition 5.2.4. *Let G be a bipartite graph. The following linear system*

$$\sum_{j:(i,j) \in E} x_{i,j} \leq 1 \ \ \forall\, i \in V(G) \tag{5.1}$$

$$x \geq 0 \tag{5.2}$$

in the vector of variables $x \in \mathbb{R}^{|E(G)|}$, defines the matching polytope of G.

Proposition 5.2.5. *If G is the line graph of a bipartite graph G', then*

$$\text{xc}(\text{STAB}(G)) \leq |V(G)| + |E(G)| \ \textit{and} \ \text{xc}(\text{STAB}(\bar{G})) \leq 2 \cdot |V(G)| + |E(G)|.$$

Proof. An independent set in G corresponds to a matching in G'. Thus we have that $\mathrm{xc}(\mathrm{STAB}(G)) \leq |V(G)| + |E(G)|$, using Proposition 5.2.3 and 5.2.4. The inequality $\mathrm{xc}(\mathrm{STAB}(\bar{G})) \leq 2 \cdot |V(G)| + |E(G)|$ follows from Lemma 5.2.2. $\square$

5.3 Double-Split Graphs

If $x \in \mathbb{Z}$ is a strictly positive integer, we denote $\{1, \ldots, x\}$ by $[x]$.

Definition 5.3.1. [6] *A graph $G = (V, E)$ is called a* double-split *graph if its vertex set can be partitioned into two sets V_1 and V_2 such that:*

(i) $G(V_1)$ is a disjoint union of edges, where

$$V_1 = \{a_i, b_i : i \in [p]\} \text{ and } E_1 = \{\{a_i, b_i\} : i \in [p]\};$$

(ii) $G(V_2)$ is the complement of a disjoint union of edges, where

$$V_2 = \{x_i, y_i : i \in [q]\} \text{ and } E_2 = \{\{x_i, y_j\} : i \neq j \text{ and } i, j \in [q]\};$$

(iii) For every $i \in [p]$ and $j \in [q]$, precisely one of the following is true

$$\{\{a_i, x_j\}, \{b_i, y_j\}\} \subseteq E \text{ or } \{\{a_i, y_j\}, \{b_i, x_j\}\} \subseteq E.$$

Let p, q be positive integers. Let $L_i \subseteq [q]$ be a subset of $[q]$, for every $i \in [p]$. Then we define $G_{p,q,L}$ as a double split graph such that there exist a vertex partition $(V_1; V_2)$ satisfying $G(V_1)$ is a disjoint union of p edges, $G(V_2)$ is the complement of a disjoint union of q edges, and the edges between vertex sets V_1 and V_2 are given by $E(V_1, V_2) = \cup_{i=1}^{p} \{\{a_i, x_j\} : j \in L_i\}$.

Lemma 5.3.2. [6] *Double-split graphs are perfect.*

We shall provide an upper bound on the extension complexity of the stable set polytope of any double-split graph G. Towards this end, we consider its slack matrix S_G. Thus we will list all the maximal cliques and the stable sets in G below.

Given a subset $I \subseteq [q]$, we define $x_I := \{x_i \in V_2 : i \in I\}$ and $y_{\bar{I}} := \{y_i \in V_2 : i \in \bar{I}\}$. The set $\mathcal{C}_k$ of the maximal cliques in $G(V_k)$ $(k = 1, 2)$ are given by

$$\mathcal{C}_1 := \{\{a_i, b_i\} : i \in [p]\},$$

$$\mathcal{C}_2 := \{x_I \cup y_{\bar{I}} : \forall\, I \subseteq [q]\}.$$

We also need the mixed maximal clique constraints. For every $i \in [p]$, define $L_i := \{j : \{a_i, x_j\} \in E\} \subseteq [q]$. The set of the mixed maximal cliques is given by

$$\mathcal{MC} := \{\{a_i\} \cup x_{L_i} \cup y_{\bar{L}_i},\ \{b_i\} \cup x_{\bar{L}_i} \cup y_{L_i} : \forall i \in [p]\}.$$

Finally, let $\mathcal{R}_i$ be the collection of the non-negative constraints and the maximal clique constraints of the stable set polytope of $G(V_i)$ $(i = 1, 2)$. Let $\mathcal{MR}$ be the set of constraints associated to the mixed cliques in $\mathcal{MC}$.

As the set of cliques $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{MC}$ contains all the maximal cliques in G, the constraints in $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{MR}$ contains all the maximal clique constraints and the non-negativity constraints for G.

Now we consider the stable sets in G. Let $\mathcal{I}_k$ be the collection of the stable sets of G_k for $k = 1, 2$. Then

$$\mathcal{I}_1 := \{\emptyset\} \cup \{a_i, b_i : \forall i \in [p]\} \cup \{a_J \cup b_K \ \ \forall J \subseteq [p], K \subseteq \bar{J}\},$$

$$\mathcal{I}_2 := \{\emptyset\} \cup \{x_i, y_i, \{x_i, y_i\} : \forall i \in [q]\}.$$

Let $\mathcal{MI}$ be the collection of the mixed stable sets in G,

$$\begin{aligned}\mathcal{MI} :=& \{\{x_j\} \cup a_J \cup b_K : \forall J \subseteq [p], K \subseteq \bar{J} \quad \text{s.t. } j \in \bar{L}_i \ \ \forall i \in J \text{ and } j \in L_i \ \ \forall i \in K\}\\ &\cup \{\{y_j\} \cup a_J \cup b_K : \forall I \subseteq [p], K \subseteq \bar{J} \quad \text{s.t. } j \in L_i \ \ \forall i \in J \text{ and } j \in \bar{L}_i \ \ \forall i \in K\}.\end{aligned}$$

It is easy to verify that $\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{MI}$ contains precisely all the stable sets in G. Now we can prove the main result in this section.

Theorem 5.3.3. *Let $G = G_{p,q,L}$ be a double-split graph. It holds that*

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq 5p + 4q + 3.$$

Proof. As double-split graphs are perfect, STAB(G) can be described by the non-negativity constraints and the clique constraints according to Theorem 1.3.17. Thus the constraints in $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{MR}$ and the stable sets in $\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{MI}$ induce a slack matrix S_G whose non-negative rank equals the extension complexity of STAB(G), which has the following form:

$$S_G = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{MR} \end{array} \begin{array}{c} \begin{array}{ccc} \mathcal{I}_1 & \mathcal{I}_2 & \mathcal{MI} \end{array} \\ \begin{pmatrix} S_{1,1} & S_{1,2} & S_{1,3} \\ S_{2,1} & S_{2,2} & S_{2,3} \\ S_{3,1} & S_{3,2} & S_{3,3} \end{pmatrix} \end{array}.$$

Notice that S_{G_k} is the slack matrix of the graph G_k, induced by the constraints $\mathcal{R}_k$ and the stable sets $\mathcal{I}_k$ ($k = 1, 2$). As G_k is perfect, we also have that $\mathrm{xc}(\mathrm{STAB}(G_k)) = \mathrm{rank}_+(S_{G_k})$.

For $k = 1, 2$, the submatrix $S_{k,k}$ is precisely the slack matrix S_{G_k} and each column of $S_{k,3}$ is a duplication of some column in $S_{k,k}$, this implies that $\mathrm{rank}_+(\begin{bmatrix} S_{k,k} & S_{k,3} \end{bmatrix}) = \mathrm{rank}_+(S_{G_k})$.

As each row of $S_{1,2}$ and $S_{2,1}$ is either all-zeros or all-ones, it holds that $\mathrm{rank}_+(S_{2,1}) = \mathrm{rank}_+(S_{1,2}) = 1$. Finally, $\mathrm{rank}_+(\begin{bmatrix} S_{3,1} & S_{3,2} & S_{3,2} \end{bmatrix}) \leq 2p$ as this matrix has only $2p$ rows.

