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On Distance-Transitive Graphs and Involutions*

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Abstract. We present a new result on distance-transitive graphs and show how it can be used in the case where the vertex stabilizer is the centralizer of some involution.

1. Introduction

In the classification of primitive distance-transitive graphs one is confronted with the question if for a finite group G there is a suitable maximal subgroup H and a suitable suborbit leading to a distance-transitive graph. In this paper (see section 2) we present a result concerning the case where there is a suborbit on which H acts unfaithfully. As an immediate consequence one gets strong restrictions on the special case where H is the centralizer of an involution. As an application we determine in section 3 all primitive distance-transitive graphs with automorphism group G such that $S \trianglelefteq G \leq \mathrm{Aut}(S)$, where S is a sporadic simple group, that is, one of the 26 groups mentioned as such in the Atlas [6], and H the centralizer of some involution.

Throughout this paper we shall use the notation of the Atlas [6]; our terminology is standard, see Bannai & Ito [2] and Brouwer, Cohen & Neumaier [5] for general reference.

2. General Theory

In this section Γ is a graph on which G acts transitively. By G_x we denote the stabilizer of $x \in \Gamma$ in G. If G_x is the centralizer of some involution $\sigma \in \mathrm{Aut}(G)$ then the vertex set of Γ can be identified with the conjugacy class D of involutions which are G-conjugate to σ. Note that for any pair $\alpha, \beta \in D$ with α $C_G(\sigma)$-conjugate to β we have $|\alpha\sigma| = |\beta\sigma|$. The following omnibus lemma states some well known results which we shall use; (i)–(v) goes back to Taylor & Levingston [15] and (vi)–(viii) follows from the work by Gardiner, see for instance Brouwer, Cohen & Neumaier [5].

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2.1. Lemma. *Let Γ be a distance-regular graph with* $\operatorname{diam}\Gamma \geq 3$, *and, for* $\gamma \in \Gamma$, *set* $k_i = |\Gamma_i(\gamma)|$.

(i) *There are* i, j *with* $1 \leq i \leq j \leq d$ *such that* $1 < k_1 < \cdots < k_i = \cdots = k_j > \cdots > k_d$.

(ii) *If* $i \leq j$ *and* $i + j \leq d$, *then* $k_i \leq k_j$.

(iii) *If* $k_i = k_j$ *for* i, j *with* $i < j$ *and* $i + j \leq d$, *then* $k_{i+1} = k_{j-1}$.

(iv) *If* $k_i = k_{i+1}$, *then* $k_i \geq k_j$ *for all* j.

(v) *If* $k_{j-1} = k$ *for some* j *with* $3 \leq j \leq d$, *then* $k = 2$ *or* Γ *is an antipodal* 2*-cover.*

(vi) *If* $c_2 = c_3$ *then* $c_2 = c_3 = 1$.

(vii) *If* $b_1 = b_2$ *then* $c_2 = 1$.

(viii) *Given a non-bipartite distance-regular graph, there are numbers* $\underline{i}, \bar{i}$ *such that* $a_l \neq 0$ *if and only if* $\underline{i} \leq l \leq \bar{i}$. *Moreover* $\underline{i} + \bar{i} \geq d$.

(ix) $c_1 \leq c_2 \leq \cdots \leq c_d$ *and* $b_0 \geq b_1 \geq \cdots \geq b_{d-1}$.

(x) *If* $i + j \leq d$ *then* $c_i \leq b_j$. □

A useful set of inequalities was given by TERWILLIGER [16]:

2.2. Theorem. *Let Γ be a distance-regular graph with intersection array* $\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$. *If Γ contains a quadrangle, then, for all* i $(i = 1, \ldots, d)$, $c_i - b_i \geq c_{i-1} - b_{i-1} + a_1 + 2$; *in particular,* $d \leq (k + c_d)/(a_1 + 2)$. □

The next lemma tells us that in the case $G_x \leq C_G(\sigma)$, with $\sigma \in \operatorname{Aut}(G)$, we may assume $\sigma \in \operatorname{Aut}\Gamma$.

2.3. Lemma. *Let Γ be a graph on which G acts primitively distance-transitively, and denote by H the stabilizer in G of a vertex of Γ. Suppose σ is an automorphism of G.*

(i) *If σ centralizes H and* $\operatorname{diam}\Gamma \geq 3$, *then* $\sigma \in \operatorname{Aut}\Gamma$.

(ii) *If σ normalizes H and* $\operatorname{diam}\Gamma \geq 5$, *then the same conclusion holds.*

Proof. As G acts distance-transitively on Γ we can identify Γ with $\Gamma(G, H, r)$ for some $r \in G$. Clearly σ induces a permutation of the vertices of Γ, and we are done if $HrH = Hr^\sigma H$. Suppose this is not the case. Let us define $\Gamma^\sigma = \Gamma(G, H, r^\sigma)$. Clearly $\sigma\colon gH \to g^\sigma H$ is an isomorphism from Γ to Γ^σ. By assumption $r^\sigma H$ is not adjacent to H, so is at distance a say, with $a > 1$. Now we get $\Gamma^\sigma = \Gamma_a$ and, as G acts distance-transitively, $k_a = k$. It follows by TAYLOR & LEVINGSTON [15] that if $a \neq d := \operatorname{diam}\Gamma$ then $k_2 = k$ or $k_{d-1} = k$ and Γ is imprimitive. So we may assume $a = d$. As $a_d = 0$ (for else $k_2 = k$) we have $c_d = k$ and $d(rH, r^\sigma H) = d - 1$. If σ centralizes H then $H \cap rHr^{-1} = H \cap r^\sigma Hr^{-\sigma} \subseteq rHr^{-1} \cap r^\sigma Hr^{-\sigma}$. By distance-transitively we have $k = [H\colon H \cap rHr^{-1}] \geq |rHr^{-1}|/|rHr^{-1} \cap r^\sigma Hr^{-\sigma}| = k_{d-1}$ and again $k_{d-1} = k$ i.e. Γ imprimitive, a contradiction. If σ normalizes H, let $S = \Gamma(r^\sigma H) \cap \Gamma_d(rH)$, then $S^\sigma = S$, $|S| = b_{d-1}$ and the graph induced on S is a coclique.

If $|S| = 1$ then $k_{d-1} = k^2$ whence $k_{d-1} > k_2$ and from lemma 2.1 follows that if $k_j = k_2$ then $j = 2$ whence $(\Gamma_2(H))^\sigma = \Gamma_2(H)$.

If $|S| \geq 2$ take $s_1, s_2 \in S$. Now $d(s_1, s_2) = 2 = d(s_1^\sigma, s_2^\sigma)$. So $HxH = Hx^\sigma H$ where HxH is the double coset corresponding to $\Gamma_2(H)$.

Thus for a vertex gH with $d(gH, H) = 2$ we have $d(g^\sigma H, H) = 2$.

