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RESEARCH ARTICLE

On the Topology Induced by the Adjoint of a Semigroup of Operators

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The adjoint of a C_0-semigroup on a Banach space X induces a locally convex $\sigma(X, X^{\odot})$-topology in X, which is weaker than the weak topology of X. In this paper we study the relation between these two topologies. Among other things a class of subsets of X is given on which they coincide. As an application, an Eberlein-Shmulyan type theorem is proved for the $\sigma(X, X^{\odot})$-topology and it is shown that the uniform limit of $\sigma(X, X^{\odot})$-compact operators is $\sigma(X, X^{\odot})$-compact. Finally our results are applied to the problem when the Favard class of a semigroup equals the domain of the infinitesimal generator.

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0. Introduction

The aim of this paper is to unify some recent results in adjoint semigroup theory concerning the interplay between the weak topology of a Banach space X and the $\sigma(X, X^{\odot})$-topology induced by the adjoint of a C_0-semigroup on X.

If $T(t)$ is a C_0-semigroup on a Banach space X then the semigroup $T^*(t) = (T(t))^*$ on X^* is called its *adjoint*. An adjoint semigroup is weak*-continuous and is weak*-generated by A^*, the adjoint of the generator A of $T(t)$. $T^*(t)$ need not be strongly continuous and hence the definition

$$X^{\odot} = \{x^* \in X^* : \lim_{t\downarrow 0} \|T^*(t)x^* - x^*\| = 0\}$$

makes sense. It can be shown that $X^{\odot} = \overline{D(A^*)}$ and therefore $X^{\odot}$ is a weak*-dense closed subspace of X^*. It is easy to see that $X^{\odot}$ is invariant under $T^*(t)$. Hence the restrictions of $T^*(t)$ to $X^{\odot}$, denoted $T^{\odot}(t)$, define a C_0-semigroup on $X^{\odot}$. The generator of $T^{\odot}(t)$, denoted $A^{\odot}$, is the part of A^* in $X^{\odot}$. For details we refer to [2,4].

Repeating this construction starting from $T^{\odot}(t)$ one obtains successively $X^{\odot *}$, $T^{\odot *}(t)$, $X^{\odot\odot}$, $T^{\odot\odot}(t)$. The map $j : X \to X^{\odot *}$ defined by

$$\langle jx, x^{\odot}\rangle = \langle x^{\odot}, x\rangle$$

is an embedding which maps X into $X^{\odot\odot}$. If $j(X) = X^{\odot\odot}$ then X is called $\odot$-*reflexive* (with respect to $T(t)$).

$X^{\odot}$ induces a weak topology on X by taking as fundamental neighbourhoods of the origin the sets of the form

$$V(x_1^{\odot},...,x_n^{\odot};\epsilon;0) := \{x \in X : \ |\langle x_i^{\odot}, x\rangle| < \epsilon, \ \ i = 1,...,n\},$$

where $x_1^{\odot},...,x_n^{\odot} \in X^{\odot}$ and $\epsilon > 0$. Since $X^{\odot}$, being weak*-dense in X^*, separates points of X this is a locally convex topology on X which will be called the $\sigma(X,X^{\odot})$-*topology*. In general $X^{\odot}$ is a proper subspace of X^* in which case the $\sigma(X,X^{\odot})$-topology is strictly weaker than the weak topology. However, the following is always true. Let $R(\lambda,A) = (\lambda I - A)^{-1}$ be the resolvent of the generator A of the semigroup $T(t)$.

Theorem 0.1 (Phillips-de Pagter [4,8]). *The following are equivalent:*

(1) X is $\odot$-reflexive with respect to $T(t)$;
(2) $R(\lambda,A)$ is $\sigma(X,X^{\odot})$-compact;
(3) $R(\lambda,A)$ is weakly compact.

The equivalence (2)⇔(3) has a simple consequence. Letting B_X denote the closed unit ball of the Banach space X, the identity map $i : (\overline{R(\lambda,A)B_X}, weak) \to (\overline{R(\lambda,A)B_X}, \sigma(X,X^{\odot}))$ is continuous. If X is $\odot$-reflexive then by Theorem 0.1 both spaces are compact (and Hausdorff) and therefore i is actually a homeomorphism. So *if X is $\odot$-reflexive then the relative weak- and $\sigma(X,X^{\odot})$-topology on $\overline{R(\lambda,A)B_X}$ coincide*, regardless whether $X^{\odot} = X^*$ or not.

Motivated by this observation, in this paper we will study in detail the relationship between the weak- and the $\sigma(X,X^{\odot})$-topology.

Section 1 deals primarily with the question to characterise those weakly closed sets G that are $\sigma(X,X^{\odot})$-closed. We treat successively the cases G arbitrary, bounded, bounded and convex, and $G = B_X$.

In section 2 we apply the results of section 1. A class of sets, containing all sets of the form $\overline{R(\lambda,A)H}$ with H bounded, is singled out on which the weak- and the $\sigma(X,X^{\odot})$-topology always coincide, thereby generalising the equivalence of (2) and (3) in Theorem 0.1. No compactness assumption on H is needed whatsoever. As an application we show that an Eberlein-Shmulyan theorem holds for the $\sigma(X,X^{\odot})$-topology, and that the uniform limit of a sequence of $\sigma(X,X^{\odot})$-compact operators is again $\sigma(X,X^{\odot})$-compact. Finally a variant of Theorem 0.1 is proved which asserts that X is $\odot$-reflexive with respect to a C_0-semigroup if and only if the integrated semigroup is weakly compact if and only if the integrated semigroup is $\sigma(X,X^{\odot})$-compact.

In section 3 we apply some of our results to the study of the so-called Favard class of a semigroup. A characterisation is given of those semigroups for which $Fav(T(t)) = D(A)$ holds.

1. The $\sigma(X, X^{\odot})$-closure of bounded sets

In this section we will study in detail which sets are $\sigma(X, X^{\odot})$-closed. From now on $T(t)$ denotes some given C_0-semigroup with generator A on a Banach space X.

The starting point of our investigations is the following simple '$\sigma(X, X^{\odot})$ equals strong' result from [6].

Theorem 1.1. *Every closed $T(t)$-invariant convex set G is $\sigma(X, X^{\odot})$-closed.*

In fact, if $x \notin G$ we may choose t so small that the vector $x_t = \frac{1}{t}\int_0^t T(\tau)x\, d\tau$ is still not in G, hence also not in $G_t = \frac{1}{t}\int_0^t T(\tau)G\, d\tau$ which is a subset of G. By the Hahn-Banach theorem x_t can be separated from G_t by some $x^* \in X^*$. But then x and G can be separated by $\frac{1}{t}\int_0^t T^*(\tau)x^*\, d\tau$ (the integral being in the weak*-sense), an element of $D(A^*)$. It follows that G is $\sigma(X, X^{\odot})$-closed.

