
Reflexivity, the dual Radon-Nikodym property, and continuity of adjoint semigroups

by J.M.A.M. van Neerven

*Centre for Mathematics and Computer Science, P.O. Box 4079, 1009 AB Amsterdam
the Netherlands*

Communicated by Prof. A.C. Zaanen at the meeting of January 29, 1990

In this note for certain Banach spaces we give characterizations of reflexivity and the dual Radon-Nikodym property in terms of continuity of adjoint semigroups. Some applications outside the realm of semigroup theory are given.

1980 Mathematics Subject Classification: 46B15, 46B22, 47D05.

Keywords & phrases: Adjoint semigroups, C_0 -semigroups, Schauder decomposition, unconditional basis, reflexivity, Grothendieck space, Radon-Nikodym property.

0. INTRODUCTION

Let $T(t)$ be a C_0 -semigroup on a Banach space X . It is well-known that the adjoint semigroup $T^*(t) = (T(t))^*$ need not be strongly continuous on X^* . However, if X is reflexive, it is; this is a theorem of R.S. Phillips [14]. In this note we will prove the following converse.

THEOREM A. *Let X be a Banach space with a Schauder basis. The following statements are equivalent:*

- (1) X is reflexive;
- (2) For every C_0 -semigroup $T(t)$ on X , the adjoint semigroup $T^*(t)$ is strongly continuous;
- (3) For every C_0 -semigroup $T(t)$ of X , the second adjoint semigroup $T^{\odot*}(t)$ is strongly continuous.

The definition of $T^{\odot*}(t)$ is given below. The idea of this theorem consists in showing that every Banach space with a Schauder basis $\{x_n\}_{n=1}^{\infty}$ (or more

generally, with a Schauder decomposition) admits C_0 -semigroups $T(t)$ with the property that $T^*(t)$ is strongly continuous if and only if $\{x_n\}_{n=1}^\infty$ is shrinking.

It follows from the proof of Theorem A that Grothendieck spaces with the Dunford-Pettis property cannot have a Schauder decomposition. This was first observed by D.W. Dean [3]; see also [12].

The Radon-Nikodym property is in many ways a close analogue of reflexivity. Here we will show that a weak*-continuous semigroup on a dual Banach space with the Radon-Nikodym property is strongly continuous for $t > 0$. In this setting it turns out to be useful to consider Banach spaces with an unconditional basis, since on them C_0 -semigroups can be constructed in a canonical way such that, when X^* is nonseparable, the adjoint semigroup fails to be strongly continuous even for $t > 0$. These observations, together with the fact that separable duals have the Radon-Nikodym property, indicate what ideas lie behind the following theorem.

THEOREM B. *Let X be a Banach space with an unconditional basis $\{x_n\}_{n=1}^\infty$. The following statements are equivalent:*

- (1) X^* has the Radon-Nikodym property;
- (2) Every adjoint semigroup on X^* is strongly continuous for $t > 0$.

In fact, if $\{x_n\}_{n=1}^\infty$ is an unconditional basis for X , we will show that (1)-(2) hold if and only if X^* is separable if and only if $\{x_n\}_{n=1}^\infty$ is shrinking, which by a theorem of R.C. James (see [11]) is the case if and only if X does not contain a subspace isomorphic to l^1 . More generally, H.P. Lotz proved that for Banach lattices X , X^* has the Radon-Nikodym property if and only if X does not contain a subspace isomorphic to l^1 ; see [7].

This note is organized as follows. In section 1 we will give some definitions and standard results which will be used afterwards. After that, sections 2 and 3 are concerned with Theorems A and B, respectively. In section 4 our results are applied to bases in c_0 .

1. PRELIMINARIES

A one-parameter family $\{T(t)\}_{t \geq 0}$ (briefly, $T(t)$) of bounded linear mappings from a Banach space X into itself is called a *semigroup* if the following two conditions are satisfied:

- (1) $T(0) = I$ (I the identity map of X);
- (2) $T(t)T(s) = T(t+s)$ for all $t, s \geq 0$.

A *strongly continuous* semigroup (also called a C_0 -semigroup) is a semigroup that satisfies

- (3) $\lim_{t \downarrow 0} \|T(t)x - x\| = 0$ for all $x \in X$.

The *generator* A of a C_0 -semigroup $T(t)$ is defined by

$$D(A) = \left\{ x \in X : \lim_{t \downarrow 0} \frac{1}{t} (T(t)x - x) \text{ exists} \right\};$$

$$Ax = \lim_{t \downarrow 0} \frac{1}{t} (T(t)x - x) \quad (x \in D(A)).$$

A C_0 -semigroup is called *compact* if for every $t > 0$ the operator $T(t)$ is compact.

A semigroup $T^*(t)$ on a dual space X^* is called an *adjoint* semigroup if there is a C_0 -semigroup $T(t)$ on X such that $(T(t))^* = T^*(t)$ for all $t \geq 0$. An adjoint semigroup need not be strongly continuous. Therefore it makes sense to define

$$X^\circ = \{x^* \in X^* : \lim_{t \downarrow 0} \|T^*(t)x^* - x^*\| = 0\}.$$

Of course, X° depends on the particular semigroup under consideration. It is easy to see that X° is invariant under $T^*(t)$; hence the restriction $T^{\circ}(t)$ of $T^*(t)$ to X° defines a C_0 -semigroup on X° ; its adjoint on $X^{\circ*}$ will be denoted $T^{\circ*}(t)$.

We will need the following properties of C_0 -semigroups and their adjoints [2, 9, 17].

