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A pointwise criterion for controller robustness

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Abstract: We present a pointwise criterion for controller robustness with respect to stability. The term 'point' here refers to complex frequency in the right half plane. The proposed test is based on the concept of the minimal angle between subspaces determined by the plant and the compensator. The test leads to separate balls of uncertainty at each frequency, and may therefore help to reduce conservativeness in the analysis of robustness.

Keywords: Stability; robustness; gap metric; minimal angle.

1. Introduction

Given a plant and a stabilizing controller for it, one defines the robustness of the controller (with respect to stability) as the smallest perturbation of the plant which may cause the closed-loop system to become unstable. Of course, this definition depends on the measure that is chosen for the perturbations. Several distance notions for linear time-invariant systems have been proposed, of which the so-called gap metric [9,38] has gained much popularity because it is relatively easy to compute [13] and lends itself well to optimization [14,15]. However, the gap between two systems is a single number, whereas the uncertainty of a model is often seen as a frequency-dependent quantity. A certain amount of frequency dependence can be obtained by introducing suitable weight functions, as for instance in [29]. Here we shall propose a criterion which addresses the dependence on frequency directly by defining a separate ball of allowable uncertainty at every point in the closed right half plane. The proof of the criterion is very simple; nevertheless, it is suggested that the proposed test is a natural and useful tool in frequency-dependent robustness analysis.

2. Robustness of complementarity

The robustness criterion to be presented below will be based on distance notions for subspaces of a finite-dimensional unitary space. In particular, we shall be interested in conditions which will guarantee that two complementary subspaces remain complementary when one of the subspaces is perturbed. To measure the size of the perturbation, we shall use the gap function introduced in [35] and [23]. Let X be a (real or complex) Hilbert space and let Y_1 and Y_2 be closed subspaces of X. Denote the orthogonal projections onto Y_1 and Y_2 by P_1 and P_2 respectively. The *gap* $\delta(Y_1, Y_2)$ between Y_1 and Y_2 is defined by

$$\delta(Y_1, Y_2) = \| P_1 - P_2 \| \tag{2.1}$$

or equivalently by

$$\delta(Y_1, Y_2) = \max\left(\vec{\delta}(Y_1, Y_2), \vec{\delta}(Y_2, Y_1)\right) \tag{2.2}$$

where

$$\vec{\delta}(Y_1, Y_2) = \|(I - P_2)|_{Y_1}\| = \sup_{y_1 \in Y_1,\ \|y_1\| = 1} \ \inf_{y_2 \in Y_2} \|y_1 - y_2\|. \tag{2.3}$$

For a proof that the two expressions are indeed the same, see [1, §34], or [22, §15.3]. The final expression in (2.3) is used as a definition in the context of Banach spaces [21,24].

To express a result on robustness of complementarity, we need the notion of a 'minimal angle' between subspaces. The study of angles between subspaces of a Euclidean space goes back to Camille Jordan [20], who showed that the mutual position of two complementary subspaces in Euclidean space is characterized completely by a finite number of invariants which he called the *angles* between the subspaces. The subject has drawn interest not only from geometers [11,34] but also from operator theorists [16], numerical analysts (both as an analysis tool [6,7] and as a subject of computation [4,18]) and, in connection with the subject of canonical correlations, from statisticians [2,8 (Ch. 5)]. Of particular interest is the *smallest* angle between two subspaces.

Definition 2.1 [16, p. 339]. The *minimal angle* $\phi(Y, Z)$ between two nontrivial subspaces Y and Z of a Hilbert space H is defined by

$$\cos\phi(Y, Z) = \sup_{\substack{y\in Y,\ z\in Z\\ \|y\|=\|z\|=1}} |\langle y, z\rangle|,$$

$$0 \leqslant \phi(Y, Z) \leqslant \tfrac{1}{2}\pi. \tag{2.4}$$

Taking into account the fact that for each $z_0 \in Z$ we have

$$\|z_0\| = \sup_{z\in Z,\ \|z\|=1} |\langle z_0, z\rangle| \tag{2.5}$$

the expression in (2.4) can be rewritten as follows:

$$\cos\phi(Y, Z)$$

$$= \sup_{y\in Y,\ \|y\|=1}\ \sup_{z\in Z,\ \|z\|=1} |\langle P_Z y, z\rangle|$$

$$= \sup_{y\in Y,\ \|y\|=1} \|P_Z y\| = \|P_Z|_Y\|, \tag{2.6}$$

where P_Z denotes the orthogonal projection onto Z. An advantage of the definition as given in (2.4) is that it clearly shows that