It is important to observe that for this proof to work we only need the following: for $x \notin G$ there should be a $t > 0$ small enough such that x_t can be separated from G_t. For this we only need some control on the G_t as $t \downarrow 0$. This was achieved in the above theorem by imposing on G the rather strong assumptions of *invariance* and *convexity*.

Motivated by this, for a given set G we define $G_t := \{\frac{1}{t}\int_0^t T(\tau)g\, d\tau : \ g \in G\}$.

At this point we remark that most of our results can be restated in terms of $G_\lambda := \lambda R(\lambda, A)G$; one obtains the 'Laplace transforms' of the corresponding statements on G_t.

Theorem 1.2. *If $G = \bigcap_{t>0} \overline{\bigcup_{0\le s\le t} G_s}^{weak}$ then G is $\sigma(X, X^{\odot})$-closed.*

Proof: It suffices to prove that the inclusion

$$\overline{G}^{\sigma(X,X^{\odot})} \subset \bigcap_{t>0} \overline{\bigcup_{0\le s\le t} G_s}^{weak}$$

always holds. Fix any $x \notin \bigcap_{t>0} \overline{\bigcup_{0\le s\le t} G_s}^{weak}$. We must show: $x \notin \overline{G}^{\sigma(X,X^{\odot})}$. By assumption there is a $t_0 > 0$ such that $x \notin \overline{\bigcup_{0\le s\le t_0} G_s}^{weak}$. Choose norm-1 functionals $x_1^*, ..., x_n^* \in X^*$ and $\epsilon > 0$ such that the weakly open set

$$V = V(x_1^*, ..., x_n^*; \epsilon; x) = \{y \in X : \ |\langle x_i^*, x - y\rangle| < \epsilon, \quad i = 1, ..., n\}$$

which contains x is disjoint from $\bigcup_{0\le s\le t_0} G_s$. By the strong continuity of $T(t)$ we may choose $0 < t_1 \le t_0$ such that additionally we have

$$\|\frac{1}{t_1}\int_0^{t_1} T(\tau)x\, d\tau - x\| < \frac{\epsilon}{2}.$$

We claim that $\tilde{V} \cap G = \emptyset$, where

$$\tilde{V} = V(\frac{1}{t_1}\int_0^{t_1} T^*(\tau)x_1^*\, d\tau, ..., \frac{1}{t_1}\int_0^{t_1} T^*(\tau)x_n^*\, d\tau; \frac{\epsilon}{2}; x).$$

Indeed, fix any $g \in G$ and choose $i_0 \in 1,...,n$ such that

$$|\langle x_{i_0}^*, x - \frac{1}{t_1}\int_0^{t_1} T(\tau)g\ d\tau\rangle| \geq \epsilon.$$

Such an i_0 exists since $V \cap G_{t_1} = \emptyset$. Then

$$\begin{aligned}
&|\langle \frac{1}{t_1}\int_0^{t_1} T^*(\tau)x_{i_0}^*\ d\tau, x - g\rangle| \\
&\quad = |\langle x_{i_0}^*, \frac{1}{t_1}\int_0^{t_1} T(\tau)x\ d\tau - \frac{1}{t_1}\int_0^{t_1} T(\tau)g\ d\tau\rangle| \\
&\quad \geq |\langle x_{i_0}^*, x - \frac{1}{t_1}\int_0^{t_1} T(\tau)g\ d\tau\rangle| - |\langle x_{i_0}^*, \frac{1}{t_1}\int_0^{t_1} T(\tau)x\ d\tau - x\rangle| \\
&\quad \geq \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}.
\end{aligned}$$

This shows $\tilde{V} \cap G = \emptyset$ and the claim is proved. It is easy to see that $\frac{1}{t_1}\int_0^{t_1} T^*(\tau)x_i^*\ d\tau \in D(A^*)$. Therefore $\tilde{V}$ is $\sigma(X, X^{\odot})$-open, and we have $\tilde{V} \cap \overline{G}^{\sigma(X,X^{\odot})} = \emptyset$. Since $x \in \tilde{V}$ the theorem is proved. ////

Let us have a look again at Theorem 1.1. If G is convex, closed and $T(t)$-invariant, then $G_t \subset G$ for all $t \geq 0$, as is easily seen. Moreover, since convex closed sets are weakly closed, it follows that

$$\bigcap_{t>0} \overline{\bigcup_{0\leq s\leq t} G_s}^{weak} = \bigcap_{t>0} \overline{G}^{weak} = G.$$

In section 2 we will single out a class of sets which are in general not $T(t)$-invariant, but do satisfy the condition of Theorem 1.2. Indeed, the set $\overline{R(\lambda, A)B_X}$ (cf. Theorem 0.1) belongs to this class.

The content of Theorem 1.2 is that a sufficient condition for $\sigma(X, X^{\odot})$-closedness is a kind of 'infinitesimal invariance' with respect to the weak topology. The following theorem asserts that *bounded* sets are in fact characterised by this property.

Theorem 1.3. *If G is a bounded set then*

$$\overline{G}^{\sigma(X,X^{\odot})} = \bigcap_{t>0} \overline{\bigcup_{0\leq s\leq t} G_s}^{\sigma(X,X^{\odot})} = \bigcap_{t>0} \overline{\bigcup_{0\leq s\leq t} G_s}^{weak}.$$

Proof: In view of the inclusion proved in Theorem 1.2 we only have to prove the inclusion

$$\bigcap_{t>0} \overline{\bigcup_{0\leq s\leq t} G_s}^{\sigma(X,X^{\odot})} \subset \overline{G}^{\sigma(X,X^{\odot})}.$$

Suppose $x \notin \overline{G}^{\sigma(X,X^{\odot})}$. Then there are $x_1^{\odot}, ..., x_n^{\odot}$ in $X^{\odot}$ and $\epsilon > 0$ such that

$$V(x_1^{\odot}, ..., x_n^{\odot}; \epsilon; x) \cap G = \emptyset.$$

Since G is bounded there is a constant K such that $\|g\| \leq K$ for all $g \in G$. Choose $t_0 > 0$ such that for all $i = 1, ..., n$ and $0 \leq s \leq t_0$ we have

$$\|\frac{1}{s}\int_0^s T^*(\tau)x_i^{\odot}\, d\tau - x_i^{\odot}\| < \frac{\epsilon}{2K}.$$

Let $g \in G$ be arbitrary and fixed. Choose $i_0 \in 1, ..., n$ such that $|\langle x_{i_0}^{\odot}, x - g\rangle| \geq \epsilon$. Then for $0 \leq s \leq t$

$$\begin{aligned} &|\langle x_{i_0}^{\odot}, x - \frac{1}{s}\int_0^s T(\tau)g\, d\tau\rangle| \\ &\geq |\langle x_{i_0}^{\odot}, x - g\rangle| - |\langle x_{i_0}^{\odot}, g - \frac{1}{s}\int_0^s T(\tau)g\, d\tau\rangle| \\ &\geq \epsilon - |\langle \frac{1}{s}\int_0^s T^*(\tau)x_{i_0}^{\odot}\, d\tau - x_{i_0}^{\odot}, g\rangle| \\ &\geq \epsilon - \frac{\epsilon}{2K}K = \frac{\epsilon}{2}. \end{aligned}$$