PROPOSITION 1.1. *Let $T(t)$ be a C_0 -semigroup on a Banach space X .*

- (1) *There exist real constants $M \geq 1$ and ω such that $\|T(t)\| \leq Me^{\omega t}$.*
- (2) *The adjoint semigroup $T^*(t) = (T(t))^*$ is weak*-continuous, that is,*

$$\lim_{t \downarrow 0} \langle T^*(t)x^* - x^*, x \rangle = 0$$

for all $x \in X$.

- (3) *X° is a norm-closed, weak*-dense subspace of X^* .*

PROPOSITION 1.2. *Let $T(t)$ be a semigroup on a Banach space X .*

- (1) *If the map $t \rightarrow T(t)x$ is measurable for all $x \in X$ then $T(t)$ is strongly continuous for $t > 0$.*
- (2) *If $T(t)$ is weakly continuous (that is, $\lim_{t \downarrow 0} \langle x^*, T(t)x - x \rangle = 0$ for all $x^* \in X^*$) then $T(t)$ is strongly continuous.*

A countable collection of closed subspaces $\{X_n\}_{n=1}^\infty$ of a Banach space X is called a *Schauder decomposition* of X if for every $x \in X$ there is a unique sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $x = \sum_{n=1}^\infty x_n$ and for each $n, x_n \in X_n$. A sequence $\{x_n\}_{n=1}^\infty$ in a Banach space X is called a *Schauder basis* (briefly, *basis*) if for every $x \in X$ there exists a unique sequence $\{\alpha_n\}_{n=1}^\infty$ of scalars such that $x = \sum_{n=1}^\infty \alpha_n x_n$. A basis $\{x_n\}_{n=1}^\infty$ is called *normalized* if $\|x_n\| = 1$ for all n . It is well-known that the *coordinate functionals* x_n^* defined by $\langle x_n^*, \sum_{n=1}^\infty \alpha_n x_n \rangle = \alpha_n$ are continuous. From this it is easy to see that the maps π_N and P_N defined by

$$\pi_N \sum_{n=1}^\infty \alpha_n x_n = \sum_{n=1}^N \alpha_n x_n, \quad P_N \sum_{n=1}^\infty \alpha_n x_n = \alpha_n x_n$$

are projections and $C = \sup_N \|\pi_N\| < \infty$. Hence if $\{x_n\}_{n=1}^\infty$ is normalized, then $\|x_n^*\| \leq 2C$ for all $n = 1, 2, \dots$. The constant C is called the *basis constant* of

$\{x_n\}_{n=1}^\infty$. Analogous definitions exist for Schauder decompositions. For instance, define π_N by $\pi_N \sum_{n=1}^\infty x_n = \sum_{n=1}^N x_n$. In this case the constant C will be called the *decomposition constant*.

A basis $\{x_n\}_{n=1}^\infty$ is called *shrinking* if the coordinate functionals $\{x_n^*\}_{n=1}^\infty$ form a basis of X^* . This is the case if and only if $\lim_{N \rightarrow \infty} \|x^*|_{[x_N, x_{N+1}, \dots]}\| = 0$ for every $x^* \in X^*$. Here $x^*|_{[x_N, x_{N+1}, \dots]}$ denotes the restriction of x^* to the closed linear span $[x_N, x_{N+1}, \dots]$ of $\{x_n\}_{n=N}^\infty$.

$\{x_n\}_{n=1}^\infty$ is called *boundedly complete* if the following holds: whenever the sequence $\{\|\sum_{n=1}^N \alpha_n x_n\|\}_{N=1}^\infty$ is bounded, then $\sum_{n=1}^\infty \alpha_n x_n$ actually converges to some $x \in X$ as $N \rightarrow \infty$.

$\{x_n\}_{n=1}^\infty$ is called *unconditional* if for every $x \in X$ the expansion $\sum_{n=1}^\infty \alpha_n x_n$ of x converges unconditionally, that is, for every permutation σ of the positive integers, $\sum_{n=1}^\infty \alpha_{\sigma(n)} x_{\sigma(n)}$ converges.

As an example, note that the standard unit vector basis of c_0 is unconditional and shrinking but not boundedly complete.

PROPOSITION 1.3. *Let $\{x_n\}_{n=1}^\infty$ be a basis of a Banach space X .*

(1) *$\{x_n\}_{n=1}^\infty$ is boundedly completely if and only if $\{x_n^*\}_{n=1}^\infty$ is a shrinking basis for its closed linear span $[x_n^*]$;*

(2) (M. Zippin [18]) *A Banach space X with a basis is reflexive if and only if every basis of X is shrinking if and only if every basis of X is boundedly complete;*

(3) *If $\{x_n\}_{n=1}^\infty$ is unconditional, then there is a constant $K > 0$ such that for every $t \in l^\infty$ and $x = \sum_{n=1}^\infty \alpha_n x_n \in X$,*

$$\left\| \sum_{n=1}^\infty t_n \alpha_n x_n \right\| \leq K \left(\sup_n |t_n| \right) \left\| \sum_{n=1}^\infty \alpha_n x_n \right\|.$$

Proofs may be found in [11] and [16].

A Banach space X is called a *Grothendieck space* if weak*-sequential convergence and weak sequential convergence in X^* coincide. Every reflexive space is trivially Grothendieck. It follows from Theorem A combined with Prop. 1.1 (2) and 1.2 (2) that Grothendieck spaces with a Schauder basis are reflexive. More generally, W.B. Johnson [10] proved that Grothendieck spaces with a Markusevich basis are reflexive, hence in particular separable Grothendieck spaces are reflexive.

A Banach space is said to have the *Dunford-Pettis property* if the following holds: whenever $\{x_n\}_{n=1}^\infty$ and $\{x_n^*\}_{n=1}^\infty$ are sequences in X and X^* respectively, such that $x_n \rightarrow 0$ weakly and $x_n^* \rightarrow 0$ weakly, then $\langle x_n^*, x_n \rangle \rightarrow 0$.