If $d \geq 5$ we have $0 = p_{2d}^2 = p_{21}^2 = a_2$, $a_d = 0$ and by lemma 2.1 $a_1 = 0$, i.e. $b_1 = k - 1$. From the equality $\sum_l p_{ij}^l p_{ls}^m = \sum_r p_{ir}^m p_{js}^r$ with $m = s = d$ and $i = j = 1$ it follows:

$$b_0 p_{0d}^d + a_1 p_{1d}^d + c_2 p_{2d}^d = c_d p_{1d}^{d-1} + a_d p_{1d}^d$$

i.e. $k + c_2 b_1 = k b_{d-1}$. Thus $c_2(k-1) = k(b_{d-1} - 1)$; as $c_2 \leq k$ and $b_{d-1} \leq k$ this contradicts $\operatorname{diam}\Gamma \geq 5$. □

2.4. Lemma. *Let Γ be a finite graph, $x, y \in \Gamma$ with $x \sim y$ and let H be a subgroup of $\operatorname{Aut}\Gamma$ such that the graphs induced on y^H and x^H are both cliques. If Γ contains no quadrangle then the graph induced on $y^H \cup x^H$ is also a clique.*

Proof. Without los of generality we may assume $|y^H| \geq 2$ and $|x^H| \geq 2$. If $x \in y^H$ then $y^H \cup x^H = y^H$ is a clique and we are done. So we may assume $x \notin y^H$. Let $h \in H$ with $hy \neq y$, we want to show that $x \sim hy$. If $hx = x$ then $x = hx \sim hy$ and we are done therefore assume $hx \neq x$ and by way of contradiction $x \not\sim hy$. Now $y \sim hy \sim hx \sim x \sim y$. As there are no quadrangles we get $hx \sim y$.

We claim that $h^i x \sim y$ for all i. We prove the claim with induction. Suppose $h^r x \sim y$, $r \geq 1$. If $h^{r+1} y = y$ then $h^{r+1} x \sim h^{r+1} y = y$ done. If $h^{r+1} x = x$ then $x = h^{r+1} x = hh^r x \sim hy$ a contradiction. Thus we have $y \sim h^r x \sim h^{r+1} x \sim h^{r+1} y \sim y$. As $x \not\sim hy$ we have $h^r x \not\sim h^{r+1} y$ thus we must have $y \sim h^{r+1} x$ proving the claim.

In particular we have $h^{-1} x \sim y$ i.e. $x \sim hy$, a contradiction. Thus $x \sim hy$ for all $h \in H$. And interchanging the role of x and y we find $y \sim hx$ for all $h \in H$ proving the lemma. □

2.5. Proposition. *Let Γ be a graph of diameter d on which the group G acts distance-transitively as a group of automorphisms. For a vertex $x \in \Gamma$, denote by G_x^i the kernel of the action of G_x on $\Gamma_i(x)$. If, for some $i \geq 1$, we have $G_x^i \neq 1$, then*

$$G_x^i \subset G_x^{i-1} \subset \cdots \subset G_x^1 \quad \text{or} \quad G_x^i \subset G_x^{i+1} \subset \cdots \subset G_x^d.$$

Proof. First we shall prove $G_x^i \subseteq G_x^{i-1} \subseteq \cdots \subseteq G_x^1$ or $G_x^i \subseteq G_x^{i+1} \subseteq \cdots \subseteq G_x^d$. Suppose not. Then there is a $i \geq 1$ with $G_x^i \neq 1$ and there are $j_1 < i < j_2$ with $G_x^i \not\subseteq G_x^{j_1}$ and $G_x^i \not\subseteq G_x^{j_2}$. Choose j_1 maximal and j_2 minimal with this property. Thus there are $y_1 \in \Gamma_{j_1}(x)$, $y_2 \in \Gamma_{j_2}(x)$ and $h_1, h_2 \in G_x^i$ with $h_1 y_1 \neq y_1$ and $h_2 y_2 \neq y_2$. Suppose $h_2 y_2 \not\sim y_2$. Then $h_2 y_2$ and y_2 have at least c_{j_2} common neighbours i.e. $c_{j_2} \leq c_2$. If $j_2 = 2$ then $i = 1, j_1 = 0$ and $G_x^{j_1} = G_x$, a contradiction. So $j_2 \geq 2$ and $c_2 = \cdots = c_{j_2} = 1$ whence $G_x^i \subseteq G_x^{i-1} \subseteq \cdots \subseteq G_x^1$, a contradiction. Thus $h_2 y_2 \sim y_2$. Without los of generality we may assume $hy_2 = y_2$ or $hy_2 \sim y_2$ for all $h \in G_x^i$ and $y_2 \in \Gamma_{j_2}(x)$. Consequently $b_{j_2 - 1} \geq 2$ and $c_{j_2} \leq \lambda$. If $h_1 y_1 \not\sim y_1$ then $h_1 y_1$ and y_1 have at least b_{j_1} common neighbours, i.e. $b_{j_1} \leq c_2$. If $j_1 < d - 2$ then $b_{j_1} \geq c_3$, so $c_3 = c_2 = 1$ whence $b_{j_1} = 1$ and $1 = b_{j_1} \geq b_{j_2} \geq 2$, a contradiction. Therefore $j_1 \geq d - 2$ and so $j_1 = d - 2$, $i = d - 1$, $j_2 = d$ and $b_{d-2} = c_2$. If Γ contains a quadrangle then, by TERWILLIGER [16]

$$c_r - b_r \geq c_{r-1} - b_{r-1} + \lambda + 2 \quad \text{for all} \quad 1 \leq r \leq d.$$

Especially we find $c_d \geq c_{d-1} - b_{d-1} + \lambda + 2 \geq c_{d-2} - b_{d-2} + 2\lambda + 4$ and so we get $\lambda \geq c_d \geq c_{d-2} - c_2 + 2\lambda + 4$ i.e. $c_2 \geq c_{d-2} + \lambda + 4$ but then $\lambda \geq c_d \geq c_2 \geq \lambda + 4$, a contradiction. Thus Γ contains no quadrangle. From lemma 2.4 follows that for each $z \in \Gamma_d(x) \cap \Gamma(y_2)$ and $h \in G_x^i$ we have $z \in \Gamma_d(x) \cap \Gamma(hy_2)$. By hypotheses there is a $h \in G_x^i$ with $hy_2 \neq y_2$ i.e. hy_2 and y_2 have at least $a_d - 1 + c_d$ common neighbours. Thus $c_d + a_{d-1} \leq \lambda$ but this implies $k = \lambda + 1$, a contradiction.

Therefore $h_1 y_1 \sim y_1$. Now $b_{j_1} \leq \lambda$.

If Γ contains a quadrangle then we have $c_{i+1} - b_{i+1} \geq c_{i-1} - b_{i-1} + 2(\lambda + 2)$. But we have $c_{i+1} \leq c_{j_2} \leq \lambda$ and $b_{i-1} \leq b_{j_1} \leq \lambda$ leading to

$$\lambda \geq c_{i+1} - b_{i+1} \geq c_{i-1} - b_{i-1} + 2(\lambda + 2) \geq c_{i-1} - \lambda + 2(\lambda + 2) = c_{i-1} + \lambda + 4 \geq \lambda + 4,$$

a contradiction.

Thus Γ contains no quadrangle. Again from lemma 2.4 follows that if

$$z \in \Gamma_{j_r}(x) \cap \Gamma(y_r) \backslash h_r y_r \quad \text{then} \quad z \in \Gamma_{j_r}(x) \cap \Gamma(h_r y_r), \quad r \in \{1, 2\}.$$

Thus y_1 and $h_1 y_1$ have at least $a_{j_1} + b_{j_1} - 1$ common neighbours, and y_2 and $h_2 y_2$ have at least $a_{j_2} + c_{j_2} - 1$ common neighbours. Whence

$$\lambda \geq a_{j_1} + b_{j_1} - 1 = k - c_{j_1} - 1 \quad \text{and} \quad \lambda \geq a_{j_2} + c_{j_2} - 1 = k - b_{j_2} - 1,$$

substituting $\lambda = k - b_1 - 1$ we get $b_1 \leq b_{j_2}$. As $j_2 \geq 2$ we have $b_1 = b_2$ whence $c_2 = 1$. Also

$$k \leq \lambda + b_{j_2} + 1 \leq \lambda + b_{j_1} + 1 \leq 2\lambda + 1.$$

Let $u, v \in \Gamma(x)$ with $u \not\sim v$ then $\Gamma(u) \cap \Gamma(v) = \{x\}$ hence

$$\{u\} \cup \{v\} \cup (\Gamma(x) \cap \Gamma(u)) \cup (\Gamma(x) \cap \Gamma(v)) \subseteq \Gamma(x)$$

counting the number of vertices in these sets we get $k \geq 2 + 2\lambda$, a contradiction.

