Nongaussian linear filtering, identification of linear systems, and the symplectic group

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Consider stochastic linear dynamical systems, $dx = Axdt + Bdw, dy = Cxdt + dv, y(0) = 0, x(0)$ a given initial random variable independent of the standard independent Wiener noise processes w, v. The matrices A, B, C are supposed to be constant. In this paper I consider two problems. For the first one A, B and C are supposed known and the question is how to calculate the conditional probability density of x at time t given the observations $y(s), 0 \leqslant s \leqslant t$ in the case that $x(0)$ is not necessarily gaussian. (In the gaussian case the answer is given by the Kalman-Bucy filter). The second problem concerns identification, i.e. the A, B, C are unknown (but assumed constant so that $dA = 0, dB = 0, dC = 0$), and one wants to calculate the joint conditional probability density at time t of (x, A, B, C), again given the observations $y(s), 0 \leqslant s \leqslant t$. The methods used rely on Wei-Norman theory, the Duncan-Mortensen-Zakai equation and a "real form" of the Segal-Shale-Weil representation of the symplectic group $Sp_n(\mathbf{R})$.

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1. Introduction

Consider a general nonlinear filtering problem of the following type:

$$dx = f(x)dt + G(x)dw \quad , \quad x \in \mathbf{R}^n, \ w \in \mathbf{R}^m \tag{1.1}$$

$$dy = h(x)dt + dv \quad , \quad y \in \mathbf{R}^p, \ v \in \mathbf{R}^p \tag{1.2}$$

where f, G, h are vector and matrix valued functions of the appropriate dimensions, and the w, v are standard Wiener processes independent of each other and also independent of the initial random variable $x(0)$. One takes $y(0) = 0$.

The general non-linear filtering problem is this setting asks for (effective) ways to calculate and/or approximate the conditional density $\pi(x,t)$ of x given the observations $y(s), 0 \leqslant s \leqslant t$; i.e. $\pi(x,t)$ is the density of $\hat{x} = E[x(t)|y(s), 0 \leqslant s \leqslant t]$ the conditional expectation of the state $x(t)$.

One approach to this problem proceeds via the socalled DMZ equation which is an equation of a rather nice form for an unnormalized version $\rho(x,t)$ of $\pi(x,t)$. Here unnormalized means that $\rho(x,t) = r(t)\pi(x,t)$ for some function $r(t)$ of time alone. A capsule description of this approach is given in section 2 below. Using this approach was strongly advocated by Brockett and Mitter (cf. e.g. their contributions in [6]), and initially the approach had a number of nontrivial successes, both in terms of positive and negative results (cf. e.g. the surveys [9] and [4]). Subsequently, the approach

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became less popular; perhaps because a number of rather formidable mathematical problems arose, and because the number of systems to which the theory can be directly applied appears to be quite small. Cf [4] for a discussion of some aspects of these two points.

It is the purpose of this paper to apply this approach to two problems concerning linear systems, which do not fall within the compass of the usual Kalman-Bucy linear filtering theory. More precisely, consider a linear stochastic dynamical system

$$dx = Axdt + Bdw, \quad x \in \mathbf{R}^n, \; w \in \mathbf{R}^m \tag{1.3}$$

$$dy = Cxdt + dv, \quad y, \, v \in \mathbf{R}^p \tag{1.4}$$

where the A,B,C are matrices of the appropriate sizes. The first problem I want to consider is the filtering of (1.3)-(1.4) in the case that the initial condition $x(0)$ is a non-gaussian random variable. The second problem concerns the identification of (1.3)-(1.4); i.e. one assumes that the matrices A,B,C are constant but unknown and it is desired to calculate the conditional density $\pi(x,A,B,C,t)$ of the (enlarged) state (x,A,B,C) at time t. Technically this means that one adds to (1.3)-(1.4) the equations

$$dA = 0, \; dB = 0, \; dC = 0 \tag{1.5}$$

and one considers the filtering problem for the nonlinear system (1.3)-(1.5). Strictly speaking this problem is not well posed. Simply because A,B,C can not be uniquely identified on the basis of the observations alone. In the DMZ equation approach this shows up only at the very end in the form that $\rho(x,A,B,C,t)$ will be degenerate in the sense that $\rho(Sx,SAS^{-1},SB,CS^{-1},t)=\rho(x,A,B,C,t)$ for all constant invertible real matrices S. As a result the normalization factor $\int \rho(x,A,B,C,t)dxdAdBdC$ does not exist, and in fact $\pi(x,A,B,C,t)$ is also degenerate. One gets rid of this by passing to the quotient space (finite moduli space) $\{(x,A,B,C)\}/GL_n(\mathbf{R})$ for the action just given and/or by considering (local) canonical forms. The normalization factor can be calculated by integrating over this quotient space.

Besides the DMZ-equation, already mentioned, the tools used to tackle the two problems described above are Wei-Norman theory and something which could be called a real form of the Segal-Shale-Weil representation of the symplectic Lie group $Sp_n(\mathbf{R})$. These two topics are discussed in sections 3 and 4 below.