PROPOSITION 1.4. (H.P. Lotz [12]) *Every C_0 -semigroup on a Grothendieck space with the Dunford-Pettis property has a bounded generator.*

Let (Ω, Σ, μ) be a finite measure space. A Banach space X is said to have the *Radon-Nikodym property with respect to (Ω, Σ, μ)* if for every μ -continuous

vector-valued measure $G: \Sigma \rightarrow X$ of bounded variation there exists $g \in L^1(\mu; X)$ such that

$$G(E) = \int_E g d\mu$$

for all $E \in \Sigma$. X has the *Radon-Nikodym property* if it has the Radon-Nikodym property with respect to every finite measure space.

A bounded linear operator $S: L^1[0, 1] \rightarrow X$ is called *Riesz-representable* if there exists a $g \in L^\infty([0, 1]; X)$ such that

$$Sf = \int_0^1 f(t)g(t)dt \quad \text{for all } f \in L^1[0, 1].$$

We will need the following result [4, Thm III.1.5; Cor. V.3.8].

PROPOSITION 1.5. *X has the Radon-Nikodym property if and only if each bounded linear operator $S: L^1[0, 1] \rightarrow X$ is Riesz-representable.*

2. REFLEXIVITY AND SCHAUDER DECOMPOSITIONS

The main result of this section is Theorem 2.2 below. It asserts that in a Banach space with a Schauder decomposition there exist C_0 -semigroups with properties reflecting those of the decomposition in terms of which they are defined. Their construction is based on the following lemma, which is in [16, Thm. II.15.4].

LEMMA 2.1. *Let X be a Banach space with a Schauder decomposition $\{X_n\}_{n=1}^\infty$ with decomposition constant C . Let (γ_n) be a sequence of scalars such that*

$$\sum_{n=1}^\infty |\gamma_{n+1} - \gamma_n| < \infty.$$

Put $\gamma = \lim_n |\gamma_n|$. Then for all $x = \sum_{n=1}^\infty x_n \in X$ we have

$$\left\| \sum_{n=1}^\infty \gamma_n x_n \right\| \leq C \cdot \|x\| \cdot \left(\sum_{n=1}^\infty |\gamma_{n+1} - \gamma_n| + \gamma \right).$$

Let P_N be the canonical projection defined in section 1 and let $[P_n^* X^*]$ denote the closed linear span of the spaces $P_n^* X^*: n = 1, 2, \dots$

THEOREM 2.2. *Let X be a Banach space with a Schauder decomposition $\{X_n\}_{n=1}^\infty$ with decomposition constant C . Let $0 \leq k_1 < k_2 < \dots \rightarrow \infty$ be any sequence of numbers. Then*

$$T(t)x_n = e^{-k_n t} x_n \quad (x_n \in X_n)$$

defines a compact C_0 -semigroup on X which moreover satisfies:

- (a) $\|T(t)\| \leq C$ for all $t > 0$;
- (b) $X^\circ = [P_n^* X^*]$.

PROOF. Fix $x \in X$ of norm 1, $x = \sum_{n=1}^{\infty} x_n$ with $x_n \in X_n$. Let $\varepsilon > 0$ be arbitrary and take N such that

$$\left\| \sum_{n=N+1}^{\infty} x_n \right\| \leq \varepsilon.$$

Let $t_0 > 0$ be so small that

$$1 - e^{-k_N t_0} \leq \frac{\varepsilon}{N}.$$

Since $0 \leq k_1 < k_2 < \dots$ also

$$1 - e^{-k_n t} \leq \frac{\varepsilon}{N}, \quad (1 \leq n \leq N; 0 \leq t \leq t_0).$$

Then for $0 \leq t \leq t_0$ we have, using Lemma 2.1,

$$\begin{aligned} \|T(t)x - x\| &\leq \left\| \sum_{n=1}^N (1 - e^{-k_n t})x_n \right\| + \left\| \sum_{n=N+1}^{\infty} (1 - e^{-k_n t})x_n \right\| \\ &\leq N \cdot \frac{\varepsilon}{N} \cdot \max_{1 \leq n \leq N} \|x_n\| + C\varepsilon(1 - e^{-k_{N+1}t} + \sum_{n=N+1}^{\infty} |e^{-k_n t} - e^{-k_{n+1}t}| + 1) \\ &\leq 2\varepsilon C + 2\varepsilon C = 4\varepsilon C. \end{aligned}$$

This shows that $T(t)$ is a C_0 -semigroup on X .

Note that by Lemma 2.1 we have

$$\|T(t)\| \leq C \cdot \sum_{n=1}^{\infty} (e^{-k_n t} - e^{-k_{n+1}t}) = C \cdot e^{-k_1 t} \leq C.$$

This is (a).

It is obvious that $[P_n^* X^*] \subset X^\odot$ since on $P_n^* X^*$ we have $T^*(t)x_n^* = e^{-k_n t} x_n^*$. To prove the reverse inclusion, let $x^* = \text{weak}^* \sum_{n=1}^{\infty} x_n^*$, with $x_n^* \in P_n^* X^*$. We claim that the weak*-sum $T^*(t)x^* = \text{weak}^* \sum_{n=1}^{\infty} e^{-k_n t} x_n^*$ is actually strongly convergent for every $t > 0$. Indeed, for every $x = \sum_{n=1}^{\infty} x_n$ we have by Lemma 2.1

$$\left| \left\langle \sum_{n=N}^M e^{-k_n t} x_n^*, \sum_{n=1}^{\infty} x_n \right\rangle \right| = \left| \left\langle \sum_{n=N}^M x_n^*, \sum_{n=N}^M e^{-k_n t} x_n \right\rangle \right| \leq 2C \|x^*\| \cdot 2e^{-k_N t} \|x\|.$$

Hence

$$\left\| \sum_{n=N}^M e^{-k_n t} x_n^* \right\| \leq 4C e^{-k_N t} \|x^*\|.$$

Since $k_N \rightarrow \infty$ as $N \rightarrow \infty$ we have shown that for $t > 0$ the sequence

$$\left\{ \sum_{n=1}^N e^{-k_n t} x_n^* \right\}_{N=1}^{\infty}$$

is Cauchy in X^* . From this it follows that $T^*(t)x^* \in [P_n^* X^*]$ for $t > 0$. Now should $x^* \in X^\odot$, then $x^* = \lim_{t \downarrow 0} T^*(t)x^*$ and by the closedness of $[P_n^* X^*]$ it

follows that we must have $x^* \in [P_n^* X^*]$. This shows $X^\odot \subset [P_n^* X^*]$ and (b) is proved.