It remains to show that if $G_x^j = G_x^{j+1}$ for some j, then $G_x^j = G_x^{j+1} = \{1\}$.

Therefore suppose $G_x^j = G_x^{j+1} \neq \{1\}$.

If $G_x^{j+1} \subseteq G_x^j \subseteq \cdots \subseteq G_x^1$ then take a $y \in \Gamma_{j+1}(x)$ and a $z \in \Gamma(x) \cap \Gamma_j(y)$. Now $G_x^{j+1} \subseteq G_z^j = G_z^{j+1}$ whence $gy = y$ for all $g \in G_x^{i+1}$.

This shows $G_x^{j+1} \subseteq G_x^{j+2}$.

If $G_x^{j+2} \subseteq G_x^{j+3} \cdots \subseteq G_x^d$ then G_x^{j+1} fixes all vertices of Γ contradicting $G_x^{j+1} \neq \{1\}$, thus

$$G_x^{j+2} \subseteq G_x^{j+1} \subseteq G_x^j \subseteq \cdots \subseteq G_x^1$$

i.e. $G_x^{j+2} = G_x^{j+1} = G_x^j$. Now $G_x^j = \{1\}$ readily follows.

If $G_x^j \subseteq G_x^{j+1} \subseteq \cdots \subseteq G_x^d$ then take a $y \in \Gamma_{j-1}(x)$ and a $z \in \Gamma(x) \cap \Gamma_j(y)$. As $\Gamma_d(z) = \Gamma_d(w)$ implies $z = w$, and $j \leq d - 1$, we have $G_x^j \subseteq G_z$. Now $G_x^j \subseteq G_z^{j+1} = G_z^j$ hence $gy = y$ for all $g \in G_x^j$ this shows $G_x^j \subseteq G_x^{j-1}$.

If $G_x^j \subseteq G_x^{j-1} \subseteq \cdots \subseteq G_x^1$ then G_x^j fixes all vertices of Γ contradicting $G_x^j \neq \{1\}$, thus $G_x^{j-1} \subseteq G_x^j \subseteq \cdots \subseteq G_x^d$ i.e. $G_x^{j-1} = G_x^j = G_x^{j+1}$. Now $G_x^j = \{1\}$ readily follows. □

Define $G_x^{\leq i}$ for the kernel of the action of G_x on the union of all $\Gamma_j(x)$ for $0 \leq j \leq i$, and, likewise $G_x^{\geq i}$ for the kernel on the union of all $\Gamma_j(x)$ for $i \leq j \leq d$.

2.6. Corollary. *Let Γ be a graph of diameter d on which the group G acts distance-transitively and primitively as a group of automorphisms. Let $x \in \Gamma$.*

(i) *If $G_x^{\leq i} \neq 1$ then $|G_x^{\leq i}| > |G_x^{\geq d-i}|$.*

(ii) *Let π be a permutation of $\{1, \ldots, d\}$ such that K_i is the kernel of the action of*

$G_x^{\pi(i)}$ *on* $\Gamma_{\pi(i)}(x)$, *and* $|K_i| \geq |K_{i+1}|$ $(i = 0 \ldots d)$, *where* $|k_{d+1}| = 1$. *If* $|K_1| > |K_2| = |K_3| \neq 1$, *then* $\pi(1) = 1$.

(iii) *If* $G_x^{\leq i} \neq 1$ *and* G_x *acts trivially on* $G_x^{\leq i}/G_x^{\leq i+1}$ *then* $G_x^{\leq i+1} = \{1\}$.

Proof. (i) Let $x \in \Gamma$ and $y \in \Gamma_d(x)$ clearly $G_x^{\geq d-i} \leq G_y^{\leq i}$ thus $|G_x^{\geq d-i}| \leq |G_y^{\leq i}| = |G_x^{\leq i}|$.

If $|G_x^{\geq d-i}| = |G_x^{\leq i}|$ then $G_y^{\leq i} = G_x^{\geq d-i}$ hence $G_y^{\leq i} \trianglelefteq \langle G_x, G_y \rangle = G$. Thus $G_y^{\leq i} = \{1\}$ and also $G_x^{\leq i} = \{1\}$ a contradiction.

(ii) From the proposition it follows that if $\pi(1) = d$ then $G_x^{\geq d-1}, G_x^1 \in \{K_2, K_3\}$ but now $|G_x^{\geq d-1}| = |G_x^1|$ contradicting (i).

(iii) Take $z \in \Gamma(x)$. Then $G_z^{\leq i+1} \leq G_x^{\leq i}$, thus G_x acts trivial on $G_z^{\leq i+1}G_x^{\leq i+1}/G_x^{\leq i+1}$. Hence $\langle G_z^{\leq i+1}, G_x^{\leq i+1} \rangle \trianglelefteq G_x$. As G acts transitively on Γ, we can interchange x and z, thus $\langle G_z^{\leq i+1}, G_x^{\leq i+1} \rangle \trianglelefteq G_z$. But now $\langle G_z^{\leq i+1}, G_x^{\leq i+1} \rangle \trianglelefteq \langle G_x, G_z \rangle = G$ and thus $\langle G_z^{\leq i+1}, G_x^{\leq i+1} \rangle = \{1\}$. In particular $G_x^{\leq i+1} = \{1\}$. □

2.7. Lemma. *Let* Γ *be a distance-regular graph of diameter* d *and automorphism group* G. *If* Γ *contains no quadrangle and* $c_2 > 1$ *then* $G_x^{\leq 2} = G_x^{\geq d-2} = \{1\}$ *for all vertices* $x \in \Gamma$.

Proof. Suppose there is a $x \in \Gamma$ with $G_x^{\leq 2} \neq \{1\}$. Then there is an $i \geq 2$ with $G_x^{\leq i} \neq \{1\}$ and $G_x^{\leq i+1} = \{1\}$. Let $y \in \Gamma_{i+1}(x)$ and $g_0 \in G_x^{\leq i}$ with $g_0 y \neq y$. Now $g_0 y$ and y have at least $|\Gamma_i(x) \cap \Gamma(y)| = c_{i+1}$ common neighbours. If $g_0 y \not\sim y$ then we have $c_{i+1} \leq c_2$ whence $c_2 = c_{i+1} = 1$, a contradiction. Thus $g_0 y \sim y$. Hence we find $gz = z$ or $gz \sim z$ for all $g \in G_x^{\leq i}$ and $z \in \Gamma_{i+1}(x)$. From lemma 2.4 it follows that each $z \in \Gamma_{i+1}(x) \cap \Gamma(y) \backslash \{g_0 y\}$ also is in $\Gamma_{i+1}(x) \cap \Gamma(g_0 y) \backslash \{y\}$. Whence y and $g_0 y$ have at least $c_{i+1} + a_{i+1} - 1$ common neighbours. Thus $\lambda \geq c_{i+1} + a_{i+1} - 1$ i.e. $b_1 = b_{i+1}$. But from lemma 2.1 it follows that $b_{i+1} = b_1$ and $c_2 = 1$, a contradiction. Therefore $G_x^{\leq 2} = \{1\}$ for all $x \in \Gamma$. Suppose there is a $x \in \Gamma$ with $G_x^{\geq d-2} \neq \{1\}$. Let $y \in \Gamma_d(x)$ then $\{1\} \neq G_x^{\geq d-2} \leq G_y^{\leq 2} = \{1\}$, the final contradiction. □

2.8. Theorem. *Let* Γ *be a distance-transitive graph with distance-transitive group* G. *Suppose that the vertex set* $V\Gamma$ *of* Γ *is a conjugacy class of involutions in* G, *that* G *acts on* Γ *by conjugation and that there are elements in* $V\Gamma$ *which commute in* G. *Take* $x, y \in \Gamma$ *with* x *adjacent to* y. *Then at least one of the following statements holds.*

(i) Γ *is a polygon or an antipodal 2-cover of a complete graph.*

(ii) G *is a 2-group.*

(iii) *The order of* xy *is an odd prime, if* $a, b \in \Gamma$ *with* ab *of order* 2, *then* a *and* b *have maximal distance in* Γ, *and if* $a, b \in \Gamma$ *the order of* ab *is not* 4.