It follows that for all $0 \leq s \leq t_0$ we have $\tilde{V} \cap G_s = \emptyset$, where $\tilde{V} = V(x_1^{\odot}, ..., x_n^{\odot}; \frac{\epsilon}{2}; x)$. Since $\tilde{V}$ is $\sigma(X, X^{\odot})$-open, it follows that

$$\tilde{V} \bigcap \overline{\bigcup_{0 \leq s \leq t_0} G_s}^{\sigma(X,X^{\odot})} = \emptyset.$$

Since $x \in \tilde{V}$ the proof is finished. ////

Remark 1.4. If G is bounded then one has

$$\overline{G}^{\sigma(X,X^{\odot})} = \bigcap_{t>0} \overline{\bigcup_{0 \leq s \leq t} T(s)G}^{\sigma(X,X^{\odot})}.$$

The proof of this is similar to those of Theorems 1.2 and 1.3. The content of this identity is that every bounded $\sigma(X, X^{\odot})$-closed set is 'infinitesimally invariant' with respect to $T(t)$ in the $\sigma(X, X^{\odot})$-topology. The corresponding formula for both the weak- and the norm topology fails: in Example 1.8 below we will construct a semigroup on c_0 for which the inclusion $B_X \subset \bigcap_{t>0} \overline{\bigcup_{0 \leq s \leq t} T(s)B_X}$ is proper.

For convex sets, Theorem 1.3 assumes a particularly nice form. Let $\overline{co}G$ denote the closed convex hull of a set G.

Theorem 1.5. *If G is convex and bounded, then*

$$\overline{G}^{\sigma(X,X^{\odot})} = \bigcap_{t>0} \Big(\overline{\text{co}} \bigcup_{0\le s\le t} T(s)G\Big).$$

Proof: For every set G we have $\overline{G}^{weak} \subset \overline{\text{co}}G$. On the other hand for every $0 \le s \le t$ we have

$$G_s \subset \overline{\text{co}} \bigcup_{0\le s\le t} T(s)G.$$

Together with Theorem 1.3 this proves the inclusion

$$\overline{G}^{\sigma(X,X^{\odot})} \subset \bigcap_{t>0} \Big(\overline{\text{co}} \bigcup_{0\le s\le t} T(s)G\Big).$$

For the converse inclusion, suppose $y \in \bigcap_{t>0} \; (\overline{\text{co}}\bigcup_{0\le s\le t} T(s)G)$. This means that there is a sequence of convex combinations

$$y_i = \sum_{n=1}^{N_i} \alpha_{in} T(t_{in}) g_{in}$$

converging to y strongly, with $g_{in} \in G$ and $\max_{n=1\ldots N_i} t_{in} < i^{-1}$. Put

$$z_i = \sum_{n=1}^{N_i} \alpha_{in} g_{in}.$$

Since G is convex we have $z_i \in G$ for all i. Since G is bounded, there is a $K < \infty$ such that $\|g\| \le K$ for all $g \in G$. For fixed $x^{\odot} \in X^{\odot}$ we have

$$\begin{aligned} |\langle x^{\odot}, y_i - z_i\rangle| &= |\langle x^{\odot}, \sum_{n=1}^{N_i} \alpha_{in} T(t_{in}) g_{in} - \sum_{n=1}^{N_i} \alpha_{in} g_{in}\rangle| \\ &= |\sum_{n=1}^{N_i} \alpha_{in} \langle x^{\odot}, T(t_{in}) g_{in} - g_{in}\rangle| \\ &= |\sum_{n=1}^{N_i} \alpha_{in} \langle T^*(t_{in}) x^{\odot} - x^{\odot}, g_{in}\rangle| \\ &\le K \sum_{n=1}^{N_i} \alpha_{in} \|T^*(t_{in}) x^{\odot} - x^{\odot}\| \quad \to 0 \qquad \text{as } i \to \infty, \end{aligned}$$

since on the one hand $\max_{n=1\ldots N_i} t_{in} < \frac{1}{i}$ and on the other hand $\|T^*(t)x^{\odot} - x^{\odot}\| \to 0$ as $t \downarrow 0$. This shows that $z_i - y_i$ converges to 0 in the $\sigma(X, X^{\odot})$-topology. But $y_i \to y$ strongly, hence $z_i \to y$ in the $\sigma(X, X^{\odot})$-topology. Since $z_i \in G$ for all i it follows that $y \in \overline{G}^{\sigma(X,X^{\odot})}$. ////

The weak closure of a convex set is just the norm closure; the above theorem can be regarded as an analogue for the $\sigma(X, X^{\odot})$-closure of bounded convex sets.

Weakly convergent sequences admit norm convergent convex combinations. For $\sigma(X, X^{\odot})$-convergent sequences we get the following analogue: if $x_n \to x$ in the $\sigma(X, X^{\odot})$-topology, then for every $\delta > 0$ and $\epsilon > 0$ there are numbers $t_n \in [0, \delta]$ and $\alpha_n \geq 0$ with $\sum_n \alpha_n = 1$ such that

$$\|x - \sum_{n=1}^{\infty} \alpha_n T(t_n) x_n\| < \epsilon.$$

Indeed, take G to be the closed convex hull of (x_n). Regarding G as a subset of $X^{\odot *}$, by the uniform boundedness theorem G is bounded in $X^{\odot *}$. Since the canonical map $j : X \to X^{\odot *}$ is an isomophism into, we see that G is bounded in X and Theorem 1.5 applies. The following example, which improves [6; Cor. 1.9], shows what this means for the translation group $T(t)f(x) = f(x + t)$ on $C_0(\mathbb{R})$, the Banach space of continuous functions on $\mathbb{R}$ vanishing at infinity, equipped with the sup-norm.

Corollary 1.6. *Let (f_n) be a bounded sequence in $C_0(\mathbb{R})$ which converges a.e. (with respect to the Lebesgue measure) to some $f \in C_0(\mathbb{R})$. Then for every $\delta > 0$ and $\epsilon > 0$ there are numbers $t_n \in [0, \delta]$ and $\alpha_n \geq 0$ with $\sum_n \alpha_n = 1$ such that*

$$\|f - \sum_n \alpha_n T(t_n) f_n\| < \epsilon.$$

Proof: We have $C_0(\mathbb{R})^{\odot} = L^1(\mathbb{R})$, see e.g. [2,4]. Since by assumption (f_n) is bounded, the dominated convergence theorem shows that $f_n \to f$ pointwise a.e. implies that $f_n \to f$ in the $\sigma(C_0(\mathbb{R}), C_0(\mathbb{R})^{\odot})$-topology. Now the conclusion follows from the preceding remarks. ////

The final result of this section, which is of a somewhat different nature, describes under what conditions the closed unit ball B_X is $\sigma(X, X^{\odot})$-closed.