Finally note that for fixed $t > 0$,

$$T(t) = \lim_{N \rightarrow \infty} \sum_{n=1}^N e^{-k_n t} P_n$$

in the uniform operator topology. This is shown in the same way as we did in (b), again using Lemma 2.1. Since each P_n is compact it follows that $T(t)$ is a compact semigroup.

In 2.3, 2.4 and 4.2 we will give examples how information on bases may be derived from the semigroups defined in the above theorem.

COROLLARY 2.3. (*D.W. Dean [3]*) *Grothendieck spaces with the Dunford-Pettis property do not admit a Schauder decomposition.*

PROOF. If X has a Schauder decomposition then the C_0 -semigroup $T(t)$ defined in Theorem 2.2 has a generator A given by

$$Ax_n = -k_n x_n$$

which is unbounded, since the sequence (k_n) is unbounded. Now apply Prop. 1.4.

REMARK 2.4. A countable collection of subspaces $\{X_n\}_{n=1}^\infty$ of a Banach space X is called a *weak decomposition* of X if for every $x \in X$ there is a unique sequence $\{x_n\}_{n=1}^\infty \subset X$ with $x_n \in X_n$, such that $x = \text{weak } \sum_{n=1}^\infty x_n$. If moreover the canonical projections π_n are weakly continuous, then $\{X_n\}_{n=1}^\infty$ is called a *weak Schauder decomposition*. It is well-known (see [16]) that a weak decomposition is a weak Schauder decomposition if and only if each X_n is closed. If X is a Banach space with a weak Schauder decomposition $\{X_n\}_{n=1}^\infty$ it is possible to define weakly continuous semigroups on X as we did in Theorem 2.2. One must be somewhat more careful since for weak decompositions one cannot use Lemma 2.1. Define $\varepsilon_m = 1/(m \cdot 2^m)$ ($m = 1, 2, \dots$). Put $k_1 = 1$. Let $t_1 > 0$ be defined by

$$e^{-k_1 t_1} = 1 - \varepsilon_1.$$

Suppose k_1, k_2, \dots, k_{m-1} and t_1, t_2, \dots, t_{m-1} have been chosen. Choose $k_m \in \mathbb{N}$, $k_m \geq k_{m-1} + 1$ such that

$$\frac{e^{-k_m t_{m-1}}}{1 - e^{-t_{m-1}}} < \frac{1}{2^m}.$$

Let t_m be defined by

$$e^{-k_m t_m} = 1 - \varepsilon_m.$$

Observe that $t_1 > t_2 > \dots \rightarrow 0$ and $1 = k_1 < k_2 < \dots$. It is not difficult to check that

$$T(t)x_n = e^{-k_n t} x_n$$

defines a weakly continuous semigroup on X . By Proposition 1.2 this semigroup is actually strongly continuous. But then straightforward estimates show that

$$\|x - \sum_{n=1}^N x_n\| \leq \|T(t_N)x - x\| + \|T(t_N)x - \sum_{n=1}^N x_n\| \rightarrow 0 \quad (N \rightarrow \infty).$$

In fact we have shown that $\{x_n\}_{n=1}^\infty$ is actually a (*strong*) Schauder decomposition. This is a result of W.H. Ruckle. [15]

THEOREM A. *Let X be a Banach space with a Schauder basis. The following statements are equivalent:*

- (1) X is reflexive;
- (2) For every C_0 -semigroup $T(t)$ on X , the adjoint semigroup $T^*(t)$ is strongly continuous;
- (3) For every C_0 -semigroup $T(t)$ on X , the second adjoint semigroup $T^{\circ*}(t)$ is strongly continuous.

PROOF. (1) \Rightarrow (2) is Phillips's theorem, from which also (1) \Rightarrow (3) follows. We have to prove (2) \Rightarrow (1) and (3) \Rightarrow (1). Suppose X is nonreflexive. Applying Prop. 1.3 (2), let $\{x_n\}_{n=1}^\infty$ be a nonshrinking basis of X ; let $T(t)$ be the C_0 -semigroup on X as in Theorem 2.2. By (b) of Theorem 2.2 and the definition of a shrinking basis we have $X^\circ = [x_n^*] \neq X^*$, that is, the adjoint semigroup $T^*(t)$ is not strongly continuous on X^* . This gives (2) \Rightarrow (1). Next, again assume that X is nonreflexive and let $\{x_n\}_{n=1}^\infty$ be a nonboundedly complete basis of X ; let $T(t)$ be the C_0 -semigroup on X as in Theorem 2.2. It follows by Prop. 1.3 (1) that $\{x_n^*\}_{n=1}^\infty$ is a nonshrinking basis of $[x_n^*] = X^\circ$ and hence by the same argument $X^{\circ\circ} = [x_n^{**}] \neq X^{\circ*}$, proving (3) \Rightarrow (1).