2. The DMZ approach to nonlinear filtering

Consider again the general nonlinear system (1.1)-(1.2). These stochastic differential equations are to be considered as Ito equations. Let $\pi(x,t)$ be the probability density of $E[x(t)|y(s), 0 \leq s \leq t]$, the conditional expectation of $x(t)$. (Given sufficiently nice f,G and h if can be shown that $\pi(x,t)$ exists.) Then the Duncan-Mortensen-Zakai result [1, 10, 12] is that there exists an unnormalized version $\rho(x,t)$ of $\pi(x,t)$, i.e. $\rho(x,t)=r(t)\pi(x,t)$, which satisfies an evolution equation

$$d\rho = \pounds\rho dt + \Sigma h_k \rho dy_k(t), \; \rho(x,0) = \psi(x) \tag{2.1}$$

where $\psi(x)$ is the distribution of the initial random variable $x(0)$ and where $\pounds$ is the second-order partial differential equation

$$\pounds\phi = \frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}(GG^T)_{ij}\phi - \sum_i \frac{\partial}{\partial x_i} f_i \phi - \frac{1}{2}\sum_k h_k^2 \phi. \tag{2.2}$$

Here $h_k, y_k(t), f_i$ are components of $h, y(t)$ and f respectively and $(GG^T)_{ij}$ is the (i,j)-entry of the product GG^T of the matrix G and its transpose.

Equation (2.1) is a Fisk-Stratonovič stochastic differential equation. The corresponding Ito differential equation is obtained by removing the $-\frac{1}{2}\Sigma h_k^2 \phi$ term from (2.2).

As it stands (2.1) is a stochastic partial differential equation. However the transformation

$$\tilde{\rho}(x,t) = \exp(\Sigma h_k(x) y_k(t))\rho(x,t) \tag{2.3}$$

turns it into the equation

$$d\tilde{\rho} = (\pounds\tilde{\rho} + \Sigma\pounds_i\tilde{\rho}y_i + \tfrac{1}{2}\Sigma\pounds_{i,j}\tilde{\rho}y_iy_j)dt \tag{2.4}$$

where $\pounds_i$ is the operator commutator $\pounds_i = [h_i, \pounds] = h_i\pounds - \pounds h_i$ and $\pounds_{ij} = [h_i, [h_j, \pounds]]$. Cf. [4] for more details. In (2.4) I have explicity indicated the dependence of the various quantities on x,t to stress that here $h(x)$ should simply be seen as a known function of x and not as the time function $h(x(t))$. Equation (2.4) does not involve the derivatives dy_i anymore; it makes sense for all possible paths $y(t)$, and can be regarded as a family of PDE parametrized by the possible observation paths $y(t)$. Thus there is a robust version of (2.1) and we can work with (2.1) as a parametrized family of PDE parametrized by the $y(t)$. Note that knowledge of $\tilde{\rho}(x,t)$ (and $y(t)$) immediately gives $\rho(x,t)$ and that the conditional expectation of any function $\phi(x(t))$ of the state at time t can be calculated by

$$E[\phi(x(t))|y(s),\ 0 \leq s \leq t] = (\textstyle\int \rho(x,t)dx)^{-1} \int \phi(x)\rho(x,t)\, dx \tag{2.5}$$

Possibly the simplest example of a filtering problem is provided by one-dimensional Wiener noise linearly observed:

$$dx = dw,\ x,\ w \in \mathbf{R} \tag{2.6}$$

$$dy = xdt + dv,\quad y,\ v \in \mathbf{R}. \tag{2.7}$$

In this case the corresponding DMZ equation is

$$d\rho = (\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2\ \rho dt + x\rho\ dy \tag{2.8}$$

an Euclidean Schrödinger equation for a forced harmonic oscillator.

3. Wei-Norman theory

Wei-Norman theory is concerned with solving partial differential equations of the form

$$\frac{\partial \rho}{\partial t} = u_1A_1\rho + \cdots + u_mA_m\rho \tag{3.1}$$

where the A_i, $i = 1,...,m$ are linear partial differential operators in the space variables $x_1,...,x_n$, and the u_i, $i = 1,...,m$ are given functions of time, in terms of solutions of the simpler equations

$$\frac{\partial \rho}{\partial t} = A_i\rho\ ,\quad i = 1,...,m \tag{3.2}$$

which we write as

$$\rho(x,t) = e^{A_it}\psi(x)\ ,\ \psi(x) = \rho(x,0) \tag{3.3}$$

Originally, the theory was developed for the finite dimensional case, i.e. for systems of ordinary differential equations

$$\dot{z} = u_1A_1z + \cdots + u_mA_mz \tag{3.4}$$

where $z \in \mathbf{R}^k$, and the A_i are $k \times k$ matrices. Both in the finite dimensional case (3.4) and the infinite dimensional case (3.1) it is well known that besides in the given directions $A_1\rho,...,A_m\rho$, the to be determined function or vector can also move (infinitesimally) in the directions given by the commutators $[A_i, A_j]\rho = (A_iA_j - A_jA_i)\rho$, and in the directions given by repeated commutators $[[A_i, A_j], A_k]$, $[[A_i, A_j], [A_k, A_l]]$, etc. etc.

Let $\mathrm{Lie}(A_1,..,A_m)$ be the Lie algebra of operators generated by the operators $A_1,...,A_m$. This is the smallest vector space L of operators containing $A_1,..,A_n$ and such that if $A,B \in L$ then also $[A,B] := AB - BA \in L$. In the finite dimensional case (3.4) L is always finite dimensional, a subvector space of $gl_k(\mathbf{R})$, the vectorspace (Lie algebra) of all $k \times k$ matrices. In the infinite dimensional case the Lie algebra generated by the operators $A_1,...,A_m$ in (3.1) can easily be infinite dimensional and it often is; also in the cases coming from filtering problems via the DMZ equation. Cf. [5] for a number of examples.