Applying Lemma 2.2.3 to S_G, we conclude that the non-negative rank of S is upper bounded by

$$\begin{aligned} \mathrm{rank}_+(S_G) &\leq \mathrm{rank}_+(\begin{bmatrix} S_{1,1} & S_{1,3} \end{bmatrix}) + \mathrm{rank}_+(S_{1,2}) \\ &\quad + \mathrm{rank}_+(\begin{bmatrix} S_{2,2} & S_{2,3} \end{bmatrix}) + \mathrm{rank}_+(S_{2,1}) \\ &\quad + \mathrm{rank}_+(\begin{bmatrix} S_{3,1} & S_{3,2} & S_{3,2} \end{bmatrix}) \\ &\leq \mathrm{rank}_+(S_{G_1}) + \mathrm{rank}_+(S_{G_2}) + 2p + 2. \end{aligned}$$

As G_1 is a disjoint union of p edges and G_2 is the complement of a disjoint union of q edges, from Lemma 3.3.11 and 4.2.7, we know that $\mathrm{rank}_+(S_{G_1}) = 3p$ and $\mathrm{rank}_+(S_{G_2}) \leq 4q + 1$. Finally, applying the Factorization Theorem 2.4.4, we obtain that

$$\mathrm{xc}(\mathrm{STAB}(G)) = \mathrm{rank}_+(S_G) \leq 5p + 4q + 3.$$

This finishes the proof. □

This upper bound on the extension complexity of the stable set polytope of double-split graphs enables us to conclude the following result.

Lemma 5.3.4. *If G is a basic perfect graph, then* $\mathrm{xc}(\mathrm{STAB}(G)) \leq 2 \cdot |V(G)|^2$.

5.4 Proper 2-Join Decompositions

Definition 5.4.1. (see [6]) *A* proper 2-join *of G is a partition of the vertices of G into two sets $(X_1; X_2)$ such that there exist disjoint nonempty subsets $A_k, B_k \subseteq X_k$ for $k = 1, 2$ satisfying:*

(i) the edges between X_1 and X_2 are given by:
$\{\{u, v\} : u \in A_1, v \in A_2\} \cup \{\{u, v\} : u \in B_1, v \in B_2\}$.

(ii) for $k = 1, 2$, every component C of $G(X_k)$ satisfies that $V(C) \cap A_k \neq \emptyset$ and $V(C) \cap B_k \neq \emptyset$.

(iii) for $k = 1, 2$, if $|A_k| = |B_k| = 1$ and $G(X_k)$ is a path connecting the two vertices in A_k and B_k, then its length is odd and at least 3.

In fact, we only need the first property in the definition of proper 2-join to prove the result in this section.

Let G be a perfect graph admitting a proper 2-join. We shall give an upper bound for $\mathrm{xc}(\mathrm{STAB}(G))$ in terms of $\mathrm{xc}(\mathrm{STAB}(G(X_1)))$ and $\mathrm{xc}(\mathrm{STAB}(G(X_2)))$. Towards this end, we consider its slack matrix of S_G. Similar to the previous section, we list all the maximal cliques and the stable sets in G below.

Define $D_k := X_k \backslash (A_k \cup B_k)$ $(k = 1, 2)$. From the definition of a proper 2-join of the graph G, the following subsets of vertices are cliques of the graph G.

$$\begin{aligned}
\mathcal{C}_1 &:= \mathcal{C}(G(X_1)), \\
\mathcal{C}_2 &:= \mathcal{C}(G(X_2)), \\
\mathcal{C}_A &:= \{D_1 \cup D_2 : D_k \in \mathcal{C}(G(A_k)) \text{ for } k = 1, 2\}, \\
\mathcal{C}_B &:= \{D_1 \cup D_2 : D_k \in \mathcal{C}(G(B_k)) \text{ for } k = 1, 2\}.
\end{aligned}$$

Notice that $\mathcal{C}_k$ contains all the maximal cliques within the vertex set X_k $(k = 1, 2)$. $\mathcal{C}_A$ and $\mathcal{C}_B$ contain all the maximal cliques whose intersection with X_1 and X_2 is not empty. Thus every maximal clique of G is contained in $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_A \cup \mathcal{C}_B$.

Let $\mathcal{R}_k$ be the collection of the non-negativity constraints and the clique constraints of the graph $G(X_k)$ $(k = 1, 2)$. Let $\mathcal{MR}_l$ be the collection of the clique constraints associated to the cliques in $\mathcal{C}_l$ $(l \in \{A, B\})$. Thus the constraints in $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{MR}_A \cup \mathcal{MR}_B$ characterize the stable set polytope of G.

Now we consider the stable sets of G. The stable sets within the vertex set X_k are given by

$$\mathcal{I}_k = \{I : I \in \mathcal{I}(G(X_k))\} \text{ for } k = 1, 2.$$

Let $\mathcal{MI}_k$ $(k = 1, \ldots, 4)$ be a collection of the mixed stable sets of G defined as

$$\begin{aligned}
\mathcal{MI}_1 &:= \{I \cup J : I \in \mathcal{I}(G(D_1)),\ J \in \mathcal{I}(G(D_2 \cup A_2 \cup B_2))\}, \\
\mathcal{MI}_2 &:= \{I \cup J : I \in \mathcal{I}(G(D_1 \cup A_1)),\ J \in \mathcal{I}(G(D_2 \cup B_2))\}, \\
\mathcal{MI}_3 &:= \{I \cup J : I \in \mathcal{I}(G(D_1 \cup B_1)),\ J \in \mathcal{I}(G(D_2 \cup A_2))\}, \\
\mathcal{MI}_4 &:= \{I \cup J : I \in \mathcal{I}(G(D_1 \cup A_1 \cup B_1)),\ J \in \mathcal{I}(G(D_2))\}.
\end{aligned}$$

It is not hard to see that $\mathcal{I}_1 \cup \mathcal{I}_2 \cup (\cup_{k=1}^4 \mathcal{MI}_k)$ contains all the stable sets of G.

Theorem 5.4.2. *Let G be a perfect graph admitting a proper 2-join decomposition. Let $(X_1; X_2)$ be a proper 2-join partition of vertices of G. It holds that*

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq 3 \cdot (\mathrm{xc}(\mathrm{STAB}(G(X_1))) + \mathrm{xc}(\mathrm{STAB}(G(X_2)))) + 2.$$

Proof. As G is perfect, $\mathrm{STAB}(G)$ can be described by the non-negativity constraints and the clique constraints in according to Theorem 1.3.17. Thus the constraints in $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{MC}_A \cup \mathcal{MC}_B$ and the stable sets $\mathcal{I}_1 \cup \mathcal{I}_2 \cup (\cup_{k=1}^4 \mathcal{MI}_k)$ induce a slack matrix S_G whose non-negative rank equals the extension complexity of $\mathrm{STAB}(G)$, which has the following form,

$$S_G = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{MR}_A \\ \mathcal{MR}_B \end{array} \begin{array}{c} \begin{array}{cccccc} \mathcal{I}_1 & \mathcal{I}_2 & \mathcal{MI}_1 & \mathcal{MI}_2 & \mathcal{MI}_3 & \mathcal{MI}_4 \end{array} \\ \begin{pmatrix} S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} & S_{1,5} & S_{1,6} \\ S_{2,1} & S_{2,2} & S_{2,3} & S_{2,4} & S_{2,5} & S_{2,6} \\ S_{3,1} & S_{3,2} & S_{3,3} & S_{3,4} & S_{3,5} & S_{3,6} \\ S_{4,1} & S_{4,2} & S_{4,3} & S_{4,4} & S_{4,5} & S_{4,6} \end{pmatrix} \end{array}.$$

Let $S_{G(X_k)}$ be the slack matrix of the graph $G(X_k)$ induced by the constraints $\mathcal{R}_k$ and the stable sets $\mathcal{I}_k$ $(k = 1, 2)$. As $G(X_k)$ is perfect, we have that $\mathrm{xc}(\mathrm{STAB}(G(X_k)) = \mathrm{rank}_+(S_{G(X_k)})$ $(k = 1, 2)$.