(iv) *The elements* x *and* y *commute, and if* $z \in \Gamma_2(x)$ *then* xz *has order* 2, 4 *or an odd prime. Moreover either* $O_2(C_G(x)) \neq \langle x \rangle$ *or* $C_G(x)$ *contains a normal subgroup generated by p-transpositions.*

Proof. First suppose $[x, y] \neq 1$. If Γ has diameter 2 then there is a number m such that for any two involutions $x, y \in \Gamma$ with $x \neq y$ the order of xy is 2 or m. Clearly $m \in \{2, 4, p\}$ where p is an odd prime. Thus by Baer [1] we are in (ii) or (iii). From now on assume that Γ has diameter >2. Then $V\Gamma$ viewed as conjugacy class of involutions is a subset of $G \leq \mathrm{Aut}(\Gamma)$. By the existence of commuting involutions it follows that at least one of $G_x^{\leq i}$, $G_x^{\geq i}$ is non trivial. Suppose $d(x^y, x) = 2$. Then, as G_x acts transitively on $\Gamma_2(x)$, there is a surjection $z \to x^z$ from $\Gamma(x)$ on $\Gamma_2(x)$, and so

$k_2 \le k$. This amounts to $b_1 \le c_2$. As Γ has diameter at least 3, it follows by standard distance-regular graph theory that $b_1 = c_2$ so, $k_2 = k$, and it is readily seen that Γ is as (i) or (ii). Therefore, taking into account that $x \sim y \sim x^y$ we may assume that x^y and x are adjacent. Then $x^y x = (yx)^2$ and yx have the same order, which must be an odd number, say p. Let $r_j = (yx)^j$ $1 \le j \le p-1$, if $x^{r_i} \not\sim x$ then we get by the same argument as before a contradiction. Hence $x^{r_i} \sim x$ and p is an odd prime. Let $v \in \Gamma$ with $|xv| = 2$, $d(x, v) = j$ say. Now $v \in G_x^{\le j}$ or $G_x^{\ge j}$. As $v \notin G_x^1$ we have $v \in G_x^{\ge j}$ and if $w \in \Gamma_{\ge j}(x)$ then $|wx| = 2$. If $j \ne d$ then v has a neighbour w with $d(x, v) \ne d(x, w)$ but $|vw| = p$ hence they are conjugate in $\langle v, w \rangle \le G_x$ a contradiction. Thus $j = d$. Let $x = x_0 \sim x_1 \sim \cdots \sim x_d = v$ be a geodesic of Γ. If $|xx_j| = 4$ for some j then $d(x_j, x_j^x) = d$ and $x_1 \sim x_1^x$ whence $d \le \min(2j-1, 2(d-j))$ i.e. $2j \le d \le 2j-1$ a contradiction. Hence assertion (iii) holds. Finally suppose $[x, y] = 1$ let N be the group generated by the neighbours of x. Clearly this is a normal subgroup of $G_x = C_G(x)$ which is generated by p-transpositions if $|xz| = p$ or a 2-group otherwise. □

If $G_x = C_G(\sigma)$, σ a involution in G, then the vertices of Γ can be identified with the conjugacy class of involutions containing σ. This graph is then called a graph on involutions.

2.9. Lemma. *If Γ is a distance-transitive graph on involutions and x, $z \in \Gamma$ with $xz \in \Gamma$ then $d(x, z) = d(x, xz) = d(z, xz)$.*

Proof. As Γ is a distance-transitive graph there is an automorphism of Γ, g say, with $x^g = z$ and $z^g = x$ thus $(xz)^g = xz$ hence $d(xz, z) = d(xz, x)$, interchanging the roles of xz and z the lemma follows. □

2.10. Lemma. *Let σ be a involution of Γ, Γ as above, D its G-conjugacy class and K a conjugacy class of G. Then the number $\alpha_k(\sigma)$ of involutions $\tau \in D$ with $\sigma\tau \in K$ equals*

$$\frac{|D||K|}{|G|} \sum_{\chi} \frac{\chi(K)\chi(D)^2}{\chi(1)}$$

where χ runs through the irreducible characters of G.

Proof. It is well known (see [11]) that the number of order pairs $(g, t) \in K \times D$ with $g\tau = \sigma$ equals

$$\frac{|D||K|}{|G|} \sum_{\chi} \frac{\chi(K)\chi(D)\overline{\chi(D)}}{\chi(1)}$$

where χ runs through the irreducible characters of G. As D is a conjugacy class of involutions we have $\chi(D) = \chi(D^{-1}) = \overline{\chi(D)}$, now its clear that this number equals $\alpha_k(\sigma)$, whence the lemma. □

Note that $\alpha_k(\sigma)$ is the sum of orbit-lengths of $C_G(\sigma)$ on the involutions of D.

3. The Groups

Recall that we mean by a sporadic simple group one of the 26 groups mentioned as such in the Atlas. Our goal in this section is to prove the following:

3.1. Proposition. *Let Γ be a graph on which the group G acts primitively distance-transitively, where G has a normal sporadic simple subgroup and $G_x = C_G(x)$ for an involution $x \in \mathrm{Aut}(G)$.*

Then G, Γ is on of the following pairs of groups and graphs.

(i) $M_{22} \trianglelefteq G \leq \mathrm{aut}\, M_{22}$ *and Γ is the 2-residual of $S(5, 8, 24)$, a graph on 330 vertices.*

(ii) $G = \mathrm{aut}\, HJ$ *and Γ is the near octagon associated with HJ on 315 vertices.*

(iii) $F \trianglelefteq G \leq \mathrm{aut}\, F$, *where F is one of Fi_{22}, Fi_{23} or Fi'_{24} and Γ is the Fischer graph on the 3-transpositions or its complement.*

In the subsequent sections all 26 groups will be handled, thus providing a proof for the proposition. Par abus de language we shall identify the vertices of the graph Γ with the corresponding involutions.

3.2. *Mathieu Group M_{11}*

There is only one conjugacy class D of involutions and $C_G(D) \cong 2 \cdot S_4 \cong GL_2(3)$.

For a fixed involution x there are 24 involutions y with $|xy| = 4$ so the adjacent vertices form a orbital of involutions commuting with x. For a fixed involution x there are 12 involutions y with $|xy| = 2$, this is one orbit under $GL_2(3)$ but not of odd transpositions. Whence we cannot make a distance-transitive graph.

3.3. *Mathieu Group M_{12}*

There are 2 conjugacy classes X of involutions, of types $2A$ and $2B$ say, with $C_G(x)$ a maximal subgroup of G. Thus we have to consider 2 cases.

(1) $X = 2A$. For a fixed involution $a \in 2A$ there are

20 involutions $x \in 2A$ with $ax \in 2A$

15 involutions $x \in 2A$ with $ax \in 2B$

60 involutions $x \in 2A$ with $|ax| = 4$ (30 with $ax \in 4A$, 30 with $ax \in 4B$).