Theorem 1.7. *B_X is $\sigma(X, X^{\odot})$-closed if and only if the canonical embedding $j : X \to X^{\odot *}$ is isometric.*

Proof: Define on X the norm $\|\cdot\|'$ by

$$\|x\|' = \sup_{\|x^{\odot}\|=1} |\langle x^{\odot}, x\rangle|.$$

In other words, X is normed by $X^{\odot}$. This norm is easily shown to be an equivalent norm on X, see [4]. Clearly j is isometric if and only if $\|\cdot\|' = \|\cdot\|$. Therefore to prove the theorem it suffices to show that the equality $\overline{B_X}^{\sigma(X,X^{\odot})} = B_{X,\|\cdot\|'}$ holds. For the proof of this, first we note that it is an immediate consequence of the definitions that $B_{X,\|\cdot\|'}$ is $\sigma(X, X^{\odot})$-closed. Therefore we have to show that $B_{X,\|\cdot\|'} \subset \overline{B_X}^{\sigma(X,X^{\odot})}$.

Suppose $y \notin \overline{B_X}^{\sigma(X,X^{\odot})}$; we will show $y \notin B_{X,\|\cdot\|'}$. Since $\overline{B_X}^{\sigma(X,X^{\odot})}$ is a $\sigma(X,X^{\odot})$-closed convex set, by the Hahn-Banach separation theorem there is a norm-1 vector $x_0^{\odot} \in X^{\odot}$ which separates y and B_X, that is, there is an $\epsilon > 0$ such that

$$|\langle x_0^{\odot}, y - x\rangle| > \epsilon, \qquad \forall x \in B_X.$$

Note that we made use of the fact that X is locally convex in its $\sigma(X,X^{\odot})$-topology. Since $\|x_0^{\odot}\| = 1$ and $X^{\odot}$ is normed by B_X, for every scalar α with $|\alpha| < 1$ there is an $x_\alpha \in B_X$ such that

$$\langle x_0^{\odot}, x_\alpha\rangle = \alpha.$$

In particular,

$$|\langle x_0^{\odot}, y\rangle - \alpha| = |\langle x_0^{\odot}, y - x_\alpha\rangle| > \epsilon$$

for all $|\alpha| < 1$. It follows that $\|y\|' \geq |\langle x_0^{\odot}, y\rangle| \geq 1 + \epsilon$. ////

Inspection of the above proof shows that the same argument goes through for any subspace $Y \subset X^*$ which induces an equivalent norm in X.

The following is an example of a strongly continuous semigroup on c_0 for which B_{c_0} is not closed in the $\sigma(c_0, c_0^{\odot})$-topology, although $c_0^{\odot}$ has codimension one in $c_0^* = l^1$.

Example 1.8. Let e_n be the nth unit vector of c_0; put $x_n = \sum_{k=1}^{n} e_k$. It can be shown [5] that $\{x_n\}_{n=1}^{\infty}$ is a Schauder basis for c_0. Define a semigroup $T(t)$ on c_0 by

$$T(t)x_n = e^{-(n-1)t}x_n.$$

By [7] this is a C_0-semigroup satisfying $\|T(t)\| \leq 2$ for all $t \geq 0$. We claim that

$$2x_1 = (2,0,0,0,....) \in \bigcap_{t>0} \overline{\bigcup_{0\leq s\leq t} T(s)B_{c_0}}.$$

In view of Theorem 1.5 this implies that $2x_1 \in \overline{B_{c_0}}^{\sigma(c_0,c_0^{\odot})}$. Indeed, put $y_k = 2x_1 - x_k$. Then $y_k \in B_{c_0}$. Fix any $t > 0$. Then we have

$$\lim_{k\to\infty} T(t)y_k = \lim_{k\to\infty} (2x_1 - e^{-(k-1)t}x_k) = 2x_1.$$

We will now show that $c_0^{\odot}$ has codimension one in l^1. It is easily checked that the coordinate functionals of the basis $\{x_n\}_{n=1}^{\infty}$ of the above example are given by $x_n^* = e_n^* - e_{n+1}^*$, where e_n^* is the nth unit vector of l^1. Clearly the closed linear span of $\{x_n^*\}_{n=1}^{\infty} \cup \{e_1^*\}$ is l^1. Since by [7] we have that $c_0^{\odot}$ is the closed linear span of $\{x_n^*\}_{n=1}^{\infty}$, it follows that $c_0^{\odot}$ has at most codimension one in l^1. But for each n,

$$\|T^*(t)e_1^* - e_1^*\| \geq |\langle e_1^*, T(t)x_n - x_n\rangle| = 1 - e^{-(n-1)t},$$

so $\|T^*(t)e_1^* - e_1^*\| \geq 1$ for all $t > 0$. Consequently $e_1^* \notin c_0^{\odot}$ and hence $c_0^{\odot}$ has codimension one in l^1.

In [7] it is shown that $c_0^{\odot} = l^1$ for every C_0-semigroup on c_0 satisfying $\|T(t)\| \leq (2-\epsilon)e^{\omega t}$ for some $\epsilon > 0$ and real constant ω. The semigroup from the present example satisfies $\limsup_{t\downarrow 0} \|T(t)\| = 2$.

2. Equicontinuous sets

In this section we define a class of sets which satisfy the condition of '*pointwise* infinitesimal invariance' from Theorem 1.2. The following definition defines a kind of '*uniform* infinitesimal invariance'.

Definition 2.1. Let G be a subset of X. We will say that G is *equicontinuous* with respect to a (semi)group $T(t)$ if the collection of maps $t \mapsto T(t)g$, where g ranges over G, is equicontinuous. G will be called *weakly equicontinuous* if for each $x^* \in X^*$ the collection of maps $t \mapsto \langle x^*, T(t)g\rangle$ is equicontinuous.

If G is (weakly) equicontinuous, so are $\overline{G}$, $\overline{\text{co}}G$ and hence also $\overline{G}^{weak}$. Equicontinuous sets are weakly equicontinuous, but the converse need not be true. For example consider $C_0(\mathbb{R})$ and define the translation group $T(t)$ as in Corollary 1.6. Let f_n be the piecewise linear function defined by

$$f_n(x) = \begin{cases} 0, & x \leq n - \frac{1}{n}; \\ 1, & x = n; \\ 0, & x \geq n + \frac{1}{n}, \end{cases}$$

and which is linear on the intervals $[n - \frac{1}{n}, n]$ and $[n, n + \frac{1}{n}]$. The sequence (f_n) is equicontinuous in the classical sense but clearly not equicontinuous with respect to $T(t)$. We claim that (f_n) is weakly equicontinuous with respect to $T(t)$ however. This follows from the following proposition.