$$\phi(Y, Z) = \phi(Z, Y). \tag{2.7}$$

In view of (2.3) and (2.6), there is a simple relation between the minimal angle and the gap:

$$\vec{\delta}(Y_1, Y_2) = \cos\phi(Y_1, Y_2^{\perp}). \tag{2.8}$$

Robustness of complementarity is most conveniently expressed in terms of $\sin\phi = (1 - \cos^2\phi)^{1/2}$ rather than in terms of $\cos\phi$. Since for $\|y\| = 1$ we obviously have

$$\|P_Z y\|^2 + \|(I - P_Z)y\|^2 = 1, \tag{2.9}$$

we have from (2.6),

$$\sin\phi(Y, Z) = \inf_{y\in Y,\ \|y\|=1} \|(I-P_Z)y\|$$

$$= \inf_{y\in Y,\ \|y\|=1}\ \inf_{z\in Z} \|y - z\|. \tag{2.10}$$

The minimum of the latter expression and the same expression with the roles of Y and Z interchanged was proposed in [17] as a definition of the sine of the minimal angle between two nonzero subspaces of a Banach space.

We shall use the above definitions mainly in *finite-dimensional* spaces. In this case one may replace 'sup' by 'max' and 'inf' by 'min' everywhere. The following result on robustness of complementarity is a special case of a theorem due to Berkson [3, Thm. 5.2].

Proposition 2.2. *Let Y and Z be complementary subspaces of a finite-dimensional normed linear space X. Every subspace Y' that satisfies*

$$\delta(Y, Y') < \sin\phi(Y, Z) \tag{2.11}$$

is complementary to Z.

The proof of this special case is simple. Note first of all that (2.11) implies $\delta(Y, Y') < 1$, so that $\dim Y = \dim Y'$ (see [21, Cor.IV.2.6]). So it is sufficient to show that Y' and Z have nonzero intersection. Suppose to the contrary that Y' and Z would intersect nontrivially; then there would be a $z_0 \in Y' \cap Z$ with $\|z_0\| = 1$. But then one would have, by (2.10) and (2.3),

$$\sin\phi(Y, Z) \leqslant \inf_{y\in Y} \|z_0 - y\| \leqslant \delta(Y, Y'). \tag{2.12}$$

We shall now even further specialize our discussion and consider unitary spaces. It is then not difficult to show that the bound given by Berkenson is sharp.

Theorem 2.3. *Let Y and Z be complementary nontrivial subspaces of a unitary space X. We have*

$$\inf_{Y'\cap Z\neq\{0\}} \delta(Y, Y') = \sin\phi(Y, Z). \tag{2.13}$$

Proof. It follows from Proposition 2.2 that the left hand side in the above equation cannot be less than the right hand side; so it suffices to construct a subspace Y' that has a nontrivial intersection with Z and whose distance to Y, as measured by the gap, is equal to $\sin \phi(Y, Z)$. This will then also show that the infimum in (2.13) is actually a minimum.

Take $z_0 \in Z$ with $\| z_0 \| = 1$ such that

$$\| z_0 - P_Y z_0 \| = \sin \phi(Y, Z). \tag{2.14}$$

Define $y_0 = P_Y z_0$, and $Y_0 = \operatorname{span}\{y_0\}$. Let Y_1 denote the orthogonal complement of Y_0 in Y. For every $y_1 \in Y_1$, we have

$$\langle z_0, y_1 \rangle = \langle y_0, y_1 \rangle + \langle (I - P_Y) z_0, y_1 \rangle = 0 \tag{2.15}$$

so that z_0 is orthogonal to Y_1. Write $Z_0 = \operatorname{span}\{z_0\}$ and define

$$Y' = Y_1 \oplus Z_0. \tag{2.16}$$

Because

$$\begin{aligned} \ker(P_Y - P_{Y'}) &\supset \left((Y^\perp \cap (Y')^\perp) + (Y \cap Y') \right) \\ &= (Y + Z_0)^\perp + Y_1 \end{aligned} \tag{2.17}$$

we have

$$(\ker(P_Y - P_{Y'}))^\perp \subset (Y + Z_0) \cap Y_1^\perp = Y_0 + Z_0. \tag{2.18}$$

(Actually equality holds, as can be easily seen.) The mapping $P_Y - P_{Y'}$ is self-adjoint and so we have

$$\| P_Y - P_{Y'} \| = \| (P_Y - P_{Y'}) |_{Y_0 + Z_0} \| = \| z_0 - y_0 \|. \tag{2.19}$$

In view of (2.14), this completes the proof.