This is the essential difference between (3.1) and (3.4). Accordingly, here I shall assume that the Lie algebra $L = \mathrm{Lie}(A_1,...,A_m)$ generated by the operators $A_1,...,A_m$ in (3.1) is finite dimensional. For a discussion of various infinite dimensional versions of Wei-Norman theory cf. [4]. Hence, granting this finite dimensionality property, by setting, if necessary, some of the $u_i(t)$ equal to zero, and by combining other $u_j(t)$ in the case of linear dependence among the operators on the RHS of (3.1), without loss of generality, we can assume that we are dealing with an equation

$$\frac{\partial \rho}{\partial t} = u_1 A_1 \rho + \cdots + u_n A_n \rho \tag{3.5}$$

with the additional property that

$$[A_i, A_j] = \sum_k \gamma_{ij}^k A_k \quad ;i,j = 1,...,n \tag{3.7}$$

for suitable real constants γ_{ij}^k; $i,j,k = 1,...,n$.

The central idea of Wei-Norman theory is now to try for a solution of the form

$$\rho(t) = e^{g_1(t)A_1} e^{g_2(t)A_2} \cdots e^{g_n(t)A_n} \psi \tag{3.8}$$

where the g_i are still to be determined functions of time. The next step is to insert the Ansatz (3.8) into (3.5), to obtain

$$\begin{aligned} \dot\rho = \dot g_1 A\, e^{g_1A_1} \cdots e^{g_nA_n}\psi + e^{g_1A_1}\dot g_2 A_2 e^{g_2A_2} \cdots e^{g_nA_n}\psi + \cdots \\ + e^{g_1A} \cdots e^{g_{n-1}A_{n-1}} \dot g_n A_n e^{g_nA_n}\psi \end{aligned} \tag{3.9}$$

Now, for $i = 2,...,n$ insert a term

$$e^{-g_{i-1}A_{i-1}} \cdots e^{-g_1A_1} e^{g_1A_1} \cdots e^{g_{i-1}A_{i-1}}$$

just behind $\dot g_i A_i$ in the i-th term of (3.9). Then use the adjoint representation formula

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]]] + \cdots \tag{3.10}$$

and (3.7)) repeatedly, and use the linear independence of the $A_1,...,A_n$ to obtain a system of ordinary differential equations for the $g_1,...,g_n$ (with initial conditions $g_1(0)=0=g_2(0)=...=g_n(0)$).

These equations are always solvable for small time. However they may not be solvable for all time, meaning that finite escape time phenomena can occur.

Let's consider an example, viz. the example afforded by the DMZ equation (2.8). One calculates that

$$[\tfrac{1}{2}\frac{d^2}{dz^2} - \tfrac{1}{2}x^2, x] = \frac{d}{dx}\,, \; [\tfrac{1}{2}\frac{d^2}{dx^2} - \tfrac{1}{2}x^2, \frac{d}{dx}] = x$$

$$[\frac{d}{dz}, x] = 1, \;\; [A,1] = 0$$

where A is any linear combination of the four operators $\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2, x, \frac{d}{dx}, 1$. Applying the recipe sketched above to the equation

$$\dot\rho = (\tfrac{1}{2}\frac{d^2}{dx^2} - \tfrac{1}{2}x^2)\rho + x\rho u(t) + \frac{d}{dx}\rho 0 + 1\rho 0 \tag{3.11}$$

one finds the equations

$$\dot g_1 = 0\,, \; \cosh(g_1)\dot g_2 + \sinh(g_1)\dot g_3 = u(t), \tag{3.12}$$

$$\sinh(g_1)\dot{g}_2 + \cosh(g_1)\dot{g}_3 = 0,\ \dot{g}_4 = \dot{g}_3 g_2$$

which are solvable for all time.

This fact and the form of the resulting equations: straightforward quadratures and one set of linear equations $B(t)g=b(t)$, with $B(t),b(t)$ known and $B(t)$ invertible, is typical for the case that the Lie algebra $L=\oplus\mathbf{R}A_i$ spanned by the $A_1,\ldots,A_n$ is solvable. This means the following. Let $[L,L]$ be the subvectorspace of L spanned by all the operators of the form $[A,B]$,$A,B\in L$. It is easily seen that this is again a Lie algebra. Inductively let $L^{(n)}=[L,L^{(n-1)}]$ be the subvectorspace of L spanned by all operators of the form $[A,B]$, $A\in L,B\in L^{(n-1)}$, $L^{(0)}=L$. These are all sub Lie algebras of L.

The Lie algebra of L is called nilpotent if $L^{(n)}=0$ for n large enough. It is called solvable if $[L,L]$ is nilpotent. The phenomenon alluded to above, i.e. solvability of the Wei-Norman equations for all time, always happens in case L is solvable [11]. (And it is no accident that these algebras have been called solvable. Though this is not the result which gave them that name.)