Proposition 2.2. *Let $T(t)$ be the translation group on $C_0(\mathbb{R})$. A bounded sequence (f_n) is weakly equicontinuous with resect to $T(t)$ if and only if (f_n) is equicontinuous (in the classical sense).*

Proof: If (f_n) is weakly equicontinuous, then for each x the maps

$$t \mapsto \langle \delta_x, T(t)f_n\rangle = T(t)f_n(x) = f_n(x+t)$$

are equicontinuous. Hence (f_n) is equicontinuous in the classical sense. Conversely, suppose (f_n) is equicontinuous in the classical sense. It clearly suffices to prove weak equicontinuity at $t = 0$. Fix $\epsilon > 0$ arbitrarily and let K be such that $\|f_n\| \leq K$ for all n. Let $\mu \in (C_0(\mathbb{R}))^*$ be arbitrary. By the Riesz Representation Theorem, μ is a regular Borel measure on $\mathbb{R}$. In particular, there is an $r > 0$ such that

$$|\mu|(\mathbb{R}\backslash[-r, r]) < \epsilon.$$

By the equicontinuity of (f_n), for each $x \in [-r, r]$ there is a $\delta(x) > 0$ such that $|x - y| < \delta(x)$ implies $|f_n(x) - f_n(y)| < \epsilon$ for all n. The open sets $B(x; \delta(x))$ form an open covering of the compact interval $[-r, r]$. Let $B_1, ..., B_N$ be some finite subcovering and let λ be its Lebesgue number. By definition this means that for each $x \in [-r, r]$

there is an $i \in 1, ..., N$ such that $B(x;\lambda) \subset B_i$. Note that if $y_1, y_2 \in B(x,\lambda)$ then $|f_n(y_1) - f_n(y_2)| < 2\epsilon$ for all n. For $|t| < \lambda$ we find

$$\begin{aligned}
|\langle \mu, T(t)f_n - f_n\rangle| &= |\int_{-\infty}^{\infty} \Big(f_n(x+t) - f_n(x)\Big)\, d\mu| \\
&\le \left(\int_r^{\infty} + \int_{-\infty}^{-r}\right)|f_n(x+t) - f_n(x)|\, d\mu + \int_{-r}^{r} |f_n(x+t) - f_n(x)|\, d\mu \\
&\le \epsilon \cdot 2K + 2\epsilon \cdot |\mu|([-r,r]) \le 2\epsilon \cdot (K + \|\mu\|).
\end{aligned}$$

////

It is an easy consequence of Definition 2.1 that for an equicontinuous set G we have $\overline{G} = \bigcap_{t>0} \overline{\bigcup_{0\le s\le t} G_s}$. That this formula also holds with respect to the weak topology is the content of the following theorem.

Theorem 2.3. *If G is weakly equicontinuous, then* $\overline{G}^{weak} = \bigcap_{t>0} \overline{\bigcup_{0\le s\le t} G_s}^{weak}$.

Proof: Fix any $x \notin \overline{G}^{weak}$. We must show: $x \notin \overline{\bigcup_{0\le s\le t_0} G_s}^{weak}$ for some $t_0 > 0$. There are norm-1 functionals $x_1^*, ..., x_n^* \in X^*$ and $\epsilon > 0$ such that the weakly open set

$$V = V(x_1^*, ..., x_n^*; \epsilon; x) = \{y \in X : \ |\langle x_i^*, x - y\rangle| < \epsilon, \ \ i = 1, ..., n\}$$

which contains x is disjoint from G. By the weak equicontinuity of G we may choose $t_0 > 0$ such that for every $0 \le s \le t_0$, every $g \in G$ and $i = 1, ..., n$ we have

$$|\langle x_i^*, T(s)g - g\rangle| < \frac{\epsilon}{2}.$$

In particular we get for every $0 \le s \le t_0$, $g \in G$ and $i = 1, ..., n$

$$|\langle x_i^*, \frac{1}{s}\int_0^s T(\tau)g\, d\tau - g\rangle| < \frac{\epsilon}{2}.$$

Now the proof may be finished by estimates similar to those in the proof of Theorem 1.3. ////

Corollary 2.4. *The weak- and the $\sigma(X, X^{\odot})$-closure of weakly equicontinuous sets are equal. In particular weakly closed weakly equicontinuous sets are $\sigma(X, X^{\odot})$-closed.*

Just combine Theorems 1.2 and 2.3. Since subsets of weakly equicontinuous sets are weakly equicontinuous, we obtain:

Corollary 2.5. *The relative weak- and $\sigma(X, X^{\odot})$-topology coincide on weakly equicontinuous sets.*

Proof: Let G be weakly equicontinuous and suppose that $H \subset G$ is relatively weakly closed. Let $\tilde{H}$ be the weak closure of H in X then $\tilde{H} \cap G = H$. Moreover, $\tilde{H}$ is $\sigma(X, X^{\odot})$-closed by Corollary 2.4, so $H = \tilde{H} \cap G$ is relatively $\sigma(X, X^{\odot})$-closed in G. ////

Corollary 2.6. *A weakly equicontinuous sequence in X is weakly convergent if and only if it is $\sigma(X, X^\odot)$-convergent.*

Proof: Suppose (x_n) is $\sigma(X, X^\odot)$-convergent to x. Put $G = \{x_n\}_{n=1}^\infty \cup \{x\}$. Then G is weakly equicontinuous as well. Let V be a weakly open neighbourhood of x in X. Then $V \cap G$ is relatively weakly open in G, hence relatively $\sigma(X, X^\odot)$-open in G by Corollary 2.5. It follows that all but finitely many x_n lie in $V \cap G \subset V$, which was to be shown. ////

In particular weakly equicontinuous $\sigma(X, X^\odot)$-convergent sequences admit norm convergent convex combinations.

Example 2.7. Let $T(t)$ be the translation group on $C_0(\mathbb{R})$. Let (f_n) be a bounded equicontinuous sequence in $C_0(\mathbb{R})$ converging pointwise *almost everywhere* to some $f \in C_0(\mathbb{R})$. Then as in Corollary 1.6, $f_n \to f$ in the $\sigma(C_0(\mathbb{R}), C_0(\mathbb{R})^\odot)$-topology. By Proposition 2.2 (f_n) is weakly equicontinuous with respect to $T(t)$, and Corollary 2.6 now shows that $f_n \to f$ weakly, in particular $f_n \to f$ *pointwise.*

Of course the conclusion we drew in this example is easily proved by an $\frac{\epsilon}{3}$-argument, but it is a nice illustration of what is happening in Corollary 2.6.

After Theorem 0.1 we noted that the relative weak- and $\sigma(X, X^\odot)$-topologies coincide on $\overline{R(\lambda, A)B_X}$ in case this set is weakly compact. The following proposition shows that the compactness assumption can be omitted and that the conclusion holds for every set of the form $\overline{R(\lambda, A)H}$ with H bounded.

Proposition 2.8. *If H is bounded then $\overline{R(\lambda, A)H}$ is equicontinuous.*

Just note that $T(t)R(\lambda, A)h - R(\lambda, A)h = \int_0^t T(\tau)AR(\lambda, A)h \, d\tau$ and use that $AR(\lambda, A)$ is bounded.