Theorem A does not hold for arbitrary Banach spaces. For instance, let $X = L^\infty[0, 1]$ or more generally any Grothendieck space with the Dunford-Pettis property. Since every C_0 -semigroup on X has a bounded generator, it is obvious that the adjoint of such a semigroup is strongly continuous and has a bounded generator as well. Note that these spaces always are nonseparable [10]. Therefore one still may ask whether Theorem A holds for arbitrary *separable* Banach spaces X , since not every separable Banach space has a basis [6]. For instance, it is known [11] that c_0 and l^1 contain subspaces Y without a basis. In these two cases however the answer is easy, since Y contains a complemented subspace Z isomorphic to c_0 or l^1 respectively [11]. On Z we may construct a C_0 -semigroup whose adjoint is not strongly continuous; this semigroup can be extended to Y by putting it identically 1 on the complement of Z . Hence, Theorem A holds for closed subspaces of c_0 and l^1 .

By a theorem of A. Pelczynski [13] a Banach space is reflexive if and only if every closed subspace with a basis is. This, in combination with Theorem A, gives the following corollary.

COROLLARY 2.5. *A Banach space X is reflexive if and only if for every closed subspace Y of X , every C_0 -semigroup $T(t)$ on Y has a strongly continuous adjoint $T^*(t)$ on Y^* .*

3. THE RADON-NIKODYM PROPERTY AND UNCONDITIONAL BASES

LEMMA 3.1. *Every weak*-continuous semigroup $T(t)$ on a dual Banach space X^* with the Radon-Nikodym property is strongly continuous for $t > 0$.*

PROOF. Fix an arbitrary $x^* \in X^*$. By the uniform boundedness theorem, there is an $M < \infty$ such that $\|T(t)x^*\| \leq M$ for all $t \in [0, 1]$. Define $S: L^1[0, 1] \rightarrow X^*$ by

$$Sg = \text{weak}^* \int_0^1 g(t)T(t)x^* dt.$$

Since $\langle T(t)x^*, x \rangle$ is continuous for each $x \in X$, it follows that $\langle g(t)T(t)x^*, x \rangle \in L^1[0, 1]$ for all $x \in X$, and the above integral is well-defined. S is bounded:

$$\|Sg\| = \sup_{\|x\|=1} \left| \int_0^1 \langle g(t)T(t)x^*, x \rangle dt \right| \leq \sup_{\|x\|=1} \int_0^1 |g(t)| |\langle T(t)x^*, x \rangle| dt \leq M \|g\|_1.$$

Since X^* has the Radon-Nikodym property, by Proposition 1.5 there is an $h \in L^\infty([0, 1]; X^*)$ such that

$$Sg = \int_0^1 g(t)h(t) dt$$

for all $g \in L^1[0, 1]$. For $0 \leq t < 1$ and $\varepsilon > 0$ small enough, let $E = [t, t + \varepsilon]$ and put $g = (1/\varepsilon)\chi_E$, where χ is the characteristic function. It follows that

$$\text{weak}^* \int_t^{t+\varepsilon} \frac{1}{\varepsilon} T(\tau)x^* d\tau = \int_t^{t+\varepsilon} \frac{1}{\varepsilon} h(\tau) d\tau.$$

By the Lebesgue differentiation theorem, for almost all $t \in [0, 1]$ the right-hand side converges to $h(t)$ as $\varepsilon \rightarrow 0$. Hence, for such t we have

$$\frac{1}{\varepsilon} \int_t^{t+\varepsilon} \langle T(\tau)x^*, x \rangle d\tau \rightarrow \langle h(t), x \rangle \quad (\varepsilon \rightarrow 0)$$

for all $x \in X$. But the integrand on the left-hand side is continuous, and therefore the integral converges to $\langle T(t)x^*, x \rangle$. So $T(t)x^* = h(t)$ a.e. In particular, $T(t)x^*$ is measurable on $[0, 1]$, hence on $[0, \infty)$. It follows from Prop. 1.2 (1) that $T(t)$ is strongly continuous for $t > 0$.

If $T(t)$ in Lemma 3.1 is an *adjoint* semigroup, the above result is implicit in

W. Arendt [1], where it is obtained by an entirely different method of proof.

It is classical result of N. Dunford and B.J. Pettis [5] that separable duals have the Radon-Nikodym property. For such spaces the above lemma is much easier to prove. Indeed, by Pettis's measurability theorem [4, Cor. II.1.4], for each $x^* \in X^*$ the map $t \rightarrow T^*(t)x^*$ is strongly measurable. Now apply Prop. 1.2 (1).

Every nonreflexive Banach space X with a basis admits a C_0 -semigroup whose adjoint is strongly continuous *precisely* for $t > 0$. In fact, the semigroup from the proof of Theorem A, (2) \Rightarrow (1), will do, as is easily verified. However, this is a rather non-constructive example. The following example is adapted from [1], where it is credited to H.P. Lotz.

EXAMPLE 3.2. Let J be the James space consisting of all sequences of scalars $x = (a_1, a_2, \dots)$ for which

$$\|x\| = \sup[(a_{p_1} - a_{p_2})^2 + (a_{p_2} - a_{p_3})^2 + \dots + (a_{p_{m-1}} - a_{p_m})^2 + (a_{p_m} - a_{p_1})^2]^{1/2} < \infty$$

and

$$\lim_{n \rightarrow \infty} a_n = 0,$$

where the sup is taken over all possible choices of integers m and $p_1 < p_2 < \dots < p_m$. Let x_n denote the n th unit vector. On J define a C_0 -semigroup $T(t)$ by

$$T(t)x_n = e^{-nt}x_n.$$

Since $\{x_n\}_{n=1}^\infty$ is a shrinking basis for J , the unit vectors x_n^* of J^* form a basis for J^* and we have $J^\odot = J^*$. One can show that J^{**} is isomorphic to $J \oplus \mathbb{C}e$, where $e = (1, 1, \dots)$. Consequently J^{**} is separable and therefore has the Radon-Nikodym property. Hence $T^{**}(t)$ is strongly continuous for $t > 0$ by Lemma 3.1. It follows from Theorem 2.2 (b) (applied to the C_0 -semigroup $T^*(t)$ on J^*) that $e \notin J^{\odot}$. Therefore $T^{**}(t)$ is not strongly continuous at $t = 0$.