Note that the DMZ equation (2.1) corresponding to a nonlinear filtering problem (1.1)-(1.2) is of the type (3.1) (with $u_h(t)=dy_k(t)$). Thus the Lie Algebra generated by the operators $\mathcal{L}$, $h_1(x),\ldots,h_p(x)$ occuring in (2.1) clearly has much to say about how difficult the filtering problem is. This Lie algebra is called the *estimation Lie algebra* of the system (1.1)-(1.2) and it can be used to prove a variety of positive and negative results about the filtering problem [4, 5, 9].

4. The Segal-Shale-Weil representation and a 'real form'

Let J be the standard symplectic matrix $J=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}$, where I_n the $n\times n$ unit matrix. Consider the vector space of $2n\times 2n$ real matrices defined by

$$sp_n(\mathbf{R}) = \{M\colon JM + M^TJ = 0\}. \tag{4.1}$$

Writing M as a 2×2 block matrix, $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$, the conditions on the $n\times n$ blocks A,B,C,D become

$$B^T = B,\ C^T = C,\ D = -A^T. \tag{4.2}$$

As we shall see shortly below this set of matrices occurs naturally for filtering problems coming from linear systems (1.1)-(1.2).

The corresponding Lie group to $Sp_n(\mathbf{R})$ is the group of invertible $2n\times2n$ matrices defined by

$$Sp_n(\mathbf{R}) = \{S\in\mathbf{R}^{2n\times2n}\colon S^TJS = J\} \tag{4.3}$$

(This is a *group* of matrices in that if $S_1, S_2\in Sp_n(\mathbf{R})$ then also $S_1S_2\in Sp_n(\mathbf{R})$ and $S_1^{-1}\in Sp_n(\mathbf{R})$ as is easily verified.)

There is a famous representation of $Sp_n(\mathbf{R})$ (or more precisely of its two-field covering group $\tilde{S}p_n(\mathbf{R})$) in the Hilbert space $L^2(\mathbf{R}^n)$ called the Segal-Shale-Weil representation or the oscillator representation; cf. [8]. Here the word 'representation' means that to each $S\in Sp_n(\mathbf{R})$ there is associated a unitary operator U_S such that $U_{S_1S_2} = U_{S_1}U_{S_2}$ for all $S_1, S_2\in Sp_n(\mathbf{R})$.

For the purposes of this paper a modification of it is of importance. It can be described as follows by explicit operators associated to certain specific kinds of elements of $Sp_n(\mathbf{R})$:

(i) Let P be a symmetric $n\times n$ matrix; then to the element

$$\begin{pmatrix}I & P\\ 0 & I\end{pmatrix}\in Sp_n(\mathbf{R})$$

there is associated the operator $f(x)\mapsto\exp(x^TPx)f(x)$

(ii) Let $A\in GL_n(\mathbf{R})$ be an invertible $n\times n$ matrix. Then to the element

$$\begin{bmatrix} A & 0 \\ 0 & (A^{-1})^T \end{bmatrix} \in Sp_n(\mathbf{R})$$

there is associated the operator

$$f(x) \mapsto |\det(A)|^{1/2} f(A^T x)$$

(iii) let Q be a symmetric $n \times n$ matrix. Then to the element

$$\begin{bmatrix} I & 0 \\ Q & I \end{bmatrix} \in Sp_n(\mathbf{R})$$

there is associated the operator

$$f(x) \mapsto \mathcal{F}^{-1}(\exp(x^T Q x)\mathcal{F}f(x))$$

where $\mathcal{F}$ denotes the Fourier transform.

(The operator corresponding to the element

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \in Sp_n(\mathbf{R})$$

is in fact the Fourier transform itself).

Except for one snag to be discussed below, this suffices to describe the operator which should be associated to any element $S \in Sp_n(\mathbf{R})$. Indeed let

$$S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \in Sp_n(\mathbf{R}) \tag{4.4}$$

then there is an $s>0, s \in \mathbf{R}$ such that $S_1 + sS_2$ is invertible and we have a factorisation

$$\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ (S_3+sS_4)(S_1+sS_2)^{-1} & I \end{bmatrix} \begin{bmatrix} I & S_2(S_1+sS_2)^T \\ 0 & I \end{bmatrix} \begin{bmatrix} S_1+sS_2 & 0 \\ 0 & (S_1^T+sS_2^T)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -s & I \end{bmatrix} \tag{4.5}$$

(It is easily verified that all four factors on the right are in fact in $Sp_n(\mathbf{R})$.

Now assign to the operator S the product of the four operators corresponding to the factors on the RHS of (4.5) according to the recipe (i)-(iii) given above. There is a conceivable second snag here in that it seems a priori possible that different factorisations could give different operators. This in fact does not happen precisely because the 'representation' described by (i)-(iii) is a 'real form' of the oscillator representation $Sp_n(\mathbf{R}) \to \mathrm{Aut}(L^2(\mathbf{R}^n))$. The relation between the oscillator representation and (i)-(iii) above is given by the substitution $x_k \mapsto \sqrt{i}x_k$ where $i = \sqrt{-1}$. (The possible sign ambiguity which could come from the fact that the oscillator representation is really a representation of the covering $\tilde{S}p_n(\mathbf{R})$ rather than $Sp_n(\mathbf{R})$ itself also seems not to happen; if would in any case be irrelevant for the applications dicussed below.)