For $k = 1, 2$, the submatrix $S_{k,k}$ coincides with the slack matrix $S_{G(X_k)}$ of $\mathrm{STAB}(G(X_k))$ and each column in the submatrix $\begin{bmatrix} S_{k,3} & \cdots & S_{k,6} \end{bmatrix}$ is a duplication of some column in $S_{k,k}$. Thus $\mathrm{rank}_+(\begin{bmatrix} S_{k,k} & S_{k,3} & S_{k,4} & S_{k,5} & S_{k,6} \end{bmatrix}) = \mathrm{rank}_+(S_{G(X_k)})$, by Lemma 2.2.3.

As each row of the submatrices $S_{1,2}$ and $S_{2,1}$ is either all-zeros or all-ones, we have that $\mathrm{rank}_+(S_{1,2}) = \mathrm{rank}_+(S_{2,1}) = 1$.

The submatrix $S_{3,1}$ is also a submatrix of $S_{1,1}$. Each column of the submatrices $S_{3,4}$ and $S_{3,6}$ is a duplication of some column in $S_{3,1}$. By Lemma 2.2.3, this implies

$$\mathrm{rank}_+(\begin{bmatrix} S_{3,1} & S_{3,4} & S_{3,6} \end{bmatrix}) \leq \mathrm{rank}_+(S_{1,1}) = \mathrm{rank}_+(S_{G(X_1)}).$$

Similarly we have

$$\begin{aligned} \mathrm{rank}_+(\begin{bmatrix} S_{3,2} & S_{3,3} & S_{3,5} \end{bmatrix}) &\leq \mathrm{rank}_+(S_{G(X_2)}), \\ \mathrm{rank}_+(\begin{bmatrix} S_{4,1} & S_{4,5} & S_{4,6} \end{bmatrix}) &\leq \mathrm{rank}_+(S_{G(X_1)}), \\ \mathrm{rank}_+(\begin{bmatrix} S_{4,2} & S_{4,3} & S_{4,4} \end{bmatrix}) &\leq \mathrm{rank}_+(S_{G(X_2)}). \end{aligned}$$

Applying Lemma 2.2.3 to S_G, we conclude that the non-negative rank of S is upper bounded by

$$\begin{aligned}\operatorname{rank}_+(S_G) &\leq \operatorname{rank}_+(\begin{bmatrix} S_{1,1} & S_{1,3} & S_{1,4} & S_{1,5} & S_{1,6}\end{bmatrix}) + \operatorname{rank}_+(S_{1,2}) \\ &+ \operatorname{rank}_+(\begin{bmatrix} S_{2,2} & S_{2,3} & S_{2,4} & S_{2,5} & S_{2,6}\end{bmatrix}) + \operatorname{rank}_+(S_{2,1}) \\ &+ \operatorname{rank}_+(\begin{bmatrix} S_{3,1} & S_{3,4} & S_{3,6}\end{bmatrix}) + \operatorname{rank}_+(\begin{bmatrix} S_{3,2} & S_{3,3} & S_{3,5}\end{bmatrix}) \\ &+ \operatorname{rank}_+(\begin{bmatrix} S_{4,1} & S_{4,5} & S_{4,6}\end{bmatrix}) + \operatorname{rank}_+(\begin{bmatrix} S_{4,2} & S_{4,3} & S_{4,4}\end{bmatrix}) \\ &\leq 3 \cdot (\operatorname{rank}_+(S_{G(X_1)}) + \operatorname{rank}_+(S_{G(X_2)})) + 2.\end{aligned}$$

From the Factorization Theorem 2.4.4, we know that $\operatorname{xc}(\operatorname{STAB}(G')) = \operatorname{rank}_+(S_{G'})$ for any perfect graph G', and thus the claim is proven. □

Finally, we use the above theorem to provide an upper bound for perfect graphs which can be decomposed into basic perfect graphs using some proper 2-join decompositions. Recall that basic perfect graphs are (i) the bipartite graphs and their complements; (ii) line graphs of bipartite graphs and their complements; (iii) double-split graphs.

Theorem 5.4.3. *Assume G is a perfect graph which can be decomposed into basic perfect graphs, using proper 2-join decompositions. Let d denote the depth of a decomposition tree representing a decomposition of G into basic perfect graphs by means of proper 2-join decompositions. Then,*

$$\operatorname{xc}(\operatorname{STAB}(G)) \leq 4^d \cdot 2 \cdot |V(G)|^2 = 2^{2d+1}|V(G)|^2.$$

Proof. We use the result of Theorem 5.4.2, which implies that

$$\operatorname{xc}(\operatorname{STAB}(G)) \leq 4 \cdot (\operatorname{xc}(\operatorname{STAB}(G(X_1))) + \operatorname{xc}(\operatorname{STAB}(G(X_2)))),$$

if G has a proper 2-join decomposition with partition $(X_1; X_2)$.

Denote by $G_1, \ldots, G_L$ the basic perfect graphs corresponding to the leaves of the decomposition tree of G. By iteratively applying the above result, we get

$$\operatorname{xc}(\operatorname{STAB}(G)) \leq 4^d \sum_{l=1}^{L} \operatorname{xc}(\operatorname{STAB}(G_l)).$$

Now by Lemma 5.3.4,

$$\sum_{l=1}^{L} \operatorname{xc}(\operatorname{STAB}(G_l)) \leq 2 \cdot \sum_{l=1}^{L} |V(G_l)|^2 \leq 2 \cdot (\sum_{l=1}^{L} |V(G_l)|)^2 \leq 2 \cdot |V(G)|^2,$$

since the vertex sets of the graphs G_l partition the vertex set of G. This gives the result. □

5.5 Skew Partitions

Definition 5.5.1. [8] *A* skew partition *of some graph G is a partition $(A_1; B_1; A_2; B_2)$ of $V(G)$ such that all edges between A_1 and A_2 are present in G and no edges between B_1 and B_2.*

The decomposition used in the proof of the strong perfect graph theorem adds more conditions to skew partitions, namely balanced skew partitions. We say P is an anti-path of G if P is a path in $\bar{G}$.

Definition 5.5.2. [6] *A skew partition is said to be* balanced *if*

(i) every induced path in G between nonadjacent vertices in $A_1 \cup A_2$ with interior in $B_1 \cup B_2$ is of even length;

(ii) every induced antipath in G between adjacent vertices in $B_1 \cup B_2$ with interior in $A_1 \cup A_2$ is of even length;

Let G be a graph admitting a skew partition, say $(A_1; B_1; A_2; B_2)$. In this section, we show that the extension complexity of G can be linearly upper bounded by the extension complexity of some subgraphs of G. Similar to the previous section, we shall list a set of cliques of G containing all the maximal cliques and the stable sets in G first.

The following subsets of vertices are the cliques of G,

$$\begin{aligned}
&\mathcal{C}_1 = \mathcal{C}(G(A_1 \cup B_1)),\\
&\mathcal{C}_2 = \mathcal{C}(G(A_2 \cup B_2)),\\
&\mathcal{C}_3 = \{C \cup C' \in \mathcal{C}(G) : C \in \mathcal{C}(G(A_2)), C' \in \mathcal{C}(G(A_1 \cup B_1))\},\\
&\mathcal{C}_4 = \{C \cup C' \in \mathcal{C}(G) : C \in \mathcal{C}(G(A_1)), C' \in \mathcal{C}(G(A_2 \cup B_2))\}.
\end{aligned}$$

The following subsets of vertices are the stable sets of G,

$$\begin{aligned}
&\mathcal{I}_1 = \mathcal{I}(G(A_1 \cup B_1)),\\
&\mathcal{I}_2 = \mathcal{I}(G(A_2 \cup B_2)),\\
&\mathcal{I}_3 = \{I \cup I' \in \mathcal{I}(G) : I \in \mathcal{I}(G(B_2)), I' \in \mathcal{I}(G(A_1 \cup B_1))\},\\
&\mathcal{I}_4 = \{I \cup I' \in \mathcal{I}(G) : I \in \mathcal{I}(G(B_1)), I' \in \mathcal{I}(G(A_2 \cup B_2))\}.
\end{aligned}$$

By definition, all edges between A_1 and A_2 are present in G and no edges between B_1 and B_2, thus $\cup_{k=1}^4 \mathcal{C}_k$ contains all the cliques in G and $\cup_{k=1}^4 \mathcal{I}_k$ contains all the stable sets in G.