It follows that Corollaries 2.5 and 2.6 hold for such sets. This 'explains' the equivalence (2)⇔(3) of Theorem 0.1.

As an illustration of this proposition let us derive an Eberlein-Shmulyan type theorem for the $\sigma(X, X^\odot)$-topology from the standard Eberlein-Shmulyan Theorem.

Corollary 2.9. *A set is $\sigma(X, X^\odot)$-compact if and only if it is $\sigma(X, X^\odot)$-sequentially compact.*

Proof: Suppose G is $\sigma(X, X^\odot)$-compact and let (x_n) be a sequence in G. Since $R(\lambda, A)$ is continuous in the $\sigma(X, X^\odot)$-topology, also $R(\lambda, A)G$ is $\sigma(X, X^\odot)$-compact. By Corollary 2.5 $R(\lambda, A)G$ is weakly compact. Hence by the Eberlein-Shmulyan Theorem there is a subsequence (x_{n_i}) and an $x \in G$ such that $R(\lambda, A)x_{n_i} \to R(\lambda, A)x$ weakly. So for every $x^* \in X^*$ we have

$$\langle R(\lambda, A^*)x^*, x_{n_i}\rangle = \langle x^*, R(\lambda, A)x_{n_i}\rangle \to \langle x^*, R(\lambda, A)x\rangle = \langle R(\lambda, A^*)x^*, x\rangle.$$

Since $R(\lambda, A^*)X^* = D(A^*)$ is norm-dense in $X^\odot$ it follows that $x_{n_i} \to x$ in the $\sigma(X, X^\odot)$-topology.

Conversely, assume that G is $\sigma(X, X^\odot)$-sequentially compact. Let $j : X \to X^{\odot *}$

be the canonical embedding. Then jG is weak*-sequentially compact. Since jG is bounded (by the uniform boundedness theorem) it follows that the weak*-closure of jG up in $X^{\odot *}$ is weak*-compact. Therefore it suffices to show that we have $\overline{jG}^{weak^*} = jG$. Let $x^{\odot *}$ be any element of $\overline{jG}^{weak^*}$ and choose a net $x_\alpha \subset G$ such that jx_α is weak* convergent to $x^{\odot *}$. Consider the net $R(\lambda, A)x_\alpha$. Since the $\sigma(X, X^\odot)$-sequential compactness of G and the $\sigma(X, X^\odot)$-continuity of $R(\lambda, A)$ imply that also $R(\lambda, A)G$ is $\sigma(X, X^\odot)$-sequentially compact, it follows from Corollary 2.6 that $R(\lambda, A)G$ is weakly sequentially compact, hence weakly compact by the Eberlein-Shmulyan Theorem. Hence the net $R(\lambda, A)x_\alpha$ has a weakly convergent subnet, say with limit $R(\lambda, A)x$. This forces that $jx = x^{\odot *}$ and the corollary is proved. ////

There are general results supplying sufficient conditions on a locally convex space for the Eberlein-Shmulyan Theorem to hold. These concern the so-called Mackey topology, see e.g. [9]. Although Corollary 2.9 might possibly be deduced from such results, the above proof seems to be by far the simplest approach.

Implicit in the proof of Corollary 2.9 is the following:

Corollary 2.10. *A bounded set G is $\sigma(X, X^\odot)$-compact if and only if $R(\lambda, A)G$ is $\sigma(X, X^\odot)$-compact.*

Corollary 2.10 fails for the weak topology. To see this, let $X = l^1$ and define a contraction semigroup $T(t)$ on X by $T(t)y_n = e^{-nt}y_n$, where y_n is the nth unit vector of l^1. We have $(l^1)^\odot = c_0$ and $l^1 = (l^1)^{\odot *}$, as is easily seen. In particular we have $R(\lambda, A) = R(\lambda, A^{\odot *}) = (R(\lambda, A^\odot))^*$, so $R(\lambda, A)$ is an adjoint operator and therefore it is continuous in the weak*-topology of l^1. It follows that $R(\lambda, A)B_{l^1}$ is weak*-compact, since B_{l^1} is. But this means that $R(\lambda, A)B_{l^1}$ is $\sigma(l^1, (l^1)^\odot)$-compact, since $c_0 = (l^1)^\odot$. By Corollary 2.5 it follows that $R(\lambda, A)B_{l^1}$ is weakly compact. Clearly B_{l^1} is not weakly compact, since l^1 is not reflexive.

Corollary 2.10 is related to the fact that $R(\lambda, A)$ is weakly compact if and only if $R(\lambda, A)^2$ is weakly compact [8]. More generally we see from Corollaries 2.5 and 2.10:

Corollary 2.11. *Suppose G is a bounded, weakly equicontinuous set. Then $R(\lambda, A)G$ is weakly compact if and only if G is weakly compact.*

The following theorem is an easy consequence of Corollary 2.9.

Theorem 2.12. *If $\|T_n - T\| \to 0$ in the uniform operator topology and each T_n is $\sigma(X, X^\odot)$-compact, then also T is $\sigma(X, X^\odot)$-compact.*

Proof: Let (x_k) be a bounded sequence, say $\|x_k\| \leq 1$ for all k. By Corollary 2.9 we must show that there is a subsequence (x_{k_i}) and a $y \in X$ such that $\langle x^\odot, Tx_{k_i} - y\rangle \to 0$ for all $x^\odot \in X^\odot$. Since each T_n is $\sigma(X, X^\odot)$-compact, by Corollary 2.9 a simple diagonal argument produces a subsequence (x_{k_i}) such that for each n there is a $y_n \in X$ such that for all $x^\odot \in X^\odot$,

$$\lim_{i\to\infty} \langle x^\odot, T_n x_{k_i} - y_n\rangle = 0.$$

We claim that the sequence (y_n) is norm-Cauchy. Indeed, since for all i and $x^\odot \in X^\odot$ we have

$$|\langle x^\odot, y_n - y_m\rangle| \le |\langle x^\odot, T_n x_{k_i} - T_m x_{k_i}\rangle| + |\langle x^\odot, T_n x_{k_i} - y_n\rangle| + |\langle x^\odot, T_m x_{k_i} - y_m\rangle|$$

it follows that for all $x^\odot \in X^\odot$,

$$|\langle x^\odot, y_n - y_m\rangle| \le \|x^\odot\| \|T_n - T_m\|.$$

But $\|T_n - T_m\| \to 0$ as $n, m \to \infty$. Since $X^\odot$ induces an equivalent norm the claim follows. Let y be the norm-limit of (y_n) and fix some $x^\odot \in X^\odot$. Then for all n and i we have

$$|\langle x^\odot, Tx_{k_i} - y\rangle| \le |\langle x^\odot, Tx_{k_i} - T_n x_{k_i}\rangle| + |\langle x^\odot, T_n x_{k_i} - y_n\rangle| + |\langle x^\odot, y_n - y\rangle|.$$

Let $\epsilon > 0$ be arbitrary. Then we may n_0 choose large enough such that for all i,

$$|\langle x^\odot, Tx_{k_i} - y\rangle| \le |\langle x^\odot, T_{n_0} x_{k_i} - y_{n_0}\rangle| + 2\epsilon.$$

Hence

$$\lim_{i\to\infty} |\langle x^\odot, Tx_{k_i} - y\rangle| \le 2\epsilon$$

and the theorem is proved. ////

In case each T_n^* leaves $X^\odot$ invariant the proof of the above result becomes much easier: in fact, if S is a bounded operator whose adjoint leaves $X^\odot$ invariant, then S is $\sigma(X, X^\odot)$-compact if and only if $S^{\odot *}$ maps $X^{\odot *}$ into X. Here $S^\odot$ denotes the restriction of S^* to $X^\odot$. From this Theorem 2.12 immediately follows.