This example is interesting for another reason. There are many examples of C_0 -semigroups on Banach spaces X such that $\dim X^*/X^\odot = \infty$. The above example shows that X^\odot can also have any *finite* codimension in X^* :

COROLLARY 3.3. *For each $n \in \mathbb{N}$ there exists a Banach space X and a C_0 -semigroup $T(t)$ on X such that $\dim X^*/X^\odot = n$.*

PROOF. If $n = 0$, let $T(t)$ be any C_0 -semigroup on a reflexive space. Otherwise, consider the C_0 -semigroup $T^*(t)$ on J^* from Example 3.2. Since $J^{**} = J \oplus \mathbb{C}e = J^{\odot} \oplus \mathbb{C}e$ we see that $\dim J^{**}/J^{\odot} = 1$. Let $X = J^* \times J^* \times \dots \times J^*$, n times, together with the 'product' semigroup obtained from n copies of $T^*(t)$.

THEOREM B. *Let X be a Banach space with an unconditional basis $\{x_n\}_{n=1}^\infty$. The following statements are equivalent:*

- (1) X^* has the Radon-Nikodym property;
 (2) Every adjoint semigroup on X^* is strongly continuous for $t > 0$.

PROOF. (1) \Rightarrow (2) follows from Lemma 3.1. It therefore remains to be shown that (2) \Rightarrow (1) holds. We already remarked that separable duals have the Radon-Nikodym property. Hence it suffices to show the unconditional basis $\{x_n\}_{n=1}^\infty$ of X is shrinking. Suppose the contrary is true. Then there are $x_0^* \in X^*$, $\|x_0^*\| = 1$ and $0 < \varepsilon < 1$ such that

$$\lim_N \|x_0^*|_{[x_N, x_{N+1}, \dots]}\| > \varepsilon.$$

Choose inductively a sequence of integers $0 = N_0 < N_1 < \dots$ and a sequence $\{y_k\}_{k=1}^\infty \subset X$ of norm-1 vectors as follows. Let $z_1 = \sum_{n=1}^\infty \alpha_{1n} x_n$ be any norm-1 vector such that

$$|\langle x_0^*, z_1 \rangle| > \varepsilon.$$

Choose N_1 sufficiently large such that

$$|\langle x_0^*, \sum_{n=1}^{N_1} \alpha_{1n} x_n \rangle| > \varepsilon.$$

Put $y_1 = \sum_{n=1}^{N_1} \alpha_{1n} x_n$. We may, by choosing N_1 large enough, multiply y_1 with an appropriate scalar so as to make a norm-1 vector of it without affecting the above inequality. Choose $z_2 = \sum_{n=N_1+1}^\infty \alpha_{2n} x_n \in [x_{N_1+1}, x_{N_1+2}, \dots]$ of norm 1 such that

$$|\langle x_0^*, z_2 \rangle| > \varepsilon.$$

Choose N_2 such that

$$|\langle x_0^*, \sum_{n=N_1+1}^{N_2} \alpha_{2n} x_n \rangle| > \varepsilon.$$

Define $y_2 = \sum_{n=N_1+1}^{N_2} \alpha_{2n} x_n$ and again assume without loss of generality that y_2 has norm 1. Continue in this way. By construction of the y_n we have for all n ,

$$|\langle x_0^*, y_n \rangle| > \varepsilon.$$

For $N_{m-1} < n \leq N_m$ define

$$T(t)x_n = e^{imt}x_n,$$

where x_n is the n th basis vector. By Prop. 1.3 (3), there is a $K > 0$ such that $\|T(t)\| \leq K$ for all $t \geq 0$. From this it is easy to see that $T(t)$ is a C_0 -semigroup on X . Now let $t > 0$ be arbitrary and fixed. We will show that $T^*(t)x_0^* \notin X^\odot$. Let $m \in \mathbb{N}$, $m \geq 1$. By the irrationality of the number π , we can find a positive integer k such that

$$|1 - e^{i(k/m)}| > 2 - \varepsilon.$$

We have the following estimates.

$$\left\| T^*\left(t + \frac{1}{m}\right)x_0^* - T^*(t)x_0^* \right\| \geq \left| \left\langle T^*\left(t + \frac{1}{m}\right)x_0^* - T^*(t)x_0^*, y_k \right\rangle \right| = \\ |e^{ik(t + 1/m)} - e^{ikt}| \cdot |\langle x_0^*, y_k \rangle| \geq (2 - \varepsilon) \cdot \varepsilon.$$

This proves Theorem B.

It is natural to ask whether an analogue of Corollary 2.5 holds for Banach spaces whose dual have the Radon-Nikodym property. H.P. Lotz's theorem on l^1 in Banach lattices [7] shows that for Banach lattices this is indeed the case: If the dual of a Banach lattice does not have the Radon-Nikodym property, then X contains a copy of l^1 ; on l^1 we have a C_0 -semigroup whose adjoint is not strongly continuous for $t > 0$ by Theorem B. For general Banach spaces we remark that J. Hagler [8] proved that a separable Banach space with a nonseparable dual has a subspace with a basis whose dual is nonseparable. Therefore it would be enough to prove Theorem B, (2) \Rightarrow (1), without the assumption that the basis of X should be unconditional. (note that we made a rather crude step at this stage in just using that the basis of a space with nonseparable dual necessarily must be nonshrinking). The following theorem shows that in order to solve this problem, it suffices to construct a C_0 -semigroup on X whose adjoint has a nonseparable orbit.