Let $\mathcal{R}_k$ be the collection of the non-negativity constraints and the clique constraints associated to cliques in $\mathcal{C}_k$ ($k = 1, 2$). Let R_k be the collection of clique constraints associated to cliques in $\mathcal{C}_k$ ($k = 3, 4$).

Theorem 5.5.3. *Let G be a perfect graph admitting some skew partitions. Let $(A_1; B_1; A_2; B_2)$ be a skew partition of vertices of G. Then*

$$\begin{aligned}\mathrm{xc}(\mathrm{STAB}(G)) \leq 3 \cdot (&\mathrm{xc}(\mathrm{STAB}(G(A_1 \cup B_1))) + \mathrm{xc}(\mathrm{STAB}(G(A_2 \cup B_2)))) \\ &+ \mathrm{xc}(\mathrm{STAB}(G(A_1 \cup B_2))) + \mathrm{xc}(\mathrm{STAB}(G(A_2 \cup B_1))) + 2.\end{aligned}$$

Proof. As G is perfect, $\mathrm{STAB}(G)$ can be described by the non-negativity constraints and the clique constraints in according to Theorem 1.3.17. Thus the constraints in $\cup_{k=1}^4 \mathcal{R}_k$ and the stable sets in $\cup_{k=1}^4 \mathcal{I}_k$ induce a slack matrix S_G whose non-negative rank equals the extension complexity of $\mathrm{STAB}(G)$, which has the following form,

$$S_G = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{I}_1 & \mathcal{I}_2 & \mathcal{I}_3 & \mathcal{I}_4 \end{array} \\ \begin{pmatrix} S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} \\ S_{2,1} & S_{2,2} & S_{2,3} & S_{2,4} \\ S_{3,1} & S_{3,2} & S_{3,3} & S_{3,4} \\ S_{4,1} & S_{4,2} & S_{4,3} & S_{4,4} \end{pmatrix} \end{array}.$$

It is easy to see that $S_{i,i} = S_{G(A_i \cup B_i)}$, i.e., it coincides with the slack matrix of $\mathrm{STAB}(G(A_i \cup B_i))$, and $S_{i,3}, S_{i,4}, S_{3,i}, S_{4,i}, S_{i+2,i+2}$ are submatrix of $S_{G(A_i \cup B_i)}$, for $i = 1, 2$. Furthermore, $S_{1,2}$ and $S_{2,1}$ are matrices whose rows are either all-ones or all-zeros.

Finally, $S_{3,4}$ is a submatrix of $S_{G(A_1 \cup B_2)}$, i.e., a submatrix of the slack matrix of $\mathrm{STAB}(G(A_1 \cup B_2))$. Because the clique and the stable set defining the corresponding entries in $S_{3,4}$ only intersect within $A_1 \cup B_2$. Similarly, one can conclude that $S_{4,3}$ is a submatrix of $S_{G(A_2 \cup B_1)}$.

Applying Lemma 2.2.3 to S_G, we obtain the following upper bounded on the non-negative rank of S,

$$\begin{aligned}
\operatorname{rank}_+(S_G) \le\ & \operatorname{rank}_+(\begin{bmatrix} S_{1,1} & S_{1,3} & S_{1,4} \end{bmatrix}) + \operatorname{rank}_+(S_{1,2}) \\
& + \operatorname{rank}_+(\begin{bmatrix} S_{2,1} & S_{2,3} & S_{2,4} \end{bmatrix}) + \operatorname{rank}_+(S_{2,2}) \\
& + \operatorname{rank}_+(\begin{bmatrix} S_{3,1} & S_{3,3} \end{bmatrix}) + \operatorname{rank}_+(S_{3,2}) + \operatorname{rank}_+(S_{3,4}) \\
& + \operatorname{rank}_+(\begin{bmatrix} S_{4,1} & S_{4,4} \end{bmatrix}) + \operatorname{rank}_+(S_{4,2}) + + \operatorname{rank}_+(S_{4,3}) \\
& = 3 \cdot (\operatorname{rank}_+(S_{G(A_1 \cup B_1)}) + \operatorname{rank}_+(S_{G(A_2 \cup B_2)})) \\
& + \operatorname{rank}_+(S_{G(A_1 \cup B_2)}) + \operatorname{rank}_+(S_{G(A_2 \cup B_1)}) + 2.
\end{aligned}$$

From the Factorization Theorem 2.4.4, we know that $\operatorname{xc}(\mathrm{STAB}(G')) = \operatorname{rank}_+(S_{G'})$ for any perfect graph G', and thus the claim is proven. □

Now the theorem above provides an upper bound for perfect graphs which can be decomposed into basic perfect graphs using some skew partition decompositions.

Theorem 5.5.4. *Assume G is a perfect graph which can be decomposed into basic perfect graphs, using skew partition decompositions. Let d denote the depth of a decomposition tree representing a decomposition of G into basic perfect graphs by means of skew partitions. Then,*

$$\operatorname{xc}(\mathrm{STAB}(G)) \le 4^{2d} \cdot 2 \cdot |V(G)|^2 = 2^{4d+1} \cdot |V(G)|^2.$$

Proof. We use Theorem 5.5.3 which implies that

$$\operatorname{xc}(\mathrm{STAB}(G)) \le 4 \cdot (\operatorname{xc}(\hat{G}_1) + \operatorname{xc}(\hat{G}_2) + \operatorname{xc}(\hat{G}_3) + \operatorname{xc}(\hat{G}_3)),$$

where $\hat{G}_1, \hat{G}_2, \hat{G}_3, \hat{G}_4$ are induced subgraphs of G such that $\sum_{i=1}^4 |V(G_i)| = 2 \cdot |V(G)|$.

Let $G_1, \ldots, G_L$ denote the basic perfect graphs corresponding to the leaves of the decomposition tree of G. Then,

$$\operatorname{xc}(\mathrm{STAB}(G)) \le 4^d \sum_l \operatorname{xc}(\mathrm{STAB}(G_l)) \le 2 \cdot 4^d \sum_l |V(G_l)|^2 \le 2 \cdot 4^d (\sum_l |V(G_l)|)^2.$$

Next we claim that $\sum_l |V(G_l)| \le 2^d \cdot |V(G)|$. Indeed if the graphs G'_1, G'_2, G'_3, G'_4 are the children of the graph G' in the decomposition tree and u is a vertex of the graph G', then u is in the vertex sets of precisely two of the graphs G'_1, G'_2, G'_3, G'_4. Combining we get:

$$\operatorname{xc}(\mathrm{STAB}(G)) \le 2 \cdot 4^{2d} |V(G)|^2 = 2^{4d+1} \cdot |V(G)|^2.$$

□

The following theorem is a direct consequence of Theorem 5.5.4 and Theorem 5.4.3.

Theorem 5.5.5. *Assume G is a perfect graph which can be decomposed into basic perfect graphs, using 2-join or skew partition decompositions. Let d denote the depth of a decomposition tree representing a decomposition of G into basic perfect graphs by means of 2-join or skew partition decompositions. Then,*

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq 2^{4d+1} \cdot |V(G)|^2.$$

Corollary 5.5.6. *Let G be a perfect graph. If G is decomposed into basic perfect graphs, using 2-join or skew partition decompositions; and the depth d of a decomposition tree is logarithmic in $|V(G)|$, namely $d \leq c \cdot \log |V(G)|$ for some constant $c > 0$, then the extension complexity of* $\mathrm{STAB}(G)$ *satisfies*

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq c' \cdot |V(G)|^{4c+2},$$

for some constant $c' > 0$.

5.6 Meyniel Graphs

As it is still an open question whether there exists a polynomial-sized extended formulation for the stable set polytope of perfect graphs, it is interesting to look at some subclasses of perfect graphs. In this section we consider Meyniel graphs, which form a subclass of perfect graphs. Meyniel [20] introduced Meyniel graphs and showed that every Meyniel graph is perfect. We provide a cubic upper bound for the extension complexity of the stable set polytope of any decomposable Meyniel graph.