The proof of Theorem 2.12 goes through for any subspace $Y \subset X^*$ that induces an equivalent norm in X and for which the Eberlein-Shmulyan theorem holds in the $\sigma(X, Y)$-topology.

We close with another consequence of the preceding results which is in a sense the 'Laplace transform' of Theorem 0.1. It states that X is $\odot$-reflexive if and only if the so-called integrated semigroup [1] is weakly compact.

Corollary 2.13. *Define $S(t)x = \int_0^t T(\tau)x\, d\tau$ $(t > 0)$. The following are equivalent:*

(1) X is $\odot$-reflexive with respect to $T(t)$;
(2) $S(t)$ is $\sigma(X, X^\odot)$-compact;
(3) $S(t)$ is weakly compact.

Proof: (1) $\Rightarrow$ (3): If X is $\odot$-reflexive with respect to $T(t)$ then $\overline{R(\lambda, A)B_X}$ is a weakly compact set and the formula

$$\frac{1}{t}\int_0^t T(\tau)x\, d\tau = R(\lambda, A)[\frac{\lambda}{t}\int_0^t T(\tau)x\, d\tau - \frac{1}{t}(T(t)x - x)]$$

shows that $\overline{S(t)B_X}$ is contained in some multiple of it. Since $\overline{S(t)B_X}$ is convex and closed it is weakly closed and therefore weakly compact.

(2) $\Leftrightarrow$ (3): It follows from the above observation that $\overline{S(t)B_X}$ is equicontinuous.

(3) $\Rightarrow$ (1): For fixed $\lambda > 0$ sufficiently large define the operators R_n by

$$R_n = \sum_{i=0}^{n^2} e^{-\lambda \frac{i}{n}} T(\frac{i}{n}) S(\frac{1}{n}).$$

Then R_n is weakly compact. Since we have

$$R_n x = \sum_{i=0}^{n^2} e^{-\lambda \frac{i}{n}} T(\frac{i}{n}) \int_0^{\frac{1}{n}} T(\tau)x \, d\tau = \sum_{i=0}^{n^2} \int_{\frac{i}{n}}^{\frac{i+1}{n}} e^{-\lambda \frac{i}{n}} T(\tau)x \, d\tau$$

it follows that $R_n \to R(\lambda, A)$ in the uniform operator topology. Since the weakly compact operators form a closed ideal in the space of bounded linear operators, $R(\lambda, A)$ is weakly compact. ////

3. The Favard class of a C_0-semigroup

Let $T(t)$ be a C_0-semigroup on a Banach space X and let A be its infinitesimal generator. The *Favard class* $Fav(T(t))$ of A is the set of elements $x \in X$ such that the orbit $t \mapsto T(t)x$ is locally Lipschitz continuous:

$$Fav(T(t)) = \{x \in X : \limsup_{t \downarrow 0} \frac{1}{t} \|T(t)x - x\| < \infty\}.$$

The estimate

$$\|T(t)x - x\| = \| \int_0^t T(t)Ax \, dt\| \leq t \cdot \sup_{0 \leq \tau \leq t} \|T(\tau)\| \cdot \|Ax\|, \quad (x \in D(A))$$

shows that $D(A) \subset Fav(T(t))$.

It is well-known that $Fav(T(t)) = D(A^{\odot *}) \cap X$, see [2]. In particular, if X is reflexive we have $Fav(T(t)) = D(A)$. In this section we will give a characterisation of those semigroups for which $Fav(T(t)) = D(A)$ holds.

Recall that $\| \cdot \|'$ denotes the equivalent norm on X induced by $X^{\odot}$, see the proof of Theorem 1.7. If $T(t)$ is a *contraction* semigroup, then $\| \cdot \|' = \| \cdot \|$. For the easy proof of this we refer to [4]. We need the following lemma.

Lemma 3.1. *We have the following inclusions:*

$$\overline{R(\lambda, A)B_{X,\|\cdot\|'}} \subset R(\lambda, A^{\odot *})B_{X^{\odot *}} \cap X \subset \bigcup_{n\in\mathbb{N}} n \cdot \overline{R(\lambda, A)B_{X,\|\cdot\|'}}.$$

Proof: By the Banach-Alaoglu theorem, $B_{X^{\odot *}}$ is weak*-compact. Therefore $R(\lambda, A^{\odot *})B_{X^{\odot *}}$ is also weak*-compact, so in particular norm-closed. The first inclusion now follows easily from the fact that the canonical embedding $j : X \to X^{\odot *}$ is an *isometry* from $(X, \|\cdot\|')$ into $X^{\odot *}$ and using the fact that $R(\lambda, A)x = R(\lambda, A^{\odot *})jx$ for all $x \in X$. The second inclusion follows from the equality

$$\frac{1}{t}\int_0^t T(\tau)x\ d\tau = R(\lambda, A)[\frac{\lambda}{t}\int_0^t T(\tau)x\ d\tau - \frac{1}{t}(T(t)x - x)].$$

Indeed, we have $R(\lambda, A^{\odot *})B_{X^{\odot *}} \cap X \subset D(A^{\odot *}) \cap X = Fav(T(t))$. Therefore if $x \in R(\lambda, A^{\odot *})B_{X^{\odot *}} \cap X$, then the righthand side $\frac{\lambda}{t}\int_0^t T(\tau)x\ d\tau - \frac{1}{t}(T(t)x - x)$ remains bounded as $t \downarrow 0$ whereas the lefthand side converges to x. ////

Theorem 3.2. *$Fav(T(t)) = D(A)$ if and only if $R(\lambda, A)B_{X,\|\cdot\|'}$ is norm-closed.*

Proof: Suppose $Fav(T(t)) = D(A)$. Let $y \in \overline{R(\lambda, A)B_{X,\|\cdot\|'}}$, say $y = R(\lambda, A^{\odot *})x^{\odot *}$ for some $x^{\odot *} \in B_{X^{\odot *}}$, using Lemma 3.1. Since $Fav(T(t)) = D(A^{\odot *}) \cap X$, $y \in Fav(T(t))$ and hence by assumption there is an $x \in X$ such that $y = R(\lambda, A)x$. But $R(\lambda, A)x = R(\lambda, A^{\odot *})jx$ and since $R(\lambda, A^{\odot *})$ is injective, we have $jx = x^{\odot *}$. But j is an isometry from $B_{X,\|\cdot\|'}$ into $B_{X^{\odot *}}$ which forces $x \in B_{X,\|\cdot\|'}$. Hence $y \in R(\lambda, A)B_{X,\|\cdot\|'}$ as was to be shown.