The minimal angle between two given subspaces can be computed as follows.

Proposition 2.4. *Let Y and Z be complementary nontrivial subspaces of $\mathbb{C}^{m+p}$, with* $\dim Y = m$ *and* $\dim Z = p$. *Let A and B be normalized image and kernel representations for Y and Z respectively; that is, we require*

$$A : \mathbb{C}^m \to \mathbb{C}^{m+p}, \quad A^*A = I_m, \quad \operatorname{im} A = Y \tag{2.20}$$

and

$$B : \mathbb{C}^{m+p} \to \mathbb{C}^m, \quad BB^* = I_m, \quad \ker B = Z. \tag{2.21}$$

Under these conditions, we have

$$\sin \phi(Y, Z) = \sigma_{\min}(BA) \tag{2.22}$$

where $\sigma_{min}(M)$ denotes the smallest singular value of a matrix M.

Proof. Note that the elements of norm 1 in Y are exactly those of the form Au where $u \in \mathbb{C}^m$ has norm 1. Moreover, we have $I - P_Z = B^*B$. Since it follows from (2.10) that

$$\sin \phi(Y, Z) = \sigma_{\min}((I - P_Z)|_Y) \tag{2.23}$$

we can write

$$\begin{aligned} \sin \phi(Y, Z) &= \sigma_{\min}(B^*BA) = \sqrt{\sigma_{\min}(A^*B^*BA)} \\ &= \sigma_{\min}(BA). \end{aligned} \tag{2.24}$$

Further alternative expressions for the minimal angle may be obtained; for instance, the reader may find it amusing to derive the following formula for $\cot \phi = (\sin^{-2}\phi - 1)^{1/2}$ in terms of the (generally skew) projection onto Z along Y, which we denote by P_Z^Y.

Proposition 2.5. *If Y and Z are complementary nontrivial subspaces of a unitary space X, then*

$$\cot \phi(Y, Z) = \| P_Z^Y |_{Z^\perp} \|. \tag{2.25}$$

This is close to Jordan's [20] original definition of the minimal angle. To be precise, Jordan assumed (essentially without loss of generality) that the subspaces Y and Z are not only complementary but also of equal dimension, so that the operator $P_Z^Y |_{Z^\perp}$ is invertible, and defined the angles between Y and Z as the angles whose tangents are what we now call the singular values of $(P_Z^Y |_{Z^\perp})^{-1}$. Jordan was also aware of the characterization (2.6). A final characterization of the minimal angle is attributed to Ljance [26] in [16, p.339].

Proposition 2.6. *If Y and Z are complementary nontrivial subspaces of a Hilbert space X, then*

$$\sin \phi(Y, Z) = \| P_Z^Y \|^{-1}. \tag{2.26}$$

3. Main result

Let us now consider the problem of stabilization by feedback for linear time-invariant finite-dimensional systems. Following the framework of [37], we shall represent such systems in the form

$$R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0 \tag{3.1}$$

where $R(s) \in \mathbb{R}^{p \times q}[s]$ is a polynomial matrix of full row rank. The vector w contains input and output variables, but for our main result there is no need to be specific about which components of w are considered as inputs and which are considered as outputs. We will consider the application of a dynamic compensator simply as an operation of adding equations for the external variables:

$$Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0 \tag{3.2}$$

where $Q(s) \in \mathbb{R}^{m \times q}[s]$ has full row rank, and $m = q - p$. The closed-loop system is given by

$$\begin{pmatrix} R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{pmatrix} w = 0. \tag{3.3}$$

The feedback loop is said to be *stable* if the square polynomial matrix

$$\begin{bmatrix} R^{\mathrm{T}}(s) & Q^{\mathrm{T}}(s) \end{bmatrix}^{\mathrm{T}}$$

is nonsingular and has no zeros in the closed right half plane. We shall also formulate a criterion for *well-posedness* in the present framework. Since well-posedness of feedback connections is usually studied in an input/output setting, let us first assume that the plant is described by