Definition 5.6.1. [20] *A graph is* Meyniel *if every odd cycle of length at least 5 contains at least 2 chords.*

Theorem 5.6.2. [20] *Every Meyniel graph is perfect.*

In Section 4, we have already defined the amalgam composition of two graphs G_1 and G_2. Conversely, we have also defined the amalgam decomposition of G_1, if such a decomposition exists.

In what follows, we use the same symbol as in the definition of the amalgam operation (see Definition 4.3.1). In addition, R_k denotes $N(v_k)\backslash C_k$ and N_k denotes

$V_k \backslash (C_k \cup \{v_k\})$ $(k = 1, 2)$. Furthermore, V_{0k} denotes $(V_k \backslash C_k) \cup C$ $(k = 1, 2)$. So V_{0k} contains the vertex v_k and the vertices of G_0 generated from G_k.

Given a clique K in G_0, we say K is mixed if $K \cap N_1 \neq \emptyset$ and $K \cap N_2 \neq \emptyset$, otherwise K is not mixed. Notice that if K is mixed, then $K = C \cup A_1 \cup A_2$ for some non-empty subsets $A_1 \subseteq R_1$, $A_2 \subseteq R_2$; if K is not mixed, then $K \subseteq N_1 \cup C$ or $K \subseteq N_2 \cup C$.

Definition 5.6.3. [4] *An amalgam decomposition of G_0 into G_1 and G_2 is said to be* proper *if $|V(G_k)| < |V(G)|$ $(k = 1, 2)$. Notice that $|V(G_k) \backslash C_k| \geq 3$ $(k = 1, 2)$ in this case.*

Burlet and Fonlupt [4] introduced the following notion of basic Meyniel graph and showed that certain Meyniel graphs can be properly decomposed into basic Meyniel graphs.

Definition 5.6.4. [4] *A* basic Meyniel graph *G is a connected graph, where $V(G)$ can be partitioned into a triple $(A; K; S)$ such that*

(i) $G(A)$ is a 2-connected bipartite graph containing at least one cycle;

(ii) $G(K)$ is a clique and $(u, v) \in E$ for every $u \in A$ and $v \in K$;

(iii) $G(S)$ is a stable set and $|N(v) \cap A| \leq 1$ for every $v \in S$.

Definition 5.6.5. *Let G be a Meyniel graph. If one can decompose G into basic Meyniel graphs using a number of proper amalgam decompositions, then G is said to be a* decomposable Meyniel graph.

Notice that a basic Meyniel graph is a decomposable Meyniel graph. Furthermore if G_1 and G_2 are a proper amalgam decomposition of a decomposable Meyniel graph, then G_1 and G_2 are also decomposable Meyniel graphs. Given a decomposable Meyniel graph G, there can be different ways to decompose it into basic Meyniel graphs. Thus we introduce so-called decomposition tree of G as follows.

Definition 5.6.6. *A* decomposition tree *of G_0 is a binary tree whose nodes are associated to some graphs such that:*

(i) the root node is associated to G_0;

(ii) each leaf node is associated to a basic Meyniel graph;

(iii) if node k is associated to the graph G_k $(k = 0, 1, 2)$ and nodes $1, 2$ are the children of node 0, then G_1 and G_2 are a proper amalgam decomposition of the graph G_0.

Clearly if we have a procedure to decompose G into basic Meyniel graphs using some proper amalgam decompositions, then it induces a decomposition tree of G. The converse is also true. Notice that the number of internal nodes in a decomposition tree of G is precisely the number of proper amalgam decompositions used in the decomposition process.

In the rest of this section, we prove that the extension complexity of the stable set polytope of any decomposable Meyniel graph is polynomial in the number of its vertices. The proof relies on the fact that there exists a decomposition tree of G whose size is not too large.

Lemma 5.6.7. *Let G be a basic Meyniel graph. It holds that*

(i) $\omega(G) = |K| + 2$;

(ii) $\mathrm{xc}(\mathrm{STAB}(G)) \leq |V(G)| + |S| + |E(G(A))|$.

Proof. Let A, K, S be as defined in Definition 5.6.4. Let C be a maximal clique of G. Consider the following two cases depending on the cardinality of $C \cap S$:

(a) $C \cap S \neq \emptyset$: Let $\{x\} = C \cap S$. Then $C = \{x\} \cup N(x)$. As $|N(x) \cap A| \leq 1$ and $|N(x) \cap K| \leq |K|$, it holds that $|C| \leq |K| + 2$. Thus there are precisely $|S|$ maximal cliques meeting the set S.

(b) $C \cap S = \emptyset$: We must have $C = K \cup \{a, b\}$ for some $\{a, b\} \in E(G(A))$ and $|C| = |K| + 2$. Thus there are precisely $|E(G(A))|$ maximal cliques disjoint from the set S.

The first claim (i) follows directly from the analysis above. Recall that S_G is the slack matrix of $\mathrm{STAB}(G)$. Since the number of maximal cliques in G is precisely $|S| + |E(G(A))|$, from Lemma 5.2.1, it follows that $\mathrm{rank}_+(S_G) \leq |V(G)| + |S| + |E(G(A))|$. Applying Theorem 2.4.4, we have that $\mathrm{xc}(\mathrm{STAB}(G)) = \mathrm{rank}_+(S_G)$ and the second claim follows. □

Lemma 5.6.8. *Let G be a basic Meyniel graph. It holds that*

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq |V(G)| + |E(G)| - \binom{|K|}{2} - |A| \cdot |K|.$$

Proof. Let us show that $|S| + |E(G(A))| \leq |E(G)| - \binom{|K|}{2} - |A| \cdot |K|$. Consider the set of edges in G: there are $\binom{|K|}{2}$ edges in $G(K)$, $|E(G(A))|$ edges in $G(A)$, $|A| \cdot |K|$ crossing edges between A and K, and at least $|S|$ edges going out of S as G is connected and S is a stable set. Thus $|E(G)| \geq \binom{|K|}{2} + |E(G(A))| + |A| \cdot |K| + |S|$. The claim then follows from (ii) in Lemma 5.6.7. □

Let us introduce a graph parameter which is essential to provide a good upper bound on the extension complexity of STAB(G), where G is a decomposable Meyniel graph. Define the following graph parameter based on the number of vertices and edges in G, and the clique number of G,

$$f(G) := |V(G)| + |E(G)| - \binom{\omega(G) - 2}{2} - 4 \cdot (\omega(G) - 2).$$

We will see that if G is a decomposable Meyniel graph, then $\omega(G) \geq 2$ (See Lemma 5.6.16) and thus the function f is well-defined. Let us show some properties of the graph parameter $f(G)$ first.

Lemma 5.6.9. *Let G be a basic Meyniel graph. It holds that* $\mathrm{xc}(\mathrm{STAB}(G)) \leq f(G)$.

Proof. As G is a basic Meyniel graph, we have $\omega(G) = |K| + 2$ and $|A| \geq 4$ as $G(A)$ is a 2-connected bipartite graph containing at least one cycle. The claim then follows from Lemma 5.6.8. □

Lemma 5.6.10. *Let G_1 and G_2 be an amalgam decomposition of the graph G_0. It holds that*

$$f(G_1) + f(G_2) \leq f(G) + 6.$$

Proof. By the definition of amalgam decompositions, there exist two triples (G_k, v_k, C_k) $(k = 1, 2)$ satisfying the conditions (i),(ii),(iii) in the Definition 4.3.1, such that $G = (G_1, v_1, C_1) \, \Phi \, (G_2, v_2, C_2)$.