As $C_{M_{12}}(a) \cong 2 \times S_5$ and $C_{M_{12}.2}(a) \cong (2^2 \times A_5):2$, it readily follows that we can identify the 20 involution x with $xa \in 2A$ with the transpositions y in S_5 and ay.

Also it is clear that if $u \sim a$ then $au \in 2A$ (for A_5 is not generated by odd transpositions). If these 20 involution are in one orbit of $C_{M_{12}.2}(a)$ then we get a contradiction by calculating orders in $\Gamma(a)$. If its not no orbital then there are 3 distances Γ with the property of having a as kernel hence also a contradiction.

(2) $X = 2B$. For a fixed involution $b \in 2B$ there are 30 involutions $x \in 2B$ with $xb \in 2B$ and $96(= 48 + 48)$ with $|xb| = 4$,

$$H = C_{M_{12}}(b) \cong 2_+^{1+4}.S_3$$

besides b this group contains 30 $2B$ and 12 $2A$ involutions. If b, $b' \in 2B$ with $|bb'| = 2$ then $bb' \in 2B$. These 30 involutions split in 2 orbits under H of size 6 and 24. As $O_2(H) - \{b\}$ contains 18 involutions (6 $2B$ and 12 $2A$), it readily follows that we get a graph of valency 6 and non existence of a distance-transitive graph follows by Faradjev, Ivanov & Ivanov [7] or Gardiner & Praeger [8].

3.4. *Mathieu Group M_{22}*

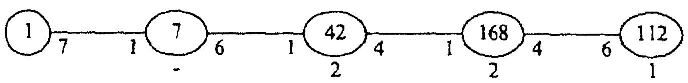
There is only one maximal subgroup of M_{22} which is the centralizer of an involution. This involution is contained in $\mathrm{Aut}\, M_{22} \backslash M_{22}$. By lemma 2.3 we may assume that these involutions are contained in $\mathrm{Aut}(\Gamma)$. Now $C_{M_{22}.3}(c) \cong 2^3 : L_3(2) \times 2$ and for a fixed involution c

there are $49(= 7 + 42)$ involutions x with $xc \in 2A$

168 involutions x with $xc \in 4A$

112 involutions x with $xc \in 3A$

leading to a graph with distance distribution diagram



$v = 330.$

3.5. *Mathieu Groups M_{23}, M_{24}*

These groups do not contain maximal subgroups which are centralizers of involutions.

3.6. *Higman-Sims Group HS*

In this group all involutions give rise to a maximal subgroup. So we have to consider 3 cases.

Type $2A$.

Let $a \in 2A$ then $C_{HS}(a) \cong 4 \cdot 2^4 : S_5$ and $C_{HS.2}(a) \cong 2_+^{1+6} : S_5$. For this fixed a there are

110 involutions $x \in 2A$ with $ax \in 2B$

480 involutions $x \in 2A$ with $ax \in 4B$

960 involutions $x \in 2A$ with $ax \in 4C$

640 involutions $x \in 2A$ with $ax \in 3A$

128 involutions $x \in 2A$ with $ax \in 5B$

1536 involutions $x \in 2A$ with $ax \in 5C$

1920 involutions $x \in 2A$ with $ax \in 6B$.

Now $O_2(C_{HS}(a))$ contains only 30 involutions and $O_2(C_{HS}(a)) \cap a^{HS} \neq \{a\}$.

The 110 involutions commuting with a split in 2 orbitals of size 30 and 80 the others are all orbitals, as one can check by using Cayley. Whence $k_1 = 30$ and $k_d = 80$ as $\Gamma_1(a)$ contains two involutions α, β with $|\alpha, \beta| = 4$. Now $480 = k_2 \leq k_i$

$i \neq d,\ d-1$ so $k_{d-1} = 128$ but if $v \in \Gamma_{d-1}(a)$ the $|v^a| = 5$ contradiction with $d(v^a, a) \leq 2$.

Type $2B$.

For a fixed involution $b \in 2B$ there are 75 involutions $x \in 2B$ with $bx \in 2A$ and 72 involutions $x \in 2B$ with $bx \in 2B$.

So if $g \sim b$ the $|yb| = 2$. Now $\langle \Gamma_1(b) \rangle \trianglelefteq C_{HS}(b)$ but $A_6 \cdot 2^2 \cong P\Gamma L_2(9)$ is not generated by p-transpositions hence the graph thus obtained is not distance-transitive.

Type $2C$.

These are involutions of $\mathrm{Aut}(HS)\backslash HS$, with $C_{HS.2}(c) \cong 2 \times S_8$.

For a fixed involution $c \in 2C$ there are

105 involutions $x \in 2C$ with $xc \in 2A$

280 involutions $x \in 2C$ with $xc \in 2B$

336 involutions $x \in 2C$ with $xc \in 3A$

630 involutions $x \in 2C$ with $xc \in 4B$.

So if $y \in \Gamma(c)$ then $|yc| = 2$ and by the usual arguments $yc \in 2B$. So there are $y, z \in \Gamma(c)$ with $|yz| = 3$, thus the 105 involutions x with $xc \in 2A$ are at maximal distance d. As the permutation rank equals 5, we can find a path $c \sim x_1 \sim x_2 \sim x_3 \sim x_4$ with

$$|cx_1| = 2, \quad |cx_2| = 3, \quad |cx_3| = 4, \quad |cx_4| = 2$$

but then $d(x_3, x_3^c) = 2$ and $x_3 x_3^c \in 2A$, contradicting distance-transitively.

3.7. *First Janko Group J_1*

There is only one conjugacy class of involutions in J_1. The centralizer of an involution a is isomorphic with $2 \times A_5$ and it readily follows that we cannot turn it into a distance-transitive graph.

Remark. One can identify the involutions with the edges of the Levingston-graph, and thus obtain a non-multiplicity free permutation character for this subgroup.

3.8. *Second Janko Group J_2 (Hall-Janko HJ)*

There are 2 maximal subgroups which are the centralizer of an involution. The involutions of type $2A$ have centralizers isomorphic with $2^{1+4}: A_5$ and $2^{1+4}.S_5$ in HJ and $\mathrm{Aut}\,HJ$ respectively.

For a fixed involution $a \in 2A$ there are

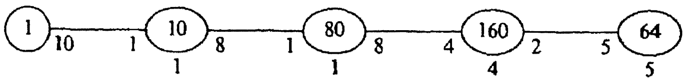
10 involutions $x \in 2A$ with $ax \in 2A$

80 involutions $x \in 2A$ with $ax \in 4A$

160 involutions $x \in 2A$ with $ax \in 3B$

64 involutions $x \in 2A$ with $ax \in 5A$ or $5B$ ($64 = 32 + 32$ in HJ)

leading to a distance-transitive graph with distance distribution diagram



$v = 315.$

The involutions of type $2C$ have centralizers isomorphic with $2 \times L_3(2):2$.

For a fixed involution $c \in 2C$ there are

21 involutions x with $cx \in 2A$

28 involutions x with $cx \in 2B$

126 involutions x with $cx \in 4A$.

So from theorem 2.8 it follows that the involutions corresponding to two adjacent vertices commute. But $L_3(2):2$ is not generated by odd transpositions and $O_2(C_{M_{22}.2}(c)) = \langle c \rangle$. This contradicts the existence of a distance-transitive graph.

3.9. *Third Janko Group* J_3

There are two conjugacy classes of involutions, $2A$ and $2B$. Those of $2B$ are contained in $\mathrm{Aut}\, J_3 \backslash J_3$.