It remains to discuss the first snag mentioned just above (5.4) and why the words 'representation' and 'real form' above have been placed in quotation marks. The trouble lies in part (iii) of the recipe. Taking a Fourier transform and than multiplying with a quadratic exponential may well take one out of the class of functions which are inverse Fourier transformable. Another way to see this is to observe that the operator described in (iii) assigns to a function ψ the value in $t=1$ of the solution of the evolution equation

$$\frac{\partial \rho}{\partial t} = ((\frac{\partial}{\partial x})^T Q \frac{\partial}{\partial x})\rho \, , \; \rho(x,0) = \psi(x) \tag{4.6}$$

and if Q is not nonnegative definite this involves anti-diffusion components for which the solution at $t=1$ may not exist. Additionally, - but this is really the same snag - applying recipe (i) to a function may well result in a function that is not Fourier transformable.

What we have in fact is not a representation of all of $Sp_n(\mathbf{R})$ but only a representation of a certain sub-semi-group cone in $Sp_n(\mathbf{R})$.
For the applications to be described below this means that we must be careful to take factorizations such that applying the various operators successively continues to make sense. The factorization (5.5) does not seem optimal in that respect and we shall for the special elements of $Sp_n(\mathbf{R})$ which come from filtering problems use a different one.

Incidentally, one says that two structures over $\mathbf{R}$ are real forms of one another if after tensoring with $\mathbf{C}$ (= extending scalars to $\mathbf{C}$) they become isomorphic (over $\mathbf{C}$). It is in this sense that the 'representation' described by the recipe (i)-(iii) is a 'real form' of the oscillator representation.

5. Propagation of non-gaussian initials

Now, finally, after all this preparation, consider a known linear dynamical system

$$dx = Axdt + Bdw,\ Cxdt + dv_i,\ x \in \mathbf{R}^n, w \in \mathbf{R}^m,\ y,v \in \mathbf{R}^p. \tag{5.1}$$

with a known, not necessarily Gaussian, initial random variable $x(0)$ with probability distribution $\psi(x)$.

The DMZ equations in this case is as follows

$$d\rho = \pounds\rho dt + \sum_{j=1}^{p}(Cx)_j dy_j(t) \tag{5.2}$$

where $(Cx)_j$ is the j-th component of the p-vector Cx. The operator £ in this case has the form

$$\pounds = \frac{1}{2}\sum_{i,j}(BB^T)_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i,j}A_{ji}x_j\frac{\partial}{\partial x_i} - \mathrm{Tr}(A) - \frac{1}{2}\sum_j (Cx)_j^2 \tag{5.3}$$

Taking brackets of the multiplication operators $(Cx)_j$ with £ yields a linear combination of the operators

$$x_1,\ldots,x_n;\ \frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n};\ 1. \tag{5.4}$$

This is a straightforward calculation to check. Moreover, the bracket (= commutator product) of £ with any of the operators in (5.4) again yields a linear combination of the operators listed in (5.4). It follows that for linear stochastic dynamical systems (5.1) the associated estimation Lie algebra (= the Lie algebra generated by £, $(Cx)_1,\ldots,(Cx)_p$) is always solvable of dimension $\leqslant 2n+2$.

As a mather of fact it is quite simple to prove that the system (5.1) is completely reachable and completely observable if and only if the dimension of the estimation Lie algebra is precisely $2n+2$ so that a basis of the algebra is formed by the $(2n+1)$ operators of (5.4) and £ itself.

In all cases Wei-Norman theory is applicable (working perhaps with a slightly larger Lie algebra than strictly necessarily makes no real difference).

Thus we can calculate effectively the solutions of the unnormalized density equation (5.2) provided we have good ways of calculating the expressions.

$$e^{t\pounds}\psi,\ e^{tx_i}\psi,\ e^{t\frac{\partial}{\partial x_i}}\psi,\ e^{t}\psi \tag{5.5}$$

for arbitrary initial data ψ. The last three expressions of (5.5) cause absolutely zero difficulties $(\exp(t\frac{\partial}{\partial x_i})\psi = \psi(x_1,\ldots,x_{i-1},x_i+t,x_{i+1},\ldots,x_n))$. Thus it remains to calculate the $e^{t\pounds}\psi$ where £ is an operator of the form (5.3). It is at this point that the business of the Segal-Shale-Weil representation of the previous section comes in. As a matter of fact the Segal-Shale-Weil representation itself, not the 'real form' described in section 4 above, is a representation of the Lie algebra spanned by the operators

$$i\,x_k x_j,\; x_k \frac{\partial}{\partial x_j} + \tfrac{1}{2}\delta_{k,j},\; i\frac{\partial^2}{\partial x_k \partial x_j},\; i = \sqrt{-1} \tag{5.6}$$

and apart form multiples of the identity (which hardly matter) and the occurence of $\sqrt{-1}$ these are the constituents of the operators £ in (5.3). It is to remove the factors $\sqrt{-1}$ that we have to go to a real form. Cf. [3] for more details on the Segal-Shale-Weil representation itself, and what it, and its real form, have to do with Kalman-Bucy filters.

It is convenient not to have to worry about multiples of the identity. To this end note that if $£' = £ + aI$ then $\exp(t£')\psi = \exp(ta)\exp(t£)\psi$, so that neglecting multiples of the identity indeed matters hardly.