Let ω_k denote $\omega(G_k)$ $(k = 0, 1, 2)$. The following relations hold:

$$|V(G_1)| + |V(G_2)| - |V(G)| = |K| + 2,$$

$$|E(G_1)| + |E(G_2)| - |E(G)| = |R_1| + |R_2| - |R_1| \cdot |R_2| + \binom{|K|}{2} + 2|K|.$$

Using the relations above, we obtain that

$$f(G_1) + f(G_2) - f(G) = 3 \cdot |K| + |R_1| + |R_2| - |R_1| \cdot |R_2| - 4 \cdot (\omega_1 + \omega_2 - \omega) + 10 + \binom{|K|}{2} + \binom{\omega - 2}{2} - \binom{\omega_1 - 2}{2} - \binom{\omega_2 - 2}{2}.$$

We can check that $g(|R_1|, |R_2|) := |R_1| + |R_2| - |R_1| \cdot |R_2| \leq 1$. The inequality is true if $|R_1| = |R_2| = 0$. Now consider the case when $|R_1| \geq 1$ and $|R_2| \geq 1$. We can rewrite the function to get $g(|R_1|, |R_2|) = |R_1| \cdot (1 - |R_2|) + |R_2|$, and thus $g(|R_1|, |R_2|)$ is a linear function in $|R_1|$ if $|R_2|$ is a constant. As $1 - |R_2| \leq 0$, the slope of this linear function is non-positive and thus it is a decreasing function in $|R_1|$. By symmetry, the same result holds if we fix $|R_1|$. Thus $g(|R_1|, |R_2|) \leq g(1, 1) \leq 1$.

Let K be a maximum clique of G_0. We distinguish the following two cases depending on whether K is mixed:

(i) K is not mixed, i.e., say $K \subseteq V_{01}$. Then there exists a clique of size $|K|$ in G_1 and thus $\omega_1 \geq \omega$. It also holds that $\omega_1 \leq \omega + 1$. Indeed, if this is not the case, then there exist a clique K' in G_1 whose cardinality is $\omega + 2$ and thus $K' \backslash \{v_1\}$ is clique of size at least $\omega + 1$ in G_1. This implies that there exist a clique in G_0 whose cardinality is at least $\omega + 1$, and we have a contradiction. Hence, it holds that $\omega_1 = \omega$ or $\omega_1 = \omega + 1$. We also note that $|C| \leq \omega_2 - 1$ as $C \cup \{v_2\}$ is a clique in G_2.

If $\omega_1 = \omega$, then

$$\begin{aligned} f(G_1) + f(G_2) - f(G) &\leq 3 \cdot (\omega_2 - 1) - 4 \cdot \omega_2 + 11 + \binom{\omega_2 - 1}{2} - \binom{\omega_2 - 2}{2} \\ &= 6. \end{aligned}$$

If $\omega_1 = \omega + 1$, then

$$\begin{aligned} f(G_1) + f(G_2) - f(G) &\leq 3 \cdot (\omega_2 - 1) - 4 \cdot (\omega_2 + 1) + 11 \\ &\quad + \binom{\omega_2 - 1}{2} + \binom{\omega - 2}{2} - \binom{\omega - 1}{2} - \binom{\omega_2 - 2}{2} \\ &= 4 - \omega \\ &\leq 3. \end{aligned}$$

The last inequality follows from $\omega \geq 1$.

(ii) K is mixed. In this case, $K = C \cup A_1 \cup A_2$ for some non-empty subsets $A_1 \subseteq R_1$ and $A_2 \subseteq R_2$. Thus $|C| = \omega - |A_1| - |A_2|$. Let $B_k := R_k \backslash A_k$ for $k = 1, 2$. So $|R_k| = |A_k| + |B_k|$ for $k = 1, 2$. As $(K \backslash A_2) \cup \{v_1\}$ is a clique in G_1, it holds that $\omega_1 \geq \omega - A_2 + 1$. Similarly $\omega_2 \geq \omega - A_1 + 1$. Now we have

$$
\begin{aligned}
& f(G_1) + f(G_2) - f(G) \\
= & 3 \cdot (\omega - |A_1| - |A_2|) + |R_1| + |R_2| - |R_1| \cdot |R_2| - 4 \cdot (\omega - |A_1| + 1 + \omega - |A_2| + 1 - \omega) + 10 \\
& + \binom{\omega - |A_1| - |A_2|}{2} + \binom{\omega - 2}{2} - \binom{\omega - |A_1| - 1}{2} - \binom{\omega - A_2 - 1}{2} \\
= & -1 \cdot (\omega - |A_1| - |A_2|) + |R_1| + |R_2| - |R_1| \cdot |R_2| + 2 \\
& + \frac{1}{2}(\omega^2 + |A_1|^2 + |A_2|^2 - 2 \cdot \omega \cdot |A_1| - 2 \cdot \omega \cdot |A_2| + 2 \cdot |A_1| \cdot |A_2| - \omega + |A_1| + |A_2|) \\
& + \frac{1}{2}(\omega^2 - 5 \cdot \omega + 6) \\
& - \frac{1}{2}(\omega^2 - 2 \cdot \omega \cdot |A_1| + |A_1|^2 - 3 \cdot \omega + 3 \cdot |A_1| + 2) \\
& - \frac{1}{2}(\omega^2 - 2 \cdot \omega \cdot |A_2| + |A_2|^2 - 3 \cdot \omega + 3 \cdot |A_2| + 2) \\
= & -\omega + |A_1| \cdot |A_2| + |R_1| + |R_2| - |R_1| \cdot |R_2| + 3 \\
\leq & 3.
\end{aligned}
$$

The last inequality uses the fact that $-\omega + |A_1| \cdot |A_2| + |R_1| + |R_2| - |R_1| \cdot |R_2| \leq 0$. As $\omega \geq |A_1| + |A_2|$, it suffices to show the inequality

$$|R_1| + |R_2| - |R_1| \cdot |R_2| \leq |A_1| + |A_2| - |A_1| \cdot |A_2|.$$

To this end, we show the optimal value of the following optimization problem is at least $|R_1| + |R_2| - |R_1| \cdot |R_2|$.

$$\min_{|A_1|,|A_2|} \{|A_1| + |A_2| - |A_1| \cdot |A_2| : 1 \leq |A_k| \leq |R_k| \text{ and } A_k \in \mathbb{Z}(k = 1, 2)\}.$$

Since there are only finitely many possible choices for the value of the pair $(|A_1|, |A_2|)$, the minimum must exists. In fact, the minimum is attained at $(|A_1|, |A_2|) = (|R_1|, |R_2|)$. Assume this not the case, say $|A_1| < |R_1|$, then the pair $(|A_1| + 1, |A_2|)$ yields strictly smaller objective value.

In both cases, $f(G_1) + f(G_2) - f(G)$ is upper bounded by 6. This finishes the proof. □

As mentioned earlier, we need to provide an upper bound on the number of proper amalgam decompositions needed to decompose G into basic Meyniel graphs. Following Burlet and Fonlupt [4], we introduce the following graph parameter.

Definition 5.6.11. *Let G be a decomposable Meyniel graph. Define $N(G)$ to be the largest number of internal nodes among all the decomposition trees of G.*

By definition, if G is a decomposable Meyniel graph, then any proper amalgam decompositions of G into basic Meyniel graphs involve at most $N(G)$ proper amalgam decompositions.

Lemma 5.6.12. *Let G be a decomposable Meyniel graph. Let T be a decomposition tree of G with $N(G)$ internal nodes. Let $G_1, \ldots, G_L$ be the basic Meyniel graphs associated to the leaves of T. It holds that*

$$\sum_{l=1}^{L} f(G_l) \leq f(G) + 6 \cdot N(G).$$

Proof. We prove the statement by induction on $N(G)$. If $N(G) = 0$, then G is a basic Meyniel graph. If $N(G) = 1$, then G is amalgam decomposed into two basic Meyniel graphs G_1 and G_2, and the result follows from Lemma 5.6.10. (This case is in fact not needed, we include it for clarity.)

Assume now $N(G) \geq 2$. Let G_1 and G_2 be children of G in T. So G is amalgam decomposed into G_1 and G_2 in our decomposition process. We have

$$N(G) = N(G_1) + N(G_2) + 1. \tag{5.3}$$

Then each basic Meyniel graph G_l corresponds to a leaf in the subtree rooted at G_1 or to a leaf in the subtree rooted at G_2. For $k = 1, 2$, let L_k denote the set of indices l for which G_l corresponds to a leaf in the subtree rooted at G_k. Then $L = |L_1| + |L_2|$.