For a fixed involution $b \in 2B$ there are 918 involutions $x \in 2B$ with $|bx| = 4$. Hence if $y \in 2B$ and $y \sim b$ then $|yb| = 2$. As $C_{J_3.2}(b) \cong 2 \times L_2(17)$ and $L_2(17)$ is not generated by p-transpositions we get that no distance-transitive graph can arise here.

For a fixed involution $a \in 2A$ there are 130 involution $x \in 2A$ with $ax \in 2A$ they fall in two orbitals of size 10 an 120 respectively. It is clear that we must have $k_1 = 10$ but there exists no distance-transitive graph on 26163 points with $k_1 = 10$ c.f. IVANOV & IVANOV [12].

3.10. *Fourth Janko Group* J_4

J_4 contains two conjugacy classes of involutions, $2A$, $2B$. The $2B$ involutions have a centralizer $2^{11}:(M_{22}:2)$ contained in the maximal subgroup $2^{11}:M_{24}$. So we only have to look at the $2A$ involutions. These have centralizers H with $H \cong 2_+^{1+2} \cdot 3M_{22}:2$. The involutions in this group are represented by the following involutions (and H-orbit lengths).

z	1
e	1386
f	2772
t'	18480
zt'	18480
t_1	110880
t_2	221760

with $z, e, f \in O_2(H)$ by Janko [13], and z, e, t', t_1, are conjugate in J_4. Thus there are three orbitals of commuting involutions under H with the same kernel. Whence there is no distance-transitive graph on these involutions.

3.11. M^c Laughlin Group M^cL

M^cL contains only one conjugacy class of involutions, $2A$ say, with centralizer isomorphic to $2 \cdot A_8$ in M^cL.

For a fixed involution $a \in 2A$ there are 210 involutions $x \in 2A$ with $ax \in 2A$ and 5040 with $ax \in 4A$. As $2 \cdot A_8$ is not generated by p-transpositions the usual argument leads to the non-existence of a distance-transitive graph on these involutions.

$\mathrm{Aut}(M^cL)$ contains an other conjugacy class of involutions, $2B$ say, with centralizer isomorphic to $2 \times M_{11}$. Fix an involution $x \in 2B$. Clearly if $y \in M_{11}$ an involution then xy conjugate to x. As M_{11} contains involutions y_1, y_2 with $|y_1 y_2| = 4$ we have $|y_1 x y_2 x| = |y_1 y_2| = 4$. M_{11} contains only one conjugacy class of involutions, thus two involutions are adjacent if and only if they commute, hence by the usual arguments there is no distance-transitive graph on involutions.

3.12. Suzuki Group Suz

Here we have three conjugacy classes of involutions to consider one contained in Suz two in $\mathrm{Aut}(Suz)\backslash Suz$.

$2A$. The centralizer in Suz of a $a \in 2A$ involution is isomorphic to $2_-^{1+6} \cdot U_4(2)$. For a fixed involution a there are 414 involutions $x \in 2A$ with $ax \in 2A$ and 1728 with $ax \in 4A$. Thus the neighbours of a commute with a and $\langle \Gamma(a) \rangle \trianglelefteq 2_-^{1+6} \cdot U_4(2)$ is a group generated by p transpositions or a 2-group. Clearly we are in the second case. It is a straightforward calculation that 2_-^{1+6} contains 54 involutions different from a. Now $\Gamma(a)/\langle a \rangle$ can be viewed as the orthogonal geometry where the involutions are in one to one correspondence with the isotropic points, two involutions commute if and only if the corresponding points are on an isotropic line. Hence we can find a quadrangle in $\Gamma(a)$. Thus we may apply Terwilliger with $k = 54, \lambda = 201$, leading to $\mathrm{diam}(\Gamma) \leq 4$. But computations with the character table learns us that there are $x \in 2A$ with $ax \in \{4A, 4C, 3C, 3B, 6D\}$ showing that the diameter of Γ is larger than 5.

$2C$. The centralizer of a $2C$ involution is $Suz.\,2$ is isomorphic to $J_2 : 2 \times 2$. For a involution $c \in 2C$ there are 315 involutions $x \in 2C$ with $xc \in 2A$ and 1800 involutions $x \in 2C$ with $xc \in 2B$. As J_2 is not generated by p-transpositions, we cannot get a distance-transitive graph.

$2D$. The centralizer of a $2D$ involution in $Suz.\,2$ is isomorphic to $M_{12} : 2 \times 2$. For a fixed involution $d \in 2D$ there are 495 involutions $x \in 2D$ with $xd \in 2A$ so these involutions xd correspond to the $2B$ involutions of $M_{12} : 2$. In $M_{12} : 2$ there are involutions α, β of type $2B$ with $|\alpha\beta| = 4$. As $M_{12} : 2$ is not generated by p-transpositions, we again cannot turn it in to a distance-transitive graph.

3.13. Lyons Group Ly

This group contains only one conjugacy class of involutions, $2A$, with $H = C_{Ly}(a) \cong 2\cdot A_{11}$ where $a \in 2A$. It is clear that there are conjugates of a, c and d say, with $|cd| = 4$. Hence, as $2\cdot A_{11}$ is not generated by odd transpositions, no distance-transitive graph can arrise from this class of involutions.

3.14. Held Group He

Here we have two conjugacy classes to consider, $2B$ and $2C$. But the corresponding centralizers do not have a multiplicity free permutation character c.f. V. Bon, Cohen & Cuypers [4].

3.15. Rudvalis Group Ru

Only one conjugacy class of involutions leads to a maximal subgroup of Ru, namely $2A$.

Note that the class $2B$ has no fixed points on the graph on 4060 vertices, and a $2A$ involution fixes 92 points. Thus the involutions of ${}^2F_4(2)$ are all conjugate in Ru.

${}^2F_4(2)$ contains 2 conjugacy classes of involutions of sizes 1755 and 11700 respectively. As Ru contains 593775 involutions of class $2A$ and $593775 - 13455 = 580320 = 2^5 . 3^2 . 5 . 13 . 31$ does not divide the order of ${}^2F_4(2)$, we see that ${}^2F_4(2)$ has at least 4 orbits on the involutions of class $2A$. Whence the inner product of the corresponding permutation characters exceeds 4 and so the permutation character corresponding to the centralizer of a $2A$ involution cannot be multiplicity free and a corresponding graph cannot be distance-transitive.

3.16. O'Nan Group O'N

$O'N$ contains only one conjugacy class of involutions $2A$, and its centralizer is isomorphic to $4\cdot L_3(4):2$.

For a fixed involution $a \in 2A$ there are 1750 involutions $x \in 2A$ with $xa \in 2A$ and 1240 involutions $x \in 2A$ with $xa \in 4A$. As this involution centralizer is not generated by p-transpositions, we cannot turn it in a distance-transitive graph.

$\mathrm{Aut}(O'N)\backslash O'N$ contains also one conjugacy class of involutions, $2B$. Its centralizer is isomorphic to $J_1 \times 2$. As J_1, is not generated by odd transpositions and as there are 2926 involutions $x \in 2B$ with $xb \in 4A$, for a fixed $b \in 2B$, we again get a contradiction with the existance of a distance-transitive graph.

3.17. Conway Group Co_3

Both conjugacy classes of involutions, $2A$ and $2B$, give rise to a maximal subgroup of Co_3. These are $2\cdot S_6(2)$ and $2 \times M_{12}$.

For the $2B$ involutions it follows from the usual argument that it is sufficient for the proof of the non-existence of a distance-transitive graph to show that for a fixed involution $b \in 2B$ there are at least two orbitals of $2B$ involutions who commute with b. Using the character table one easely finds 495 involutions $x \in 2B$ with $xb \in 2A$ and 792 involutions $x \in 2B$ with $xb \in 2B$.