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u(t), \tag{3.4a}$$

$$y(t) = C_1 x_1(t) + D_1 u(t), \tag{3.4b}$$

and that the compensator is described by

$$\dot{x}_2(t) = A_2 x_2(t) + B_2 y(t), \tag{3.5a}$$

$$u(t) = C_2 x_2(t) + D_2 y(t). \tag{3.5b}$$

The usual criterion for well-posedness in this context is that the square matrix $I - D_1 D_2$ should be nonsingular. Equivalently, we may require that the subspaces

$$\mathrm{im}\begin{bmatrix} D_1^{\mathrm{T}} & I \end{bmatrix}^{\mathrm{T}} \quad \text{and} \quad \mathrm{im}\begin{bmatrix} I & D_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

are complementary. It is not difficult to show (cf. [25]) that if (3.4) and (3.1) represent the same behavior, then

$$\mathrm{im}\begin{bmatrix} D_1 \\ I \end{bmatrix} = \lim_{s \to \infty} \ker R(s) \tag{3.6}$$

where the limit is taken in the natural manifold topology of the Grassmannian $G^m(\mathbb{C}^q)$, which is the same as the topology induced by the gap metric. Likewise, we have

$$\mathrm{im}\begin{bmatrix} I \\ D_2 \end{bmatrix} = \lim_{s \to \infty} \ker Q(s). \tag{3.7}$$

It is convenient to introduce the notation

$$\ker R(\infty) = \lim_{s \to \infty} \ker R(s) \tag{3.8}$$

for any given polynomial matrix $R(s)$, even if the left hand side can of course not be interpreted as the result of inserting $s = \infty$ in the subspace-valued function $s \mapsto \ker R(s)$. We now define well-posedness for polynomial representations as follows.

Definition 3.1. The feedback connection (3.3) of the two systems (3.1) and (3.2), in which $R(s) \in \mathbb{R}^{p \times q}(s)$ and $Q(s) \in \mathbb{R}^{m \times q}(s)$ are polynomial matrices of full row rank, is said to be *well-posed* if $\ker R(\infty)$ and $\ker Q(\infty)$ are complementary subspaces of $\mathbb{R}^q$.

It can be shown, using results in [33] and [25], that the condition of the definition is necessary and sufficient for preservation of proper intput/output structure under the feedback connection.

We can now proceed to our main result, which has a very simple proof. We shall denote the closed right half plane by

$$\mathbb{C}_+ = \{s \in \mathbb{C} \mid \mathrm{Re}\, s \geqslant 0\} \cup \{\infty\}.$$

Theorem 3.2. *Let a linear system be given by* (3.1), *and suppose that the system is stabilized in a well-posed feedback connection by the compensator* (3.2). *The same compensator will also stabilize the system given by*

$$\tilde{R}(s) w = 0 \tag{3.9}$$

($\tilde{R}(s) \in \mathbb{R}^{p \times q}[s]$ *of full row rank*), *and the feedback connection of* (3.2) *and* (3.9) *will be well-posed, provided the following condition is satisfied*:

$$\delta(\ker R(s), \ker \tilde{R}(s)) < \sin \phi(\ker R(s), \ker Q(s)) \quad \text{for all } s \in \mathbb{C}_+. \tag{3.10}$$

Proof. For each finite s, the matrix

$$\begin{bmatrix} \tilde{R}^{\mathrm{T}}(s) & Q^{\mathrm{T}}(s) \end{bmatrix}^{\mathrm{T}}$$

is nonsingular if and only if the subspaces $\ker \tilde{R}(s)$ and $\ker Q(s)$ are complementary. The complementarity of $\ker \tilde{R}(\infty)$ and $\ker Q(\infty)$ is by definition equivalent to the well-posedness of the feedback connection of (3.2) and (3.9). The result is therefore immediate from Proposition 2.2.

It follows from Theorem 2.3 that (3.10) is the best bound that can be given at s in terms of the gap metric.

4. Relations with other criteria

Theorem 3.2 is of the following type: closed-loop stability is guaranteed if 'uncertainty' is less than 'tolerance'. In most approaches to stability robustness, both uncertainty and tolerance are expressed as single numbers rather than as functions. To investigate the relation of the criterion (3.10) with 'global' criteria, it is natural to introduce the following quantities:

$$d(R, \tilde{R}) = \max_{s \in \mathbb{C}_+} \delta(\ker R(s), \ker \tilde{R}(s)) \tag{4.1}$$

and

$$s(R, Q) = \min_{s \in \mathbb{C}_+} \sin \phi(\ker R(s), \ker Q(s)). \tag{4.2}$$

To see that we indeed have a maximum and a minimum here, note as in [27] that the mapping $s \mapsto \ker R(s)$, for a stabilizable system given by $R(s)$, is a continuous mapping from $\mathbb{C}_+$ to the Grassmannian manifold $G^m(\mathbb{C}^q)$. Using this, one can verify that both sides of the inequality (3.10) represent continuous functions on $\mathbb{C}_+$. Since $\mathbb{C}_+$ is compact, the two functions must indeed have a maximum and a minimum on $\mathbb{C}_+$.