By the induction assumption, it holds that $\sum_{l \in L_k} f(G_l) \leq f(G_k) + 6 \cdot N(G_k)$ $(k = 1, 2)$. Summing these two inequalities gives

$$\begin{aligned}\sum_{l=1}^{L} f(G_l) &\leq f(G_1) + f(G_2) + 6 \cdot (N(G_1) + N(G_2)) \\ &\leq f(G) + 6 \cdot (N(G_1) + N(G_2) + 1) \\ &= f(G) + 6 \cdot N(G).\end{aligned}$$

Here, the last two lines use Lemma 5.6.10 and Equation (5.3), respectively. This concludes the proof. □

Theorem 5.6.13. *Let G be a decomposable Meyniel graph. It holds that*

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq f(G) + 6 \cdot N(G).$$

Proof. Let $G_1, \ldots, G_L$ be the basic Meyniel graphs that are associated to the leaves in the decomposition tree of G. By iteratively applying Theorem 4.3.3, we have that $\text{xc}(G) \le \sum_{l=1}^{L} \text{xc}(G_l)$. Combining with Lemma 5.6.9, we get that $\text{xc}(G) \le \sum_{l=1}^{L} f(G_l)$. Now we can use Lemma 5.6.12 to derive that $\text{xc}(G) \le f(G) + 6 \cdot N(G)$. □

Now we already obtained an upper bound on $\text{xc}(\text{STAB}(G))$ in terms of $f(G)$ and $N(G)$. The term $f(G)$ is clearly upper bounded by $|V(G)|^2$. It remains to provide an upper bound for $N(G)$. To this end, we define the graph parameter $r(G) := V(G) - \omega(G)$. In what follows, we use the notation $r_k := r(G_k)$ and $\omega_k := \omega(G_k)$ for the graph G_k $(k = 0, 1, 2)$. Now the next lemma shows some results of a proper amalgam decomposition and the graph parameter $r(G)$.

Lemma 5.6.14. [4] *If the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are a proper amalgam decomposition of the graph $G_0 = (V_0, E_0)$, then*

(i) $|V_k| < |V_0|$, $\omega_k \le \omega_0$, $r_k \le r_0$ $(k = 1, 2)$;

(ii) $r_1 + r_2 \le r_0 + 1$.

Proof. We follow the same notation as in Definition 4.3.1. Recall that R_k denotes $N(v_k) \backslash C_k$ and N_k denotes $V_k \backslash (C_k \cup v_k)$ $(k = 1, 2)$. Furthermore, V_{0k} denotes $(V_k \backslash C_k) \cup C$ for $k = 1, 2$.
(i): The inequalities $|V_k| < |V_0|$ $(k = 1, 2)$ follow directly from the definition of proper amalgam decomposition. As G_1 and G_2 are a proper amalgam decomposition of G_0, it holds that $|N_k| \ge 3$ $(k = 1, 2)$ and thus $\omega_k \le \omega_0$ $(k = 1, 2)$.

Now we prove that $r_k \le r_0$ $(k = 1, 2)$. Let K be a maximum clique of G_0. Consider the following cases depending on whether K is mixed.

(a) K is mixed: Then $K = C \cup A_1 \cup A_2$ for some non-empty subsets $A_1 \subseteq R_1$ and $A_2 \subseteq R_2$, and thus $C_1 \cup A_1 \cup \{v_1\}$ is a clique in G_1. This implies that $\omega_1 \ge |C_1| + |A_1| + 1 = |C| + |A_1| + 1$. We have

$$r_0 = |V_0| - \omega_0 = |C| + |N_1| + |N_2| - (|C| + |A_1| + |A_2|),$$
$$r_1 = |V_1| - \omega_1 \le |C| + |N_1| + 1 - (|C| + |A_1| + 1).$$

Thus $r_0 - r_1 \ge |N_2| - |A_2| \ge 0$, as $A_2 \subseteq R_2 \subseteq N_2$.

(b) K is not mixed and $K \cap N_1 = \emptyset$: Then $K \subseteq C \cup N_2$. As $C_1 \cup \{v_1\}$ is a clique in G_0, $\omega_1 \geq |C_1| + 1 = |C| + 1$. We have

$$r_0 = |V_0| - \omega_0 = |C| + |N_1| + |N_2| - |K \cap (C \cup N_2)),$$
$$r_1 = |V_1| - \omega_1 \leq |C| + |N_1| + 1 - (|C| + 1).$$

Thus $r_0 - r_1 \geq |C| + |N_2| - |K \cap (C \cup N_2)| \geq 0$.

(c) K is not mixed and $K \cap N_1 \neq \emptyset$: Then G_1 has a clique of size $|K|$ and thus $\omega_0 \leq \omega_1$. As $|V_1| < |V_0|$, We have $r_0 - r_1 = (|V_0| - |V_1|) - (\omega_0 - \omega_1) \geq 1$.

This proves that $r_1 \leq r_0$. Similarly it holds that $r_2 \leq r_0$.

(ii): Let K be a maximum clique of G_0. Consider the following cases depending on whether K is mixed.

(a) K is mixed: In this case, $K = C \cup A_1 \cup A_2$ for some non-empty subsets $A_1 \subseteq R_1$ and $A_2 \subseteq R_2$. Thus $\omega_k \geq |A_k| + |C| + 1$ as $A_k \cup C_k \cup \{v_k\}$ is a clique in G_k, and $|C_k| = |C|$ $(k = 1, 2)$. Thus we have

$$r_i = |V_i| - \omega_i \leq |V_i| - |A_k| - |C| - 1, \quad (k = 1, 2).$$

This gives

$$\begin{aligned} r_1 + r_2 &\leq (|V_1| - |A_1| - |C| - 1) + (|V_2| - |A_2| - |C| - 1) \\ &= (|V_1| + |V_2| - |C| - 2) - (|A_1| + |A_2| + |C|) \\ &= |V_0| - \omega_0 \\ &= r_0. \end{aligned}$$

(b) K is not mixed. Then $K \subseteq N_1 \cup C$ or $K \subseteq N_2 \cup C$, say $K \subseteq N_1 \cup C$. Hence there is a clique of size $|K|$ in G_1 and thus $\omega_1 \geq \omega_0$. As $C_2 \cup \{v_2\}$ is a clique in G_2 and $|C_2| = |C|$, it holds that $\omega_2 \geq |C| + 1$. Finally, we also note that $|V_k| = |N_k| + |C| + 1$ $(k = 1, 2)$ and $|V_0| = N_1| + |N_2| + |C|$. Thus we have

$$r_1 = |V_0| - \omega_1 \leq |N_1| + |C| + 1 - \omega_0,$$
$$r_2 \leq |V_2| - |C| - 1 = |N_2|.$$

This gives

$$\begin{aligned} r_1 + r_2 &\leq (|N_1| + |C| + 1 - \omega_0) + |N_2| \\ &= (N_1| + |N_2| + |C|) - \omega_0 + 1 \\ &= |V_0| - \omega_0 + 1 \\ &= r_0 + 1. \end{aligned}$$

□

Lemma 5.6.15. [4] *If G is a basic Meyniel graph, then $\omega(G) \geq 2$ and $r(G) \geq 2$.*

Proof. By definition, the vertex set $V(G)$ can be partitioned into a triple $(A; K; S)$, where $G(A)$ is a 2-connected bipartite graph containing at least one cycle. Thus A is non-empty and it must contain at least 4 vertices. This implies that $\omega(G) \geq 2$ and $|A| \geq 4$.