Conversely, if $R(\lambda, A)B_{X,\|\cdot\|'}$ is closed, then by Lemma 3.1 we have

$$R(\lambda, A^{\odot *})B_{X^{\odot *}} \cap X \subset \bigcup_{n\in\mathbb{N}} n \cdot R(\lambda, A)B_{X,\|\cdot\|'} = D(A).$$

Since $Fav(T(t)) = D(A^{\odot *}) \cap X$ it follows that $Fav(T(t)) \subset D(A)$, as was to be shown. ////

If $T(t)$ is a C_0-semigroup on a *reflexive* space X, then we have $X^{\odot} = X^*$, so $B_{X,\|\cdot\|'} = B_X$ and this set is weakly compact. Hence $R(\lambda, A)B_X$ is weakly compact as well; in particular $R(\lambda, A)B_X$ is closed. From Theorem 3.2 it now follows that $Fav(T(t)) = D(A)$.

Reflexivity is not needed in order to have $Fav(T(t)) = D(A)$. In fact, this equality is trivially true for any uniformly continuous semigroup. A more interesting example is the semigroup $T(t)$ on l^1 constructed after Corollary 2.10. There it was shown that $R(\lambda, A)B_{l^1}$ is weakly compact. This in particular implies that this set is norm-closed. On the other hand, since $T(t)$ is a contraction semigroup we have $B_{l^1} = B_{l^1,\|\cdot\|'}$. Hence $Fav(T(t)) = D(A)$ by Theorem 3.2.

Our next result describes the $\odot$-reflexive case.

Theorem 3.3. *Suppose X is $\odot$-reflexive with respect to $T(t)$. The following are equivalent:*

(a) $Fav(T(t)) = D(A)$;
(b) j maps X onto $X^{\odot *}$;
(c) $R(\lambda, A)B_{X,\|\cdot\|'}$ is weakly compact;
(d) $R(\lambda, A)B_{X,\|\cdot\|'}$ is $\sigma(X, X^{\odot})$-compact;
(e) $B_{X,\|\cdot\|'}$ is $\sigma(X, X^{\odot})$-compact.

Proof: (a)⇔(c): Since $\|\cdot\|'$ is an equivalent norm, this follows from Theorems 0.1 and 3.2.

(c)⇒(b): By assumption X is $\odot$-reflexive and $R(\lambda, A)B_{X,\|\cdot\|'}$ is closed. Hence from Theorem 3.2 and from the inclusions $D(A^{\odot *}) \subset X^{\odot\odot} = X$ we have $D(A^{\odot *}) = D(A^{\odot *}) \cap X = Fav(T(t)) = D(A) = D(A^{\odot\odot})$. Since $A^{\odot\odot}$ is the part of $A^{\odot *}$ in $X^{\odot\odot}$, it follows that $X^{\odot\odot} = X^{\odot *}$. Since X is $\odot$-reflexive with respect to $T(t)$, this is the desired result.

(b)⇒(e): $B_{X^{\odot *}}$ is weak*-compact. By assumption we may identify $B_{X,\|\cdot\|'}$ with $B_{X^{\odot *}}$ and (e) follows.

(e)⇔(d)⇔(c): Combine Corollaries 2.10, 2.5 and Proposition 2.8. ////

Remark 3.4. In the proof of Theorem 1.7 we saw that $\overline{B_X}^{\sigma(X,X^{\odot})} = B_{X,\|\cdot\|'}$. Therefore (a)-(e) remain equivalent if in (c), (d) and (e) one replaces $B_{X,\|\cdot\|'}$ by B_X, provided 'compact' is replaced by 'relatively compact'.

If $T(t)$ is a semigroup on a space X such that one of the equivalent conditions of Theorem 3.3 holds, then by (b) X must be isomorphic to a dual space. One can prove more, viz. that X must have the so-called Radon-Nikodym property (RNP), see [3] for the definition. This is an easy consequence of the facts that separable dual Banach spaces have the RNP and that a Banach space has the RNP if and only each of its *separable* closed subspaces has this property [3]. Indeed, suppose we have $X = X^{\odot *}$ and let Y be a separable closed subspace of X. We must show that Y has the RNP. By considering the closed linear span of the set

$$\{T(t)y : t \geq 0, y \in Y\}$$

we may assume that Y is $T(t)$-invariant. Then the restrictions $T_Y(t)$ of $T(t)$ to Y define a C_0-semigroup on Y, say with generator A_Y. Now by Theorem 3.3 we have $Fav(T(t)) = D(A)$ and $X = X^{\odot\odot}$. It is easily seen that these properties are inherited by Y, that is, $Fav(T_Y(t)) = D(A_Y)$ and $Y = Y^{\odot\odot}$ (for the latter use Theorem 0.1). Therefore by Theorem 3.3 we get $Y = Y^{\odot *}$, so the separable space Y is isomorphic to a dual space and consequently has the RNP.

Many common non-reflexive spaces, such as $L^1[0,1]$ and $C[0,1]$, fail to have the RNP. Thus, if such a space is $\odot$-refexive with respect to some C_0-semigroup, then one can a priori conclude that its Favard class is strictly larger than $D(A)$.

Until now we were concerned with one single semigroup. Our final result considers all possible semigroups on a given space. It gives a partial converse to the fact that $Fav(T(t)) = D(A)$ on reflexive spaces. For the terminology on bases we refer to [5].

Theorem 3.5. *Let X have a Schauder basis $\{x_n\}_{n=1}^{\infty}$. Then X is reflexive if and only if for every C_0-semigroup on X we have $Fav(T(t)) = D(A)$.*

Proof: Suppose X is nonreflexive. By Zippin's theorem [10] there is a non-boundedly complete basis $\{y_n\}_{n=1}^{\infty}$ for X. Define $T(t)y_n = e^{-nt}y_n$. By [7] these operators extend to a C_0-semigroup on X. We claim that the Favard class of this semigroup is strictly larger than $D(A)$. Let $\{y_n^*\}_{n=1}^{\infty}$ be the coordinate functionals of $\{y_n\}_{n=1}^{\infty}$; they form a non-shrinking basis for their closed linear span $[y_n^*]$. Let $\{y_n^{**}\}_{n=1}^{\infty}$ be the coordinate functionals of this basis. By [7] we have $X^{\odot} = [y_n^*]$ and $X = X^{\odot\odot} = [y_n^{**}]$, so X is $\odot$-reflexive with respect to $T(t)$. But since $\{y_n^*\}_{n=1}^{\infty}$ is non-shrinking, $[y_n^{**}] = X$ is strictly smaller than $[y_n^*]^* = X^{\odot *}$. Now apply Theorem 3.3. ////

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