The quantity $d(R, \tilde{R})$ has been introduced recently as a distance measure for linear systems in [32] (see also [31] for the scalar case). It was shown there that the metric on plants given by the distance measure (4.1) is topologically equivalent to the graph metric introduced by Vidyasagar [36], which in its turn is topologically equivalent to the gap metric of Zames and El-Sakkary, as shown in [39]. The quantity $s(R, Q)$ can be related to the H_∞ theory of robustness of stability in the following way. We can find RH_∞-matrices $X(s)$ and $Y(s)$ such that, for all $s \in \mathbb{C}_+$,

$$\ker R(s) = \operatorname{im} X(s), \quad \ker Q(s) = \operatorname{im} Y(s). \tag{4.3}$$

The closed-loop system formed by $R(s)$ and $Q(s)$ is well-posed and stable if and only if the compound matrix $[X(s) Y(s)]$ is RH_∞-unimodular. Supposing this is the case, let us write

$$\begin{bmatrix} U(s) \\ V(s) \end{bmatrix} = [X(s) \quad Y(s)]^{-1}. \tag{4.4}$$

It is easy to verify that, for each $s \in \mathbb{C}_+$, the mapping $P(s)$ defined by $P(s) = X(s)U(s)$ is the projection onto $\operatorname{im} X(s)$ along $\operatorname{im} Y(s)$. Let us now consider the situation in H_2-terms. Associate the following subspaces of H_2^q to $R(s)$ and $Q(s)$:

$$G(R) = \{X(s)f(s) \mid f \in H_2^m\}, \tag{4.5a}$$

$$G(Q) = \{Y(s)f(s) \mid f \in H_2^p\}. \tag{4.5b}$$

It can be verified that $G(R)$ and $G(Q)$ are indeed uniquely determined by $R(s)$ and $Q(s)$, in spite of the non-uniqueness of the representations (4.3). The projection along $G(Q)$ onto $G(R)$ is given by

$$P : f(s) \mapsto X(s)U(s)f(s) \quad (f \in H_2^q) \tag{4.6}$$

as is trivially verified (use $X(s)U(s) + Y(s)V(s) = I$). Using Proposition 2.6, we can therefore conclude that $s(R, Q)$ is the sine of the minimal angle in H_2^q between $G(R)$ and $G(Q)$:

$$\begin{aligned} &\sin \phi(G(R), G(Q)) \\ &\quad = \| P \|^{-1} = (\max_{s \in \mathbb{C}_+} \| P(s) \|)^{-1} \\ &\quad = \min_{s \in \mathbb{C}_+} \| P(s) \|^{-1} = s(R, Q). \end{aligned} \tag{4.7}$$

It also follows from this interpretation that the minimum in (4.2) will be achieved on the imaginary axis or at infinity.

We obtain, as an immediate corollary of Theorem 3.2, the following result due to Qiu and Davison [32].

Corollary 4.1. *Let a linear system be given by* (3.1), *and suppose that the system is stabilized in a well-posed feedback connection by the compensator* (3.2). *The same compensator will also stabilize the system given by*

$$\tilde{R}(s)w = 0 \tag{4.8}$$

($\tilde{R}(s) \in \mathbb{R}^{p\times q}[s]$ *of full row rank*), *and the feedback connection of* (3.2) *and* (3.9) *will be well-posed, provided the following condition is satisfied*:

$$d(R, \tilde{R}) < s(R, Q). \tag{4.9}$$

It is shown in [32] that the above result is the best that one can get in terms of the d-metric. Notwithstanding this, it is clear that (4.9) is conservative with respect to the pointwise criterion (3.10) since the maximum in (4.1) will in general be reached at another point in $\mathbb{C}_+$ than the minimum in (4.2).

Another related criterion is the one given in [12] (see also [30]):

$$\delta\big(G(R), G(\tilde{R})\big) < s(R, Q) \tag{4.10}$$

(the gap at the left hand side being taken in the sense of H_2^q). This is again a global criterion but it incorporates analyticity information, and so it seems unlikely that a statement is possible about (4.10) being conservative with respect to (3.10) or vice versa.