For the $2A$ involutions we find 630 involutions $x \in 2B$ with $xa \in 2A$ for a fixed $a \in 2A$ and no with $xa \in 2B$. Thus the $2A$ involutions of Co_3 correspond with the $2B$ involutions of $2 \cdot S_6(2)$. The $2B$ involutions of $2 \cdot S_6(2)$ are not the transpositions of $S_6(2)$. Thus we are done if we have shown that involutions corresponding to adjacent vertices commute. But one can find at least 30240 involutions $x \in 2A$ with $|xa| = 4$, so we are done by theorem 2.8.

3.18. Conway Group Co_2

Of the three conjugacy classes of involutions contained in Co_2 only two of them give rise to a maximal subgroup of Co_2, viz the classes $2A$ and $2B$.

The conjugacy classes of involutions of $U_6(2):2$ behave as follows:

The $2A$ and $2D$ involutions fuse to the class of $2A$ involutions, the $2C$ and $2E$ involutions fuse to the class of $2C$ involutions and the $2B$ of $U_6(2):2$ lift to $2B$ involutions of Co_2. The smallest class, $2A$, has involution centralizer isomorphic to $2_+^{1+8}:S_6(2)$. Let $a \in 2A$ and $H = C_{Co_2}(a)$ then $a^H \cap O_2(H) = \langle a \rangle$ c.f. SMITH [14]. There are involutions a' and $a'' \in 2A$ with $aa' \in 2B$ and $aa' \in 2C$. Hence if $x \in \Gamma_1(a)$ then $|xa| = 2$. Now $\langle \Gamma_1(a) \rangle \trianglelefteq H$ and must be generated by 3-transpositions, but $\langle \Gamma_1(a) \rangle = H$ a contradiction.

Let $b \in 2B$ and $H = C_{Co_2}(b) \cong (2_+^{1+6} \times 2^4).A_8$. Now there are involution $b, b' \in 2B$ with $b'b \in 2A$, $b'b \in 2B$ and $b'b \in 2C$. Hence $\langle \Gamma_1(b) \rangle \trianglelefteq H$ must be elementary abelian and a contradiction follows. So in both cases we do not get a distance-transitive graph.

3.19. Conway Group Co_1

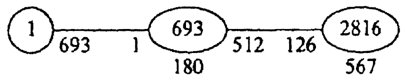
Only the conjugacy class of $2A$ involutions leads to a maximal subgroup of Co_1. Let $a \in 2A$. Then $H = C_{Co_1}(a) \cong 2_+^{1+8} \cdot O_8^+(2)$. As $a^H \cap O_2(H) \neq \{a\}$ we can find involutions a', $a'' \in 2A \cap O_2(H)$ with $|aa''| = 4$. By standard arguments we must have $\langle \Gamma(a) \rangle = O_2(H)$. But now Terwilliger's diameter bound yields $\operatorname{diam}(\Gamma) < 4$, which readily leads to a contradiction.

3.20. Fischer Group Fi_{22}

There are 3 conjugacy classes of involutions where the involution centralizer is a maximal subgroup of Fi_{22}. These involutions, $2A$, $2B$ and the outer $2D$ have centralizer (in Fi_{22}) of the form

$$2 \cdot U_6(2), \quad (2 \times 2_+^{1+8}:U_4(2)):2 \quad \text{and} \quad O_8^+(2):S_3.$$

The $2A$ involutions are the 3-transpositions leading to a graph Γ with distance distribution diagram



$v = 3510$

(where two involutions are adjacent if and only if they commute).

Γ has 1216215 edges and these correspond to the $2B$ involutions. Hence the permutation character of the 2nd group is not multiplicity free and so no distance-transitive graph can arise.

The $2D$ involutions give a rank 4 permutation representation. Fix $\partial \in 2D$ if $\partial' \in 2D$ with $|\partial\partial'| = 2$. Then $\partial\partial'$ is a involution of $O_8^+(2):S_3$. Now by counting it follows that these involutions are of type $2A$ in $O_8^+(2):S_3$.

Two $2A$ involutions have order 2, 3 of 4 whence there are $2D$ involutions is $\mathrm{Aut}(Fi_{22})$ with order 2, 3, 4. Thus two $2D$ involutions are adjacent if and only if they commute and the graph induced on the neighbours of δ is the graph on the $2A$ involutions of $O_8^+(2):S_3$, but this is not a class of odd transpositions. Thus again we cannot turn it into a distance-transitive graph.

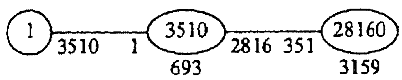
3.21. *Fischer Group Fi_{23}*

There are 3 conjugacy classes of involutions in Fi_{23}, $2A$, $2B$, $2C$ with involution centralizer isomorphic to

$$2\cdot F_{22},\quad 2^2\cdot U_6(2)\cdot 2,\quad (2^2\times 2_+^{1+8}).(3\times U_4(2))\cdot 2$$

respectively.

The $2A$ involutions are the 3-transpositions leading to a distance transitive graph Γ with distance distribution



$v = 31671.$

Thus Γ contains $3^7.5.13.17.23$ edges and $3^8.5.7.11.13.17.13$ 3-cliques who are in one to one correspondence with the $2B$ and $2C$ involutions. It readily follows that the permutation characters of $2B$ and $2C$ involutions are not multiplicity free.

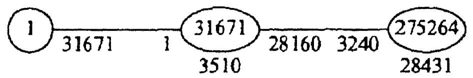
Hence no other distance-transitive graph arises.

3.22. *Fischer Group Fi_{24}*

There are 4 conjugacy classes of involutions, $2A$, $2B$, $2C$, $2D$ with involution centralizers (in Fi_{24})

$$(2\times 2\cdot Fi_{22}):2,\quad 2_+^{1+12}\cdot 3U_4(3).2^2,\quad Fi_{23}\times 2,\quad (2\times 2^2\cdot U_6(2)):S_3.$$

The $2C$ involutions are the 3-transpositions with corresponding distribution diagram



$v = 306936$

hence there are $2^2.3^7.7^2.17.23.29$ edges of Γ and $2^3.3^9.5.7^2.13.17.23.29$ 3-cliques of Γ corresponding with the $2A$, $2D$ involutions respectively. So the existence of a corresponding distance graph by these involutions fails on the permutation character.

Let x_1, x_2, x_3, x_4 be $2C$ involutions forming a 4-clique in Γ then $x_1x_2x_3x_4$ is a $2B$ involution.

Thus if x_1, x_2, x_3, y and x_1, x_2, z_1, z_2 are also 4-cliques $y, z_1, z_2 \notin \{x_1, \ldots, x_4\}$ with $|x_4y| = |x_3x_4z_1z_2| = 2$ then they give $2B$ involutions b and b' with $bb' \in 2A$ and $bb' \in 2B$; ($\Gamma(x_1) \cap \Gamma(x_2)$ is just the graph on involution of Fi_{22}. As the maximal set of commuting involutions is that graph has size 22 the existence of these 4-cliques follows). $3U_4(3)$ acts irreducible on 2^{12} so there are at most 2 orbitals of involutions commuting with a fixed $b \in 2B$. But, by using CAS, one can find for a given $b \in 2B$ $69552 = 2^4.3^3.7.23$ and $2997162 = 2.3^5.7.881$ $b' \in 2B$ with $bb' \in 2A$, $2B$ respectively. It readily follows that there are more than two orbitals on the involutions commuting with b, so we are done in this case.