The first observation is now that, modulo multiples of the identity operator, if £ and £′ are two operators of the form (5.3) then their commutator difference $[£,£'] = ££' - £'£$ is again of the same form. (To make this exact replace £ in (5.3) by $£ + \frac{1}{2}\mathrm{Tr}(A)$ and similarly for £′.) Thus these operators actually form a finite dimensional Lie algebra and this is, of course, the symplectic Lie algebra $sp_n(\mathbf{R})$. The correspondence is given by assigning to $£(=£(A,B,C))$ the $2n \times 2n$ matrix

$$£(A,B,C) \to \begin{bmatrix} -A^T & -C^TC \\ -BB^T & A \end{bmatrix} \tag{5.7}$$

(If you want to be finicky it is the operator $£(A,B,C) + \frac{1}{2}\mathrm{Tr}(A)$ which corresponds to the matrix on the right of (5.7).)

In terms of a basis on the left and right side the correspondence (i.e. the isomorphism of Lie algebras) is given as follows. Let E_{ij} be the $n \times n$ matrix with a 1 in spot (i,j) and zero everywhere else. Then

$$\frac{\partial^2}{\partial x_i \partial x_j} \mapsto \begin{bmatrix} 0 & 0 \\ -E_{ij}-E_{ji} & 0 \end{bmatrix} \tag{5.8}$$

$$x_i \frac{\partial}{\partial x_j} + \tfrac{1}{2}\delta_{i,j} \mapsto \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{bmatrix} \tag{5.9}$$

$$x_i x_j \mapsto \begin{bmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{bmatrix} \tag{5.10}$$

It is now straightforward to check that this does indeed define an isomorphism of Lie algebras from the Lie algebra of all operators $£(A,B,C) + \frac{1}{2}\mathrm{Tr}(A)$ where £ is as in (5.3) and the algebra $sp_n(\mathbf{R})$ described and discussed in section 4 above. For example one has

$$[\frac{\partial^2}{\partial x_1 \partial x_2}, x_2 x_3] = x_3 \frac{\partial}{\partial x_1} \tag{5.11}$$

which fits perfectly with

$$[\begin{bmatrix} 0 & 0 \\ -E_{12}-E_{21} & 0 \end{bmatrix}, \begin{bmatrix} 0 & E_{23}+E_{32} \\ 0 & 0 \end{bmatrix}] = \begin{bmatrix} E_{31} & 0 \\ 0 & -E_{13} \end{bmatrix} \tag{5.12}$$

It is precisely the correspondence (5.8) - (5.10) or, modulo multiples of the identity, (5.7), plus the fact that 'real form' described in section 4 of the SSW representation is precisely the way to remove the $\sqrt{-1}$ factors, plus, again, the fact that the SSW is really a representation, which makes it possible to use finite dimensional calculations to obtain expressions for

$$\exp(t(\pounds(A,B,C)+\tfrac{1}{2}\mathrm{Tr}(A))\psi \tag{5.13}$$

for arbitrary initial conditions.

Basically the recipe is as follows. Take $\pounds(A,B,C)+\frac{1}{2}\mathrm{Tr}(A)$. Let $M\epsilon\ sp_n(\mathbf{R})$ be its associated matrix as defined by the RHS of (5.7). Calculate $\exp(tM)=S(t)$. Write $S(t)$ as a product of matrices as in (i), (ii), (iii) in section 4. Apply successively the operators associated to the factors. The result, if defined, will be an expression for (5.13). One factorisation which can be used is that of (4.5) above. It does not, however, seem to be very optimal and it is difficult to show that everything is well defined.

It is better and more efficient to use a preliminary reduction. Consider the algebraic Riccati equation

$$A^TP + PA - PBB^TP + C^TC = 0 \tag{5.14}$$

determined by the triple of matrices (A,B,C). It is easy to check that for any solution P one has

$$\begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} M \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} = \begin{bmatrix} -\tilde{A}^T & 0 \\ -BB^T & \tilde{A} \end{bmatrix} \tag{5.15}$$

where $\tilde{A}=A-BB^TP$. Given this it becomes useful to know when (5.14) has a solution and to know some properties of the solutions. These will also be important for the next section. In fact the function $\mathrm{rc}(A,B,C)$ that assigns to the triple (A,B,C) under suitable conditions the unique positive definite solution of (5.14) is important enough to be considered a standard named function which should be available in accurate tabulated form much as say the Airy function or Bessel functions. I know of no such tables. The symbol 'rc' of course stands for Riccati.

Let A^* be the adjoint of the complex $n\times n$ matrix A, i.e. the conjugated transpose of A, so, if A is real, $A^*=A^T$. Consider the equation (algebraic Riccati equation)

$$A^*P + PA = PBB^*P - C^*C \tag{5.16}$$

(Here A is an $n\times n$ matrix, B an $n\times m$ matrix, C an $p\times n$ matrix.) Some facts about (5.16) are then as follows:

(5.17) If (A,B) is stabilizable, i.e. if there exists an F such that $A-BF$ has all eigenvalues with negative real part, then there is a solution of (5.16) which is positive semidefinite ($P\geqslant 0$) (and for this solution $\tilde{A} = A - BB^*P$ is stable).

(So in particular if (A,B) is completely reachable there is a solution of (5.14).)