A basic feature of the analysis in this paper is that systems are studied through an associated subspace-valued function, rather than through the transfer matrix. This point of view is by no means new (cf. [5,27]). There are two cases in which there is a simple relation between the transfer matrix and the function $s \mapsto \ker R(s)$ from $\mathbb{C}_+$ to $G^m(\mathbb{C}^q)$. If

$$R(s) = [D(s) \quad -N(s)] \tag{4.11}$$

where $D(s)$ is invertible and the transfer matrix $G(s) = D^{-1}(s)N(s)$ is proper and stable, then

$$\ker R(s) = \operatorname{im}\begin{bmatrix} G(s) \\ I \end{bmatrix} \tag{4.12}$$

for all $s \in \mathbb{C}_+$. In the scalar case the same representation may even be used for unstable systems. This is due to the fact that $G^1(\mathbb{C}^2)$ is homeomorphic to the Riemann sphere via the identifications

$$\operatorname{span}\begin{bmatrix} \alpha \\ 1 \end{bmatrix} \mapsto \alpha, \quad \operatorname{span}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \infty. \tag{4.13}$$

The metric on the Riemann sphere corresponding to the gap on $G^1(\mathbb{C}^2)$ is known as the *chordal metric* (see for instance [19]). Robustness theory for the scalar case using essentially the identification (4.13) was developed in [10,31]. A similar approach, however, does not seem feasible for the general (unstable and multivariable) case.

Obviously, an input/output system represented by (4.11) is stable if and only if it is stabilized by the compensator given by $Q(s) = [0 \quad I]$. The criterion (3.10) can therefore be used to obtain results on robustness of *open-loop* stability. In this connection it is useful to note the following consequence of Proposition 2.4: if $G(s) = D^{-1}(s)N(s)$ is proper and stable, then for all $s \in \mathbb{C}_+$ one has

$$\sin\phi(\ker[D(s) \quad -N(s)], \ker[0 \quad I]) \\ = \| I + (G(s))^* G(s) \|^{-1/2}. \tag{4.14}$$

In the scalar case we write $g(s)$ rather than $G(s)$, and we get the following result due to El-Sakkary [10] as a corollary of Theorem 3.2. We use χ to denote the chordal metric on the Riemann sphere.

Corollary 4.2. *Let the rational functions* $g(s)$ *and* $h(s)$ *represent scalar stabilizable systems. If* $g(s)$ *is stable and*

$$\chi(g(s), h(s)) < \frac{1}{\sqrt{1 + |g(s)|^2}} \tag{4.15}$$

for all s *with* Re $s \geq 0$, *then* $h(s)$ *is also stable.*

5. Further research

We have seen that (3.10) gives the sharpest possible pointwise bound in terms of the gap metric on $G^m(\mathbb{C}^q)$. In specific applications, however, there may be good reasons to use a different metric. In that case one would of course also be interested in obtaining a sharpest bound as in Theorem 2.3. At the present, little seems to be known in this direction. Modifications of the theorem could be made in at least the following respects.

(i) Use of the gap with a different vector norm. If the norm does not correspond to an inner product, the Banach space version of the definition of the gap has to be used.

(ii) Use of a different distance notion on $G^m(\mathbb{C}^q)$. An example is the distance notion proposed in [28]: for subspaces Y and Y' of equal dimension, define

$$r_0(Y, Y') = \inf\{ \| I - A \| \mid A : \mathbb{C}^q \mapsto \mathbb{C}^q \text{ invertible and } AY' = Y\}. \tag{5.1}$$

Note that this definition uses an operator norm, for which again various choices are possible.

(iii) In some applications it may be reasonable to take the infimum in (2.13) over a restricted set of subspaces. This happens for instance when we know that the perturbed system is dissipative.

All modifications may depend on s. A great deal of flexibility can already be achieved within the framework provided by (3.10) if one uses inner products of the form $\langle \Phi(s) \cdot, \cdot \rangle$, where $\Phi(s)$ should be positive definite Hermitian and may depend non-continuously on s.

Carrying over the material from this paper to the discrete-time case is straightforward. Extension to infinite-dimensional systems is perhaps less straightforward, but seems certainly possible. Uncertainty on both plant and controller can be incorporated by an obvious modification of Proposition 2.2 (Y' and Z' are complementary if

$$\delta(Y, Y') + \delta(Z, Z') < \sin \phi(Y, Z)).$$

Among the many subjects of further study that suggest themselves, let us just mention here the effects of *performance constraints* on the pointwise minimal angle between plant and controller.

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