3.23. Harada-Norton Group HN

Only the conjugacy classes of involutions denoted with $2A$ and $2B$ give a maximal subgroup of HN. The type $2A$ involutions with centralizer $2.HS.2$ can be dealt with in the usual way, and no distance-transitive graph can arise.

The type $2B$ involutions have centralizers $2_+^{1+8}.(A_5 \times A_5).2$ i.e. $2_+^{1+8}.SO_4(4)$. Let $b \in 2B$, $H = C_{HN}(b)$.

The 270 noncentral involutions of $O_2(H)$ fall in to 2 orbitals (of sizes 150 conjugate to b and 120 not conjugate to b) c.f. [9]. Also are there involutions $b' \in 2B$ with $bb' \in 2A$. Hence if $x \sim b$ then x and b commute. Now clearly $\langle \Gamma(b) \rangle \leq O_2(H)$ so $k = 150$.

In $O_2(H)/\langle b \rangle$ the 75 conjugates of b bear the structure of an orthogonal geometry over $GF(4)$, hence each vertex is colinear with 26 others i.e. $\lambda \geq 52$. Now Terwilliger's diameter bound yields a contradiction, as the permutation rank is at least 10.

3.24. Thompson Group Th

This group contains only one conjugacy class of involutions, $2A$, with $H = C_{Th}(a) \cong 2_+^{1+8} \cdot A_9$ where $a \in 2A$.

Clearly H has at least two orbits on the involutions that commute with a and by the usual arguments we must have $b \in 2_+^{1+8}$ if $b \sim a$. This group contains 270 involutions different from a. If H acts transitively on these involutions then $\Gamma(a)$ is the graph on involutions of 2_+^{1+8} with 2 are adjacent if and only if they commute, which leads to a contradiction with Terwilliger's diameter bound. If H acts intransitively on 2_+^{1+8} then there are at least 3 orbitals of involutions commuting with a. As $H/O_2(H)$ does not stabilize a singular subspace of the $O_8^+(2)$-geometry, $\Gamma_1(a)$ can not be elementary abelian and hence there are at least 3 orbitals with only a as kernel contradicting the existence of a distance-transitive graph.

3.25. Baby Monster Group B

The Baby Monster group contains 4 conjugacy classes of involutions, $2A$, $2B$, $2C$ and $2D$ with centralizers

$$2\cdot({}^2E_6(2)):2,\quad 2_+^{1+22}\cdot Co_2,\quad (2^2\times F_4(2)):2,\quad (2\times 2^8)2^{16}.D_4(2).2.$$

The last one is not maximal in B so we only have to look at the $2A$, $2B$ and $2C$ involutions.

The $2A$ involutions form a class of $(3,4)$ transpositions with $ab\in\{2B,2C,3A,4B\}$ for a, $b\in 2A$ $a\neq b$, with orbital lengths 3968055, 23113728, 2370830336 and 11174042880 respectively. ${}^2E_6(2)$ is not generated by p-transpositions, and hence no distance-transitive graph exist. (See also Higman [10] for the rank 5 graph).

The $2C$ involutions are in 1-1 correspondence with the edges of the graph on $2A$ involutions obtained by calling a, $b\in 2A$ adjacent if and only if $ab\in 2C$. If we fix $a\in 2A$ the involutions of the form $aa'\in 2C$ with $a'\in 2A$ are the $2D$ involutions of ${}^2E_6(2)$. It is clear that we can find 2 involutions $a'a''$ with $|a'a''|=4$ (for $|aa'\cdot aa''|=|a'a''|\in\{2,3,4\}$ and ${}^2E_6(2)$ is not generated by odd transpositions. Note that the $2D$ involutions of ${}^2E_6(2)$ are not the $\{3,4^+\}$ transpositions). As $F_4(2)$ is not generated by odd transpositions the nonexistence of a distance-transitive graph follows.

The $2B$ involutions have centralizer isomorphic to $H\cong 2_+^{1+22}\cdot Co_2$. Call $V=O_2(H)/ZO_2(H)$, where $Z=ZO_2(H)$. It is well known that Co_2 acts irreducible on V. The nontrivial orbits of Co_2 on V are as follows.

length	point stabilizer	
$2300=2^2.5^2.23$	$U_6(2):2$	
$46565=3^4.5^2.23$	$2^{10}:M_{22}:2$	
$24049300=2^2.3^4.5^2.11.23$	$2_+^{1+8}:S_8$	
$476928=2^8.3^4.23$	$HS:2$	(corresponding to $4A$ in B)
$1619200=2^8.5^2.11.23$	$U_4(3).D8$	(corresponding to $4B$ in B)

$2^{22}=4194304=1+2098175+2096128$ so the first three represent the isotropic and the last two the nonisotropic points of $V-\{0\}$ viewed as $O_{22}^+(2)$ module.

The isotropic points become involutions in $O_2(H)$. It is easy to see that the first class of involutions correspond with those of type $2A$. From Bierbrauer [3] it follows that the others represents $2B$ and a third class, call the first orbital Δ. Now

Co_2 cannot fix a singular subspace of V so there are points $x, y \in V \cap D$ with xy not isotropic. Hence there are involutions a, b conjugate to z with $|ab| = 4$.

It follows that the orbital of involutions in $O_2(H)$ of size 2×46565, must correspond to the neighbours of z, and that there are at most 2 orbitals of involutions commuting with z. Again we find, by using CAS, for a given $b \in 2B$ $7379550 = 2.3^2.5^2.23^2.31$ and $262310400 = 2^9.3^4.5^2.11.23$ $b' \in 2B$ with $bb' \in 2B$, $2D$ respectively. Looking at the prime-numbers the existence of more than two orbitals of involutions commuting with b follows. Thus we cannot turn this class into a distance-transitive graph.

3.26. *Monster Group M*

There are only 2 conjugacy classes of involutions in M, $2A$ and $2B$. The corresponding centralizers are

$$2 \cdot B \quad \text{and} \quad 2_+^{1+24} \cdot Co_1.$$

Let us fix a $2A$ involution a. If $b \in 2A$ with $|ab| = 2$ then $b \in 2 \cdot B$. There exist a $2A$ pure subgroup in M. This group of order 4 has normalizer $2^2.({}^2E_6(2)) : S_3$.

Hence there exist a $2 \cdot B$ orbit of $2A$ involutions who commute with a and correspond to the $2A$ involutions of B. Thus we can find two $2A$ involutions a', a'' in B with $|a'a''| = 4$. Thus the neighbours of a are involutions that commute with a and as $O_2(2 \cdot B) = \langle a \rangle$ and as B is not generated by p-transpositions no distance-transitive graph arises.

Let us fix $2B$-involution b with centralizer $H \cong 2_+^{1+24} \cdot Co_1$. Call $V = O_2(H)/\langle b \rangle$, then $|V| = 2^{24}$ and Co_1 acts irreducible on V and has three nontrivial orbits on V. One orbit corresponding with the nonisotropic points with pointstabilizer Co_3 (type 3). Two orbits on the isotropic points with pointstabilizers Co_2 and $2^{11} : M_{24}$, (type 2 and 4). Also it is known that there are lines with points of types 222, 223, 224, and 442, 443, 444. If $a \in O_2(H)$ and $a \neq b$ then ab is conjugate in M to a, this follows from the existence of only 2-orbits, which are of different length. The involutions is $O_2(H)$ are of type $2A$ or $2B$, both occur. Now its straightforward that there are involutions $a_1, a_2, a_3, a_4 \in O_2(H) \cap z^M$ commuting with z and where $|a_1 a_2| = 4$, $a_1 a_3 \in 2A$ and $a_1 a_4 \in 2B$. As we may assume that these are in one orbit under H, nonexistence of a distance-transitive graph follows.

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