THEOREM 3.4. *Let $T(t)$ be a C_0 -semigroup on a Banach space X . Let $x^* \in X^*$. The orbit $\{T^*(t)x^*: t \geq 0\}$ is separable if and only if $t \rightarrow T^*(t)x^*$ is strongly continuous for $t > 0$ if and only if $t \rightarrow T^*(t)x^*$ is weakly continuous for $t > 0$.*

PROOF. It is obvious that strong continuity implies weak continuity. If $t \rightarrow T^*(t)x^*$ is weakly continuous for $t > 0$ then it is certainly weakly separable, which is the same as strongly separable. Suppose $\{T^*(t)x^*: t \geq 0\}$ is separable. The proof that the map $t \rightarrow T^*(t)x^*$ is strongly continuous for $t > 0$ is a slight modification of the argument given in [9, Thm 10.3.2]. Choose numbers $0 < \alpha < \tau < \beta < \xi$ and let η be so small that $\beta < \xi - \eta$. Now $T^*(\xi)x^* = T^*(\tau)T^*(\xi - \tau)x^*$ is independent of τ , hence certainly integrable on $[\alpha, \beta]$ with respect to τ . Therefore

$$(\beta - \alpha)[T^*(\xi \pm \eta) - T^*(\xi)]x^* = \int_{\alpha}^{\beta} T^*(\tau)[T^*(\xi \pm \eta - \tau) - T^*(\xi - \tau)]x^* d\tau.$$

The norm of the integrand is majorized by $2M\|x^*\|$, where M is such that $\|T^*(t)\| = \|T(t)\| \leq M$ on $[0, \xi + \eta]$. Since $\tau \rightarrow [T^*(\xi \pm \eta - \tau) - T^*(\xi - \tau)]x^*$

is measurable (by Pettis' measurability theorem), so is $\| [T^*(\xi \pm \eta) - T^*(\xi - \tau)]x^* \|$. This gives

$$(\beta - \alpha) \| [T^*(\xi \pm \eta) - T^*(\xi)]x^* \|$$

$$\leq M \int_{\xi - \beta}^{\xi - \alpha} \| [T^*(\sigma \pm \eta) - T^*(\sigma)]x^* \| d\sigma \rightarrow 0 \quad (\eta \rightarrow 0);$$

see [9, Thm 3.8.3].

THEOREM 3.5. *Let $T(t)$ be a C_0 -semigroup on a Banach space X . Let $x^* \in X^*$. Then $t \rightarrow T^*(t)x^*$ is strongly continuous for $t \geq 0$ if and only if $t \rightarrow T^*(t)x^*$ is weakly continuous for $t \geq 0$.*

PROOF. We only have to prove the 'if' part. If $T^*(t)$ is an adjoint semigroup, then there is a positive M such that $\| T^*(t) \| \leq M$ in a neighbourhood of $t = 0$ (since such an estimate holds for its predual $T(t)$). Now the proof can be finished in exactly the same way as in [17, Ch. IX, 1].

These two theorems can be considered as the 'orbitwise' analogous for adjoint semigroups of Prop. 1.2. The point of their proofs is that we have bounds on $T^*(t)$ beforehand, since we are dealing with *adjoint* semigroups.

4. NONSHRINKING BASES IN c_0

Theorem A guarantees the existence of a C_0 -semigroup without strongly continuous adjoint on the nonreflexive space c_0 (and, more generally, on every separable Banach space containing c_0 , since by A. Sobczyk's theorem [11], c_0 is complemented in such spaces). The following theorem shows that it can be hard to give an explicit example of such a semigroup.

THEOREM 4.1. *Let $T(t)$ be a C_0 -semigroup on c_0 ; $\| T(t) \| \leq Me^{\omega t}$. If $M < 2$, then $T^*(t)$ is strongly continuous on l^1 .*

PROOF. Choose $\varepsilon > 0$ such that $M - 1 + \varepsilon < 1$. Let $x_0 = \sum_n \alpha_n e_n \in l^1$ be arbitrary (e_n denoting the n th unit vector of l^1); $\| x_0 \| = 1$. Let N be such that $\| \sum_{n=N+1}^{\infty} \alpha_n e_n \| < \varepsilon/5$. Choose $t_1 > 0$ so small that $\| T^*(t_1)x_0 \| \leq M + \varepsilon/5$ and $| (T^*(t_1)x_0 - x_0)_n | \leq \varepsilon/(5N)$ ($n = 1, 2, \dots, N$). Such t_1 exists by the weak*-continuity of the map $t \rightarrow T^*(t)x_0$ and by the estimate $\| T(t) \| \leq Me^{\omega t}$. We have

$$\sum_{n=1}^N | (T^*(t_1)x_0)_n | \geq \sum_{n=1}^N | (x_0)_n | - \sum_{n=1}^N | (T^*(t_1)x_0 - x_0)_n |$$

$$\geq 1 - \frac{\varepsilon}{5} - N \cdot \frac{\varepsilon}{5N} = 1 - \frac{2\varepsilon}{5}.$$

Therefore

$$\begin{aligned} \|x_0 - T^*(t_1)x_0\| &= \sum_{n=1}^N |(T^*(t_1)x_0 - x_0)_n| + \sum_{n=N+1}^{\infty} |(T^*(t_1)x_0 - x_0)_n| \\ &\leq \frac{\varepsilon}{5} + \sum_{n=N+1}^{\infty} |(T^*(t_1)x_0)_n| + \sum_{n=N+1}^{\infty} |(x_0)_n| \\ &\leq \frac{\varepsilon}{5} + \left(\|T^*(t_1)x_0\| - \left(1 - \frac{2\varepsilon}{5}\right) \right) + \frac{\varepsilon}{5} \leq M - 1 + \varepsilon. \end{aligned}$$