From Lemma 5.6.7, we know that $\omega(G) = |K| + 2$ and thus

$$\begin{aligned} r(G) &= |V(G)| - \omega(G) \\ &= (|A| + |K| + |S|) - (|K| + 2) \\ &= |A| + |S| - 2 \\ &\geq 2. \end{aligned}$$

The last inequality uses the fact that $|A| \geq 4$. □

Lemma 5.6.16. [4] *If G is a decomposable Meyniel graph, then $\omega(G) \geq 2$ and $r(G) \geq 2$.*

Proof. By definition, G can be properly amalgam decomposed into basic Meyniel graphs $G_1, \ldots, G_L$. Applying (i) or (ii) in Lemma 5.6.14, and Lemma 5.6.15, we obtain that $\omega(G) \geq 2$ and $r(G) \geq 2$. □

It remains to provide an upper bound for $N(G)$. For this we follow Burlet and Fonlupt [4] and show that the number of nodes in a decomposition tree of G is small. Indeed, $N(G)$ is equal to the number of internal nodes of the decomposition tree which is no greater than the number of nodes in the tree.

Definition 5.6.17. [4] *Let G be a decomposable Meyniel graph. If T is a decomposition tree of G, then $\mathcal{G}^T$ is the family of graphs associated to the nodes of T. We define $k(G)$ to be the cardinality of the largest family $\mathcal{G}^T$ among all decomposition trees T of G.*

Theorem 5.6.18. [4] *Let G_0 be a decomposable Meyniel graph. It holds that*

$$k(G_0) \leq |V_0| \cdot (r_0^2 + 1).$$

Proof. We prove this by induction on the number of vertices $|V_0|$. The statement is clearly true if $|V_0| = 1$ or $|V_0| = 2$. Assume $|V_0| \geq 3$ and the statement is true for any graph whose number of vertices is less than $|V_0|$.

Consider a proper amalgam decomposition of G_0 into basic Meyniel graphs. Let $\mathcal{G}_0^T$ be the family of graphs induced by the decomposition. If G_0 is not a basic Meyniel graph, we denote the children of G_0 in the decomposition by G_1 and G_2. In this case, we have that $k(G_0) = k(G_1) + k(G_2) + 1$ as $\mathcal{G}_0^T = \{G_0\} \cup \mathcal{G}_1^T \cup \mathcal{G}_2^T$.

If G_0 is a basic Meyniel graph, then $k(G_0) = 1$ and the theorem follows.

If G_0 is not a basic Meyniel graph and precisely one of its child, say G_2, is a basic Meyniel graph, then $k(G_2) = 1$, and $k(G_1) \leq |V_1| \cdot (r_1^2 + 1)$ by induction hypothesis. Since G_1 is in the family $\mathcal{G}_0^T$, it is a decomposable Meyniel graph. From (ii) in Lemma 5.6.14, it holds that $r_1 \geq 1$. We have that,

$$k(G_0) = k(G_1) + k(G_2) + 1 \leq 2 + |V_1| \cdot (r_1^2 + 1) \leq (|V_1| + 1) \cdot (r_1^2 + 1).$$

Since $|V_1| < |V_0|$ and $r_1 \leq r_0$,

$$k(G_0) \leq |V_0| \cdot (r_0^2 + 1).$$

If G_0 is not a basic Meyniel graph and both G_1 and G_2 are not basic Meyniel graphs. Applying the induction hypothesis, we have that

$$\begin{aligned} k(G_0) &= k(G_1) + k(G_2) + 1 \\ &\leq |V_1| \cdot (r_1^2 + 1) + |V_2| \cdot (r_2^2 + 1) + 1 \\ &\leq |V_0| \cdot (r_1^2 + 1) + |V_0| \cdot (r_2^2 + 1) \\ &\leq |V_0| \cdot (r_1^2 + r_2^2 + 2). \end{aligned}$$

Here we have used the relation $|V_k| < |V_0|$ $(k = 1, 2)$ and $|V_0| \geq 3$.

From (ii) in Lemma 5.6.14, we know that $2 \leq r_1 \leq r_0$, $2 \leq r_2 \leq r_0$, and thus

$$r_1 r_2 - r_1 - r_2 \geq 0.$$

From Lemma 5.6.16, we know that $r_1 + r_2 - 1 \leq r_0$, as both sides are non-negative, we can take squares on both sides and rearrange the terms to get

$$r_1^2 + r_2^2 + 1 + 2 \cdot (r_1 r_2 - r_1 - r_2) \leq r_0^2.$$

Therefore we obtain that $r_1^2 + r_2^2 + 2 \leq r_0^2 + 1$ as $r_1 r_2 - r_1 - r_2 \geq 0$. Thus $k(G_0) \leq |V_0| \cdot r_0^2 + 1$ and this proves the claim. □

Theorem 5.6.19. [4] *Let G be a decomposable Meyniel graph. It holds that*

$$N(G) \leq |V(G)|^3.$$

Proof. Recall that $r(G) = |V(G)| - |\omega(G)|$, and $\omega(G) \geq 2$, thus

$$\begin{aligned} r(G)^2 + 1 &= (|V(G)| - \omega(G))^2 + 1 \\ &\leq (|V(G)| - 1)^2 + 1 \\ &= |V(G)|^2 - 2 \cdot (|V(G)| - 1) \\ &\leq |V(G)|^2. \end{aligned}$$

As $N(G) \leq k(G)$, the claim follows from Theorem 5.6.18 that

$$\begin{aligned} N(G) &\leq k(G) \\ &\leq |V(G)| \cdot (r(G)^2 + 1) \\ &\leq |V(G)|^3. \end{aligned}$$

□

The above theorem provides a cubic upper bound on $N(G)$, i.e., the largest number of proper amalgam decompositions needed to decompose G into basic Meyniel graphs. This result allows us to derive the following cubic upper bound on the extension complexity of any decomposable Meyniel graph.

Theorem 5.6.20. *Let G be a decomposable Meyniel graph. It holds that*

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq 7 \cdot |V(G)|^3.$$

Proof. It is not hard to see that $f(G) \leq |V(G)|^2$. From Theorem 5.6.19, we know that $N(G) \leq |V(G)|^3$. Applying Theorem 5.6.13, we obtain that $\mathrm{xc}(\mathrm{STAB}(G)) \leq f(G) + 6 \cdot N(G) \leq 7 \cdot |V(G)|^3$. □

A *chordal graph* is one in which every cycle of length four or greater has a chord. Chordal graphs are Meyniel graphs. The extension complexity of the stable set polytopes of chordal graphs are polynomial, as since a chordal graph on n vertices has at most n maximal cliques. We point out to the reader that chordal graphs are not captured by Theorem 5.6.20. For example, a clique of size 3 is not a basic Meyniel graph and it also does not have a proper amalgam decomposition. This is possibly the reason why the notion of basic Meyniel graph was later modified. In particular, chordal graphs have been added as a new class of basic graphs (see e.g. Chapter 10 in [11]). However we point out that it is not clear whether the result from Theorem 5.6.19 extends to this setting (since the inequality $r(G) \geq 2$ does not hold for general chordal graphs). It follows from more general results in the recent work [10] that the stable set polytope of general Meyniel graphs has polynomial extension complexity.

Chapter 6

Conclusion

In this thesis, we provided an upper bound for the extension complexity of the stable set polytope of G_0 in terms of $\text{xc}(\text{STAB}(G_1))$ and $\text{xc}(\text{STAB}(G_2))$, where the graph G_0 is composed from the graphs G_1 and G_2 via some graph operation. We considered the following graph operations: graph substitution, graph amalgamation and clique sum. We showed that $\text{xc}(\text{STAB}(G_0)) \leq \text{xc}(\text{STAB}(G_1)) + \text{xc}(\text{STAB}(G_2))$, for all these operations. Furthermore, this upper bound can be improved slightly for some special graphs, e.g., K_r or $\overline{K_2}$.

As an application of the link between communication complexity and extension complexity, we showed that the extension complexity of the stable set polytope of the disjoint union of p edges is precisely $3p$.

We also showed that the extension complexities of the stable set polytopes of all the basic perfect graphs are polynomial. Furthermore, if a perfect graph G is decomposed into basic perfect graphs via proper 2-join and balanced skew partitions, then we provide an upper bound on $\text{xc}(\text{STAB}(G))$ in terms of the number of vertices of G and the depth of the induced decomposition tree. Finally, we considered a subclass of perfect graphs, known as Meyniel graphs and we showed that the extension complexities of the stable set polytopes of every decomposable Meyniel graphs are polynomial.

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