(5.18) Suppose (5.16) has a solution $P\geqslant 0$ and suppose that (A,C) is completely observable. Then P is the only nonnegative definite solution of (5.16) and $P>0$.

(5.19) If (A,B,C) is *co* and *cr* then there is a unique $P>0$ which solves (5.16).

This last property is the essential one for this section. For the next one we need something better. Let $L^{co,cr}_{m,n,p}(\mathbf{R})$ be the space of all triples of real matrices (A,B,C) such that (A,B) is completely reachable and (A,C) is completely observable. Let $\mathrm{rc}(A,B,C) := P$ be the unique solution P of (5.16) such that $P>0$ (the matrix P is positive definite and selfadjoint). Then

(5.20) The function $\mathrm{rc}(A,B,C)$ from $L^{co,cr}_{m,n,p}(\mathbf{R})$ to the space of selfadjoint matrices is real analytic (and so in particular C^∞ (= smooth)

Moreover

(5.21) $\mathrm{rc}(TAT^{-1},\ TB,\ CT^{-1}) = (T^*)^{-1}\mathrm{rc}(A,B,C)T^{-1}$

(5.22) $\mathrm{rc}(-A^*),\ \pm C^*,\pm B^*) = \mathrm{rc}(A,B,C)^{-1}$

Property (5.21) is important in section 6; more precisely it will be important when these results are really implemented for multi-input multi-output systems. The point is that the matrices (A,B,C) are not determinable from the observations alone, simply because the systems (A,B,C) and (TAT^{-1}, TB, CT^{-1}) for $T \in GL_n(\mathbb{R})$ produce exactly the same input-output behaviour. For completely reachable and completely observable systems this is also the only indeterminacy. Property (5.21) guarantees that the whole analysis of these two section 5 and 6 'descends' to the moduli space (quotient manifold) $L^{co,cr}_{m,n,p}(\mathbb{R})/GL_n(\mathbb{R})$.

Having all this available it is tempting (and natural) to play the trick embodied by (5.15) again, this time using conjugation by a 2×2 block matrix with identities on the diagonal, a zero in the upper right hand corner and a Riccati equation solution Q in the lower left hand corner. This, however, is no particular good because this will introduce both the two factors

$$\begin{bmatrix} I & 0 \\ -Q & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ Q & I \end{bmatrix}$$

in the factorisation of $S(t)=\exp(tM)$, and at least one will cause difficulties with inverse and direct Fourier transforms; cf. part (iii) of the recipe of section 4.

Instead, writing

$$\exp(t\begin{bmatrix} -\tilde{A}^T & 0 \\ -BB^T & \tilde{A} \end{bmatrix}) = \begin{bmatrix} \exp(-t\tilde{A}^T) & 0 \\ -R & \exp(t\tilde{A}) \end{bmatrix} \tag{5.23}$$

one uses the factorisation

$$\exp(t\begin{bmatrix} -\tilde{A}^T & 0 \\ -BB^T & \tilde{A} \end{bmatrix}) = \begin{bmatrix} I & 0 \\ -R\exp(t\tilde{A}^T) & I \end{bmatrix} \begin{bmatrix} \exp(-t\tilde{A}^T) & 0 \\ 0 & \exp(t\tilde{A}) \end{bmatrix} \tag{5.24}$$

giving the following total factorisation for $S(t)=\exp(tM)$

$$S(t) = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -R\exp(t\tilde{A}^T) & I \end{bmatrix} \begin{bmatrix} \exp(-t\tilde{A}^T) & 0 \\ 0 & \exp(t\tilde{A}) \end{bmatrix} \begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} \tag{5.25}$$

Except for possibly the second factor on the right hand side of (5.25) applying the recipe of section 4 is a total triviality.

As to that second factor observe that

$$\begin{aligned} \frac{d}{dt}(\exp(t\begin{bmatrix} -\tilde{A}^T & 0 \\ -BB^T & \tilde{A} \end{bmatrix}) &= \begin{bmatrix} -\exp(-t\tilde{A})A^T & 0 \\ -\frac{d}{dt}R & \exp(t\tilde{A})\tilde{A} \end{bmatrix} \\ &= \begin{bmatrix} \exp(-t\tilde{A}^T) & 0 \\ -R & \exp(t\tilde{A}) \end{bmatrix} \begin{bmatrix} -\tilde{A}^T & 0 \\ -BB^T & \tilde{A} \end{bmatrix} \end{aligned} \tag{5.26}$$

from which it follows that

$$\frac{dR}{dt} = -R\tilde{A}^T + \exp(t\tilde{A})BB^T. \tag{5.27}$$

As a result

$$\begin{aligned} \frac{d}{dt}(R\exp(t\tilde{A}^T)) &= -R\tilde{A}^T\exp(t\tilde{A}^T) + \exp(t\tilde{A})BB^T\exp(t\tilde{A}^T) + R\tilde{A}^T\exp(t\tilde{A}^T) \\ &= \exp(t\tilde{A})BB^T\exp(t\tilde{A}^T) \geqslant 0 \end{aligned} \tag{5.28}$$

and it follows that

$$R\exp(t\tilde{A}^T) \geqslant 0 \quad \text{all } t \tag{5.29}$$

which means that applying part (iii) of the recipe of section 4 (= part (iii) of the definition of the real form of the SSW representation) just involves solving a diffusion equation (no anti diffusion component); or, in other words that the inverse Fourier transformation involved will exist. Note also that if the initial condition ψ is Fourier transformable then, if P is nonnegative definite, the result of applying the parts of the recipe corresponding to the third and fourth factors on the RHS of 5.25 will still be a Fourier transformable function.