Put $x_1 = x_0 - T^*(t_1)x_0$. In the same way, there is an $t_2 > 0$ such that

$$\|x_1 - T^*(t_2)x_1\| \leq (M - 1 + \varepsilon) \|x_1\| \leq (M - 1 + \varepsilon)^2.$$

Put $x_2 = x_1 - T^*(t_2)x_1$. Proceed with the construction inductively in the obvious way. After n steps, we have $t_1, t_2, \dots, t_n > 0$ and vectors x_1, x_2, \dots, x_n such that $x_n = x_{n-1} - T^*(t_n)x_{n-1}$ and

$$\begin{aligned} \|x_0 - T^*(t_1)x_0 - T^*(t_2)x_1 - \dots - T^*(t_n)x_{n-1}\| \\ = \|x_{n-1} - T^*(t_n)x_{n-1}\| \leq (M - 1 + \varepsilon)^n \end{aligned}$$

Since l^1 has the Radon-Nikodym property, by Lemma 3.1 we get that $T^*(t_i)x_{i-1} \in (c_0)^\circ$ for all $i = 1, 2, \dots$. Since $(M - 1 + \varepsilon)^n \rightarrow 0$ as $n \rightarrow \infty$ we have proved that x_0 is in the closure of $(c_0)^\circ$. By 1.1 (3), $(c_0)^\circ$ is closed and therefore $x_0 \in (c_0)^\circ$. Hence $(c_0)^* = l^1 = (c_0)^\circ$, as was to be shown.

We noted that the standard unit vector basis of c_0 is shrinking. Of course, this basis has basis constant $C = 1$. By M. Zippin's theorem we are told that there exists a nonshrinking basis for c_0 , since c_0 is nonreflexive. What can be said of the basis constant of such a basis?

COROLLARY 4.2. *Every nonshrinking basis of c_0 has basis constant $C \geq 2$.*

PROOF. Let $\{x_n\}_{n=1}^\infty$ be nonshrinking basis of c_0 with basis constant C . Let $T(t)$ be the C_0 -semigroup, defined with respect to $\{x_n\}_{n=1}^\infty$, as in Theorem A. Then $T^*(t)$ is not strongly continuous. By Theorem 2.2, $\|T(t)\| \leq C$. Now by Theorem 4.1 we must have $C \geq 2$.

The results of Theorem 4.1 and Corollary 4.2 are optimal: let z_i denote the i th unit vector of c_0 and put $y_n = \sum_{i=1}^n z_i$, then the basis $\{y_n\}_{n=1}^\infty$ is nonshrinking and has basis constant 2. Moreover, the semigroup $T(t)$ as defined in Theorem 2.2 satisfies $\|T(t)\| \leq 2$ and has an adjoint which is not strongly continuous. Using a very different approach, another example of such a semigroup on c_0 was constructed by A. Di Bucchianico and A.J. Stam [private communication].

ACKNOWLEDGEMENT

I would like to thank Profesor D. van Dulst for many interesting and stimulating discussions and Ben de Pagter who suggested many improvements and pointed out to me Lemma 2.1.

REFERENCES

1. Arendt, W. – Vector valued Laplace transforms and Cauchy problems, Semesterbericht Funktionalanalysis, Universität Tübingen, Band 10, 1–33 (1986).
2. Butzer, P.L. and H. Berens – Semigroups of operators and approximation, Springer Verlag, Berlin-Heidelberg-New York (1967).
3. Dean, D.W. – Schauder decompositions in (m) , Proc. Amer. Math. Soc., 18, 619–623 (1967).
4. Diestel, J. and J.J. Uhl – Vector measures, Math. Surveys nr. 15, Amer. Math. Soc., Providence, R.I. (1977).
5. Dunford, N. and B.J. Pettis – Linear operations on summable functions, Trans. Amer. Math. Soc. 47, 323–392 (1940).
6. Enflo, P. – A counterexample to the approximation problem, Acta Math. 130, 309–317 (1973).
7. Ghoussoub, N. and E. Saab – On the weak Radon-Nikodym property, Proc. Amer. Math. Soc. 81, 81–84 (1981).
8. Hagler, J. – A note on separable Banach spaces with nonseparable dual, Proc. Amer. Math. Soc. 99, 452–454 (1981).
9. Hille, E. and R.S. Phillips – Functional Analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., vol 31, Amer. Math. Soc., rev. ed., Providence, R.I. (1957).
10. Johnson, W.B. – No infinite-dimensional P space admits a Markusevich basis, Proc. Amer. Math. Soc. 26, 467–468 (1970).
11. Lindenstrauss, J. and L. Tzafriri – Classical Banach spaces I. Springer Verlag, Berlin-Heidelberg-New York (1977).
12. Lotz, H.P. – Uniform convergence of operators on L^∞ and similar spaces, Math. Z. 190, 207–220 (1985).
13. Pelczynski, A. – A note on the paper of I. Singer “Basic sequences and reflexivity of Banach spaces”, Studia Math. 21, 371–374 (1962).
14. Phillips, R.S. – The adjoint semi-group, Pac. J. Math. 5, 269–283 (1955).
15. Ruckle, W.H. – The infinite sum of closed subspaces of an F -space, Duke Math. J. 31, 543–554 (1964).
16. Singer, I. – Bases in Banach spaces I and II, Springer Verlag, Berlin-Heidelberg-New York (1970, 1981).
17. Yosida, K. – Functional analysis, Springer Verlag, Berlin-Göttingen-Heidelberg (1965).
18. Zippin, M. – A remark on bases and reflexivity in Banach spaces, Isr. J. Math. 6, 74–79 (1968).