This concludes the description of the algorithm for propagating non-gaussian initial densities.

6. Identification

Given all that has been said above, this section can be mercifully short. The problem is the following. Given a linear system

$$dx = Axdt + Bdw,\ dy = Cxdt + dv \tag{6.1}$$

with *unknown* A,B,C, but constant A,B,C, we want to calculate the joint conditional density (given the observations $y(s)$, $0 \leq s \leq t$) for A,B,C,x. This can be approached as a nonlinear filtering problem by adding the equations

$$dA = 0,\ dB = 0,\ dC = 0 \tag{6.2}$$

or, more precisely, the equations stating (locally) that the free parameters remaining after specifying a local canonical form are constant but unknown. More generally one has the same setup and problem when, say, part of the parameters of (A,B,C) are known (or, generalizing a bit more, imperfectly known).

The approach, of course, will be the calculate the DMZ unnormalized version of the conditional density $\rho(x,A,B,t)$ given the observations $y(s)$, $0 \leq s \leq t$. Writing down the DMZ equation for the system (6.1)-(6.2) gives

$$d\rho = \pounds\rho dt + \sum_{j=1}^{p}(Cx)_j dy_j(t) \tag{6.3}$$

with $\pounds$ given by (5.3); i.e. exactly the same equation as occurred in section 5 for the case of known A,B,C. And, indeed the only difference is that in section 5 the A,B,C are known, while (6.3) should be seen as a family of equations parametrized by (the unknown parameters in) the A,B,C. Thus if $\rho(x,t|A,B,C)$ denotes the solution of (5.2) and $\rho(x,A,B,C,t)$ denotes the solution of (6.3) then

$$\rho(x,t|A,B,C) = \rho(x,A,B,C,t) \tag{6.4}$$

Now the bank of Kalman-Bucy filters for $\hat{x}$ parametrized by $(A,B,C) \in L^{co,cr}_{m,n,p}$ gives the probability density

$$\pi(x,t|A,B,C) = r(t,A,B,C)^{-1}\rho(x,t|A,B,C) \tag{6.5}$$

so that the normalization factor $r(t,A,B,C)$ can be calculated as $\int\rho(x,t,A,B,C)dx$.
By Bayes

$$\pi(x,A,B,C,t) = \pi(x,t|A,B,C)\pi(A,B,C,t)) \tag{6.6}$$

so that the normalization factor $r(t,A,B,C)$ is, so to speak, precisely equal to the difference between the solution of the DMZ equation (6.3) (or (5.2)) and the bank of Kalman filters producing $\pi(x,t|A,B,C)$. I.e. the marginal conditional density

$$\pi(A,B,C,t) = \int\pi(x,A,B,C,t)dx = \int\rho(x,A,B,C,t)dx \,/\, \int\rho(x,A,B,C,t)dxdAdBdC \tag{6.7}$$

is obtainable from the unnormalized version of the bank of Kalman-Bucy filters parametrized by (A,B,C). Given the relations between this bank of filters described in [13] and briefly recalled in section 7 below this may offer further opportunities.

Be that as it may the marginal density $\pi(A,B,C,t)$ which up to a normalization factor is equal to

$\int \rho(x,A,B,C,t)dx$ can be effectively calculated by the procedure of section 5 above with the only difference that $P = \pi(A,B,C)$ now has to be treated as a function. Once $\pi(A,B,C,t)$ (or in various cases some unnormalized version $\rho(A,B,C,t)$ is available a host of well known techniques such as maximum likelyhood become available.

If it is possible (as it will be in many cases) to work with a $\rho(A,B,C,t) = r(t)\pi(A,B,C,t)$ there is no (immediate) need to descend to the quotient manifold $L_{m,n,p}^{co,cr}(\mathbf{R}) / GL_n(\mathbf{R})$.

7. On the relation between the 'real form' of the SSW representation and the Kalman-Bucy filter

We have seen that the essential difficulty in obtaining the (unnormalized) conditional density $\rho(x,t)$ lies in 'solving' $\exp(t\pounds)\psi$ where $\pounds$ is the second order differential operator (5.3). Now $\pounds$ corresponds in a fundamental way with the $2n \times 2n$ matrix

$$\begin{bmatrix} -A^T & -C^T C \\ -BB^T & A \end{bmatrix} \tag{7.1}$$

Not very surprisingly this matrix in turn is very much related to the matrix Riccati equation part of the Kalman-Bucy filter. Indeed, consider the matrix differential equation

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} -A^T & -C^T C \\ -BB^T & A \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \tag{7.2}$$

and, assuming that $X(t)$ is invertible, let

$$-P = YX^{-1}. \tag{7.3}$$

Then

$$\begin{aligned} \dot{P} &= -\dot{Y}X^{-1} + YX^{-1}\dot{X}X^{-1} = (+BB^T X - AY)X^{-1} + YX^{-1}(-A^T X - C^T CY)X^{-1} \\ &= +BB^T + AP + PA^T - PC^T CP \end{aligned}$$

which is the covariance equation of the Kalman-Bucy filter.

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